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By

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LONDON: HER MAJESTY'S STATIONERY OFFICE

1966

PRICE 6s. 6d. NET

C.P. No. 835

February 1965

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SUMMARY

In addition to the Kármán momentum-integral equation, two further equations are required for the purposes of calculating the development of the incompressible turbulent boundary layer - a "skin-friction law" and an "auxiliary equation". The problem of deriving a satisfactory form of auxiliary equation is a major one. Indeed, despite the effort devoted to this question for many years, few derived forms of auxiliary equation can be relied upon to account for the boundary-layer development in more than a restricted number of cases.

As a contribution to the elucidation of the problem an examination is made of several sets of experimental data covering different types of pressure distribution. Among the data certain basic types of boundary-layer behaviour can be distinguished. The equilibrium boundary-layer may be regarded as a datum condition and the other types of behaviour are discussed in the context of tendencies towards, or departures from, equilibrium.

Simple forms of auxiliary equation are postulated and an examination is made of the extent to which they can be reconciled, even qualitatively, with the observed types of boundary-layer behaviour. It is shown that several forms of the equation must be rejected as inadequate. The most economical form which appears to be capable of describing all the various trends of the data is a second-order differential equation involving the shape factor. A limited number of comparisons with experiment indicate that values can be ascribed to the free constants in the equation which lead to quantitatively acceptable predictions.

CONTENTS

Replaces NPL Aero Report 1137 - ARC 26 753
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NOTATION

x, y	Orthogonal coordinates, x measured along the surface
ρ	Fluid density
ν	Kinematic viscosity
p	Static pressure
u	Velocity in the x-direction
u_e	Velocity at the edge of the boundary layer
u_τ	Friction velocity $(u_\tau^2 = \tau_w/\rho)$
τ_w	Wall shear stress
C_f	Local skin-friction coefficient $\left(C_f = \frac{\tau_w}{\frac{1}{2}\rho u_e^2}\right)$
δ	Boundary-layer thickness
δ^*	Displacement thickness
θ	Momentum thickness
Re_θ	Reynolds number based on θ
H	"Geometric" shape factor $(H = \delta^*/\theta)$
G	Shape factor based on the velocity-defect profile, $\left[G = \left(\frac{2}{C_f}\right)^{\frac{1}{2}} \left(1 - \frac{1}{H}\right)\right]$.
$\hat{G}$	Value of G if the boundary-layer were in equilibrium at the local value of Π.
Π	Pressure-gradient parameter $\left(\Pi = \frac{\delta^*}{\tau_w} \frac{dp}{dx}\right)$.
ℓ	Length scale
$\bar{x}$	Nondimensional distance parameter $\left(\bar{x} = \int_{x_0}^{x} \frac{dx}{\ell}\right.$, with ℓ generally equal to $\left.\delta^*\right)$.
α, β, λ	Constants in equation (22).

$f, f_1, f_2, \bar{f}, \bar{\bar{f}}, F, H_1, H_2, \phi, \Phi$ and Ψ denote arbitrary functions.

The suffix o denotes initial values.

1. INTRODUCTION

The purpose of this paper is to discuss the two-dimensional, incompressible turbulent boundary-layer developing on a smooth, plane, impermeable wall in an adverse pressure gradient.

Basic to most methods of treating the incompressible turbulent boundary-layer is the Kármán momentum-integral equation which expresses the rate of change of momentum-defect in terms of the pressure gradient and the wall shear stress:-

$$\frac{d}{dx}(\rho u_e^2 \theta) = \delta^* \frac{dp}{dx} + \tau_w . \qquad \ldots (1)$$

If the pressure p (or the velocity, u_e, at the edge of the boundary-layer) is given as a function of x, equation (1) contains three unknowns: the momentum thickness, θ, the displacement thickness, δ^*, and the local wall shear stress, τ_w. Thus, for the purposes of calculating the development of the boundary layer, two further equations involving these quantities are required. Using the conventional nomenclature, these are referred to as the "skin-friction law", and the "auxiliary equation" or "shape-factor equation".

As usually formulated, skin-friction laws relate the local wall shear stress to a Reynolds number based on a length scale typical of the boundary-layer thickness and a parameter (such as H, $= \delta^*/\theta$) which describes the shape of the velocity profile. A brief review of some skin-friction laws in current use was made in Ref. 1.

The auxiliary equation essentially describes the effect of pressure gradients on the shape of the mean velocity profile. Attempts have been made for more than thirty years to derive a satisfactory form of auxiliary equation but recent reviews of this problem (2, 3, eg.) have shown that few forms of this equation can be relied upon to account for the boundary-layer development faithfully in more than a limited number of cases.

In the next Section a brief discussion will be made of the principal forms of auxiliary equation to be found in the literature. The point will be made that most of these were originally based on data relating to only one type of boundary-layer development. As a contribution to the general work in this field an examination is made in Section 3 of the various different types of boundary-layer behaviour which can be distinguished from an analysis of existing experimental data. Finally, in Section 4 simple forms of auxiliary equation will be discussed in relation to these types of boundary-layer behaviour, and an attempt will be made to find an equation which will account for them at least qualitatively.

2. REVIEW OF EXISTING AUXILIARY EQUATIONS

2.1 Most published forms of the auxiliary equation (see, e.g. Ref. 2) are formulated on the assumption that the mean velocity profiles in the turbulent boundary layer reduce to a two-parameter family, i.e. that they can be described adequately by a thickness parameter and a single "shape factor". Thus the auxiliary equation takes the form of an expression relating the shape factor and, usually, its derivatives to the Reynolds number and the pressure gradient.

Basically, there are two types of shape factor in general use. One type is a parameter based simply on the geometry of the velocity profile; the other type involves the wall shear stress in addition. Most of the auxiliary equations found in the literature are formulated in terms of the geometric type of shape factor. In some cases this is the ratio, H, of displacement to momentum thickness; in others the shape factor is based on such ratios as

$$\left(\frac{u}{u_e}\right)_{y=\theta} \quad \text{or} \quad \frac{\delta-\delta^*}{\theta} .$$

There is no essential difference between any of these geometric shape factors and, in the light of the assumption that the velocity profiles form a two-parameter family, they can all be related to one another (at least in principle). Accordingly, our remarks will refer explicitly to auxiliary equations based on H but they can be taken to apply to equations involving other geometric shape factors also.

This interchangeability of shape factors does not extend to those which involve the wall shear stress. Among these are Clauser's parameter, G, which is based on the velocity-defect profile[4], and Coles' wake-component coefficient[5] which is also used by Spalding[6]. Reference to shape factors of this category will be left to another Section.

2.2 Despite the confusing variety of auxiliary equations in current use, only two basic types can be distinguished. The first is represented by the method of Buri[7] and is a direct analogue of the Polhausen method for the laminar boundary-layer and the numerous later methods which owe their inspiration to the Polhausen approach (see Ref. 8, e.g.). Buri postulated that the shape factor, H, was a function of the Reynolds number and the pressure gradient:-

$$H = f_1\left(Re_\theta , \frac{\theta}{u_e}\frac{du_e}{dx}\right) . \qquad \ldots (2)$$

This form of the auxiliary equation has never commanded much attention, largely as a result of Prandtl's criticism[9] that it ignored the effects of the upstream history of the boundary layer. In the light of more recent knowledge it would be fairer to say that it is valid so long as the layer is in local equilibrium at each streamwise station. More will be said about this point later in the paper.

2.3 The second form of the auxiliary equation is represented by some thirteen methods listed by Rotta[(2)] to which must be added recent methods such as Head's[(10)]. This form can be written

$$\theta \frac{dH}{dx} = f_2 \left(H , \ Re_\theta , \ \frac{\theta}{u_e} \frac{du_e}{dx} \right) . \qquad \ldots (3)$$

By virtue of its being a differential equation of the first order in H, equation (3) requires the specification of an initial value of H with which to start the calculation. Thus in broad terms equation (3) contains a mechanism by which the upstream history can be taken into account in so far as it affects the velocity profile. Consequently, as far as this auxiliary equation is concerned, two boundary-layers with the same initial velocity profile subsequently subjected to the same pressure distribution will develop alike whether or not their previous history was the same. No provision is made for the possibility that their initial shear-stress distributions could be dissimilar. The distribution of shear stress across the boundary-layer is, of course, related not to the local velocities in the layer but, via the equation of motion, to their derivative with respect to x. Therefore some estimate of the effect of the initial shear distribution could be made by specifying, in addition to H, the initial value of dH/dx, say. Equation (3) would then need to be replaced by a second-order differential equation in H. So far as is known to the author, no attempt to do this has been reported.

It is not intended, here, to give a detailed discussion of the merits or demerits of the various auxiliary equations grouped under equation (3). Useful work has already been done in this respect by Rotta[(2)] and Thompson[(3)]. Their work has shown that the confidence with which the state of the art has been viewed in many of the text books could not be substantiated and that many of the auxiliary equations have a severely limited validity of application. Essentially they are correlations of experimental data whether or not some physical concept - for instance the kinetic energy or moment - of - moment equations (see Ref. 12, e.g.), or the entrainment equation (6,10,11) - has been invoked as the basis for the correlation. Consequently much depends on the range of types of boundary-layer development which has been examined in the correlation. Nearly all the auxiliary equations in the literature have been derived from an analysis of boundary layers of a single type, namely, where the shape factor H increases with distance downstream. This is typical of boundary layers growing on aerofoils or in diffusers. Bearing this in mind it is not difficult to see why these methods are of doubtful validity when applied to either equilibrium boundary layers (where H is approximately constant) or boundary layers where H is decreasing with x (see Refs. 2, 3). Even when applied to the same sort of boundary-layer development as the ones on which they were originally based, some auxiliary equations have exhibited limitations such as an unlikely sensitivity to initial conditions[(13)]. In this respect, however, it must be pointed out that there is little experimental data to elucidate the question of the extent to which initial conditions affect the boundary-layer development at appreciable distances downstream.

3. TYPES OF BOUNDARY-LAYER BEHAVIOUR

3.1 The equilibrium boundary layer

The first type of boundary-layer behaviour which we shall discuss, and the most important from a fundamental standpoint, is the equilibrium boundary layer. This topic has received considerable attention in recent years. The early experimental work of Clauser[4,14] and the analysis of Rotta[15] served to demonstrate that equilibrium boundary layers in non-zero pressure gradients could exist (at least in an approximate form) on a smooth surface. Their work has been added to by a number of theoretical treatments, among which those of Townsend[16] and Mellor and Gibson[17] are important in the present context, and by the recent experiments of Bradshaw[18].

The particular aspect of equilibrium boundary layers which is of prime importance in our present work is the observation that a certain type of streamwise pressure distribution can support a boundary-layer growth charafterised by similarity of the velocity-defect profiles. The pressure distribution is one of constant "severity" in so far as the ratio of pressure-gradient forces to skin-friction forces acting on an element dx of boundary layer is the same at each streamwise station. The appropriate pressure-gradient parameter which expresses this ratio is Π (see Ref. 2, e.g.) where

$$\Pi = \frac{\delta^*}{\tau_w}\frac{dp}{dx}, \qquad \ldots (4)$$

and Π is independent of x for an equilibrium boundary layer. For a particular value of this pressure-gradient parameter, the velocity-defect profile in the boundary layer has a given shape independent of Reynolds number:-

$$\frac{u_e - u}{u_\tau} = f\left(\frac{y}{\delta}\right). \qquad \ldots (5)$$

Clauser[4] has suggested that a convenient "shape factor" for describing the velocity-defect profile could be defined by

$$G = \frac{\int_0^1 f^2 \, d\left(\frac{y}{\delta}\right)}{\int_0^1 f \, d\left(\frac{y}{\delta}\right)}. \qquad \ldots (6)$$

G can also be related to the geometric shape factor H by

$$G = \left(\frac{2}{C_f}\right)^{\frac{1}{2}}\left(1 - \frac{1}{H}\right). \qquad \ldots (7)$$

The value of G is about 6.5 for the flat-plate case and tends to infinity for the equilibrium boundary layer with zero wall shear stress[19].

Thus for equilibrium boundary layers G is a unique function of Π. Some relevant experimental data is shown in fig. 1 along with the relation between G and Π indicated by the theories of Townsend[16] and Mellor and Gibson[17]. The former is restricted to values of Π greater than about 2; the latter makes no such restriction but states that no equilibrium boundary layer can exist for values of Π less than -0.5. Over the common range Mellor and Gibson predict higher values of G than are given by Townsend's theory. Judging by the experiment of Bradshaw[18] and Clauser's "Boundary layer I"[4], Townsend's theory seems the more accurate. Clauser's "Boundary layer II has a measured value of G higher than that predicted by either theory if one takes the value of Π indicated in Ref. 14, namely 7. However Mellor and Gibson found from an examination of Clauser's data that the actual value varied between about 6 and 13 over the course of development of the layer. Thus it would seem that the value of 7 is to be taken only as a guide and the discrepancy indicated in fig. 1 is of little significance. For small values of Π, Mellor and Gibson's theory appears to predict values of G which agree satisfactorily with experiment.

For the purposes of our later calculations a relation between G and Π will be required covering the whole range of Π. An empirical curve has therefore been drawn in fig. 1 representing a synthesis of experiment and theory. This curve is given by the function

$$G = 6.1\,(\Pi + 1.81)^{\frac{1}{2}} - 1.7 \; . \qquad \ldots (8)$$

3.2 <u>Tendency towards equilibrium</u>

If the pressure distribution appropriate to a particular equilibrium boundary-layer is set up but the initial value of G is not the equilibrium value, one of two things can happen. If the boundary-layer has "downstream stability" (see Ref. 2) the value of G will approach the equilibrium value $\hat{G}$, say, as the layer progresses downstream; if it is "unstable" the value of G will diverge from the equilibrium value. The only direct experimental evidence there is concerning the approach to equilibrium relates to the flat plate case. Tillmann[20] and Klebanoff and Diehl[21] carried out tests to observe the downstream behaviour of constant-pressure boundary layers which had been disturbed initially giving a value of G higher than 6.5. Recently one of Bradshaw's experiments[18] consisted of setting up an equilibrium boundary layer and subsequently (i.e., downstream of some x-position) removing the pressure gradient so that the layer could return to the flat-plate type.

It might be supposed that, at least some distance downstream of the disturbing agency, the return to equilibrium would exhibit some universality independent of the particular form of the disturbance. For instance, the rate of change of G might be uniquely related to the amount, $G-\hat{G}$, say, by which G was out of equilibrium. Such considerations lead us to inquire whether an expression of the form

$$\epsilon \frac{dG}{dx} = f(G-\hat{G}) , \qquad \ldots (9)$$

has any general validity. In equation (9) $\hat{G}$ is the particular equilibrium value of G (6.5 for the flat-plate case) and ϵ is some length scale typical of the boundary-layer thickness. Integration of equation (9) yields an expression for G:-

$$G - \hat{G} = F(\bar{x}) , \qquad \ldots (10)$$

where

$$\bar{x} = \int_{x_0}^{x} \frac{dx}{\epsilon} , \qquad \ldots (11)$$

and x_0 is an arbitrary constant. Three sets of experimental data are shown in fig. 2 in the form of a plot of G against $\bar{x}$, using δ^* for ϵ, and choosing x_0 such that all the curves pass through the point "A". Fig. 2 indicates little evidence that a universal function of the form of equations (9) or (10) exists; nor does it seem likely that a better correlation could be achieved by using some other length scale in equation (11).

The failure of this exercise casts considerable doubt on the suggestion that boundary layers with the same initial velocity profile will develop in the same way if subjected to the same pressure distribution downstream of the initial station. It will be recalled that this assertion is implicit in nearly all the auxiliary equations appearing in the literature (see Section 2, above).

Before passing to the next topic it is instructive to note from fig. 2 how long it can take for a disturbance to die out. The data of Klebanoff and Diehl, and Bradshaw indicate that G is unlikely to approach the equilibrium value closely for a distance of hundreds of times the displacement thickness. This observation strongly supports the comments of Coles in Appendix A of Ref. 22.

3.3 Departures from equilibrium

The equilibrium boundary-layer develops in a pressure gradient of constant severity (see Section 3.1 above). The parameter Π (equation 4) is constant with respect to x and the shape factor G is also constant. On the other hand, if the severity of the pressure gradient changes the boundary-layer will cease to be in equilibrium and both Π and G will be functions of x. In a sense Π can be regarded as the independent and G the dependent variable, or, to use Clauser's "black-box" terminology[14], there is a certain response in G to a given input function $\Pi(x)$.

The severity of the pressure gradient can either increase $(d\Pi/dx > 0)$ or decrease $(d\Pi(dx < 0)$. We shall now proceed to examine experimental data relating to each of these possibilities.

An increasingly "severe" pressure gradient is typical of the boundary layer developing on an aerofoil surface or in a diffuser. The actual pressure gradient dp/dx may be constant but due to the increase in δ^* and the decrease in τ_w with increasing x, the value of Π increases with x, reaching infinity at a separation point (zero τ_w). Some experimental results obtained under such conditions are shown in fig. 3* as a plot of G against Π. Since Π is increasing with x each curve represents a trajectory whose sense is indicated by the arrow head.

One of the most important features of the data in fig. 3 is the fact that the curves lie close to the curve $\hat{G}(\Pi)$ which represents the locus of all possible equilibrium boundary layers. [This curve has been drawn according to equation (8).] The significance of a trajectory which coincides with the curve $\hat{G}(\Pi)$ is not, of course, that the boundary layer is in equilibrium† but that the variation in shape factor is the same as if the layer were passing through each possible equilibrium state. This situation might be referred to as "local equilibrium" at each streamwise station. To illustrate the point further fig. 4 shows the variations in H corresponding to the spread of the data in fig. 3 about the curve $\hat{G}(\Pi)$. The dotted curves represent the loci of points for which H is the given percentage above or below the value corresponding to "local equilibrium" at a Reynolds number (Re_θ) of 10^4. It will be noted that the data lie within about 5 or 10 percent of the "equilibrium" values of H. One might expect a boundary-layer trajectory to remain close to the $\hat{G}(\Pi)$ curve so long as the value of Π was increasing very slowly. However this does not seem to be a necessary condition. The curve in fig. 3 derived from Schubauer and Klebanoff's data(23) is close to the "local equilibrium" condition although a typical value of $\delta^* d\Pi/dx$ is 0.3 (when $\Pi = 8$).

To turn to the case where the severity of the pressure gradient is decreasing, fig. 5 shows some experimental data presented in a similar way to that in fig. 3. The data of Ludwieg and Tillmann(24) relate to the case where Π first increases with x and subsequently decreases Bradshaw's boundary layer(18) is initially in equilibrium with $\Pi = 5.5$; subsequently Π falls to zero.

Compared with the data in fig. 3, that in fig. 5 gives a quite different picture. Whereas for $d\Pi/dx > 0$ the trajectories were confined to a narrow corridor about the curve $\hat{G}(\Pi)$, in the present case the trajectories diverge markedly from the equilibrium locus. This is most evident in the cases where Π is initially increasing with x; the subsequent reduction of Π is accompanied by little sympathetic response in G. The impression is gained that some kind of "inertia" effect is causing G to continue

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*To reduce the data to this form one requires values of the wall shear stress. These were found using the skin-friction law derived in Ref. 1.

†For instance, the shear-stress distribution would be expected to differ considerably from that in an equilibrium boundary layer.

increasing even after $d\Pi/dx$ has decreased to zero and is increasing negatively. Even in the case of the boundary-layer initially in equilibrium there is a "sluggish" response of G to the decrease in Π to zero. Nor is Π changing particularly rapidly. The maximum value of $-\delta^* d\Pi/dx$ in the case of Bradshaw's test was about 0.25; this may be compared with the value quoted above for the data of Schubauer and Klebanoff which lay close to the condition of "local equilibrium".

3.4 Summary

Before proceeding to the next Section it will be useful to list the main points which have emerged from this study of the data:-

(1) The equilibrium boundary layer is specified by values of G $(= \hat{G})$ and Π which are independent of x. From a synthesis of experimental data and theory the function $\hat{G}(\Pi)$ can be defined fairly precisely.

(2) The rate at which an initially-disturbed, flat-plate boundary-layer tends to equilibrium is not determined solely by the initial value of the shape factor and the boundary-layer thickness. This would appear to indicate that in more general cases also knowledge of the pressure distribution downstream of some initial station together with the initial value of the shape factor is insufficient information from which to compute the boundary-layer growth.

(3) Boundary-layers in pressure gradients of increasing "severity" $(d\Pi/dx > 0)$ remain close to the condition of "local equilibrium"; i.e. $G \approx \hat{G}(\Pi)$. This appears to be true even if Π is changing quite rapidly.

(4) Boundary-layers in pressure gradients of decreasing "severity" $(d\Pi/dx < 0)$ depart markedly from the condition of "local equilibrium". This is particularly so in the case where $d\Pi/dx$ is initially positive but subsequently changes sign, suggestive of some kind of "inertia" effect.

4. THE AUXILIARY EQUATION

4.1 We return now to the problem of the auxiliary equation. Stated briefly the problem is one of finding some algebraic or differential equation involving the shape factor which exhibits a response to various changes of pressure gradient which is similar to that observed in the experimental data. It was seen in the previous Section that certain basic trends can be distinguished in the data and that these trends can be interpreted in the context of tendencies towards, or departures from, a condition of "local equilibrium". It would seem that this way of examining the data is an important one which can make many of the observed trends meaningful and coherent. Moreover it is likely to facilitate the process of deriving a satisfactory auxiliary equation.

With the object of making maximum use of this concept of variations about an equilibrium state, G is selected as the appropriate shape factor, and Π becomes the corresponding pressure-gradient parameter. A fundamental requirement of our auxiliary equation is that a solution must exist of the form

$$\left.\begin{array}{l} G = \text{constant} \\ \Pi = \text{constant} \end{array}\right\} .$$

4.2 One possibility is the algebraic auxiliary equation:-

$$G = \hat{G}(\Pi) . \qquad \ldots (12)$$

Clearly this equation satisfies the conditions for equilibrium boundary-layers. Moreover, as was seen in Section 3.3 above, it is a reasonably good approximation in the case of boundary layers of the "aerofoil" or "diffuser" type, (i.e. for which $d\Pi/dx > 0$). If a given form for equation (12) is assumed - equation (8) for example - together with a suitable skin-friction law, the geometric shape factor, H, can be expressed as a function of Re_θ and the local pressure gradient:-

$$H = H\left(Re_\theta , \frac{\theta}{u_e}\frac{du_e}{dx}\right) , \qquad \ldots (13)$$

which is identical to equation (2). If the necessary calculations are performed one does not arrive at an expression of the same detailed form as Buri's, namely

$$H = H\left(\frac{\theta Re_\theta^{\frac{1}{4}}}{u_e}\frac{du_e}{dx}\right) , \qquad \ldots (14)$$

but one more nearly of the form

$$H = H_1(Re_\theta) + H_2\left(\frac{\theta}{u_e}\frac{du_e}{dx}\right) . \qquad \ldots (15)$$

Nevertheless, in so far as equation (13) represents his fundamental assumption Buri's work appears to be confirmed.

On the other hand equation (12) is incapable of describing the return to equilibrium following a perturbation - indeed perturbations from equilibrium are themselves inadmissible - nor can it account for the type of observed behaviour illustrated in fig. 5 for pressure gradients of decreasing "severity".

4.3 If equation (15) is equivalent to Buri's approach, an auxiliary equation corresponding to that of most other investigators (see equation (3)) would be of the form

$$\frac{dG}{d\bar{x}} = \Phi(\Pi, G) , \qquad \ldots (16)$$

with $\bar{x}$ as a non-dimensional distance defined, say, by

$$d\bar{x} = \frac{1}{\delta^*} dx \,. \qquad \ldots (17)$$

A form similar to this has been suggested by Rotta(2). If the function Φ in equation (16) is of a form which vanishes for $G = \hat{G}(\Pi)$ the equilibrium case would be taken into account. Thus we might postulate some expression like

$$\frac{dG}{d\bar{x}} = \phi(\Pi) \,.\, (G-\hat{G})^n \,. \qquad \ldots (18)$$

Equation (18) also goes some way to accounting for a return to equilibrium following some disturbance, with ϕ and n determining the degree of downstream stability. However the remarks in Section 2.2 should serve to show that no combination of values of $\phi(0)$ and n can lead to an expression which can account quantitatively for all the data in fig. 2. This is because the experimental results indicate different degrees of stability for the same value of G.

Passing to the cases where Π is a function of x this problem of fitting either equation (16) or equation (18) to the data becomes even more difficult. Figs 3 and 5 show that at, any given value of Π, the data are not even consistent as far as the sign of stability is concerned, at least so long as the "stability" is interpreted in the sense of equations (16) or (18). Furthermore equations (16) or (18) contain no mechanism for taking into account the apparent "inertia" effect suggested by the data in fig. 5.

It soon becomes clear that an auxiliary equation of the form of equation (16) is inadequate in describing the different types of behaviour which we have distinguished in the experimental data. The best that could be done with equation (16) is to make it strongly stable about the "local equilibrium" condition. In this way the advantages of the simple form, equation (15), would be retained, in that the equation would predict values of G close to $\hat{G}$ which is correct for the "aerofoil type" boundary-layers, and also in that the approach to equilibrium following an initial perturbation would be accounted for at least qualitatively. It is possible that this provides the explanation for the partial success of some of the auxiliary equations in the form of equation (3), that of Ref. 10 for example.

4.4 Some of the disadvantages of equation (16) can be minimised by the use of an auxiliary equation formed by a combination of equations (15) and (16). In a fairly general form this could be written as

$$\frac{dG}{d\bar{x}} = \Psi\left(\Pi \,,\, \frac{d\Pi}{d\bar{x}} \,,\, G\right) . \qquad \ldots (19)$$

However, this equation still cannot account for the different rates of approach to equilibrium exhibited by the data in fig. 2, nor for the apparent inertia effect suggested by the data in fig. 5. For these reasons it will not be considered further in this paper.

4.5 The two effects which it has not been found possible to account for - the return to equilibrium and the inertia effect - appear to demand that the auxiliary equation be of the second order in G. Starting with the former, it was seen in Section 3.2 that equation (9) was inadequate because the behaviour of a flat-plate boundary-layer following a disturbance did not depend solely on the initial value of G. This point was mentioned in Section 2.3 also, and it was suggested that there were good grounds for expecting that the first derivative of the shape factor might be a necessary additional starting condition. Accordingly, we postulate that the approach to equilibrium can be described by

$$\frac{dG}{d\bar{x}} = \bar{f}\left[\left(\frac{dG}{d\bar{x}}\right)_o , \; G-\hat{G}\right], \qquad \ldots (20)$$

where the suffix o denotes an initial condition. Differentiating equation (20) throughout with respect to $\bar{x}$ and eliminating $(dG/d\bar{x})_o$ between this new equation and equation (20) leads to an expression of the form

$$\frac{d^2G}{d\bar{x}^2} = \bar{\bar{f}}\left[\frac{d}{d\bar{x}}(G-\hat{G}) , \; G-\hat{G}\right], \qquad \ldots (21)$$

since $\hat{G}$ is assumed constant.

To give the function $\bar{\bar{f}}$ some definite form we suggest

$$\frac{d^2G}{d\bar{x}^2} = \lambda\left\{\frac{d}{d\bar{x}}(G-\hat{G})\right\}^{\alpha} (G-\hat{G})^{\beta} . \qquad \ldots (22)$$

By a suitable choice of the constants λ, α and β, equation (22) can, in fact, be fitted satisfactorily to the experimental data relating to the approach to constant-pressure equilibrium.* But what is more important, however, equation (22) also appears to be capable of describing, at least qualitatively, each of the other types of boundary-layer behaviour discussed in Section 3. In those latter cases, of course, $\hat{G}$ is not constant but is a function of x by way of its relation with $\Pi(x)$. Thus in the sense of equation (22) $\hat{G}$ can be regarded as equivalent to a pressure-gradient parameter.

By trial and error the values of the coefficients λ, α and β in equation (22) have been assessed to give satisfactory agreement with two or more sets of boundary-layer data from each class discussed in Section 3. When a comparison is made with more data it may be necessary to modify these assessments, but the provisional values obtained are as follows:-

$$\left.\begin{aligned} &\frac{d}{d\bar{x}}(G-\hat{G}) > 0 :- \\ &\qquad \lambda = -0.25 , \quad \alpha = 3 , \quad \beta = -2 \\ &\frac{d}{d\bar{x}}(G-\hat{G}) < 0 :- \\ &\qquad \lambda = 5 , \quad \alpha = 2 , \quad \beta = -2 \end{aligned}\right\} . \qquad \ldots (23)$$

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*It may be noted that for the case where $\hat{G}$ is independent of x, equation (22) can be solved analytically.

The distinction between the values of λ and α, depending on the sign of $d(G-\hat{G})/dx$, is of prime importance in describing the different type of response of G according to whether $d\Pi/dx$ is positive or negative.

Since β is negative, equation (22) is singular at $G = \hat{G}$. This behaviour has only local repercussions but it is an embarrassment for a number of reasons and it is suggested that the term $(G-\hat{G})^{-2}$ in equation (22) be replaced by

$$\{(G-\hat{G})^2 + a^2\}^{-1} ,$$

where a is some small number. Insufficient experience of the equation has been gained so far to estimate the precise significance of the value of a but in the calculations it has been taken as 0.1.

Some comparisons between the new auxiliary equation and experimental data are shown in figs. 6 to 11. In this exercise the measured variation of Π and δ^* with x have been assumed as data and the variation of G with x has been calculated using equation (22). In figs. 6 to 11 the solid curve represents the predicted values of G, the dots showing the intervals in the computation. The measured values of G are shown as square data points. In each case suitable initial values of G and dG/dx have had to be assumed in the calculations.

In figs. 6 and 7 two boundary-layers of the "aerofoil" type are considered. Equation (22) strongly portrays the tendency of boundary-layers of this type to remain close to the "local equilibrium" condition. The initial conditions need to be chosen fairly critically if the precise small departure from "local equilibrium" is to be correctly represented. If the initial values of G and dG/dx had been appreciably higher the predicted values of G would soon have coincided with the $\hat{G}$-curves. As it is the small departure from "local equilibrium" is somewhat exagerated. The significance of this sensitivity to initial conditions needs to be examined more carefully, but allowing for this the comparisons in figs. 6 and 7 can be regarded as satisfactory.

Fig. 8 shows Sandborn's data[25] relating to his "zero suction" conditions. The pressure distribution is of the same general form as that considered in figs. 6 and 7 but the initial value of G is higher than the local value of $\hat{G}$. Again the agreement between the predicted and the measured values of G is very encouraging.

Fig. 9 shows one of the sets of data obtained by Ludwieg and Tillmann[24]. This boundary-layer was subjected to a pressure gradient of initially increasing, and subsequently decreasing, "severity". Over the first part, with Π increasing, the value of G remains close to $\hat{G}$, as was the case in figs. 6 and 7. For larger values of x, where Π and, consequently, $\hat{G}$ are decreasing, G continues to increase - exhibiting the apparent "inertia" effect. The predicted variation of G with x is seen to represent these different types of behaviour adequately.

Another case in which Π decreases with increasing x is illustrated in fig. 10. This shows the data from Bradshaw's experiment in which a boundary-layer initially in equilibrium at a value of Π of about 5.5 is subsequently subjected to constant pressure. Π falls rapidly to zero but

G responds only slowly and would take a distance of several hundreds of times the displacement thickness to approach the new equilibrium state closely. The predicted variation of G follows the observed behaviour very well.

For values of x greater than about 65 in, $\Pi = 0$, and the data in fig. 10 correspond to the case of a perturbed flat-plate boundary-layer. Fig. 11 shows another set of data relating to this class of boundary-layers, namely the results of Klebanoff and Diehl for the 0.25 in rod (Ref. 21). Again the agreement between the measured and the predicted variation of G with x is very satisfactory.

These comparisons between calculations based on equation (22) and experimental data are far from exhaustive. Nevertheless they serve to show that the proposed form of auxiliary equation can be fitted to a range of different types of boundary-layer development, and that it is probably the most economical one which can be. There is scope for a considerable amount of further work. Comparisons must be made with a far greater number of sets of data before equation (22) can be used with confidence in making boundary-layer predictions. When further comparisons have been made it may, of course, be necessary to modify the values of the constants to give the best overall agreement.

5. CONCLUDING REMARKS

5.1 A review, in broad terms, of existing forms of the auxiliary equation used in the calculation of the incompressible turbulent boundary layer in two dimensions, reveals two basic types. One is an algebraic equation involving the shape factor (Buri); the other is the familiar first-order differential equation on which attention has been concentrated for more than thirty years.

5.2 From an analysis of the experimental data certain fundamental types of boundary-layer behaviour can be distinguished. These are

(a) The equilibrium boundary-layer which is characterised by a pressure gradient of constant "severity" and similarity of the velocity-defect profiles.

(b) The return to equilibrium conditions following an initial perturbation.

(c) The departure from equilibrium when the "severity" of the pressure gradient is changing with x. Two possibilities can be considered, according to whether the "severity" of the pressure gradient is increasing or decreasing with x.

5.3 A synthesis of experiment and theory relating to equilibrium boundary-layers enables a relation to be defined fairly accurately between a shape factor G (based on the velocity-defect profile) and a pressure-gradient parameter

$$\Pi \left(= \frac{\delta^*}{\tau_w} \frac{dp}{dx} \right).$$

The function $G = \hat{G}(\Pi)$ thus represents all possible equilibrium boundary layers.

5.4 At least in the particular case of $\Pi = 0$, the rate at which the shape factor G approaches the equilibrium value $\hat{G}$, following an initial perturbation, is not solely determined by the initial value of G and the scale of the boundary layer. From this it can be deduced that, in the general case also, knowledge of the initial velocity profile together with the subsequent pressure distribution is insufficient information from which to compute the development of the boundary-layer.

5.5 If the pressure-gradient parameter Π is a function of x the "response" of G takes alternative forms depending on the sign of $d\Pi/dx$. In cases where Π is continuously increasing G remains close to the value $\hat{G}$ corresponding to an equilibrium boundary-layer at the local value of Π. This situation might be referred to as "local equilibrium" at each streamwise station. On the other hand, if Π is decreasing with increasing x, G departs markedly from the "local equilibrium" condition This type of behaviour appears to be accentuated when Π decreases subsequent to an initial increase, i.e. when $d\Pi/dx$ is first positive and then negative. Under such conditions G can continue increasing although the local value of $\hat{G}$ is decreasing with increasing x. By analogy with dynamical systems one might attribute this to a kind of "inertia" effect.

5.6 The main points which emerge from the examination of experimental data are used as a basis for evaluating various possible forms of auxiliary equation. It is seen that for an important class of boundary-layers - including typical ones on aerofoils or in diffusers - the assumption of "local equilibrium" could lead to predictions of H which are accurate to better than 10 percent under most conditions. This assumption is equivalent to the type of algebraic auxiliary equation proposed by Buri.

The use of an algebraic auxiliary equation implies that the upstream history of the boundary-layer has no significant influence on the shape factor except by way of its effect on the thickness of the layer. To take direct account of the effect of upstream history one requires a differential equation. But in view of the comment made in 5.4, above, it would seem that a first-order auxiliary equation involving one initial condition (the initial value of the shape factor) must be inadequate. This goes some way to explaining why the use of a first-order equation is only marginally more effective in describing the various types of boundary-layer behaviour than the algebraic auxiliary equation mentioned in the previous paragraph.

The use of a second-order differential equation offers considerably more promise of success. Two initial conditions are required, and these may be regarded as specifying information about the initial velocity profile and shear-stress profile. The demand for an additional starting condition thus has a strong physical justification.

A tentative proposal is made as to a suitable form for a second-order auxiliary equation. It would seem that this new auxiliary equation is capable of describing all the types of boundary-layer behaviour listed above at least qualitatively, and a limited number of comparisons with experiment indicate that acceptable quantitative agreement can be obtained also.

The work described in this paper is at an interim stage. Further comparisons between the auxiliary equation proposed and experimental data will probably require some adjustment of the constants in the equation to maintain the best overall agreement. But in any event it is thought that the results which have already been obtained are of sufficient interest to merit presentation at this stage. Furthermore it is hoped that the paper will stimulate discussion of the more general points raised, and of their relevance and possible repercussions on the current work of other investigators.

ACKNOWLEDGEMENTS

The author is indebted to Prof. J. C. Rotta, of the Aerodynamishe Versuchsanstalt Göttingen, for making available certain unpublished A.V.A. material.

The author also wishes to acknowledge fruitful discussions with Mr. P. Bradshaw, of the Aerodynamics Division NPL, and Dr. B. G. J. Thompson, of the Engineering Laboratory, Cambridge University, during the course of this work.

The work described above has been carried out as part of the research programme of the National Physical Laboratory, and this paper is published by permission of the Director of the Laboratory.

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List of figures

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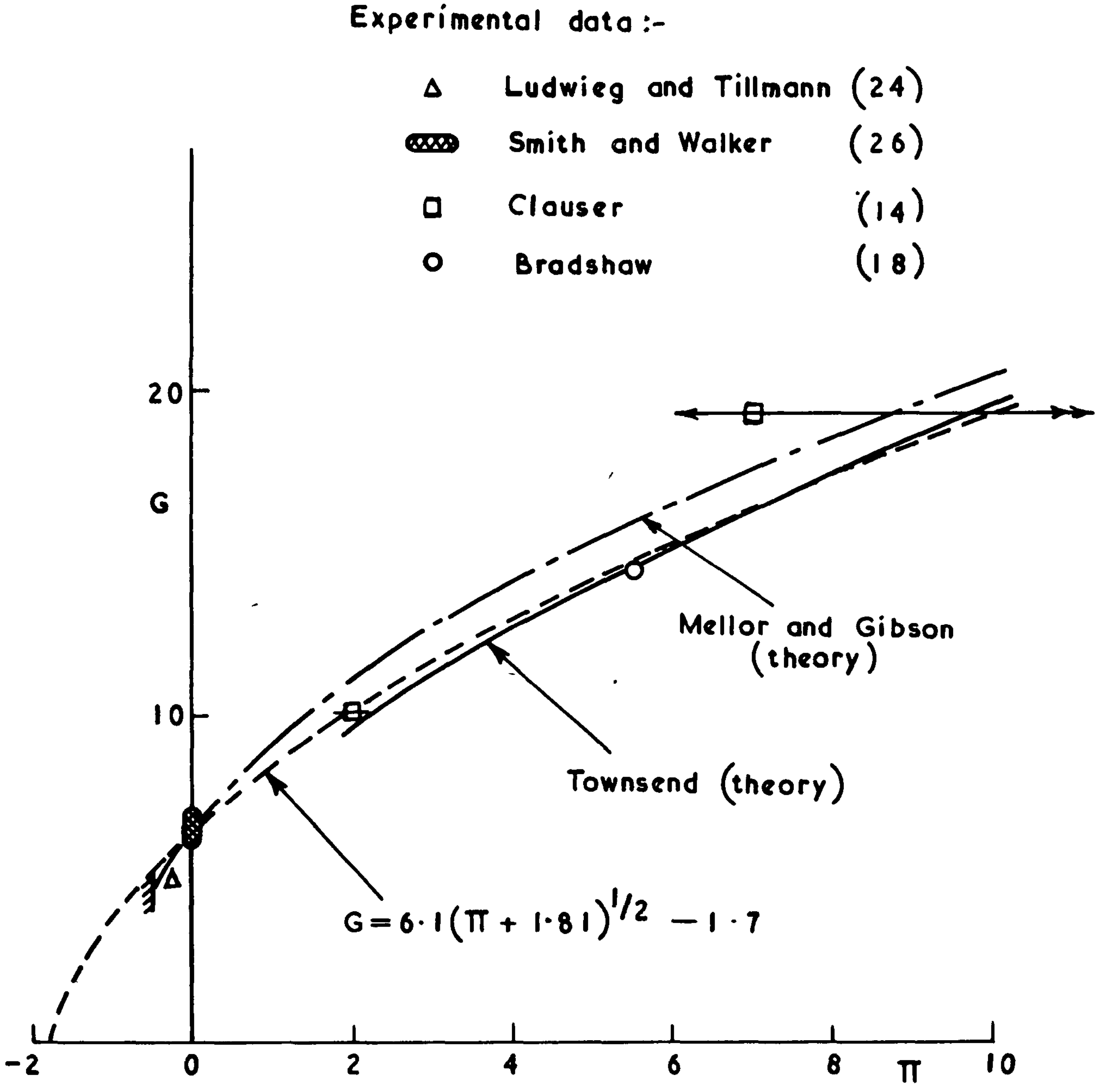
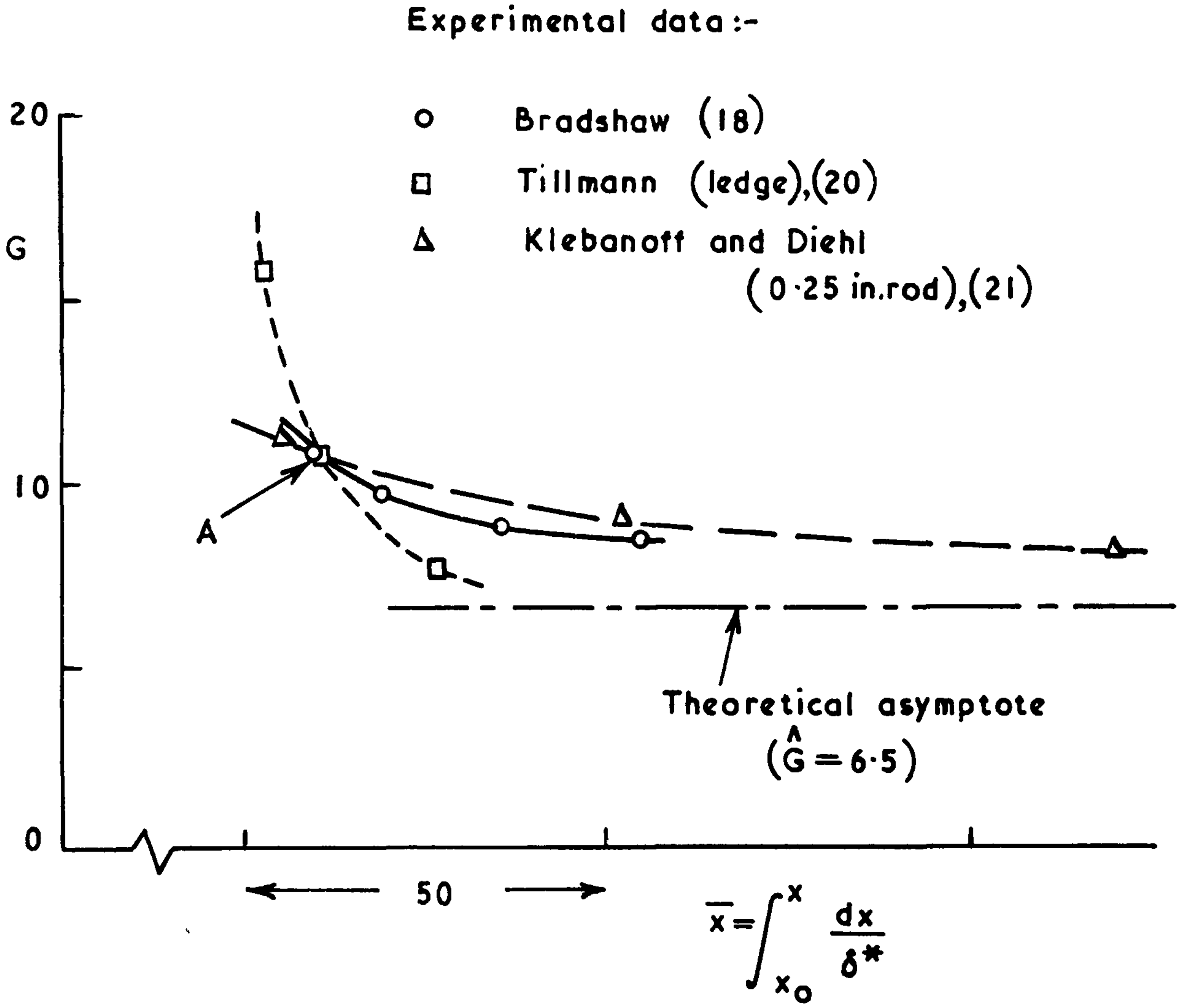
FIG. 1



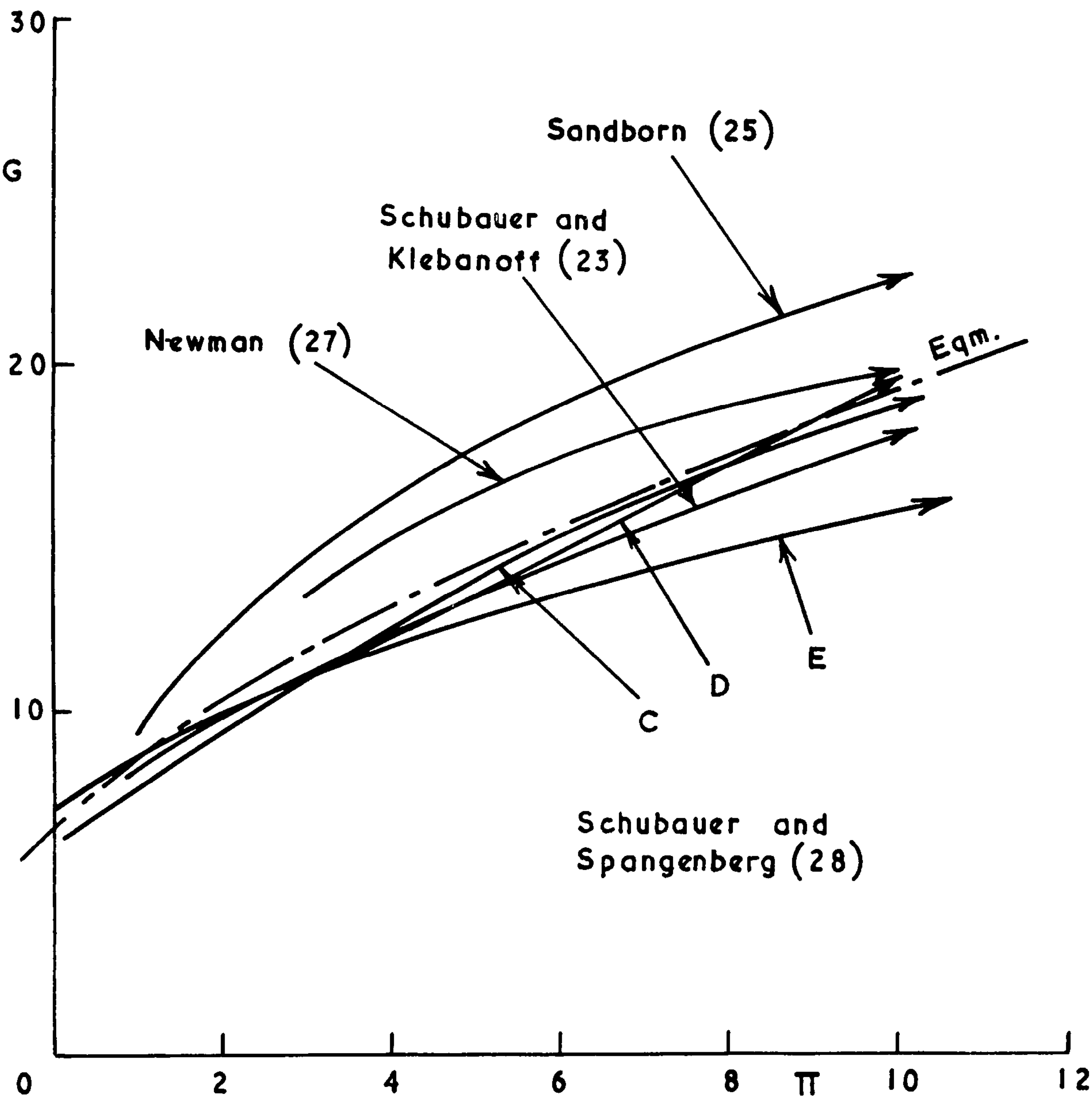
Equilibrium boundary layer — comparison between theory and experiment

FIG. 2



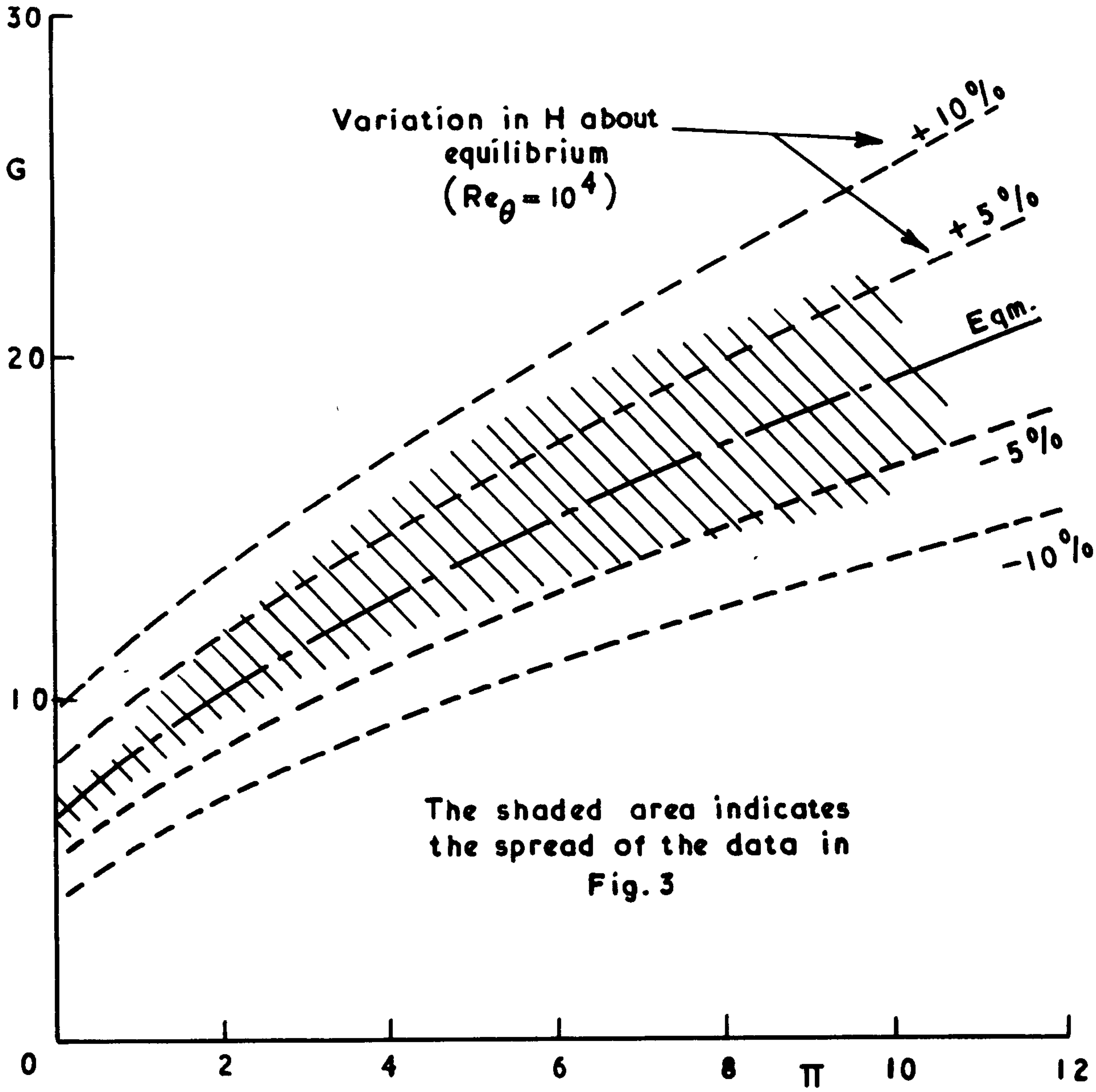
Return to equilibrium ($\Pi = 0$) following a perturbation

FIG. 3



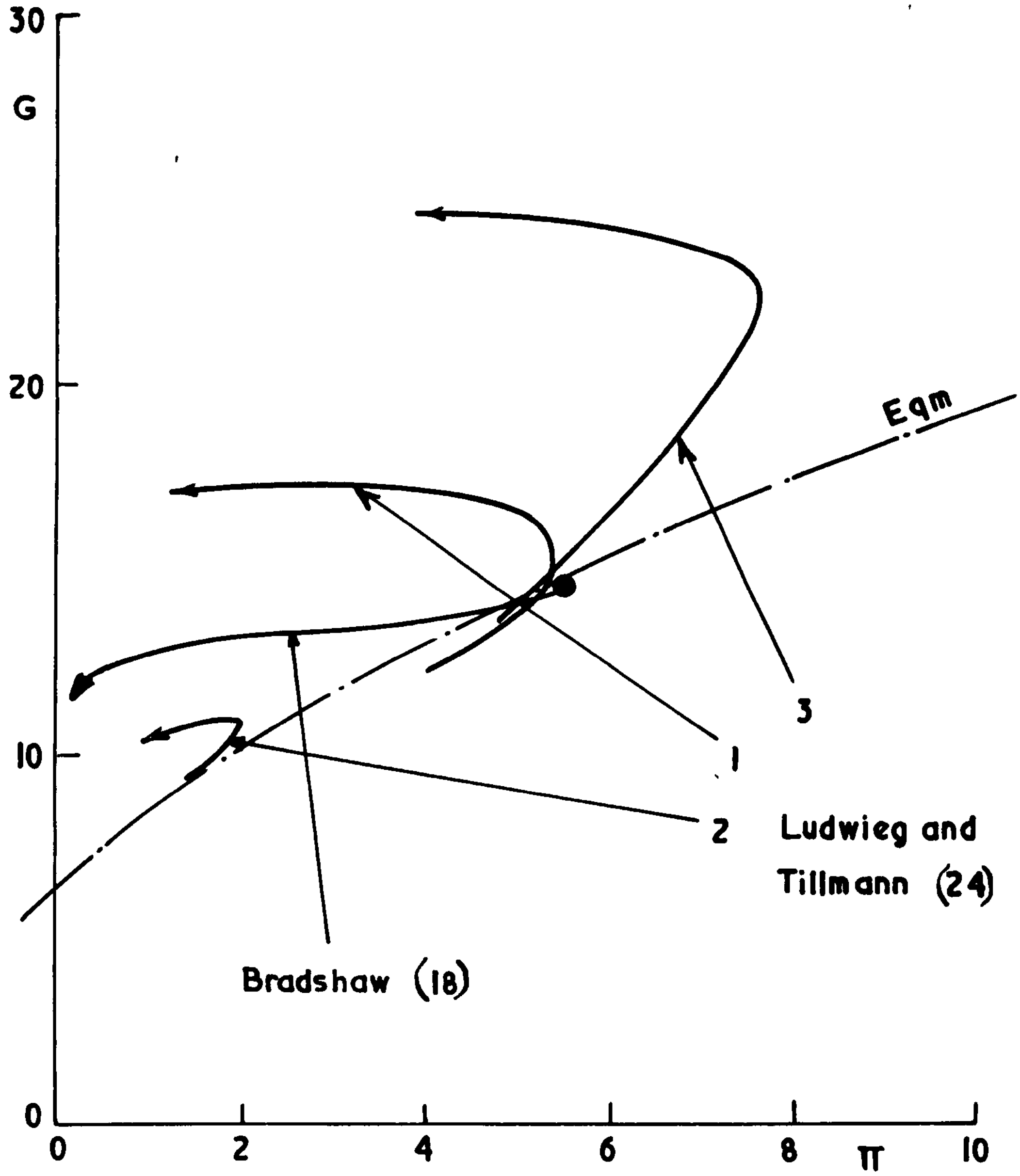
Turbulent-boundary-layer trajectories—augmentation of pressure gradient

FIG. 4



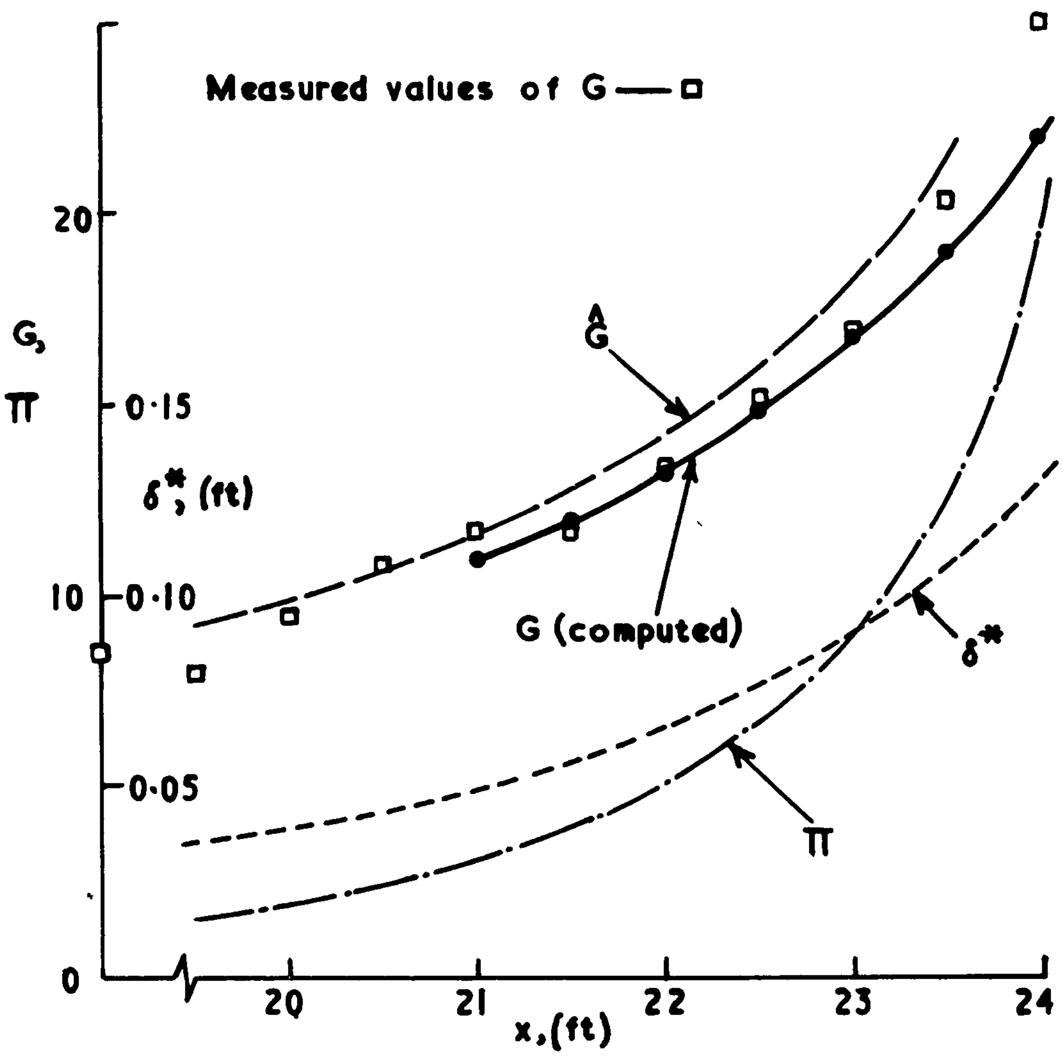
Proximity to "local equilibrium" in boundary layers with increasing pressure gradients

FIG. 5



Turbulent – boundary – layer trajectories – alleviation of pressure gradient

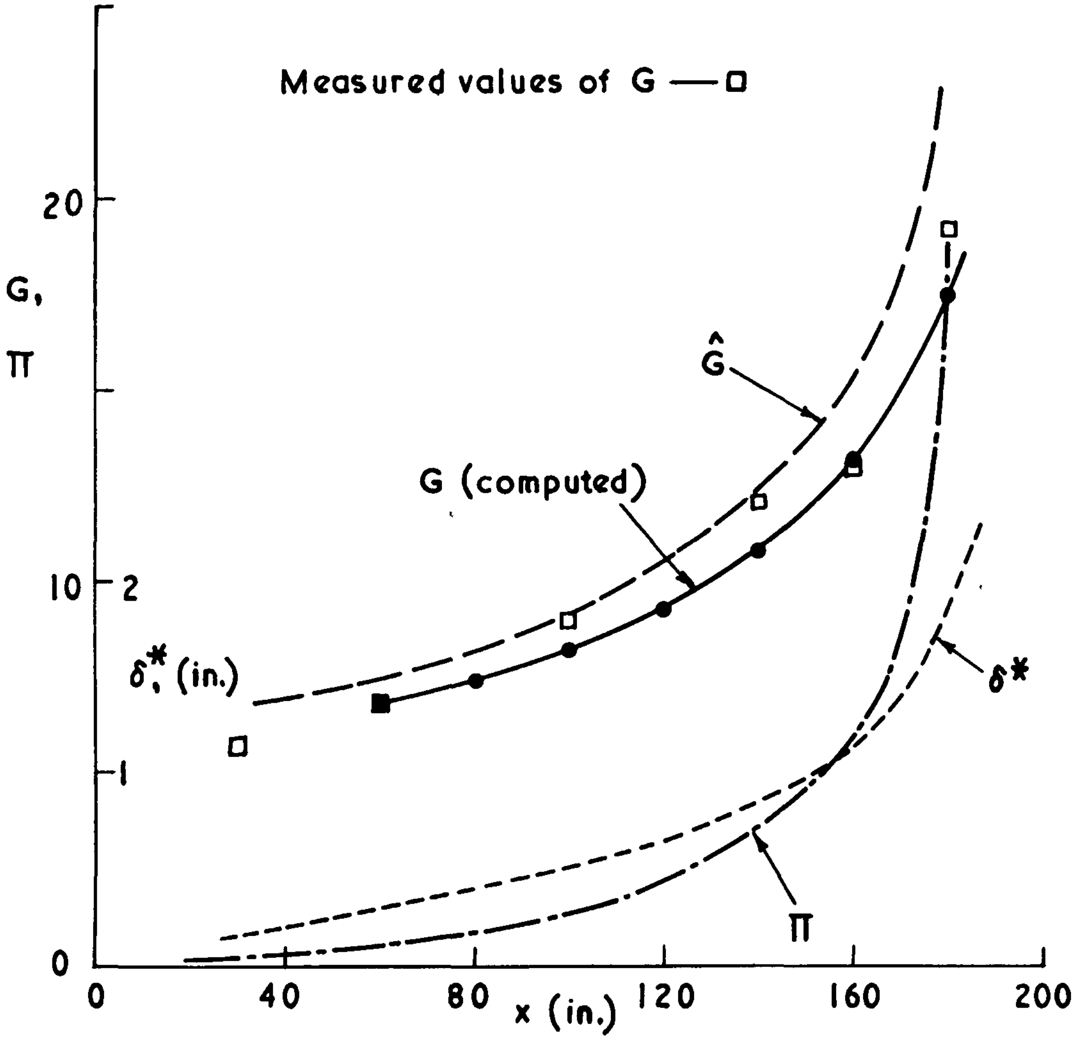
FIG. 6



Comparison between new auxiliary equation and experiment

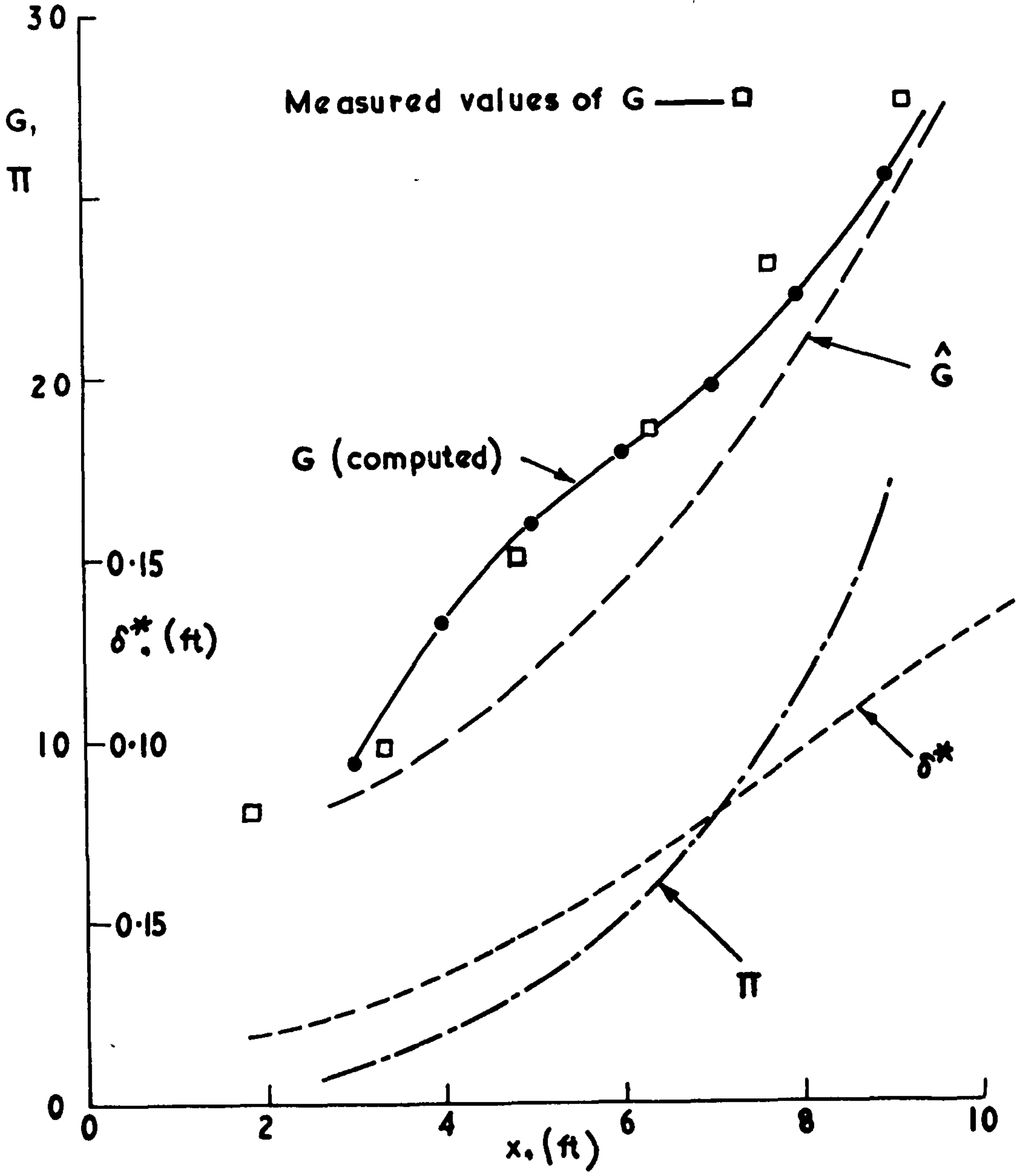
Schubauer and Klebanoff (Ref. 23)

FIG. 7



Comparison between new auxiliary equation and experiment: Schubauer and Spangenberg (E). (Ref. 28)

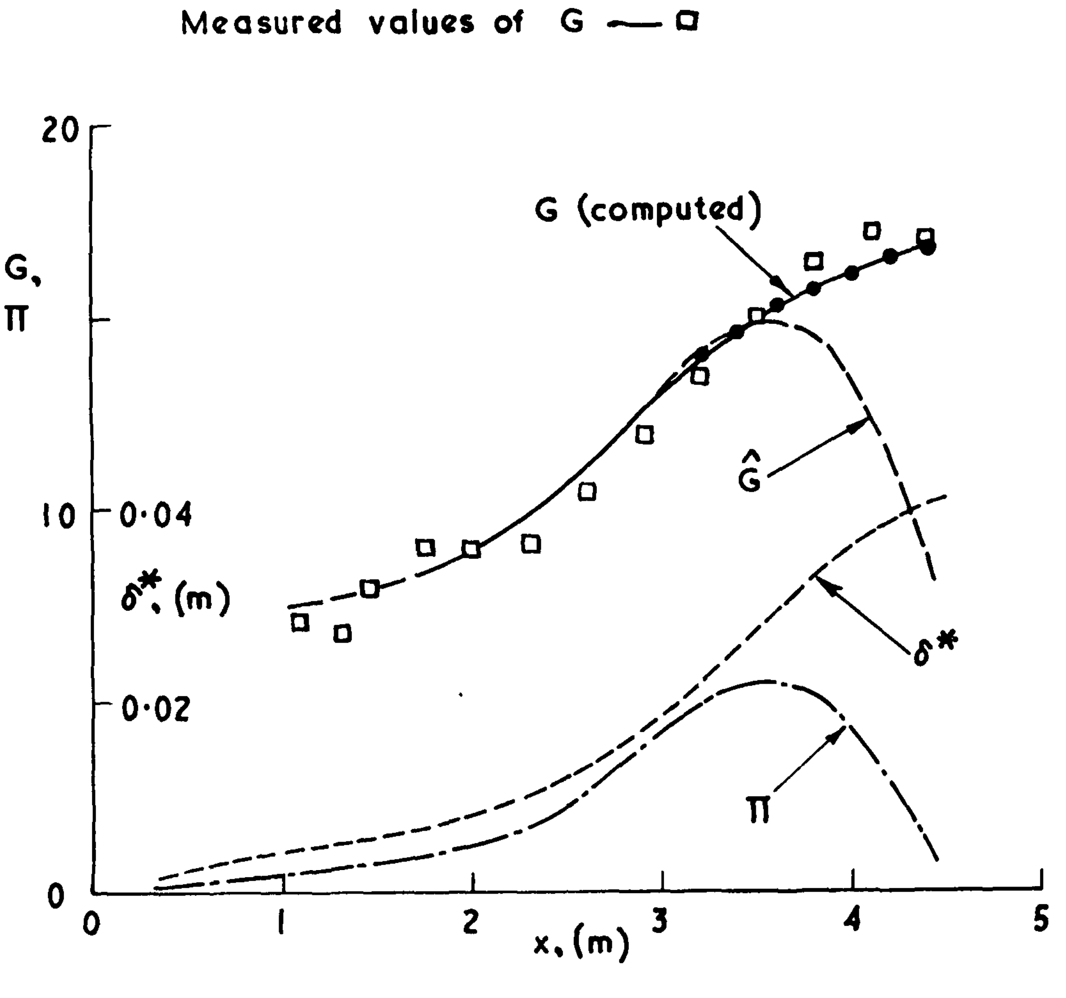
FIG. 8



Comparison between new auxiliary equation and experiment:

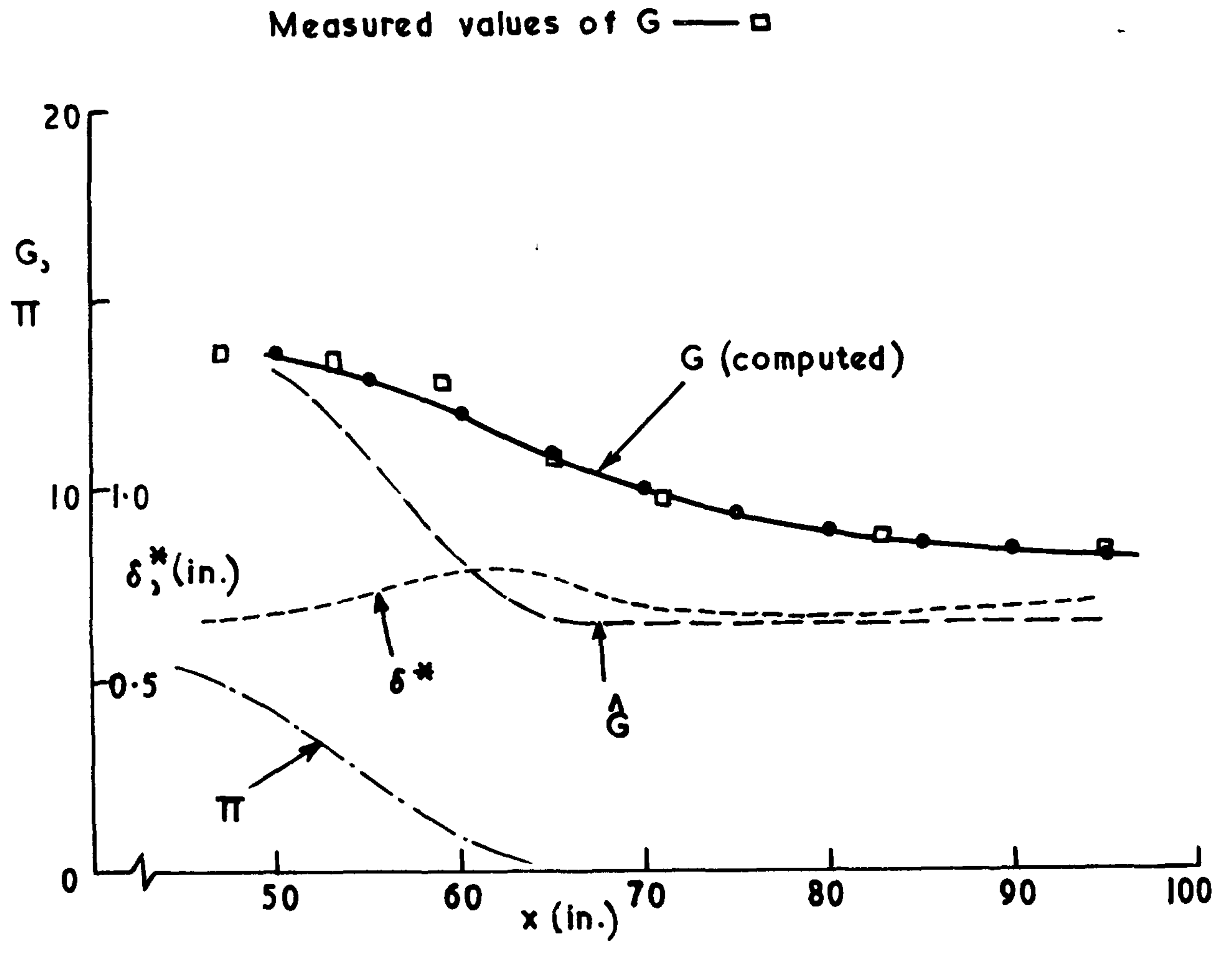
Sandborn, (Ref. 25)

FIG. 9



Comparison between new auxiliary equation and experiment: Ludwieg and Tillmann (1), Ref. 24)

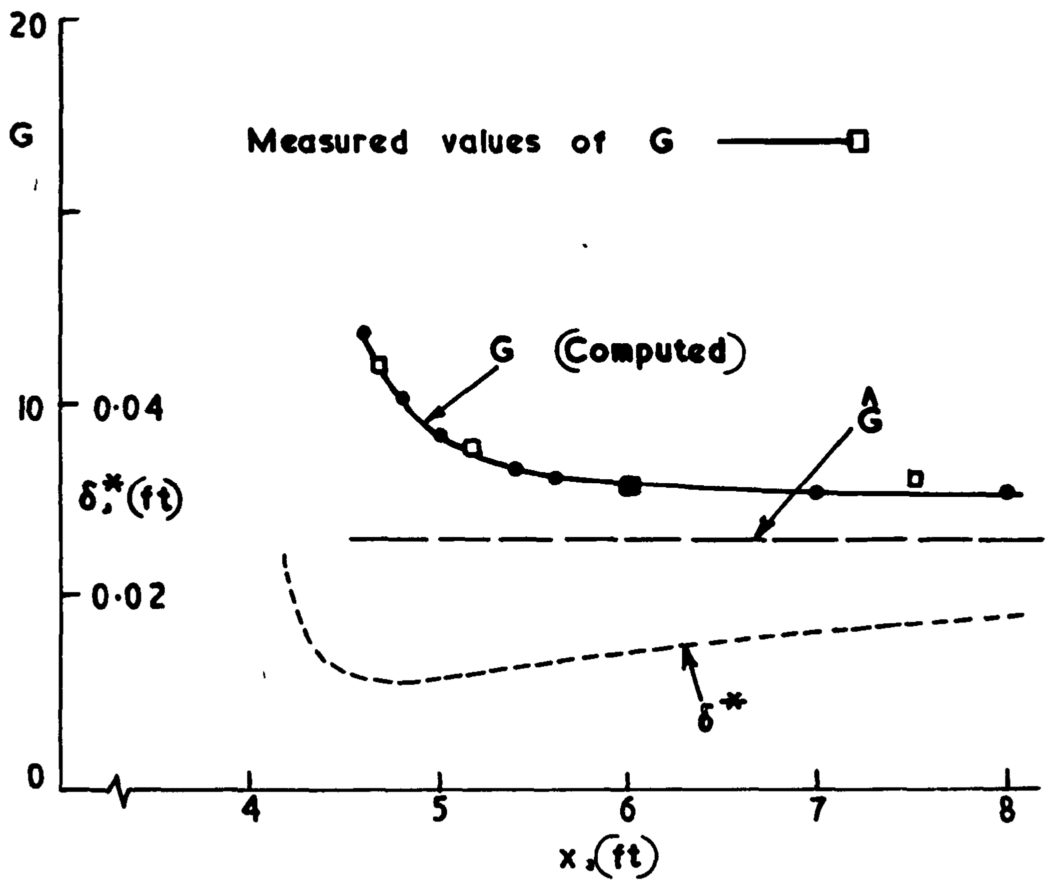
FIG. 10



Comparison between new auxiliary equation and experiment:

Bradshaw (Ref. 18)

FIG. 11



Comparison between new auxiliary equation and experiment: Klebanoff and Diehl, (Ref. 21)

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