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The Breakdown of Certain Types of Vortex

By

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SUMMARY

Under some conditions, the core of a streamwise vortex exhibits a severe disruption at a particular point along its length.

The phenomenon has previously been examined in detail for the type of vortex generated by flow separation from the leading edge of a delta wing and it is found almost invariably that the axis of the vortex deforms very abruptly from a straight line; just downstream of this there is a region of periodic flow where the vortex axis itself takes on a spiral form. A different, almost axisymmetric, form sometimes occurs intermittently. In both cases, the flow becomes turbulent further downstream.

An abrupt change can also be observed with swirling flow in a tube; this may have features similar to those of the spiral breakdown of a leading-edge vortex or, as observed by Harvey, may be steady and axisymmetric with the formation of a bubble of stagnant fluid at the axis. Certain points of similarity between the two forms are noted. Under certain transient conditions, the axisymmetric form has been observed to develop initially and then to change to the spiral form; this suggests that the spiral form should be regarded as arising from instability of the axisymmetric form.

The effect of an imposed pressure gradient on the core of a vortex is examined theoretically. The results, as well as indicating the possibility of the formation of an inner core of reversed flow, show the existence of

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*Replaces N.P.L. Aero Report 1166 - A.R.C. 27 200

critical conditions, depending on the ratio of the rotational and longitudinal velocity components, beyond which drastic changes of flow structure must occur. These theoretical results appear to be related to the observed phenomena.

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1. Introduction

During recent years, interest has been aroused in a phenomenon that has become known as vortex breakdown, or vortex bursting. Various specific aspects are dealt with in Refs. 2 to 8 whilst the recent review of Hall[1] on the structure of vortices puts these in perspective and provides a convenient starting point for anyone not already familiar with the subject.

Essentially, the phenomenon consists of an abrupt change of flow structure at some definite position along a streaming vortex. It has been observed to occur both with vortices shed from the highly swept leading edges of slender delta wings and for vortex cores occurring in swirling flow in a tube. With both kinds of vortex flow, different types of breakdown have been observed and two extreme forms can be recognized; these are clearly differentiated by the behaviour of the axial filament of the vortex. In the steady axisymmetric form, the axial filament expands to become a surface enclosing a stagnant bubble, and in the periodic spiral form, the axial filament remains as a filament but is deformed into a spiral. In both forms, the flow becomes turbulent further downstream.

Experience gained at N.P.L. from experiments mainly with low speeds of flow suggests that, with leading edge vortices, the spiral form almost invariably occurs with only rare and intermittent occurrences of the axisymmetric phenomenon. Maltby et alii[9] have found that the spiral form is present at

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much higher speeds of flow. For swirling flow in a tube, the evidence of experiments at N.P.L. suggests again that the axisymmetric form occurs only under exceptional circumstances. On the other hand, Harvey[4] in describing his observations in a vortex tube has referred only to the axisymmetric form.

Apart from its intrinsic interest, the existence of two forms has some bearing on the relative status of various theoretical predictions[3,5,6,8] some of which involve spiral disturbances whilst others do not. Lowson[10] and Jones[11] give evidence to the effect that, with leading-edge vortices, the phenomenon always shows itself initially in time as a spiralling of the axial filament, and that the axisymmetric form, which they generally find under steady conditions, is a later development of the primary spiral form. The present report, however, gives experimental evidence from swirling flow that leads to the contrary conclusion that, initially in time, the phenomenon is axisymmetric but that this form becomes unstable and changes to the spiral form. This might suggest that a theoretical approach should firstly seek to explain the axisymmetric form and then should look for conditions under which this degenerates into a spiral form.

2. Descriptions of Breakdown Observed under Steady Conditions

Two examples of breakdown are shown by Figs. 1 and 2. Fig. 1 shows a smoke filament along the axis of swirling air flow in a transparent tube of circular cross-section. The flow ahead of breakdown consists of an annulus of swirling irrotational flow surrounding a vortex core. The time exposure photograph, Fig. 1(a), shows an abrupt expansion of the region occupied by the smoke, whilst the short exposure photograph, Fig. 1(b), shows more clearly the existence of an almost spherical region at the head of a turbulent core.

Fig. 2 shows filament lines in the flow over the upper surface of a sharp-edge triangular-plate set at moderately high incidence (approx. 22°) in a water tunnel. Two vortices are formed above the plate by flow separation

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from the leading edges and dye has been introduced along both the axial filaments. At a certain point along each vortex axis there is an abrupt change from a straight axial filament.

The examples given in Figs. 1 and 2 show that the behaviour of smoke or dye introduced along the axis of a vortex is a powerful indicator of the existence of breakdown and allows a breakdown position to be defined.

The particular photograph of Fig. 2 is unusual in showing an example of each form of breakdown; an intermittent occurrence of an almost axisymmetric form for the lower vortex was present by chance at the time of the photograph. The upper vortex shows the more usual spiral type of breakdown. For both types, turbulence develops some distance downstream of breakdown and the resulting flows appear to be much the same. The breakdown phenomena of leading-edge vortices, including details of the spiral form, are described in Ref. 2, but, for the sake of completeness, a short description of the spiral mode follows.

As illustrated in Fig. 3, which shows a dye trace at a particular instant of time, the axial filament remains almost straight up to position S, downstream of which it remains intact but is deformed into a spiral which becomes irregular and breaks up after a few turns. It is important to note that this spiral is of the opposite hand to a spiral streamline of the primary vortex structure. If we consider the spiral dye trace simply as a filament line of the flow without regard to the individual behaviour of the particles that compose it, we find that it rotates as a whole about the central axis in the same sense as the primary rotation of the vortex and with a definite frequency; this gives the flow in the neighbourhood a spatial and temporal periodicity. But this statement could be misleading without some reference to particle paths - that is, without a description of the behaviour of an individual particle within the dye trace*. The observations and measurements of Ref. 2 show that such a

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* The relation between filament lines and particle paths in unsteady flow is discussed in Ref. 12.

particle, after moving along the straight portion of the filament upstream of the breakdown point, is suddenly decelerated on reaching S and is then deflected along a path, such as S P P', which hardly turns at all about the central axis. To make the matter clearer, the particular particle at P at the instant of time to which the diagram refers, will previously have travelled along S P, and, at a slightly later time, being convected along with the convolution of the dye filament of which it is part, will have moved to P'. Rotation of fluid about the deformed vortex axis continues to occur and is consistent with the reduced, or reversed, longitudinal velocity component within the spiral and with mutual interactions between the individual convolutions. These interactions lead to the eventual breakdown of the regular spiral and thus to the development of turbulence.

It is of interest to remark that Chanaud[13] refers to a similar spiralling arrangement in describing the fluid motion of the vortex whistle and the cyclone separator.

Both the spiral and the axisymmetric form have been observed in swirling flow along a tube, but, unlike the experience with leading-edge vortices, it is clear from the work of Harvey[4] that the axisymmetric form can be very steady and persistent. A comparison between the forms is now made with the aid of Fig. 4 which shows idealized diagrams representing the two types of flow. Each diagram shows the traces of flow tubes in a plane through the central axis of the system. Each of the outer flow tubes can be imagined as being formed by all the fluid particles that have passed through a stationary ring coaxial with the vortex ahead of breakdown; thus the walls of the tubes are built up from filament lines. The upper diagram, Fig. 4(a), which is based on Harvey's observations[4] refers to the axisymmetric form and represents steady motion. The axial filament A S expands and spreads over the surface of bubble B B which contains a stagnant circulatory flow resembling a vortex ring. A second bubble may appear downstream of the first,

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and certainly unsteadiness, spiralling and the development of turbulence will occur further downstream but these features are not included in the diagram. The lower diagram Fig. 4(b) refers to the spiral form, already described, and, since the flow is periodic, represents a cross-section at a particular instant of time. It shows cuts through the deformed spiral axis and an indication of unsteady reversed flow along the axis within the spiral. In spite of the widely different geometries, the two flows are similar in a number of respects:

1. Each has a stagnation, or quasi-stagnation, point S.
2. There is, in both, an expansion of the outer flow tubes.
3. Vorticity in the circumferential direction is present in the circulatory region in the axisymmetric form, and appears in the spiral form as a component of the vorticity along the deformed axis.
4. The paths of individual particles at the axes of the vortices are similar. Particle path A S B in Fig. 4(a) may be compared with particle path A S P in Fig. 3.

3. Observations of Transient Behaviour in Swirling Flow

Observations of swirling flow in a tube have thrown further light on the relation between the two forms of breakdown. The water vortex tube referred to in Ref. 2 was used in the experiments and is shown in Fig. 5. Swirl is imparted to the water by an axisymmetric arrangement of vanes in the radial entry at the top of the tube. The flow enters from a constant head device at the top and is controlled by a valve at the outlet from the lower tank. With this apparatus, vortex breakdown can be arranged to occur a short distance below the entry to the straight tube and, under conditions of steady discharge, the form of breakdown is usually a rather irregular spiralling. However, if by adjustments of the control valve, a transient change is made, first increasing the flow rate causing the already present breakdown to be swept downstream and sometimes out of the tube, and then decreasing the flow to its original value, a new breakdown appears incipiently in the neighbourhood of the original position and subsequently evolves to a steady-state condition. The initial process and

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the subsequent development have been studied with the aid of cinematography and it is found that, almost invariably, the nascent form is axisymmetric but that this rapidly changes to the spiral form. A typical time history of the development is described with reference to Figs. 6 and 7 which show the behaviour of a dye filament introduced along the axis of the tube. The photographs of Fig. 6 are chosen from 16 m.m. film and do not necessarily refer to the same transient occurrence; they have been chosen to correspond to the significant stages of development (a to f) shown in Fig. 7. The diagrams of Fig. 7 represent sections through the axis of the tube and show sections through the axial filament of dye and the surface that forms from it.

Fig. (a) On increasing the flow rate, the initial breakdown disappears and the dye forms a narrow steady filament along the centre of the tube. Then, on decreasing the flow rate, a thickening appears in the filament as shown in the diagram.

Fig. (b) The thickening rapidly develops into a surface of dye having the form of a wine-glass or tulip. At this stage, the dye in the central filament to the right of S is quite stationary, but to the left of S dye still flows along the axial filament and spreads over the surface of the tulip shape. This behaviour is consistent with the presence of a core of fluid that has no longitudinal velocity component, and which is indicated in the diagram by the broken lines; it would seem likely that this body of fluid became stationary at the same time as the initial thickening of the dye filament was first observed.

Fig. (c) By the next stage, the dye shape has become re-entrant. The behaviour of the remnant of dye filament to the right of S shows that fluid is now moving upstream within the tulip, and this combined with the movement of dye at the surface shows the presence of a vortex ring system which has the basic axisymmetric arrangement of Fig. 4(a).

Fig. (d) The onset of asymmetry marks the next stage. The circulatory, or vortex ring, flow within the tulip appears to cant to one side and to rotate and this leads to a kind of emptying process in which the trapped fluid is shed downstream along a spiral path. Observations of some transients suggest that spiralling first appears well downstream of point S and that the spiral character is propagated upstream until it reaches the vortex ring system. It is possible that this upstream propagation leads to the breakdown of the axisymmetric flow.

Fig. (e) The subsequent development was most difficult to follow but it appears that, with the fluid which was previously trapped now moving downstream, the dye forming the surface of the tulip becomes attenuated; the dye which has flowed along the axis now remains as a discrete filament which is convected downstream to give the final spiral form, Fig. (f), already described.

In short, although the details of the change-over remain obscure, the breakdown phenomenon appears initially in an axisymmetric form as a core of stationary, and later, circulatory, flow which evolves to the final spiral arrangement. The observations thus suggest that the spiral form observable under steady conditions should be considered as a derivative of the axisymmetric form. Thus, in seeking a theoretical explanation of the phenomenon, it may be necessary, firstly to provide an explanation of the axisymmetric form and then to consider the stability of such a flow with the possibility of its degeneration to another form. An attempt at the first stage is contained in the following section.

4. The Effect of an Adverse Pressure Gradient on a Vortex Core

It has already been suggested in Ref. 2 that the breakdown of a leading-edge vortex is associated with the adverse longitudinal pressure gradient which occurs in the flow field above a delta wing. It is therefore of interest to examine theoretically the behaviour of a simple vortex in an adverse pressure

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gradient or, what amounts to the same but is more convenient, to consider the effect of an imposed retardation in the streamwise flow.

The present section is the result of an attempt to predict the conditions under which stagnation of flow within the core would occur, stagnation being regarded as an essential of the observed breakdown phenomenon. However, the results of the analysis show another feature, an abrupt change of solution, which may correspond more closely to the observations than the stagnation condition originally sought.

We consider a streamwise vortex system consisting of a rotational core embedded in an external irrotational flow. The problem is as follows: Suppose a change occurs along the length of the vortex such that the longitudinal velocity component in the external flow is altered, what happens to the flow within the core? In particular, what happens when the longitudinal component is reduced? Some attempt was made to consider this problem in Ref. 2 and the present analysis can be regarded as an extension of that work.

The model vortex system is similar to that used previously and is shown in Fig. 8. At an upstream section S_1 , the flow is in radial equilibrium and has a uniform longitudinal velocity component U_1 both in the external flow and within the core; the fluid in the core is in "solid body" rotation, the rotational velocity component at the edge of the core being V. The system at S_1 is then completely defined by the values of U_1 , V and R_1 the radius of the core. A change is imposed downstream of S_1 so that at some other section S_2 the external flow has a uniform longitudinal velocity component U_2 . Such a situation could exist in practice if the flow is confined within a tube whose cross section changes along its length. It seems realistic to regard U_2 as an independent and disposable parameter representing the imposed change, provided the cross section of the tube is large in comparison with the vortex core. It is assumed that radial equilibrium has again been achieved before the flow

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reaches S_2 and that the distance between S_1 and S_2 is sufficiently short for the flow to be regarded as inviscid. By the application of Bernoulli's equation to streampaths within the core between S_1 and S_2, and the use of the equations for the conservation of mass flow and angular momentum, it is found, as shown in the Appendix, that the radial distribution of the longitudinal velocity component, u_2 within the core at S_2 is given by

$$\frac{d^2u}{d\eta^2} + \frac{1}{\eta}\frac{du}{d\eta} + k^2(u - \beta) = 0 \qquad \ldots(1)$$

where

$$u = u_2/U_2$$

$$\eta = r_2/R_2$$

$$\beta = U_1/U_2$$

$$k = 2(V/U_1)(R_2/R_1)$$

r_2 being the radial co-ordinate and R_2 the radius of the core at S_2.

The problem considered so far and the governing differential equation, above, are closely related to the discussions of Weske[14] and Burgers[15], but the treatment to be followed here is different from theirs. Also, it is of interest to note at this stage a connection with the theory of Brooke Benjamin[3]: when $\beta = 1$ (i.e., no imposed change) the equation becomes identical to that obtained by differentiation of equation (5.17) of Ref. 3.

As shown in the Appendix, the solution of equation (1), in terms of Bessel functions, is

$$u = \frac{1 - \beta}{J_o(k)} J_o(k\eta) + \beta. \qquad \ldots(2)$$

Also, it is deduced that the rotational velocity component v_2 within the core at S_2 is given by

$$v_2/V = \frac{k}{(V/U_1)}\left[\frac{(1-\beta)}{\beta k}\frac{J_1(k\eta)}{J_o(k)} + \frac{\eta}{2}\right]. \qquad \ldots(3)$$

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On the condition that the net mass flow within the core is the same at the two sections — a condition which allows the possibility of an inner core of reversed flow at S_2, a relation is found between R_1 and R_2 which leads to the following expression for k in terms of V/U_1, the initial specification of the vortex* and β, the imposed change of longitudinal velocity.

$$k\left[\frac{1}{4}-\frac{(\beta-1)}{2\beta k}\frac{J_1(k)}{J_0(k)}\right]^{\frac{1}{2}} = \frac{V}{U_1}. \qquad \ldots(4)$$

The condition for incipient stagnation at the axis is $u = 0$ when $\eta = 0$, which gives from equation (2)

$$\beta J_0(k) = \beta - 1$$

and thus, from equation (4), the critical relation between V/U_1 and k for incipient stagnation is,

$$V/U_1 = k\left[\frac{1}{4}-\frac{1}{2k}J_1(k)\right]^{\frac{1}{2}}. \qquad \ldots(5)$$

Certain aspects of the relations between k, V/U_1 and β provided by equation (4) are examined graphically in Fig. 9 which also illustrates equation (5). If, for the moment, we ignore that part of the diagram corresponding to $k > 2{\cdot}4$ approx., the first zero of J_0, we see that, for values of $\beta > 1$, two values of k are obtained for any value of V/U_1 less than that corresponding to the maximum of the curve. Those solutions represented by points lying to the right of the chain dotted curve corresponding to equation (5) have a central core of reversed flow. However, it will be noted that for values of V/U_1 greater than the critical, defined by the maximum of the curve for a given value of β, no solution can be obtained with $k < 2{\cdot}4$ approx. A value of $k \simeq 5{\cdot}5$ can be obtained from the higher branch of the curve and, because of the form of J_0 and J_1, solutions for even higher values of k exist but the resulting flows include an annular region of back flow and are regarded as physically unrealistic. For example, for $\beta = 1{\cdot}2$, the critical/

*It may be observed that the swirl ratio V/U_1 defines the helix angle of a streamline at the edge of the core.

critical value of (V/U_1) is seen to be 0·76 approx. This seems to mean that the present theory is unable to provide a realistic solution when a change represented by $\beta = 1{\cdot}2$ is imposed on a vortex for which $(V/U_1) > 0{\cdot}76$. Calculations for a few other conditions have yielded the critical relation between (V/U_1) and β shown in Fig. 10.

As shown in Fig. 9, the relation between V/U_1 and k for $\beta = 1{\cdot}0$ is represented by two straight lines which intercept at a point corresponding to $V/U_1 = 1{\cdot}2$ approx.; that given by $V/U_1 = k/2$ corresponds to the structure of the core remaining unaltered. No curve for $\beta > 1{\cdot}0$ intercepts the horizontal line corresponding to $V/U_1 = 1{\cdot}2$ approx., so that for this vortex no solution can be obtained however small the imposed retardation. This suggests that this vortex core (one for which $V/U_1 = 1{\cdot}2$) is inherently unstable, and that some change, as yet unspecified, must occur spontaneously. It is relevant to remark that Squire[5] and Brooke Benjamin[3] using different approaches reach the conclusion that, for the same vortex model, $V/U_1 = 1{\cdot}2$ approx. is the critical value for breakdown.

For the particular value $\beta = 1{\cdot}2$, the radial distributions of longitudinal and rotational velocity components have been calculated from equations (2) and (3) for several values of V/U_1. The results are shown in Fig. 11 where the component velocities, u_2 and v_2, have been normalized by division by U_1 and V respectively, and the radial co-ordinate r_2 normalized by division by R_1. It is of interest to note from the diagram that for the critical condition the longitudinal velocity component is not close to reversal and this is confirmed in Fig. 9 by the reversal line falling to the right of the maximum of the curve for $\beta = 1{\cdot}2$. It can be seen from Fig. 11 that the solution for $V/U_1 = 0{\cdot}68$, which corresponds to the higher value of k (i.e., to the greater expansion of the core), shows a reversal of the u_2 component in the neighbourhood of the axis. It can be shown from mass flow considerations/

considerations that the axial streamline at section S_1 becomes a stream cylinder at S_2 having a radius $r_2/R_1 \simeq 0{\cdot}64$ for which it can be seen, as would be expected, that the rotational component v_2 vanishes. A flow satisfying the end conditions at S_1 and S_2 is sketched in Fig. 12 with the intermediate flow suggested. Viscosity, neglected in the theory, would be expected to modify the flow in practice.

In summary, it is noted that for any particular pair of values of V/U_1 and β, a number of values of k can be found to satisfy equation (4). In other words, for a specified initial vortex flow and for a given imposed change, there are a number of resulting flows each of which theoretically satisfies the specified conditions, although some, or all, of these may be physically unrealistic. The relation between these flow solutions and the original flow appears to be similar to the relation between the conjugate flows discussed by Brooke Benjamin[3]. For the particular condition of imposed retardation examined numerically, there is a critical relation between V/U_1 and β for which physically realistic flow solutions can just occur. Below the critical boundary, there are two possible realistic solutions, one of which may involve a central region of reversed flow. There may be factors not considered in the analysis that lead, in practice, to a preference for the reversed flow, rather than the forward flow solution, so that it is this flow which corresponds to the observed breakdown. On the other hand, and this seems more likely, the absence of a realistic solution when the critical condition is exceeded may correspond to the observed phenomenon. The interpretation of the absence of a realistic theoretical solution is not at all clear. It may indicate an inadequacy of the chosen model vortex system in some particular, perhaps in the assumption that radial equilibrium can be attained after the change has been imposed. Or, the absence of a solution may mean that the flow cannot remain axisymmetric, although this possibility would not be consistent with the idea of breakdown being basically an axisymmetric phenomenon.

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In spite of these obscurities, the present analysis does suggest that some kind of radical change of flow structure must occur in practice, and, the fact that no peculiarities are obvious in the velocity distributions at the critical condition, indicates an abruptness in the predicted change which is in accord with experimental observations.

5. Concluding Remarks

Two basic types of vortex breakdown have been observed — the steady axisymmetric and the spiral form. Observations of incipient breakdown in swirling flow in a tube suggest that the occurrence of a spiral, and not an axisymmetric form is due either to an instability of the latter which develops under certain circumstances, or, as for leading-edge vortices, to an inherent asymmetry in the flow structure. It thus appears that the spiral form is to be regarded as a derivative of the axisymmetric form.

Theoretical calculations on the effect of imposing an adverse pressure gradient, or longitudinal retardation, on an inviscid model vortex show that it is possible for such a change to lead to an inner core of reversed flow. Perhaps of greater interest is the theoretical conclusion that an imposed change beyond a critical value can lead to a drastic but, as yet, unspecified change of flow structure.

6. Acknowledgements

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References/

References

No.	Author(s)	Title, etc.
1	M. G. Hall	The structure of concentrated vortex cores. To be published in the series "Progress in Aeronautical Sciences". Vol.7 Pergamon Press. 1966.
2	N. C. Lambourne and D. W. Bryer	The bursting of leading-edge vortices - Some observations and discussion of the phenomenon. A.R.C. R.& M.3282. April, 1961.
3	T. Brooke Benjamin	Theory of the vortex breakdown phenomenon. J. Fluid Mech. 14, 593-629. (1962).
4	J. K. Harvey	Some observations of the vortex breakdown phenomenon. J. Fluid Mech. 14, 585-592. (1962).
5	H. B. Squire	Analysis of the vortex breakdown phenomenon, Part I. Imperial College of Science and Technology Aeronautics Dept. Report No.102 (1960).
6	J. P. Jones	The breakdown of vortices in separated flow U.S.A.A. Report No.140. A.R.C.22 241 F.M.3003. July, 1960.
7	D. Hummel	Investigations on vortex breakdown on slender delta wings. (in German) Z. Flugwiss 13. 5. 158, May 1965. Also as Bericht 64/24 of Institut für Strömungsmechanik, T. H. Braunschweig (1964).
8	H. Ludwieg	Explanation of vortex breakdown by the stability theory for spiralling flows (in German) AVA Bericht 64 A 14 (1964).

No.	Author(s)	Title, etc.
9	R. L. Maltby P. B. Engler and R. F. A. Keating	Some exploratory measurements of leading-edge vortex positions on a delta wing oscillating in heave. A.R.C. R.& M.3410 July, 1963.
10	M. V. Lowson	Some experiments with vortex breakdown. J. Roy. Aero. Soc. Vol. 68, pp.343-346. May, 1964.
11	J. P. Jones	On the explanation of vortex breakdown. Lecture at Symposium on Concentrated Vortex Motions. Ann Arbor (1964).
12	J. T. Stuart R. C. Pankhurst and D. W. Bryer	Particle paths, filament lines and streamlines. N.P.L. Aero Report 1057 (1963).
13	R. C. Chanaud	Observations of oscillatory motion in certain swirling flows. J. Fluid Mech. Vol. 21 Part 1, pp.111-127 January, 1965.
14	J. R. Weske	The effect of stretching of a vortex core. University of Maryland, Tech. Note BN 57 (1955).
15	J. M. Burgers	The effect of stretching of a vortex core. University of Maryland, Tech. Note BN 80 (1956).

APPENDIX/

APPENDIX

The Effect of Velocity Changes in the External Flow on the Velocity Distribution within a Vortex Core

The vortex system is shown in Fig. 8. S_1 and S_2 are two stations along an axisymmetric vortex whose axis is OX, and AB is the edge of the core outside of which the flow is irrotational. The flow between S_1 and S_2 is assumed incompressible and inviscid. The latter assumption implies that the edge of the core, AB, represents a stream surface.

The flow conditions at S_1 are prescribed as follows:

(a) the longitudinal velocity component U_1 is uniform across the core and in the external flow

(b) the core has solid body rotation and the velocity is continuous at the edge; that is the rotational component at radius r_1 is

$$\left.\begin{aligned} v_1 &= V\frac{r_1}{R_1} \qquad r_1 < R_1 \\ v_1 &= V\frac{R_1}{r_1} \qquad r_1 > R_1 \end{aligned}\right\} \qquad \ldots(A.1)$$

where V is the rotational component at the edge of the core and R_1 the radius of the core.

We postulate that a change occurs downstream of S_1, but that radial equilibrium has been restored ahead of S_2. In particular, we impose a change in the longitudinal velocity component such that at S_2 the external flow has a uniform longitudinal velocity component U_2.

We consider an axisymmetric stream surface CD within the core which has radii r_1 and r_2 at S_1 and S_2 respectively; u_2 and v_2 are the longitudinal and rotational velocity components at the radial distance r_2.

Expressions/

Expressions for u_2 and v_2 in terms of V, U_1, and U_2 are derived by application of

(i) Bernoulli's equation

(ii) Condition of radial equilibrium

(iii) Conservation of angular momentum

(iv) Mass flow continuity.

<u>Bernoulli's equation</u>

Equating the total pressures at S_1 and S_2 for a streamline on the stream surface CD, we have

$$p_2 - p_1 = \tfrac{1}{2}\rho(U_1^2 + v_1^2 - u_2^2 - v_2^2) \qquad \ldots(A.2)$$

where p_1 and p_2 are the static pressures at S_1 and S_2 respectively.

<u>Condition of radial equilibrium</u>

Applying the condition at S_1 and S_2 we have

$$\left.\begin{aligned} \frac{dp_1}{dr_1} &= \frac{\rho v_1^2}{r_1} \\ \frac{dp_2}{dr_2} &= \frac{\rho v_2^2}{r_2} \end{aligned}\right\} \qquad \ldots(A.3)$$

<u>Conservation of angular momentum</u>

Equating angular momentum at S_1 and S_2 gives

$$v_2 r_2 = v_1 r_1. \qquad \ldots(A.4)$$

<u>Continuity of mass flow</u>

The mass flow across S_1 in the annulus between stream surfaces at r_1 and $r_1 + dr_1$ must equal the mass flow across S_2 in the annulus between surfaces at r_2 and $r_2 + dr_2$, that is

$$\frac{dr_1}{dr_2} = \frac{u_2 r_2}{U_1 r_1}. \qquad \ldots(A.5)$$

<u>Derivation of equations for u_2 and v_2</u>

Differentiating equation (A.2) with respect to r_2 and using equation (A.3) to eliminate p_1 and p_2, we have

$\frac{v_2^2}{r_2}$ /

$$\frac{v_2^2}{r_2} - \frac{v_1^2}{r_1}\frac{dr_1}{dr_2} = v_1 \frac{dv_1}{dr_1}\frac{dr_1}{dr_2} - u_2 \frac{du_2}{dr_2} - v_2 \frac{dv_2}{dr_2}. \qquad \ldots(A.6)$$

From equations (A.1) and (A.4) we obtain

$$\left.\begin{aligned} v_1 &= V\frac{r_1}{R_1} \\ v_2 &= v_1\frac{r_1}{r_2} = V\frac{r_1^2}{R_1 r_2} \\ \frac{dv_1}{dr_1} &= \frac{V}{R_1} \\ \frac{dv_2}{dr_2} &= \frac{V}{R_1}\left(\frac{2r_1}{r_2}\frac{dr_1}{dr_2} - \frac{r_1^2}{r_2^2}\right) \end{aligned}\right\} . \qquad \ldots(A.7)$$

Using the above expressions in equation (A.6) and substituting for $\frac{dr_1}{dr_2}$ from equation (A.5) we have, after division by $u_2 (\neq 0)$,

$$r_2 \frac{du_2}{dr_2} + 2\frac{V^2}{R_1^2 U_1}(r_1^2 - r_2^2) = 0. \qquad \ldots(A.8)$$

Differentiating with respect to r_2 and using equation (A.5) again, we obtain

$$\frac{d^2u_2}{dr_2^2} + \frac{1}{r_2}\frac{du_2}{dr_2} + 4\left(\frac{V}{U_1}\right)^2 \frac{1}{R_1^2}(u_2 - U_1) = 0 \qquad \ldots(A.9)$$

which, by writing

$$\eta = r_2/R_2$$

$$u = u_2/U_2$$

$$k = 2\frac{V}{U_1} . \frac{R_2}{R_1}$$

$$\beta = U_1/U_2$$

becomes the zero-order Bessel equation

$$\frac{d^2u}{d\eta^2} + \frac{1}{\eta}\frac{du}{d\eta} + k^2(u - \beta) = 0. \qquad \ldots(A.10)$$

Since solutions in which $u \to \infty$ when $\eta \to 0$ are not admitted, the general solution of equation (A.10) is

$$u = AJ_0(k\eta) + \beta. \qquad \ldots(A.11)$$

The condition that the flow at the edge of the core is compatible with the adjacent external flow is $u = 1$ when $\eta = 1$, which, on insertion in equation (A.11), gives

$$u = \frac{1-\beta}{J_0(k)} J_0(k\eta) + \beta. \qquad \text{...(A.12)}$$

This equation gives the required distribution of longitudinal velocity component provided $\beta (\equiv U_1/U_2)$ and $k(\equiv 2(V/U_1)(R_2/R_1))$ are specified. An expression for k is now obtained since the core radii R_1 and R_2 are related by a consideration of mass flow continuity. The mass flow within the core at S_1 must equal the <u>net</u> mass flow across S_2 for the region $r_2 < R_2$, that is*

$$R_1^2 U_1 = 2R_2^2 U_2 \int_0^1 u\eta d\eta$$

which on substituting for u from equation (A.12) yields

$$\left(\frac{R_1}{R_2}\right)^2 = \frac{2(1-\beta)}{\beta J_0(k)} \int_0^1 \eta J_0(k\eta) d\eta + 1. \qquad \text{...(A.13)}$$

Since $$\int_0^1 \eta J_0(k\eta) d\eta = \frac{1}{k} J_1(k),$$

an expression for k can be written as follows:-

$$k \left[\frac{1}{4} - \frac{\beta - 1}{2\beta k} \frac{J_1(k)}{J_0(k)}\right]^{\frac{1}{2}} = \frac{V}{U_1}. \qquad \text{...(A.14)}$$

This last equation allows k to be determined from V/U_1 the initial specification of the vortex, and the imposed velocity change β.

The rotational velocity component v_2 is determined from the equation

v_2/

---

*This condition is not restricted to cases when <u>all</u> the fluid at S_2 has passed through S_1, but allows for the possibility of reversed flow at S_2 provided we allow a fictitious but compatible flow within an inner core.

$$v_2 = V\frac{r_1^2}{R_1 r_2} = \frac{V}{\eta}\cdot\frac{r_1^2}{R_2^2}\cdot\frac{R_2}{R_1} \qquad \ldots(A.15)$$

after an expression for $(r_1/R_2)^2$ is found. Constancy of mass flow within the axisymmetric stream surface represented by CD in Fig. 8 gives

$$U_1 r_1^2 = 2U_2 R_2^2 \int_0^{\eta} u\eta d\eta$$

which, on substitution for u from equation (A.12) and after integration, yields

$$\left(\frac{r_1}{R_2}\right)^2 = \frac{2U_2}{U_1}\left[\frac{(1-\beta)}{J_0(k)}\frac{\eta}{k}J_1(k\eta) + \frac{\beta\eta^2}{2}\right].$$

Using this equation with equation (A.15) and writing

$$\frac{R_2}{R_1} = \frac{k}{2(V/U_1)}$$

we have

$$\frac{v_2}{V} = \frac{k}{(V/U_1)}\left[\frac{(1-\beta)}{\beta k}\frac{J_1(k\eta)}{J_0(k)} + \frac{\eta}{2}\right]. \qquad \ldots(A.16)$$

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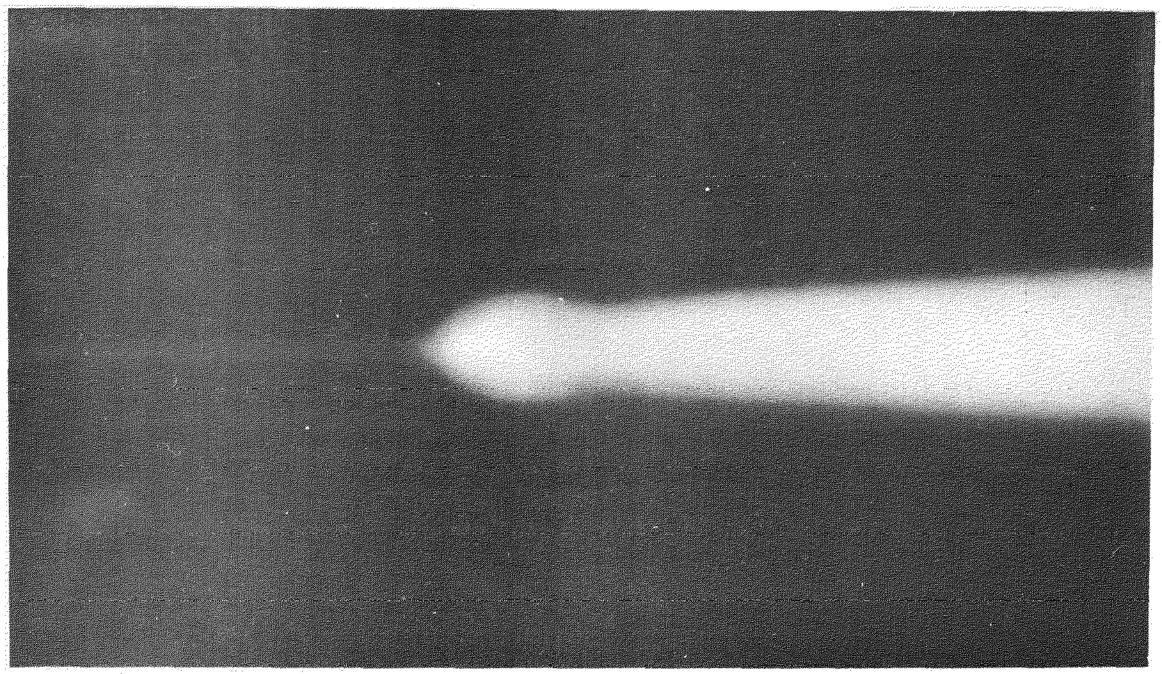
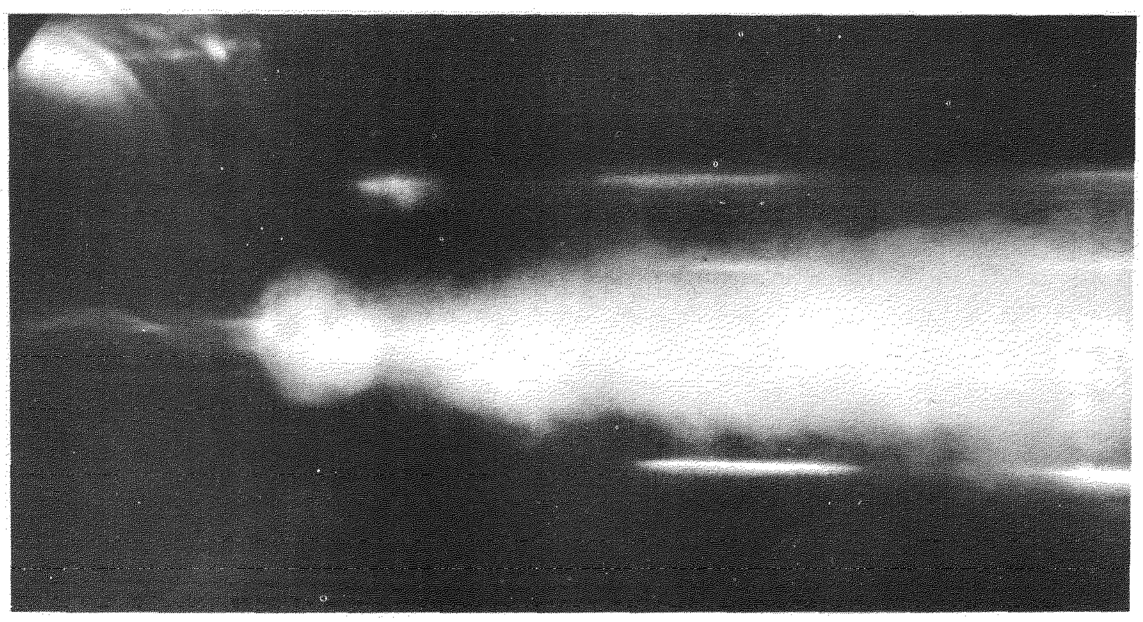
FIG. 1 (a & b)



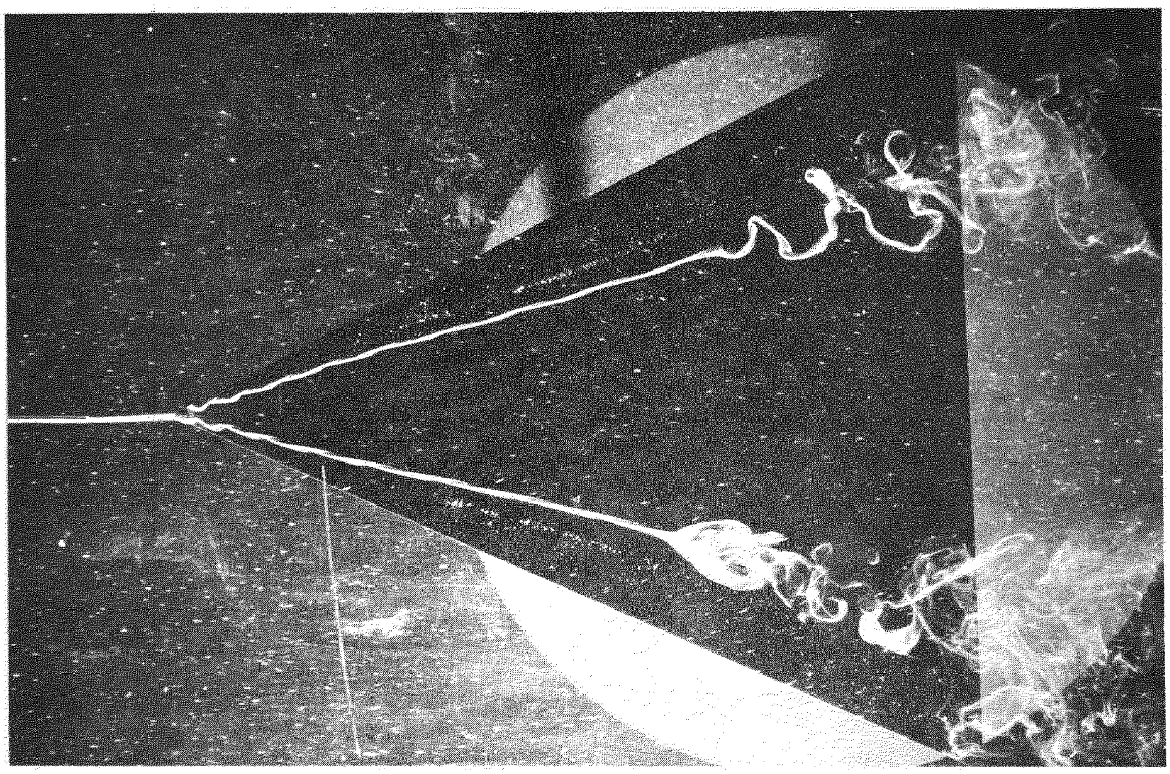
(a) Time exposure photograph



(b) Short exposure photograph

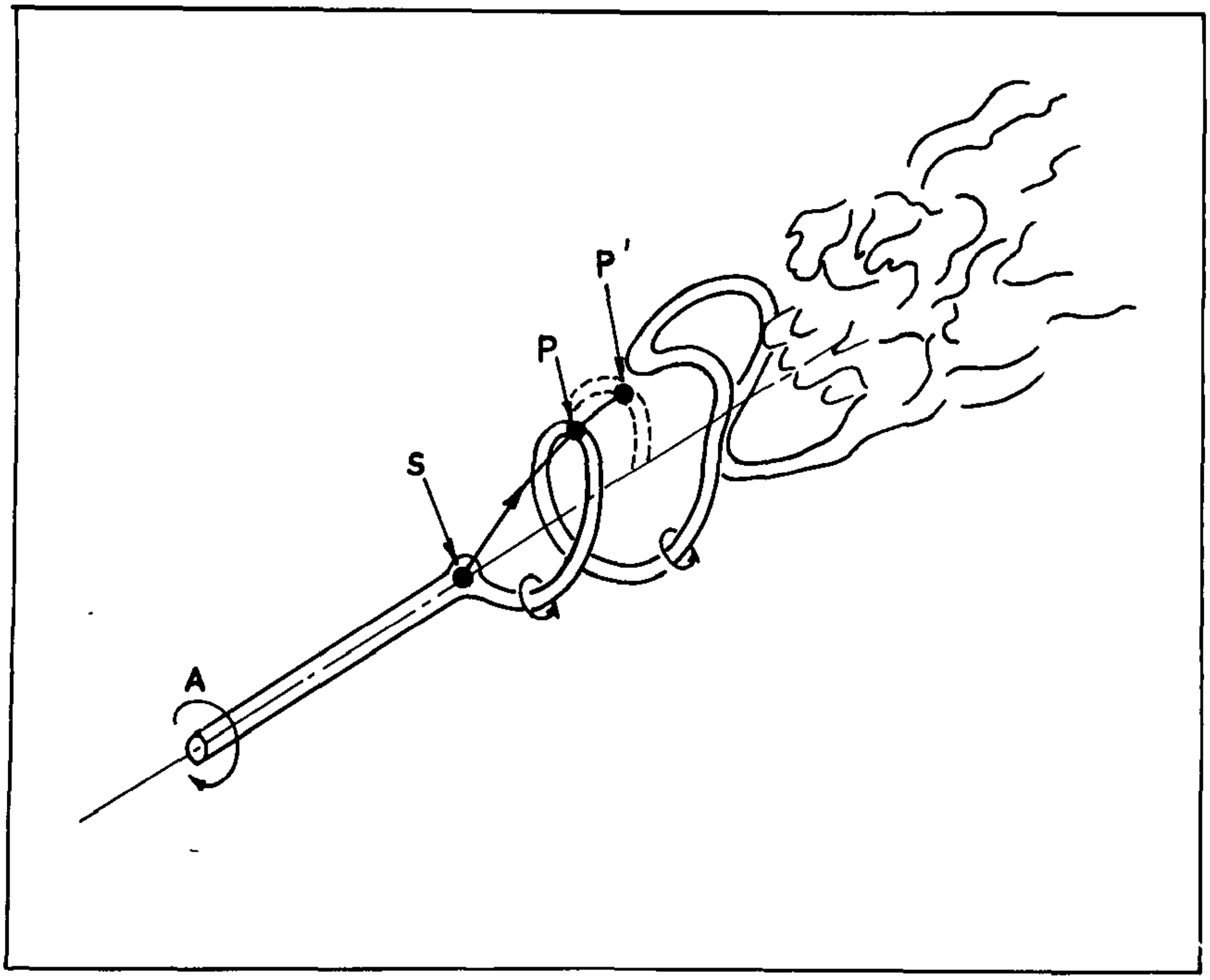
Breakdown in swirling flow along a tube

FIG. 2



Axial filament lines for a pair of leading edge vortices showing two types of breakdown. (from Ref.2)

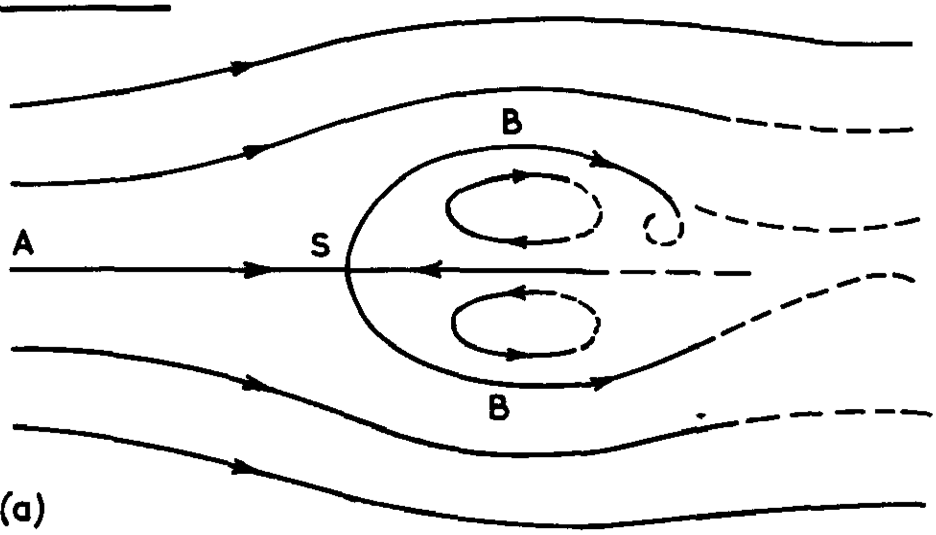
FIG. 3



Behaviour of axial filament in the spiral form of breakdown (after Ref. 2)

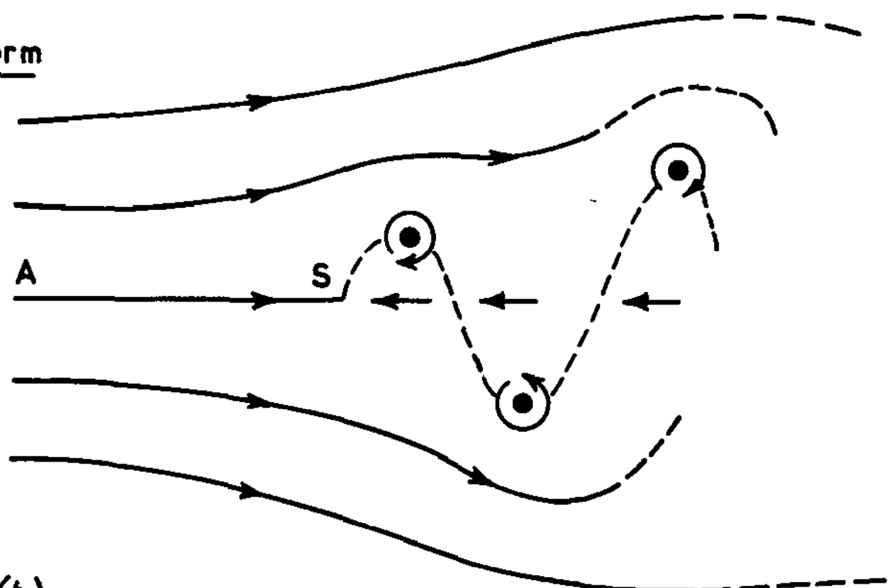
FIG. 4

Axisymmetric form



(a)

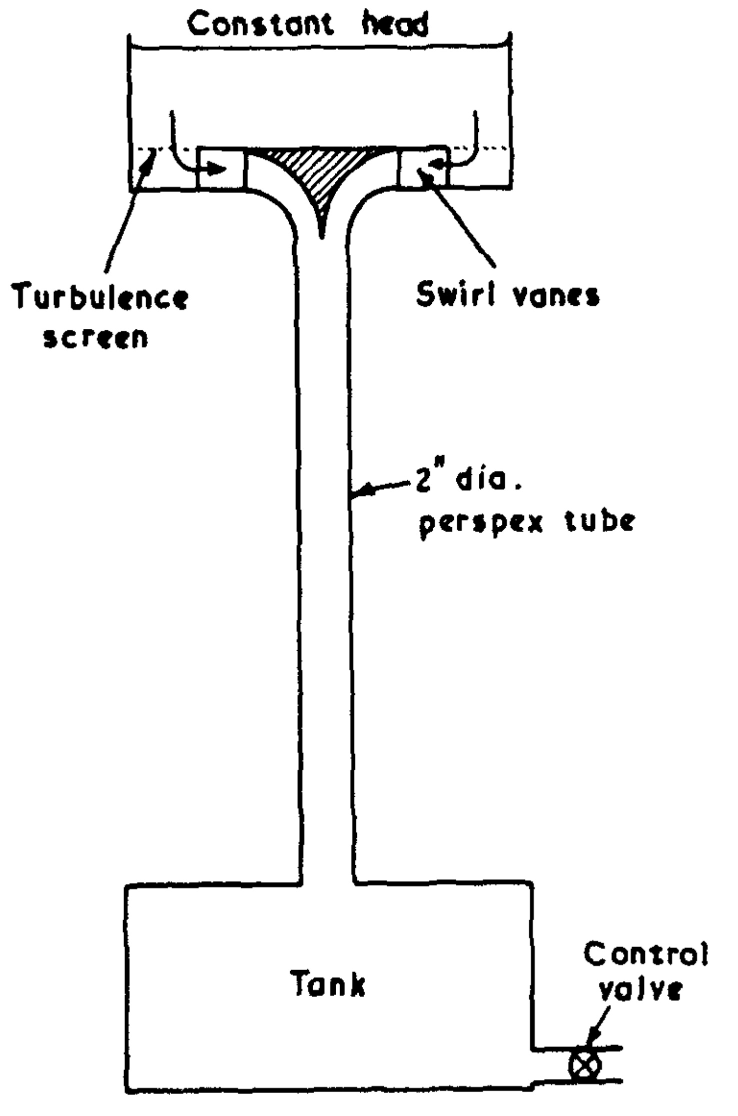
Spiral form



(b)

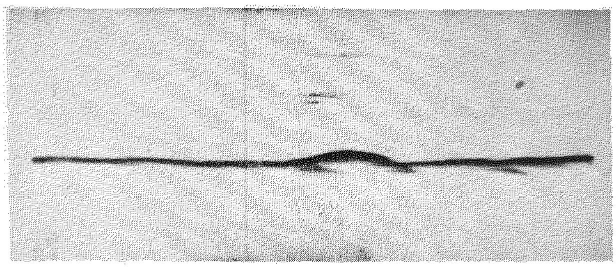
The two forms of breakdown

FIG. 5

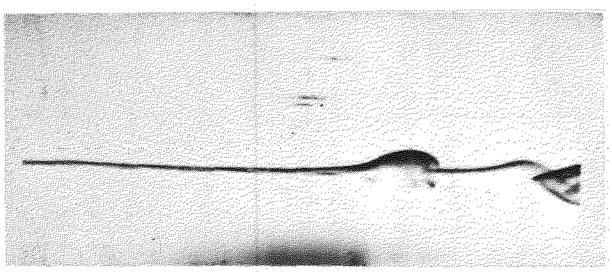


Schematic diagram of water vortex tube

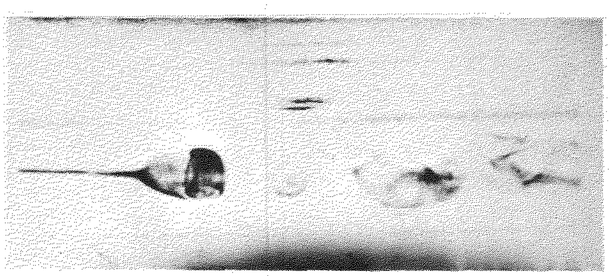
FIG. 6



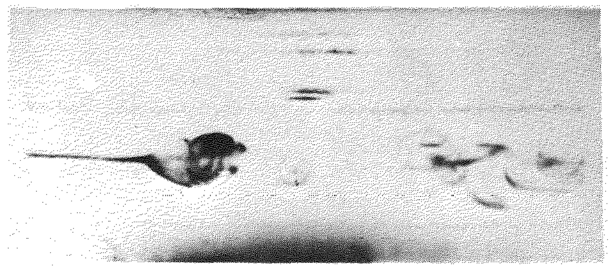
(a)



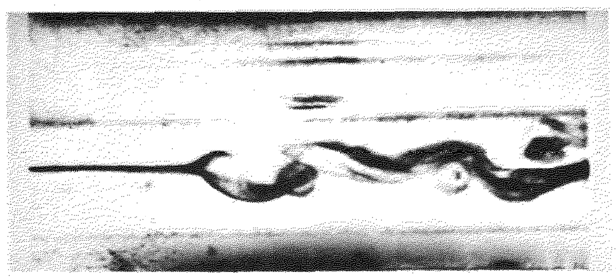
(b)



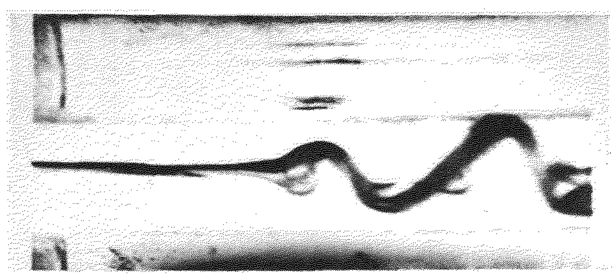
(c)



(d)



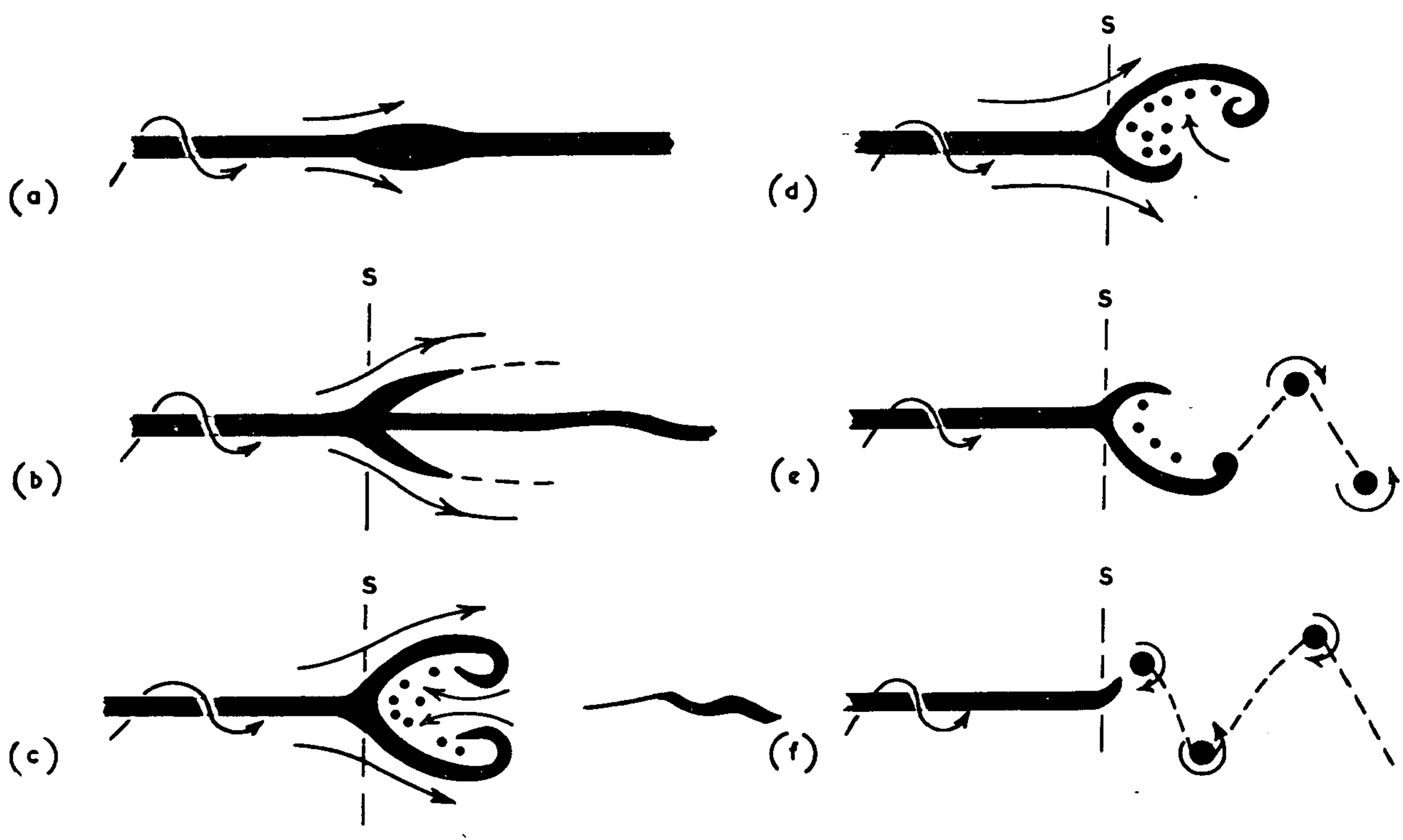
(e)



(f)

Film frames chosen from various transients to illustrate, in conjunction with Fig. 7 the development of breakdown

FIG. 7



Time history of transient development of a breakdown

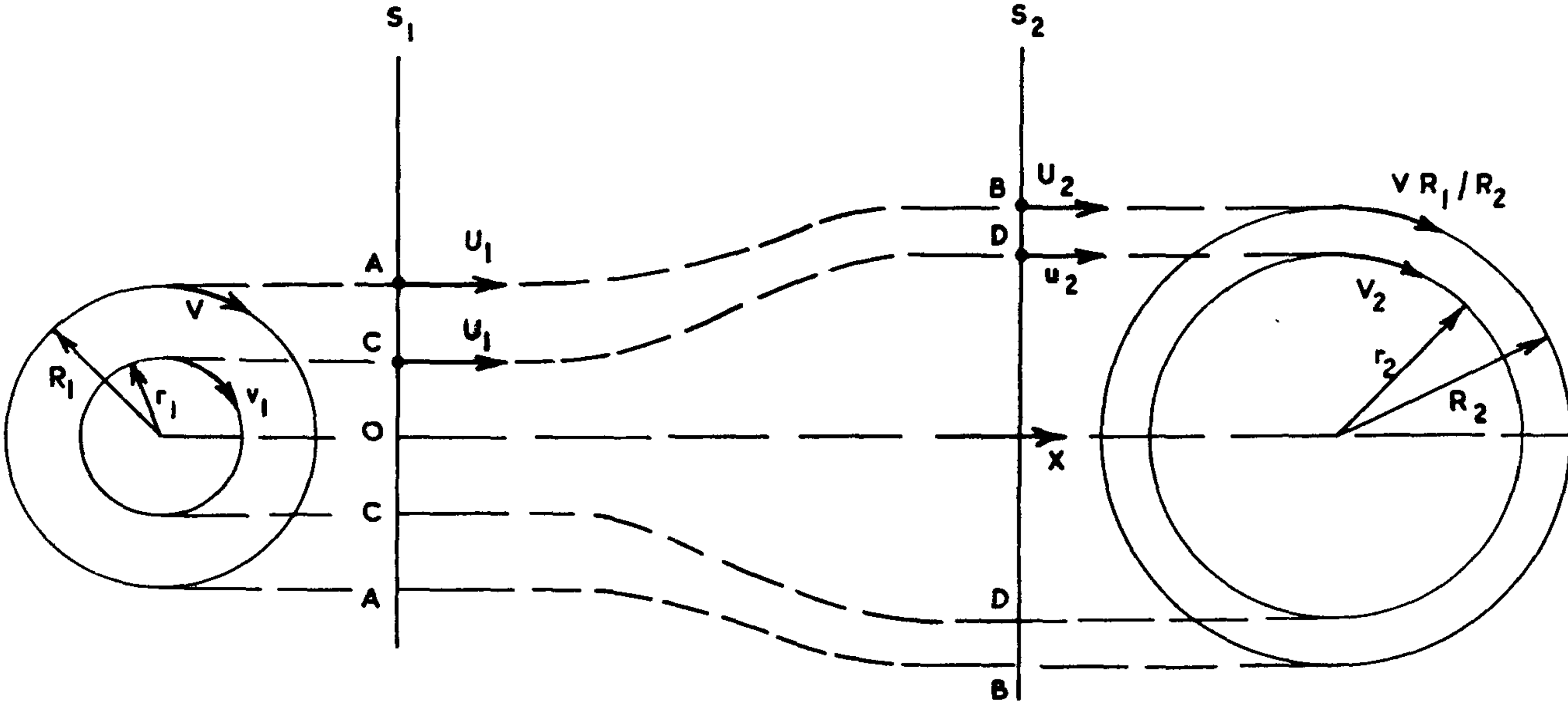
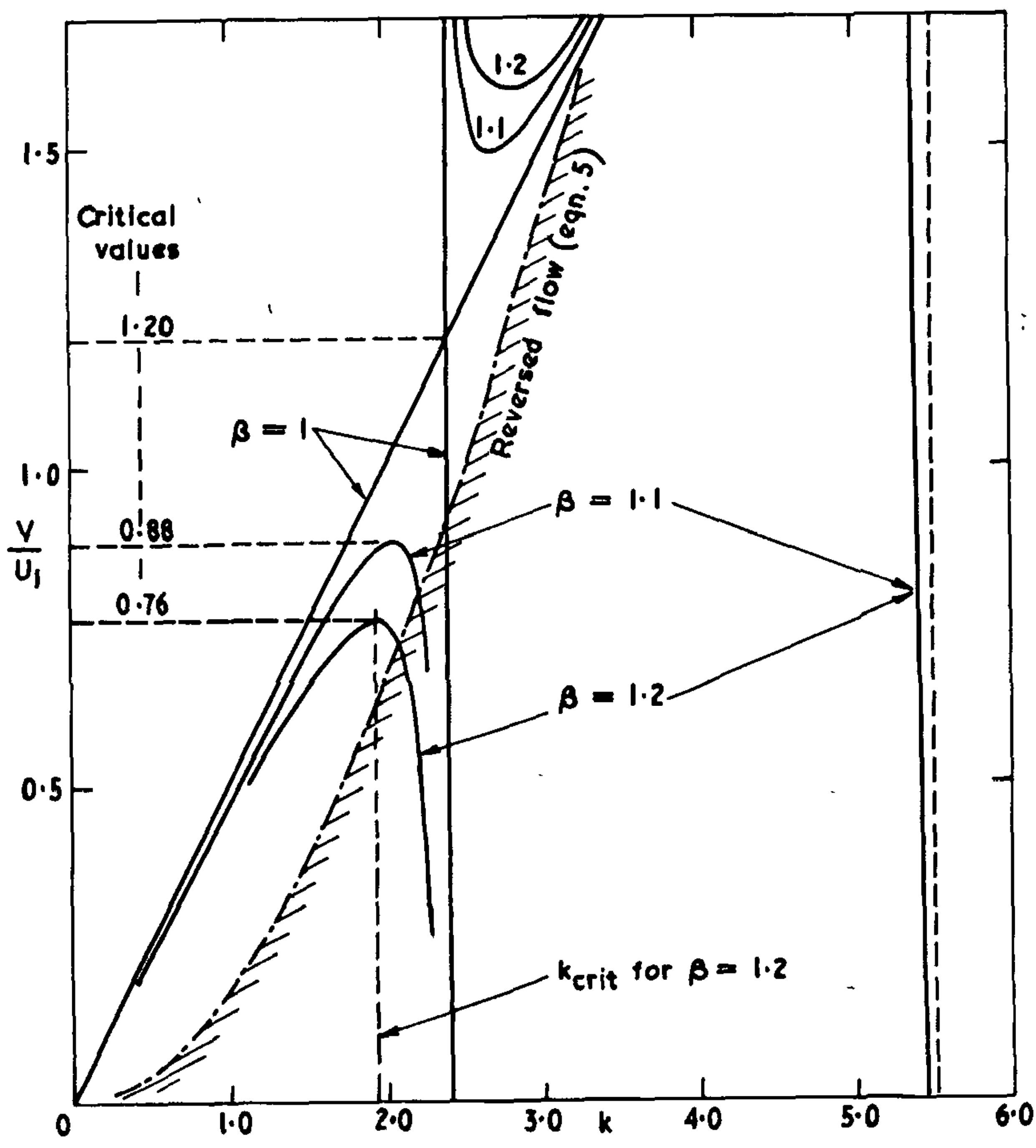


FIG. 8

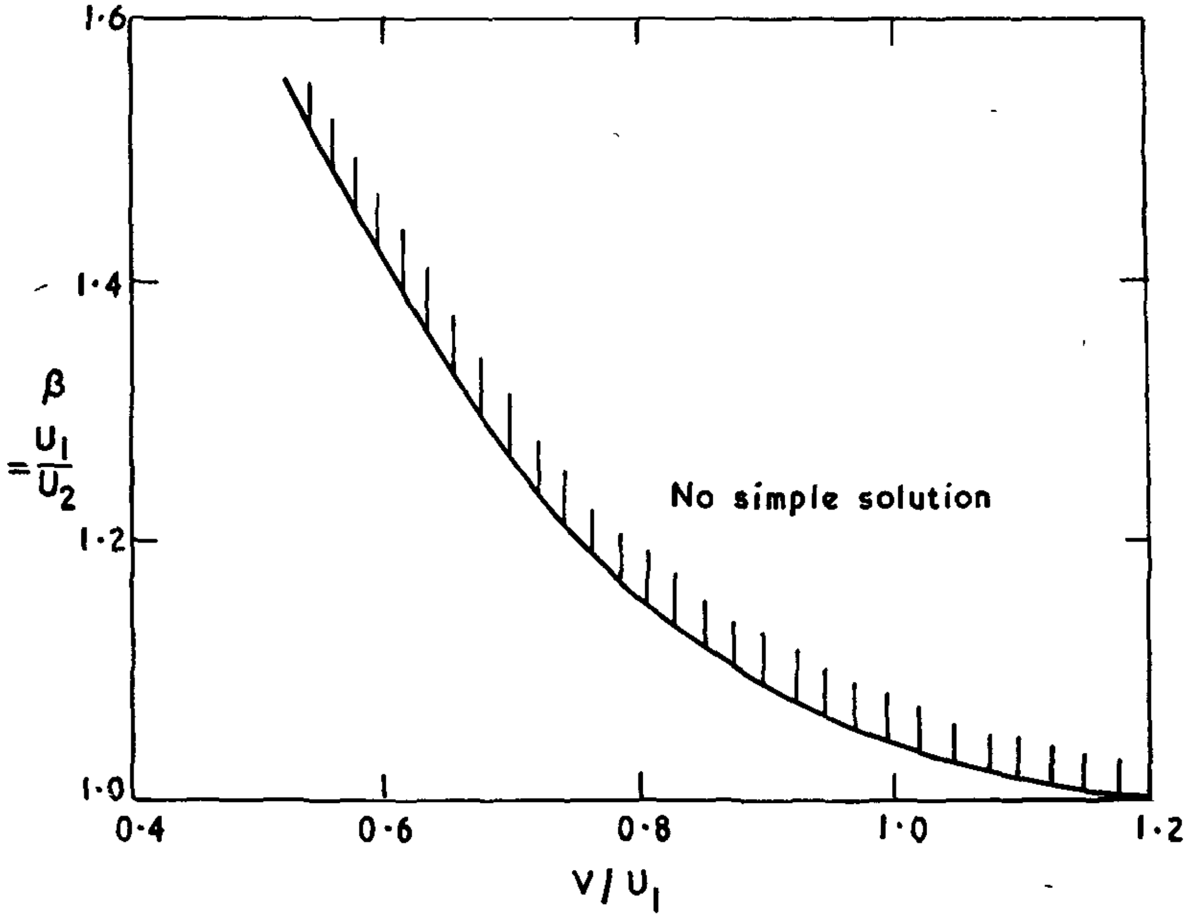
Vortex system used in analysis

FIG. 9



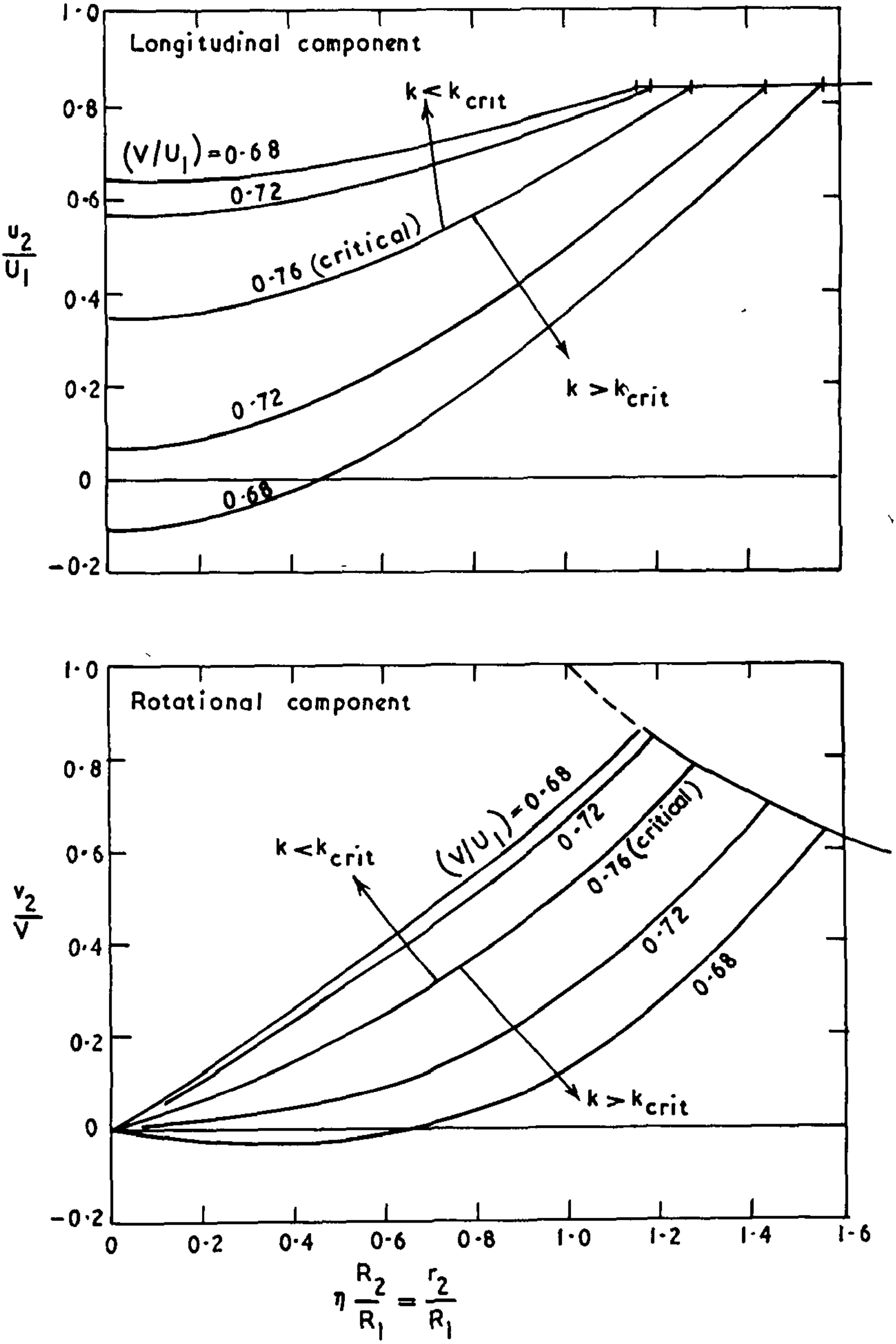
Values of k for imposed axial retardation from eqs. 4 and 5

FIG. 10



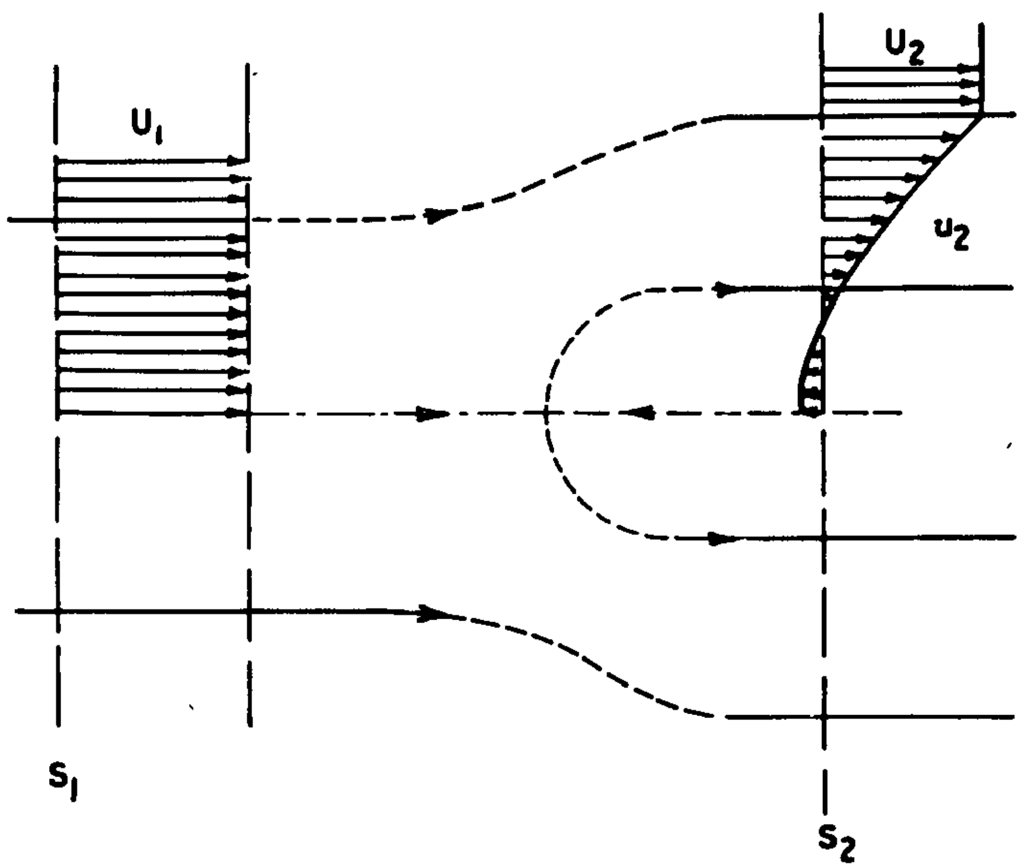
Critical values of applied retardation β, and swirl ratio V/U_1

FIG. 11



Radial distributions of longitudinal and rotational velocity components for $\beta = 1.2$

FIG. 12



Sketch of flow system for $(V/U_1) = 0{\cdot}68$ and $\beta = 1{\cdot}2$ with inner core of reversed flow

A.R.C. C.P. No. 915
September, 1965
Lambourne, N. C.

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