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On the Calculation of Cascade Flows

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SUMMARY

This paper considers the inviscid incompressible flow, uniform at infinity upstream, past a two-dimensional cascade of aerofoils, and in particular the problem of determining the flow field, given the geometrical characteristics of the cascade.

The literature on this problem is already very extensive but in spite of this, there is a need for a definitive method of solution, to any desired degree of accuracy, which can readily be applied as a routine procedure. Especially is the need felt to incorporate into such a standard method the power of the electronic computer, and the analysis needs to be positively fashioned to take full advantage of such machines.

Consideration of the inherent features of the two principal methods of tackling such Laplacian problems leads to the adoption of a method based on conformal transformations (closely akin to that of Garrick, 1944) in preference to one based on distributions of singularities. Such comparisons as have been made between the results obtained by these two approaches suggest that our preference is justified.

The problem of design is notoriously more difficult and we do not report, in this paper, any progress towards a practical and accurate design method. Perhaps, however, the thought might be expressed that design ideas may be formulated on the basis of numbers of solutions of the direct problem.

It is hoped that, if this work can be regarded as a definitive procedure for the highly idealised case of potential flow about a cascade of aerofoils, others may be encouraged to build on it methods equally valid for the more complicated flows which occur in the reality of turbo-machinery.

1. Introduction

Until recent years, the technology of gas turbine engines was not sufficiently advanced in any of its aspects to necessitate accurate methods of performance estimation; certainly, engine designers have not been accustomed to fluid-dynamic theories as accurate as those of external aerodynamics.

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However, there are many indications that the efficiency of gas turbines has reached some kind of limit according to available design methods, many of which in any case describe flow conditions only in the large, rather than in detail. This is particularly true of multi-stage compressors in which, for example, it is still virtually impossible accurately to take into account non-uniformities in the inlet flow.

A full theory of viscous compressible flow through an annulus of varying cross-sectional area in which there are a number of rotor and stator stages is clearly beyond our reach at the present time; Wu (1952) has attempted to state the general problem, but there are limitations even in his statement of it, and in any case, there is much to be done before his work yields a practical method of calculation. Also, there are as yet, in this field of engine aerodynamics, no theories, however specialised, as highly developed and accurate in application as the corresponding theories in external aerodynamics.

In this situation, therefore, it seems worthwhile to return to some of the simplest idealisations so as to establish methods of solution of certain accuracy on which to build more complicated theories. Such a procedure is unlikely to give the engine designer everything he wants in the immediate future, and he will no doubt have to continue to rely for some time on more or less ad hoc and approximate theories. But in the long-term view, there is much to be said for disposing fully of the direct and indirect problem of a single two-dimensional cascade-row in incompressible inviscid flow, before going on to consider the complications due, in turn, to viscosity, compressibility, axisymmetry and a number of stages.

This paper attempts a definitive solution to the problem of determining this most highly idealised flow past a cascade row of prescribed geometrical shape. We do not seek a solution which can in some sense be displayed analytically in closed form, partly because even here the flow is too complicated to admit of the kind of formula typical of, say, linearised aerofoil theory, and partly because in any case the electronic computer is a legitimate constituent of any modern method. Thus our aim has been to develop a theory and its associated computational techniques which, when taken together, enable a solution of any desired degree of accuracy to be obtained.

It is instructive to admit, at this point, that much of the analysis in this paper has, in one form or another, appeared in the literature over the last thirty years, and it may be asked, if this is so, whether further work on similar lines adds anything of value. The answer, in part, is that earlier theories of cascade flow - for example, those of Collar (1940), Merchant (1940), Merchant and Collar (1941), Garrick (1944), Mutterperl (1944), Carter and Hughes (1946), Katzoff, Finn and Laurence (1947), Howell (1948), Isay (1953), Murai (1955), Schlichting (1955), Mortensen (1959), Polacek (1959), Czibere (1960), Riegels (1961), and Pollard and Wordsworth (1962) - were devised with the desk calculating machine in mind. Such methods are not necessarily the most appropriate in this computer age and a modern method should be constructed so that, both in detail and in its overall features, it is wholly suitable for modern numerical techniques.

As to these overall features, there are two principal techniques available for solving the two-dimensional Laplace equation with conditions prescribed on given boundaries, namely the use of conformal transformations and representation by a suitable distribution of singularities.

The representation of the flow by singularities within the body contour has, of course, been highly developed in isolated aerofoil theory and is in effect the basis of linearised theories; the first applications to cascade flow seems to have been made by Betz (1942) and Katzoff, Finn and Laurence (1947). In their work the first approximation to the flow about a

cascade/

cascade of blades is taken to be the flow about one of the blades in an infinite stream and this is represented by a distribution of discrete singularities within that blade's contour. This distribution of singularities is then assumed to be repeated in each of the remaining blades, and Betz gives a figure from which the effect of the row of equally-spaced singularities on the first-chosen blade can be calculated. In this way the original flow past the single blade can be modified and an iteration set up. This method was improved by Schlichting (1955) who gave the analytical expression for the velocity potential due to the series of equally-spaced singularities; a singular integral equation results.

Pollard and Wordsworth (1962) tackle the problem by assuming continuous distributions of singularities corresponding to the individual terms of a Fourier series. The boundary conditions are then satisfied at a number of discrete points on one of the blades to give a set of simultaneous equations for the Fourier coefficients. A number of simplifying approximations, similar to those of linearised theories, are employed and they restrict the validity of the method to blades of small camber. As it deals largely with simultaneous linear equations, the method is well suited to electronic computers.

A further refinement of the singularities method has been introduced by Martensen (1959) who transforms Schlichting's singular integral equation into a regular Fredholm equation of the second kind. Also, in his method, the singularities are placed on the blade contour itself, permitting consideration of a wider range of blade shapes.

Turning now to methods involving the use of conformal transformations, the first step normally consists of an initial transformation of the infinite series of aerofoils into a single closed contour in the whole plane by some periodic transformation. Simple standard transformations are then usually used to obtain a near-circle. Finally, the near-circle is transformed to an exact circle by Theodorsen's method. The flow past the cascade can now be derived from the known flow about a circle.

The practical success of such methods obviously depends on the transformations used; in particular, it is desirable to minimise the number of transformations and to avoid the creation of highly irregular shapes. A favourite initial transformation has been $\zeta = \tanh z$, used in this country notably by Howell (1948) and by Pollard and Wordsworth (1962); but it usually results in S-shaped figures which both authors had to treat with several successive Joukowski transformations in each of which it was necessary to choose the co-ordinate axes. Further, although all methods tend to become more difficult to apply as the pitch-chord ratio decreases, the tanh-transformation is particularly unsuitable in this respect.

A rather different method was put forward by Mutterperl (1944) in which the cascade of blades is transformed into a cascade of flat plates, the chord length being kept constant. The flat plates are then transformed into a circle by an equation similar to that used by Murai (1955). The first transformation is, of course, initially unknown and has to be determined by an iterative procedure; but a peculiarity of the method is that the final transformation to the circle is involved in the iteration and on the whole this gives an unfavourable comparison with the methods of, say, Howell or Murai in which each transformation can be made independently of the others.

There are naturally many variants of the two principal lines of attack which we have exemplified by quoting a few of the best-known papers, but enough may have been said to support our view that the singularity method is the less valid of the two when it comes to laying down a definitive form. There are two main reasons for this view. First, any method based on a finite number of linearly-dependent unknown quantities is vulnerable to the

effects/

effects of the choice of pivotal points and to the possibility of ill-conditioned equations; and what is more, there are no known methods of guarding against these effects. Second, the algebraic complications become such as to make highly desirable some sort of linearised approximation in which the boundary conditions and values are not necessarily satisfied exactly on the boundary.

Transformation methods, however, are not completely free from disadvantages; their difficulties arise mainly in the choice and number of the transformations and especially of the co-ordinate axes.

This paper proposes the use of only two transformations (apart from a trivial translation), the first being that used by Garrick (1944) and the second that of Theodorsen (1932). Indeed the whole procedure given here follows Garrick's very closely with, however, two significant improvements. First, great attention is paid to the choice of axes and parameters involved in the first transformation so that the given blade contours are mapped into as smooth a near-circular shape as possible. Second, the computation of Theodorsen's theory is improved by the treatment given by Thwaites (1963) which, by using various formulae given first by Watson (1945), avoids the explicit determination of the coefficients of the Laurent series in the transformation.

2. Notation

(a) Geometrical properties of the given cascade

β	stagger angle
$\delta\beta$	parameter in T_1; its derivation is given in Section 7
c	chord length of a cascade blade
δc	parameter in T_1; its derivation is given in Section 7
d	pitch
γ	parameter in T_1 given by equation (6)
$(x_n, y_n), n = 0, 1, \ldots 2N-1$	co-ordinates of the cascade referred to the axes of Fig. 4b.
ρ_L	radius of curvature of the leading edge of a cascade blade
ϵ_L	angle between the directions of the blade chord and the normal at the blade leading edge
ρ_T	radius of curvature of the trailing edge of a cascade blade
ϵ_T	angle between the directions of the blade chord and the normal at the blade trailing edge

(b) Other notation

$z = x + iy$	complex variable in the plane of the cascade
$z' = x' + iy'$	alternative complex variable in the plane of the cascade

$\zeta = \xi + i\eta$	complex variable in the plane of the near-circle
$e^{\psi+i\theta}$	point on the near-circle in the ζ-plane
$\sigma = \psi + i\theta$	
$e^{i\theta_T}$	point in the ζ-plane corresponding to the sharp trailing edge of the given aerofoil
$\bar{\zeta}$	weighted mean of the points of the near-circle in the ζ-plane
$\zeta' = \xi' + i\eta' = \zeta - \bar{\zeta}$	complex variable in the plane of the displaced near-circle
$e^{\psi'+i\theta'}$	point on the near-circle in the ζ'-plane
$Z = X + iY$	complex variable of the plane of the exact circle
$e^{\bar{\Psi}+i\phi}$	point on the circle in the Z-plane, whose radius is therefore $e^{\bar{\Psi}}$
$R = e^{\bar{\Psi}}$	
a_n, $n = 1,2, \ldots$	complex coefficients in Theodorsen's transformation
T_1	transformation from the z-plane to the ζ-plane
T_2	transformation from the ζ-plane to the ζ'-plane
T_3	transformation from the ζ'-plane to the Z-plane (Theodorsen's transformation)
W	velocity on the circle in the Z-plane
f,g,h	functions used in calculating W
$V_1 e^{i\alpha_1}$	velocity at infinity upstream of the cascade
$V_2 e^{i\alpha_2}$	velocity at infinity downstream of the cascade
v	fluid speed on a cascade blade
p	pressure on a cascade blade
ρ_1	fluid density.

3. Outline of General Procedure

We refer first to Figs. 1 and 2. In the z-plane it is assumed that the given shape of the cascade blades will be expressed as a list of randomly distributed co-ordinates of the surface.

The transformation T_1 is designed to produce a reasonably regular curve approximating to a circle in the ζ-plane; the details are given in Sections 6 and 7. For the moment it is sufficient to remark that, unlike some of the other methods, T_1 is entirely determined by the various geometrical properties of the cascade, and thus that a set of points is then known on the contour in the ζ-plane.

Now for the efficient application of Theodorsen's transformation, it is desirable that the variation of radial distance from the origin to the

near-circle/

near-circle contour is, in some sense, minimised. Thus the ζ-plane contour is displaced by a constant $\bar{\zeta}$, of which the derivation is given in Section 5, so that in the ζ'-plane the radial distance - now $e^{\psi'}$ - is more nearly uniform. Thus in this plane too, a set of points is known.

At the final stage, however, the transformation into a circle in the Z-plane is best done when the points are given at equal intervals of θ'. Thus an interpolation is carried out in the ζ'-plane to obtain these equally-spaced values of $\psi'(\theta')$.

The final transformation is arranged so that $|d\zeta'/dZ|$ is evaluated at these equally-spaced points in the ζ'-plane and so that the corresponding points on the Z-plane circle are known.

Finally it is necessary to relate the now equally-spaced ζ'-values to points on the cascade but this only involves straightforward calculation through the transformations T_1 and T_2.

We go on to discuss the general features of the three transformations.

4. T_3: Near-Circle to Circle

Theodorsen's theory is well known and depends on the fact that, on the boundaries, the transformation given in equation (3) of Fig. 2 reduces to the form

$$\psi'(\theta') - \bar{\Psi} - i(\phi - \theta') \;=\; \text{value on the circle } |Z| = e^{\bar{\Psi}} \text{ of the function } \sum_{n=1}^{\infty} a_n Z^{-n} \text{ which is analytic outside the circle.}$$

Hence $(\psi'(\theta') - \bar{\Psi})$ and $(\phi - \theta')$ are related by Poisson's integrals and the correspondence between the two planes is thus established. The analytical details and the computational procedure adopted in this paper are given by Thwaites (1963) with the improvement that this differentiation matrix is replaced by one derived from a Lagrangian partial range differentiation The method requires the values of ψ' to be given at equally-spaced values of θ'. To obtain these from the randomly-sited values of ψ' which result from the previous transformation, seventh-order Lagrangian interpolation polynomials are used, and these are found to produce sufficient accuracy.

In our present problem an additional result is required, as will be seen later, namely the calculation of the point Z in the Z-plane which corresponds to a given point ζ' exterior to the near-circle. Following Appendix C of Garrick (1944) we apply Cauchy's formula to the function

$f(z) = \sum_{n=1}^{\infty} a_n Z^{-n}$ in equation (3) to give

$$\log\left(\frac{\zeta'}{Z}\right) = -\frac{1}{\pi}\int_0^{2\pi} \Psi(\phi)\{1 - (Ze^{-\bar{\Psi}})e^{-i\phi}\}d\phi = f(Z) \qquad \ldots(4)$$

where $\Psi(\phi) = \psi'(\theta')$.

This equation (4) is an implicit equation for Z and can be simply solved by an iteration in which the n^{th} value, Z_n, of Z is inserted in the integral to give an improved value, Z_{n+1}, where $Z_{n+1} = \zeta'\exp(-f(Z_n))$. Z_1 may be taken as ζ'.

5./

5. T_2: Displacement of Near-Circle

The ζ-plane contour will, it is hoped, be approximately circular, but there is no reason for the origin in the plane to be near the centre of the approximating circle. There is obviously no exact criterion by which to determine this centre, and so some rough and ready procedure is sufficient, especially since the method of computing T_3 is by no means critically dependent on the choice of origin.

The method employed here is to determine $\bar{\zeta}$ as something in the nature of a weighted mean of the given points in the ζ-plane. If these are denoted by ζ_n, n = 0, 1, ... (2N-1), then the weighting given to the point ζ_n is $\frac{1}{2}r_n(\theta_{n+1} - \theta_{n-1})$ in the notation of Fig. 3. As a result, the contour will be more evenly spaced round the origin in the $(\zeta-\bar{\zeta})$-plane, that is the ζ'-plane, than in the ζ-plane.

6. T_1: Cascade to Near-Circle

We refer to Fig. 1 again for some basic notation and to Fig. 4 for further details of T_1 given by equation (1).

T_1 is a modification of equation (6b) given by Garrick (1944). Its characteristic property is the mapping of the exterior of $|\zeta| = 1$ onto the whole z-plane cut along an infinite number of straight, finite periodically-spaced lines.

To be more precise, the central cut is the straight line L'T joining the point $-(c-\delta c)e^{-i\delta\beta}$ to the origin in the z-plane. The ζ-points corresponding to T and L' are $\zeta = \pm e^{i\theta_T}$ respectively where

$$\tan\theta_T = \tanh\gamma \tan(\beta-\delta\beta) \qquad \ldots(5)$$

and θ_T is an acute angle. Here γ is a convenient parameter which can be thought of as depending primarily on the gap-chord ratio.

In fact, the equation connecting the geometrical quantities d, (c-δc), β-δβ and γ is

$$\frac{\pi}{2}\left(\frac{c-\delta c}{d}\right) = \cos(\beta-\delta\beta)\log\left[\frac{\cos(\beta-\delta\beta) + \{\cosh^2\gamma - \sin^2(\beta-\delta\beta)\}^{\frac{1}{2}}}{\sinh\gamma}\right]$$
$$+ \sin(\beta-\delta\beta)\tan^{-1}\left[\frac{\sin(\beta-\delta\beta)}{\{\cosh^2\gamma - \sin^2(\beta-\delta\beta)\}^{\frac{1}{2}}}\right] \qquad \ldots(6)$$

which is easily verified by expressing the length (c-Sc) as the distance L'T between the singular points of T_1, at which $dz/d\zeta = 0$. Thus in principle - and the practical problem will be discussed in Section 7 - the value of γ may be found from this equation (6) from the geometrical properties of the cascade.

Let us now look at some typical sets of contours in the z-plane which correspond to circles in the ζ-plane and, to begin with, we might consider the set of concentric circles $|\zeta| = e^{\psi} > 1$, together with the orthogonal set of radial lines θ = constant. Fig. 5 gives a typical picture in the case $(\beta-\delta\beta) = -30°$ and $(c-\delta c)/d = 0{\cdot}3866$.

If a given blade profile from a cascade of the same gap/chord ratio and stagger angle as those of the figure were to conform to one of the full

lines,/

lines, then the solution follows at once. But in practice the problem will involve placing a given cascade shape, on a suitable scale, in the z-plane so that, as has already been discussed, the ζ-plane contour is closely circular. Now the ψ-constant curves in Fig. 5 appear substantially different from typical blade shapes, but this in itself does not imply that realistic shapes cannot be transformed into ζ-plane circles; it implies only that such circles will certainly not have their centres at the origin. Therefore it is of interest to examine the z-plane contours corresponding to other sets of ζ-plane circles.

Now most real turbine blades are more or less cusped at the trailing edge; such a trailing edge clearly should be located at one of the singularities of the transformation, namely at T (which has been arranged for convenience to lie at the origin of the z-plane). The other singularity L' of the transformation should lie within the blade contour, and so we might consider the family of circles shown in Fig. 6. The z-plane contours corresponding to these circles are shown as full lines in Fig. 7, and the contour corresponding to the circle which passes through T is of a more realistic shape than the curves in Fig. 5. Conversely, one is led to hope that a given cascade could in fact be placed in the Z-plane to give a ζ-plane contour reasonably close to a circle.

The first step then for blades cusped at the trailing edge, is to place the trailing edge at one of the singularities of the transformation. There are then two parameters $\delta\beta$ and δc appearing in T_1 which need to be determined, and this is done by matching the radius of curvature and normal direction of the appropriate ψ-constant curve with the radius of curvature and camber line direction of the given blade at its leading edge. This procedure is carried through in Section 7.

However, for blades with rounded trailing edges, the position of T the trailing edge must also be moved away from T' the corresponding singularity of the transformation, and the analysis for this is briefly given as part of the worked example in Section 9(b).

7. Calculation of $\delta\beta$ and δc

The matching problem at the leading edge is attempted by an approximate analysis, linear in $\delta\beta$ and $\delta c/c$, which is justified a posteriori on the grounds that these two parameters are found to be of order ρ_L/c, the ratio of the leading-edge radius of curvature to the blade chord length. On the assumption that $\delta\beta$ and $\delta c/c$ are $O(\rho_L/c)$, equation (1) gives

$$F(\zeta) = F_o(\zeta)(1 + O(\rho_L/c)) \qquad \ldots(7)$$

$$\text{where} \quad F_o(\zeta) = \frac{d}{2\pi}\left\{e^{-i\beta}\log\frac{e^{\gamma}+\zeta}{e^{\gamma}-\zeta} + e^{i\beta}\log\frac{\zeta+e^{\gamma}}{\zeta-e^{\gamma}}\right\} - \tfrac{1}{2}c. \qquad \ldots(8)$$

We will write

$$F(\zeta) = F_o(\zeta) \qquad \ldots(9)$$

on the understanding that the errors in equations (9), (10) and (12) can be represented by a factor $(1 + O(\rho_L/c))$.

We/

We have, at the singularity L',

$$\left.\begin{aligned} &\left(\frac{dF}{d\zeta}\right)_{L'} = 0 \\ \text{and also} \quad &\left(\zeta^2 \frac{d^2F}{d\zeta^2}\right)_{L'} = -Bd \end{aligned}\right\} \qquad \ldots(10)$$

where

$$B = \frac{1}{2\pi} \frac{(\cosh 2y + \cos 2\beta)^2}{(\cosh^2 y - \sin^2\beta)\sinh^2 2y} \qquad \ldots(11)$$

and is real. In terms of the variable σ, where $\zeta = e^{\psi+i\theta} = e^{\sigma}$, equations (10) are

$$\left(\frac{dF}{d\sigma}\right)_{L'} = 0; \quad \left(\frac{d^2F}{d\sigma^2}\right)_{L'} = -Bd. \qquad \ldots(12)$$

With these values, an expansion of z, in terms of σ, may be made in the neighbourhood of L'.

At this stage it is, in fact, more convenient to develop the analysis in terms of the two parameters A and δ, shown in Fig. 4, rather than $\delta\beta$ and δc. The point L' is then given by

$$z = z_{L'} = -c\{1 - Ae^{i\delta}(\rho_L/c)\}. \qquad \ldots(13)$$

Once A and δ have been found, the required values of $\delta\beta$ and δc follow easily.

Thus, near L', we have, from equations (1), (12) and (13), that

$$z = -c + \rho_L Ae^{i\delta} - \frac{Bd}{2}\{\sigma - i(\theta_T+\pi)\}^2 \left[1 + 0\left(\frac{\rho_L}{c}\right) + 0(\sigma - i(\theta_T+\pi))\right]. \qquad \ldots(14)$$

Thus the value, σ_L, of σ at the leading edge (where $z = -c$) is given by

$$\{\sigma_L - i(\theta_T+\pi)\}^2 = \frac{2}{Bd}\rho_L Ae^{i\delta}\left[1 + 0\left(\frac{\rho_L}{c}\right) + 0(\sigma - i(\theta_T+\pi))\right]. \qquad \ldots(15)$$

Equation (15) shows that $\{\sigma_L - i(\theta_T+\pi)\} = 0(\rho_L/c)^{\frac{1}{2}}$. Hence, sufficiently close to the leading edge, the errors in equations (14) and (15) can be represented by the factor $(1 + 0(\rho_L/c)^{\frac{1}{2}})$, which is now understood to be the error in subsequent equations.

To calculate δ we use the condition that the tangent to the blade camber line at the leading edge should be parallel to the normal to the appropriate ψ-constant curve, or to the tangent to the orthogonal θ-constant curve. Now, for a θ-constant curve,

$$\frac{dz}{d\sigma}\Big/$$

$$\frac{dz}{d\sigma} = \frac{dz}{d\psi} = -Bd\{\sigma - i(\theta_T+\pi)\} \qquad \ldots(16)$$

from equation (14). Thus the required condition is

$$\arg dz = \arg[-Bd\{\sigma - i(\theta_T+\pi)\}] = \arg\{\sigma - i(\theta_T+\pi)\} \qquad \ldots(17)$$

since B is real. Thus with the value at the leading edge given by equation (15),

$$\arg dz = \frac{\delta}{2}$$

while from Fig. 4, $\arg dz = \epsilon_L$. Thus

$$\delta = 2\epsilon_L. \qquad \ldots(18)$$

Once δ has been found, the condition that the radius of curvature of the appropriate ψ-constant curve and the radius of curvature of the leading edge should be the same, can be used to find A. The curve in the z-plane in the neighbourhood of L' which corresponds to a small arc of a circle, centre the origin, in the ζ-plane is given by

$$x = \left\{ -c + \rho_L A \cos\delta - \frac{Bd}{2}\psi^2 \right\} + \frac{Bd}{2}(\theta - \theta_T - \pi)^2,$$

$$y = \rho_L A \sin\delta - Bd\,\psi(\theta - \theta_T - \pi), \qquad \ldots(19)$$

and its radius of curvature is given by

$$\frac{[\{Bd(\theta-\theta_T-\pi)\}^2 + (-Bd\psi)^2\}^{\frac{3}{2}}}{-(-Bd\psi)Bd} = \frac{Bd\{\psi^2 + (\theta-\theta_T-\pi)^2\}^{\frac{3}{2}}}{\psi}. \qquad \ldots(20)$$

Now at the leading edge, equation (15) gives the values

$$\psi = \psi_L = \left(\frac{2\rho_L A}{Bd}\right)^{\frac{1}{2}} \cos\frac{\delta}{2}; \quad \theta = \theta_L = \theta_T + \pi + \left(\frac{2\rho_L A}{Bd}\right)^{\frac{1}{2}} \sin\frac{\delta}{2}.$$

Equation (20) then yields

$$\rho = Bd\left(\frac{2\rho_L A}{Bd}\right)^{\frac{3}{2}} \Bigg/ \left\{\left(\frac{2\rho_L A}{Bd}\right)^{\frac{1}{2}} \cos\frac{\delta}{2}\right\}$$

$$= 2\rho_L A \sec \epsilon_L, \quad \text{from equation (18)},$$

which gives

$$A = \tfrac{1}{2}\cos\epsilon_L. \qquad \ldots(21)$$

Now that A and δ have been found it is a simple matter to calculate, from Fig. 4, the values of the parameters $\delta\beta$ and δc in terms of the blade geometry. They are

$\delta\beta$/

$$\left.\begin{aligned} \delta\beta &= \frac{\rho_L}{2c}\cos\epsilon_L \sin 2\epsilon_L, \\ \frac{\delta c}{c} &= \frac{\rho_L}{2c}\cos\epsilon_L \cos 2\epsilon_L \end{aligned}\right\} \qquad \ldots(22)$$

8. The Flow about the Cascade

We proceed to describe the method by which the flow about the cascade is calculated now that the transformations which map a given cascade onto a circle have been completely determined.

Consider the flow in the Z-plane about the circle $|Z| = R$ given by the complex potential

$$\Omega(Z) = \frac{Vd}{2\pi}\left[e^{i\alpha}\log\frac{\beta_1+Z}{\beta_2-Z} + e^{-i\alpha}\log\frac{Z+R^2/\bar{\beta}_1}{Z-R^2/\bar{\beta}_2}\right] - \frac{i\Gamma}{4\pi}\log\frac{(Z+R^2/\bar{\beta}_1)(Z-R^2/\bar{\beta}_2)}{(Z+\beta_1)(\beta_2-Z)} \qquad \ldots(23)$$

which arises from concentrated sources, sinks and vortices placed as shown in Fig. 8, where $-\beta_1$, β_2 are the points corresponding to the points at infinity in the plane of the cascade.

$W = |d\Omega/dZ|$, the speed on the circle*, can be calculated as

$$W = f\cdot\left(\frac{Vd\cos\alpha}{\pi}\right) + g\cdot\left(\frac{Vd\sin\alpha + \frac{\Gamma}{2}}{\pi}\right) + h\cdot\left(\frac{Vd\sin\alpha - \frac{\Gamma}{2}}{\pi}\right) \qquad \ldots(24)$$

$$\text{where}\quad f = \frac{1}{2R}\left\{\frac{\sin\phi_1}{\frac{1}{2}\left(m_1+\frac{1}{m_1}\right)+\cos\phi_1} + \frac{\sin\phi_2}{\frac{1}{2}\left(m_2+\frac{1}{m_2}\right)-\cos\phi_2}\right\}, \qquad \ldots(25)$$

$$g = -\frac{1}{4R}\left\{\frac{m_1-\frac{1}{m_1}}{\frac{1}{2}\left(m_1+\frac{1}{m_1}\right)+\cos\phi_1}\right\},$$

$$h = +\frac{1}{4R}\left\{\frac{m_2-\frac{1}{m_2}}{\frac{1}{2}\left(m_2+\frac{1}{m_2}\right)-\cos\phi_2}\right\},$$

and/

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*A similar expression for W was given by Howell (1948) and it may be useful to note that the source strengths are $\pm Va\pi$ in his notation where we have $\pm Vd\cos\alpha$, and the vortex strengths are $Vw_1\pi$ and $Vw_2\pi$ in his notation corresponding to our values $(Vd\sin\alpha + \Gamma/2)$ and $(Vd\sin\alpha - \Gamma/2)$.

and $$m_1 = |\beta_1| R^{-1}, \quad m_2 = |\beta_2| R^{-1}. \qquad \ldots(26)$$

Fig. 8 should be consulted for the definitions of ϕ_1 and ϕ_2.

We must now discover the interpretation that should be placed on the variables V and α appearing in equations (23) and (24), in terms of the physical properties of the flow through the cascade. It will be shown that, in fact, $Ve^{i\alpha}$ is the vector mean of the inlet and outlet velocities, $V_1 e^{i\alpha_1}$ and $V_2 e^{i\alpha_2}$; the angles α, α_1 and α_2 being measured from the line perpendicular to the stagger line (see Fig. 9).

The velocity in the z-plane is obtained from the equation

$$\frac{d\Omega}{dz} = \frac{d\Omega}{dZ} \cdot \frac{dZ}{d\zeta'} \cdot \frac{d\zeta'}{d\zeta} \cdot \frac{d\zeta}{dz'} \cdot \frac{dz'}{dz}, \qquad \ldots(27)$$

the terms of which can be evaluated using equations (1), (2), (3) and (23). Considering the flow at infinity upstream of the cascade we have from equation (27), since we are taking the direction of flow to be from left to right,

$$\left[\frac{d\Omega}{dz}\right]_{-\infty} \equiv V_1 e^{i(-\alpha_1+\beta)} = Ve^{i(\alpha+\beta)} + \frac{i\Gamma}{2d} e^{i\beta}. \qquad \ldots(28)$$

Similarly the outlet velocity vector is

$$\left[\frac{d\Omega}{dz}\right]_{+\infty} \equiv V_2 e^{i(-\alpha_2+\beta)} = Ve^{i(\alpha+\beta)} - \frac{i\Gamma}{2d} e^{i\beta}. \qquad \ldots(29)$$

Addition of equations (28) and (29) shows that the mean of the inlet and outlet velocities has magnitude V and is at an angle $(\alpha+\beta)$ to the x-axis. We can also show that the inlet and outlet angles are given by

$$\alpha_1 = -\tan^{-1}\left(\frac{\sin\alpha + \frac{\Gamma}{2Vd}}{\cos\alpha}\right); \quad \alpha_2 = -\tan^{-1}\left(\frac{\sin\alpha - \frac{\Gamma}{2Vd}}{\cos\alpha}\right). \qquad \ldots(30)$$

In the practical problem of finding the flow about a cascade we will be given the inlet velocity $V_1 e^{i\alpha_1}$ and the Joukowski condition at the trailing edge of the blade, from which the outlet velocity $V_2 e^{i\alpha_2}$ is calculated as follows. Rewrite equation (24), using equations (30), as

$$W = \frac{V_1 d\cos\alpha_1}{\pi}(f - g\tan\alpha_1 - h\tan\alpha_2) \qquad \ldots(31)$$

in which $V\cos\alpha$ has been replaced by $V_1\cos\alpha_1$, from the real part of equation (28). Then, for a stagnation point at the trailing edge, we have

$$f_T - g_T\tan\alpha_1 - h_T\tan\alpha_2 = 0, \qquad \ldots(32)$$

where the suffix T denotes values at the point on the Z-plane circle which corresponds to the trailing edges of the blades. Thus the outlet angle is given by

α_2/

$$\alpha_2 = \tan^{-1}\left(\frac{f_T - g_T \tan\alpha_1}{+h_T}\right). \qquad \ldots(33)$$

Also, from equations (28) and (29), the outlet velocity is given by

$$V_2 = V_1 \cos\alpha_1 \sec\alpha_2. \qquad \ldots(34)$$

The speed, W, on the circle can now be evaluated from equation (31) since α_2 is known. It is clear that, in our case, it will be most convenient to calculate W at the points on the circle with the angular displacements ϕ which result from Theodorsen's transformation; that is, corresponding to the interpolated, equally-spaced points on the near circle in the ζ'-plane. The points on the cascade corresponding to these can then be traced back through transformations T_2 and T_1. The speed, v, at these points on the cascade can then be calculated using the modulus of equation (27), which becomes

$$\left|\frac{d\Omega}{dz}\right| = v = W\left|\frac{dZ}{d\zeta'}\right| \cdot \left|\frac{d\zeta}{dz'}\right|, \qquad \ldots(35)$$

since $|d\zeta'/d\zeta| = 1$, T_2 being merely a translation. $|dZ/d\zeta'|$ is calculated as part of Theodorsen's method (see Thwaites, 1963); $|d\zeta/dz|$ can be derived from equation (1) and is given by

$$\left|\frac{dz'}{d\zeta}\right| = \frac{2d}{\pi|z'|} \times$$

$$\left[\frac{\cos^2(\beta-\delta\beta)\cosh^2\gamma(\cosh^2\psi-\cos^2\theta)+\sin^2(\beta-\delta\beta)\sinh^2\gamma(\cosh^2\psi-\sin^2\theta)-\frac{1}{4}\sin2(\beta-\delta\beta)\sinh2\gamma\sin2\theta}{\cosh^2 2\gamma - 2\cosh2\gamma\cosh2\psi\cos2\theta + \cos^2 2\theta + \sinh^2 2\psi}\right]^{\frac{1}{2}}.$$

$$\ldots(36)$$

The pressure on an aerofoil is usually given in terms of the pressure coefficient C_p, defined as $(p-p_1)/\frac{1}{2}\rho_1 V_1^2$, where p_1 is the free-stream pressure. Thus, by Bernoulli's equation

$$C_p = 1 - \frac{v^2}{V_1^2}. \qquad \ldots(42)$$

In the case of a profile with a cusped trailing edge, W and $|dz/d\zeta|$ are of course both zero at the trailing edge, so that v has to be calculated as a limit. It is easily found that

$$v_T = \left|\frac{d\Omega}{dz}\right|_T = \left|\frac{dZ}{d\zeta'}\right|_T^2 \left|\frac{d^2\Omega}{dZ^2}\right|_T \Bigg/ \left|\frac{d^2 z'}{d\zeta^2}\right|_T, \qquad \ldots(37)$$

where, from equation (1),

$$\left|\frac{d^2 z}{d\zeta^2}\right|_T = \frac{d\{\cosh^2\gamma - \sin^2(\beta-\delta\beta)\}^{\frac{3}{2}}}{\pi(\sinh\gamma\ \cosh\gamma)^2}, \qquad \ldots(38)$$

and from equations (23) and (30),

$$\left|\frac{d^2\Omega}{dZ^2}\right|_T = \frac{V_1 d\ \cos\alpha_1}{2\pi}(P^2+Q^2)^{\frac{1}{2}}. \qquad \ldots(39)$$

In/

In equation (39), P and Q are given by

$$P = -A_1 + A_2 - A_3 + A_4 - \tan\alpha_1(B_1 - B_2) + \tan\alpha_2(B_2 - B_4),$$
$$Q = -B_1 + B_2 - B_3 + B_4 + \tan\alpha_1(A_1 - A_3) - \tan\alpha_2(A_2 - A_4), \quad \ldots(40)$$

and
$$A_1 = A(m_1, \phi_{1T}), \quad B_1 = B(m_1, \phi_{1T}),$$
$$A_2 = A(-m_2, \phi_{2T}), \quad B_2 = B(-m_2, \phi_{2T}),$$
$$A_3 = A(1/m_1, \phi_{1T}), \quad B_3 = B(1/m_1, \phi_{1T}),$$
$$A_4 = A(-1/m_2, \phi_{2T}), \quad B_4 = B(-1/m_2, \phi_{2T}),$$

where
$$\left.\begin{aligned} A(m, \phi) &= \frac{\frac{1}{2}\left(\frac{1}{m} + m\cos 2\phi\right) + \cos\phi}{2mR^2\left\{\cos\phi + \frac{1}{2}\left(m + \frac{1}{m}\right)\right\}^2} \\ B(m, \phi) &= \frac{\sin\phi + \frac{1}{2}m\sin 2\phi}{2mR^2\left\{\cos\phi + \frac{1}{2}\left(m + \frac{1}{m}\right)\right\}^2} \end{aligned}\right\} . \quad \ldots(41)$$

9. Test Examples

(a) Merchant and Collar Cascade

To test the foregoing theory, a flow was required whose properties can be determined exactly, but which was not derived by Garrick's transformation. The most convenient method of producing such a flow seemed to be that of Merchant and Collar (1941) which has also been used for comparison purposes by other recent workers, especially by Gostelow (1963).

Merchant and Collar's analysis is not reproduced here, but the blade shown in Fig. 10 was derived by a straightforward calculation using the following values of the parameters in the original notation:-

Stagger angle = 37·5°

β = 0·725

β' = 0·8

Co-ordinates of the centre of the β'-oval in the ℓ-plane = (-0·06320, +0·11251).

For the application of the present method to this cascade, it is necessary to specify the blade shape as a set of points whose co-ordinates could be used as input data. Two sets of co-ordinates were in fact used, one of 30 points and the other of 40 points; and upon these were based respectively the 20-point* and 40-point programmes. The set of 40 chosen points is indicated on Fig. 10, on which is also given the points on the blade which were found, after calculation, to correspond to the equally-spaced points in the ζ'-plane. Fig. 11 shows, for the 40-point programme, the

contour/

* A P-point programme is one which takes P equally-spaced points on the ζ'-plane contour.

contour in the ζ-plane and also this contour when displaced by the transformation T_2; again the given points and interpolated points are shown. This figure indicates how successfully the choice of parameters in T_1 has avoided the irregularities usually present in contours obtained by transformations.

The flow about this given Merchant and Collar cascade has been calculated exactly for an inlet flow angle $\alpha_1 = 53{\cdot}5^\circ$, namely that used by Gostelow (1963). The exact pressure distribution for this inlet angle is shown in Fig. 12 on which are also plotted the points calculated by the 20-point and 40-point programmes. These calculated points agree very well with the exact result and it is surprising that the 20-point calculation seems almost as satisfactory as the 40-point. No doubt, the accuracy of the calculations could be further improved by taking more than 40 points. (It may be further noted that 40-point calculations come close to using all the available storage space on the Pegasus computer used here; blade profiles specified at many more points could, of course, be handled on larger machines.)

The outlet angle has also been calculated using the 40-point and 20-point programmes and is compared in the following table with the exact value.

	Exact value	40-point	20-point
$\tan \alpha_2$	0·57793	0·57808	0·57704
% error	-	0·026	-0·154

Some of the other methods of calculating cascade pressure distributions have recently been applied to the same Merchant and Collar cascade. Fig. 13 shows results obtained by Gostelow (1963) using Schlichting's singularity method, and by Rolls-Royce Ltd.* using the singularity method of Martensen (1959), compared with the values obtained by the 40-point programme. The reader may draw his own conclusions.

(b) Cascase of 10C4/30C50 aerofoils

In a recent report, Pollard and Wordsworth (1962) have compared two theoretical methods of solving the direct problem of cascade flow. Their methods were: first, a conformal-transformation method based on Howell's transformation and second, a modified version of Schlichting's singularities method. They have calculated the pressure distributions and deviation angles for a cascade of 10C4/30C50 aerofoils with gap-chord ratio unity, for a variety of stagger angles.

Further interesting comparison can be made by applying the present method to the same examples. We have thus used a 10C4/30C50 aerofoil given at 32 points as shown in Fig. 15. These are derived from the 17 values of the C4 thickness distribution given in Howell (1946) which were also the basis of Pollard and Wordsworth's calculations.

It will be noted that the C4 profile has a non-zero trailing-edge radius of curvature. To take account of this the proceeding theory must be modified slightly at the stage when the axes of the z-plane are being fixed. The position of T, the trailing edge, must be moved away from T', the corresponding singularity of the transformation, in the same way as, at the leading edge, L was moved away from L'.

Fig. 14 shows the general position of a blade relative to the co-ordinates in the z-plane. The transformation T_1 is now given by equation (43) of Fig. 14, where δz is calculated by an argument similar to that used in Section 7 to find $\delta\beta$ and δc. δz and the now modified expressions for $\delta\beta$ and δc are given by equations (44) to (46) on Fig. 14.

Initially/

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*Not available at the present time.

Initially we consider a cascade with stagger angle 36°. The computations were based on 32 equally-spaced points interpolated in the ζ'-plane from the 32 initially specified points. The interpolated points and the points in the ζ- and z-planes corresponding to them are shown in Figs. 15 and 16. Retaining the inlet angle of 51° used by Pollard and Wordsworth, the pressure distribution has been calculated and is compared on Fig. 17 with the results obtained by their two methods.

Over the central parts of the blade, the results from the two conformal transformation methods lie, on the whole, very closely together and both differ significantly from the values given by the distributed singularities method. However, there are some striking differences between the conformal transformation methods in the neighbourhood of the leading-edge pressure peaks.

The lift coefficient has been independently calculated from the integrated pressure distribution and from the total angle through which the flow is turned, and is given in the following table, for a stagger angle of 36°:

Integrated pressure distribution	Turning angle	% difference
0·724	0·720	0·6

Pollard and Wordsworth give the difference for this calculation as ½% in the Howell transformation case and 1% in the singularities case.

One of the most disturbing results for practical design is the lack of agreement which Pollard and Wordsworth found between the curves of deviation v. stagger when calculated by Howell's and the singularities methods. The curve obtained by the present method, shown in Fig. 18, is not in close agreement with either of the other curves, and until many more cases are calculated, it is difficult to draw any firm conclusion.

10. Conclusions

A method has been produced to calculate the potential flow about a given two-dimensional cascade of aerofoils. It has been programmed for a Pegasus computer and needs no intermediate curve-plotting.

Comparisons with an exactly-known flow have been made. Excellent agreement between the exact and calculated pressure distributions is achieved; in particular the fluid outlet angle is given to three significant figures. It appears likely that the method will give results superior to those obtained from other methods of treating the problem such as those based on the work of Schlichting (1955) and Martensen (1959). The Schlichting method, in the modified form of Pollard and Wordsworth (1962), may be the least satisfactory.

With a cascade of 10C4/30C50 aerofoils, comparison between the present method and the methods developed by Pollard and Wordsworth (1962) shows that, of the latter methods, that based on Howell's transformation gives the closest agreement except for the leading-edge pressure peaks. Once again the results based on Schlichting's singularity method differ substantially from those of the transformation methods. The curve for the deviation angle as a function of stagger obtained by the present method lies a little lower than those obtained by Pollard and Wordsworth. Experimental investigation of the pressure distribution and the variation in deviation angle for this profile would be of interest.

Satisfactory results from the present method have been obtained for cascades of low cambered profiles using at most 40 point programmes which require 1½ hr computing time in the Autocode version given in this paper. A Pegasus machine orders version has also been prepared which allows calculations using up to 70 points to be carried out and which reduces the computing time to 20 min. When it is noted that this time would be reduced to under 10 sec on an Atlas computer, the possibility presents itself of using the method as a part or subroutine of an iterative design calculation.

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APPENDIX/

APPENDIX

Computing Programmes and Detailed Formulae

A description is given here of the programme used in the calculation of the pressure distribution and outlet velocity for a given cascade, and the required form in which the data must be made up, together with formulae too detailed to be suitable for presentation in the main part of the report.

The calculation has been programmed for a Ferranti Pegasus machine using the Automat programme code which allows both arithmetic and matrix instructions to be written in the same programme. Automat is a combination of the usual Pegasus Autocode and the Matrix Interpretive Scheme which has been made up by Dr. Samet of the Computation Laboratory, Southampton University. It can be used only on Pegasus machines which have a large drum.

The complete calculation programme is arranged as a series of nine sub-programmes each dealing with a distinct part of the calculation. The sub-programmes run consecutively using information calculated and stored from stage to stage, giving a completely automatic solution. The maximum number of blade points which can be dealt with is 40. The annotated programme shown at the end of this Appendix applies only to blades with cusped trailing edges although a more general programme which can be used for blades with both cusped and rounded trailing edges has also been produced. A description of the sub-programmes follows.

Sub-Programme 1. Calculation of the Transformation Parameters

Given the cascade parameters β and c/d, together with the blade characteristics ρ_L/c and ϵ_L, the parameters $\delta\beta$ and δc are calculated using equations (22). The transformation parameters $(\beta-\delta\beta)$ and $(c-\delta c)$ are then set up and the related parameter γ is calculated from equation (6) using Newton's root finding procedure. In fact it has been found simpler to perform the iterations involved to find the quantity Q which is defined by

$$Q^2 = \cosh^2\gamma - \sin^2(\beta-\delta\beta).$$

Thus equation (6) is rewritten as

$$G(Q) \equiv \frac{\sin(\beta-\delta\beta)}{Q} - \tan\left[\frac{\pi c}{2d\sin(\beta-\delta\beta)} - \cot(\beta-\delta\beta)\log\left\{\frac{Q+\cos(\beta-\delta\beta)}{\{Q^2-\cos^2(\beta-\delta\beta)\}^{\frac{1}{2}}}\right\}\right] = 0. \qquad \ldots(47)$$

Also required is

$$\frac{dG}{dQ} = -\frac{\sin(\beta-\delta\beta)}{Q^2} - \sec^2\left[\frac{\pi c}{2d\sin(\beta-\delta\beta)} - \cot(\beta-\delta\beta)\log\left\{\frac{Q+\cos(\beta-\delta\beta)}{\{Q^2-\cos^2(\beta-\delta\beta)\}^{\frac{1}{2}}}\right\}\right] \cdot \frac{\cot(\beta-\delta\beta)\cos(\beta-\delta\beta)}{Q^2-\cos^2(\beta-\delta\beta)}. \qquad \ldots(48)$$

To obtain a good initial value of Q we note that, in Fig. 2 of Garrick (1944), for a given value of γ the variation of $(c-\delta c)$ with $(\beta-\delta\beta)$ is small. Thus we may put $(\beta-\delta\beta) = 0$ in equation (6) to obtain the following approximation to $\cosh\gamma$:-

cosh γ/

$$\cosh \gamma = \coth \left\{ \frac{\pi(c-\delta c)}{2d} \right\}. \qquad \ldots(49)$$

This expression is used to calculate the first approximation to Q. Once γ has been calculated, θ_T is found from equation (5).

The required input data for this programme is as follows:-

specified accuracy for the complete calculation expressed as the number of significant binary digits

β - the stagger angle

c/d - chord/pitch

ρ_L/c - leading-edge radius of curvature/chord

ϵ_L - direction of the leading-edge normal

d - pitch.

Sub-Programme 2. Reduction of the Blade Data to a Standard Form

The cartesian co-ordinates of an individual blade of the cascade with its chord line along the x-axis are first of all transformed so that the trailing edge is at the origin. The blades are now in the position shown in Fig. 4b. These co-ordinates are then transformed so that the blade is rotated by $-\delta\beta$ and the trailing edge moved to the point $(\frac{1}{2}(c-\delta c), 0)$. This is done so that the simpler form of the original Garrick transformation can be used in sub-programme 3 and the blade co-ordinates thus transformed will be in the z'-plane.

An additional part of this sub-programme can be called upon by reading in a non-zero camber angle, to generate blades made up from a thickness distribution on a circular arc camber line.

The form of the input data for this programme follows. It should be noted that the blade points or thickness distribution must be read in consecutively, starting with the trailing-edge value.

+ 1, input parameter

Camber angle

Number of given blade points

Blade co-ordinates in cartesian form or thickness distribution based on unit chord.

Sub-Programme 3. T_1 - Transformation to the ζ-plane

This sub-programme starts with the transformed trailing-edge point which is known to be $e^{i\theta_T}$, so that $\sigma_o = i\theta_T$, and a step-by-step procedure for calculating in turn the value of ζ corresponding to each value of z, is followed from there. In general, to go from a point z'_n, at which σ_n is known, to z'_{n+1}, the difference $dz'_n = z'_{n+1} - z'_n$ is formed and used in the formula

$$d\sigma_n^{(o)} = \frac{dz'_n}{e^{\sigma_n} F'(\sigma_n)}, \qquad \ldots(50)$$

where/

where $\sigma = \psi + i\theta$ and

$$F'(\sigma) = A + iB \qquad \ldots(51)$$

and

$$\frac{\pi A}{4d} = \frac{\sinh\psi[\cos(\beta-\delta\beta)\cos\theta\cosh y(\cosh^2 y-\cosh^2\psi-\sin^2\theta)+\sin(\beta-\delta\beta)\sin\theta\sinh y(\cosh^2 y+\cosh^2\psi-\sin^2\theta)]}{\cosh^2 2y - 2\cosh 2y\cosh 2\psi\cos 2\theta + \sinh^2 2\psi + \cos^2 2\theta} \qquad \ldots(52)$$

$$\frac{\pi B}{4d} = \frac{\cosh\psi[\cos(\beta-\delta\beta)\sin\theta\cosh y(\sinh^2 y+\sinh^2\psi+\sin^2\theta)+\sin(\beta-\delta\beta)\cos\theta\sinh y(\cosh^2\psi-\cosh^2 y-\sin^2\theta)]}{\cosh^2 2y - 2\cosh 2y\cosh 2\psi\cos 2\theta + \sinh^2 2\psi + \cos^2 2\theta}. \qquad \ldots(53)$$

Now $z_n'^{(0)} = F(\sigma_n + d\sigma_n^{(0)})$ is calculated. In general this will not be the same as z'_{n+1}, in which case the difference $dz_n'^{(1)} = z'_{n+1} - z_n'^{(0)}$ is formed and used in equation (50) with σ_n replaced by $\sigma_n + d\sigma_n^{(0)}$ giving $d\sigma_n^{(1)}$ and hence $z_n'^{(1)}$ which should be a closer approximation to z'_{n+1}. The sequence is repeated until the desired accuracy is obtained. The method of iteration is illustrated in Fig. 19.

There are some modifications to this scheme which come into operation near the trailing edge. Once again the difference $dz'_o = z'_1 - z'_o$ is formed. However, since $F'(\sigma_o) = 0$, equation (50) is replaced by

$$d\sigma_o = \left[\frac{2dz'_o}{F''(\sigma_o)}\right]^{\frac{1}{2}}, \qquad \ldots(54)$$

where

$$F''(\sigma_o) = \frac{d}{\pi}\,\frac{\{\cosh^2 y - \sin^2(\beta-\delta\beta)\}^{\frac{3}{2}}}{\sinh^2 y\cosh^2 y}. \qquad \ldots(55)$$

Now $z'_{o,o} = F(\sigma_o + d\sigma_o)$ is calculated and if $|z'_1 - z'_o| < |z'_1 - z_{o,o}|$ the interval (z'_o, z'_1) is halved so that the end of the step to be taken is now at $z'_o + \frac{1}{2}(z'_1 - z'_o)$. The approximation $z_{o,1}$ to this is calculated, and if $|\frac{1}{2}(z'_1 - z'_o)| < |z'_o + \frac{1}{2}(z'_1 - z'_o) - z'_{o,1}|$ the end of the step is taken as $z'_o + \frac{1}{4}(z'_1 - z'_o)$ and so on until $\left|\frac{1}{2^p}(z'_1 - z'_o)\right| > \left|z'_o + \frac{1}{2^p}(z'_1 - z'_o) - z'_{o,p}\right|$.

Once this condition has been fulfilled the iteration is allowed to continue in the way described in the previous paragraph to calculate, first of all, $\sigma_{o,p}$ corresponding to $z'_o + \frac{1}{2^p}(z'_1 - z_o)$. The programme then calculates values of σ at the end of each subinterval of length $\frac{1}{2^p}(z'_1 - z'_o)$ until z_1 is reached. The procedure has been illustrated in Fig. 20. It has been designed to combat the possible poor convergence obtained by using equations (50) and (54) in the neighbourhood of the singularity situated at the trailing edge.

The input data for this programme is:-

+1, input parameter.

Sub-Programme/

Sub-Programme 4. T_2 - Change of Origin

$\bar{\zeta}$ is calculated according to the procedure of Section 5. The contour in the ζ-plane is then transformed to the ζ'-plane by T_2.

The input data required is:-

+1, input parameter.

Sub-Programme 5. Lagrangian Interpolation of Points at Equal Angles

Values of ψ' are interpolated at equally spaced values of θ' starting with θ'_T, using Lagrangian interpolating polynomials of the seventh order based on four given points on each side of the point being interpolated. Any number of interpolated points, within the overall maximum of 40 points, can be produced.

The input data required by the programme is as follows:-

+1, input parameter.

Number of interpolated points required.

Sub-Programme 6. T_3 - Theodorsen's Transformation with Lagrangian Differentiation

The programme calculates $\Psi(\phi)$, $\bar{\Psi}$, ϵ and $|d\zeta'/dZ|$ using T_3. Thwaites (1963) should be consulted for details of the analysis. Some small changes have, however, been made to his method. First, six terms of the series for Ψ or ϵ (equations (18) and (20) of Thwaites (1963)) were found to give sufficient accuracy. Second, his differentiation matrix (D) has been replaced by one which produces Lagrangian derivatives based on three values on either side of the position at which the derivative is required. The coefficients used were taken from Kopal (1955).

The input data required for this programme is:-

+1, input parameter.

Sub-Programme 7. The Position of the Singularity Points in the Z-Plane

The position of singular points in the ζ'-plane, i.e., $\mp e^{\gamma} + \bar{\zeta}$, are calculated and then referred to the line $\arg(\zeta') = \theta'_T$. The value of $\Psi(\phi)$, found in sub-programme 6, is then used in a direct iteration of equation (4) to calculate the positions, $-\beta_1$ and β_2, of the singular points in the Z-plane. For the purposes of computation the real and imaginary parts of equation (4) are required and are as follows. If $f(Z) = p + iq$ then

$$p = \frac{1}{\pi}\int_0^{2\pi} \frac{(Xe^{-\bar{\Psi}}\cos\phi + Ye^{-\bar{\Psi}}\sin\phi - 1)\Psi(\phi)d\phi}{1 - 2(Xe^{-\bar{\Psi}}\cos\phi + Ye^{-\bar{\Psi}}\sin\phi) + (X^2 + Y^2)e^{-2\bar{\Psi}}} \qquad \ldots(56)$$

$$q = \frac{1}{\pi}\int_0^{2\pi} \frac{(Xe^{-\bar{\Psi}}\sin\phi - Ye^{-\bar{\Psi}}\cos\phi)\Psi(\phi)d\phi}{1 - 2(Xe^{-\bar{\Psi}}\cos\phi + Ye^{-\bar{\Psi}}\sin\phi) + (X^2 + Y^2)e^{-2\bar{\Psi}}} . \qquad \ldots(57)$$

The input data required for this programme is:-

+1, input parameter.

Sub-Programme 8. Interpolated Co-ordinates

The co-ordinates in the ζ-plane corresponding to the equally-spaced interpolated values in the ζ'-plane are calculated using T_2. These ζ-plane co-ordinates are then transformed by T_1 to give the interpolated points on the blade. [The programme comes to a stop just before printing out the data required for a possible re-run of sub-programme 9. A short amount of blank tape should be punched out before running on, and the punched tape which follows preserved.]

The input data for this programme is:-

+1, input parameter.

Sub-Programme 9. Velocities and Blade Pressure Distribution

With given values of the inlet angle and speed, the outlet angle and speed are calculated using equations (25), (33) and (34). W is now calculated using equations (25) and (31) and hence v and C_p are calculated from equations (35), (36) and (42). v and $\left|\frac{d\zeta}{dz}\right|$ will be printed only if handswitch 0 is depressed. The calculation of v at a cusped trailing edge is a special case and is done by evaluating equations (37) to (41).

The input data required for this programme is as follows:-

+1, input parameter

v_1 inlet speed

α_1 inlet angle.

Other values of the inlet conditions can be considered by reading in new data in the form given above when the programme has come to a stop, or the programme can be re-run at another time with the following data:-

+0, input parameter

v_1 inlet speed

α_1 inlet angle.

Special output from sub-programme 8.

Annotated Programme

The Automat programme which follows is annotated with reference to the equations of the text showing which instructions deal with each part of the calculation. It is followed by an example showing the required form for the data and also by a list of operating instructions.

Programme/

Programme Operating Instructions

Notes:- S + R = Start and run

R = Run

* = Data or parameters on a separate tape. The remainder comprises the main programme tape.

All programmes and data are read by tape reader 1.

Operating instruction	
S + R	Sub-programme 1
R	Input data (cascade and blade parameters)*
S + R	Sub-programme 2
R	Input data (blade co-ordinates)*
S + R	Sub-programme 3
R	Input parameter
S + R	Sub-programme 4
R	Input parameter
S + R	Sub-programme 5
R	Input parameter
R	Data (number of interpolated points required)*
S + R	Pre-set parameter instruction (T1600)
R	Parameter (number of points)*
R	Sub-programme 6
R	Input parameter
S + R	T1600
R	Parameter (number of points)*
R	Sub-programme 7
R	Input parameter
S + R	T1600
R	Parameter (number of points)*
R	Sub-programme 8
R	Input parameter
R	After stop and punching out some blank tape

S + R/

Operating instruction	
S + R	T1600
R	Parameter (number of points)*
R	Sub-programme 9
R	Data (inlet velocity)*
R	Further inlet velocity data, if required*.

ES

Q6600	Read in AUTOMAT from magnetic tape.
D N PR.1 CALCULATION OF TRANSFORMATION PARAMETERS	
~~J7090~~ J1.0 JXXXX	Initial orders.
STOP	
n27=TAPE	Read in accuracy required.
v100=TAPE5	Read in β, c/d, e_L/c, a_1, d.
v18=3.14159265 v19=v18/180 v10=v100Xv19 v0=v101Xv18 v17=2Xv0 n0=n27	
v0=v103Xv19 v2=2Xv0 v4=COSv2 v5=SINv2 v6=COSv0 v0=0.5Xv4 v34=v6Xv0 v0=.5Xv5 v35=v6Xv0 Xv3=v34Xv102 v4=v35Xv102 v5=v4/v19	Calculation of $\delta c/c$ and $\delta\beta$ using equations (22).
PRINTv3,3025 PRINTv5,4005	Print $\delta c/c$, $\delta\beta$.
Xv10=v10-v4 v1=1-v3 v103=v101 v101=v1Xv101	Calculation of $\beta - \delta\beta$ and $\frac{c - \delta c}{d}$.
v102=v5 v11=SINv10 v12=COSv10 v13=1/v11 v14=v12Xv13 v15=v12Xv12 v16=v12Xv14 v0=v18X0.5 v1=v101 v20=v0Xv1 v23=v20Xv13	Constants required in the calculation of γ.
PRINTv1,3025 Sv8=0 v0=v10/v19 PRINTv0,4065	Print $\frac{c - \delta c}{d}$ and $\beta - \delta\beta$.
v90=v0	Store $\beta - \delta\beta$.
v0=EXPv20 v1=1+v0 v2=v0-1 v3=v1/v2 v1=1/v3 v0=v3-v1 v0=0.5Xv0 v0=v0Xv0 v0=v0+v15 v22=SQRTv0	First approximation to $Q = \{\cosh^2\gamma - \sin^2(\beta - \delta\beta)\}^{\frac{1}{2}}$.

```
n3=0
2)n3=n3+1
v24=1/v22
v21=v22Xv22
v25=v24Xv24
v5=v21-v15
→3,0>v5
v4=SQRTv5
v1=v22+v12
v0=v1/v4
v0=LOGv0
v0=v14Xv0
v26=v23-v0
v0=TANv26

v6=v11Xv24
v6=v6-v0

v0=v26Xv26
v0=1+v0
v0=v0Xv5
v0=v16/v0
v1=-v11Xv25
v7=v1-v0

v9=-v6/v7
v22=v22+v9
→5,v22=•v8
v8=v22
→2
```

Iteration loop to calculate Q, terminating when consecutive approximations differ by less than n_0 binary digits.

```
5)v1=v5+1
v1=SQRTv1
v2=v4+v1
v0=LOGv2
```

Calculation of y.

```
PRINTv2,4065
PRINTv0,4025
PRINTn3,4000
```

Print out e^y, y and the number of iterations.

```
v91=v2
```

Store e^y.

```
→1
```

```
3)n3=n3+1
v22=SQRTv15
v22=v22+0.000001
→2,100>n3
PRINTn3,3000
→0
```

Outlet sequence when an iteration of Q implies $\cosh^2 y < 0$.

```
1)v36=v11Xv11
v0=v12Xv12
v37=v0
v38=v12Xv13

v0=1/v2
v1=v0+v2
v2=v2-v0
v0=.5Xv1
v1=v2X.5

v39=v0Xv0
v40=v1Xv1
v41=v1/v0

v0=v41/v38
v1=ARCTANv0
v42=v1/v19
```

Calculation of θ_T using equation (5).

```
PRINTv42,3065
```

Print θ_T.

```
v94=v42
v96=v103
v99=v102
v97=v104
v98=v101
```

Store parameters θ_T, c/d, $\delta\beta$, d, $(c - \delta c)/d$.

```
(→0)
*********
```

```
Z                          Stop; to enable cascade data to be read in.
--------------------------------------------------------------------------
D
N
PR.2  REDUCTION OF BLADE COORDINATES TO STANDARD FORM
--------------------------------------------------------------------------
J7090                      Initial orders.
J1.0
J7XXXX
STOP
--------------------------------------------------------------------------
n1=TAPE                    Read in input parameter.
--------------------------------------------------------------------------
→14,n1=1                   Read in parameters if calculation is started at this programme.
v96=TAPE4
--------------------------------------------------------------------------
14)v100=TAPE
n2=TAPE                    Read in camber angle (non-zero only with thickness
n3=2Xn2                    distribution), no. of given points and given coordinates.
v151=TAPEn3
--------------------------------------------------------------------------
v18=3.14159265
v19=v18/180
v110=v98Xv97
v111=v96Xv97

→10,0.1>v100               Jump to calculation of standard form.
--------------------------------------------------------------------------
PRINTv100,3060
v100=v100Xv19
n4=-1
n1=n2-1

11)n4=n4+1
n5=2Xn4
n6=n5+1
n7=n3-n6                   Calculation of blade coordinates with a circular arc
n7=n7+n3                   camber line from a given thickness distribution and
n7=n7+148                  camber angle.
n8=n7+1
n5=n5+151
n6=n6+151
v0=0.5-vn5
v0=v0Xv100
v50=SINv0
v51=COSv0
v0=0.5Xv100
v1=SINv0
v52=1/v1
v53=COSv0
v54=v111X0.5
v0=v50Xv52
v55=v100Xvn6
v1=1+v55
v2=1-v55
v1=v1Xv0
v2=v2Xv0
v1=1-v1
v2=1-v2
vn5=v1Xv54
vn7=v2Xv54

v1=1+v55
v2=1-v55
v1=v1Xv51
v2=v2Xv51
v1=v1-v53
v2=v2-v53
v0=v54Xv52
vn6=v0Xv1
vn8=v0Xv2
→11,n1>n4
--------------------------------------------------------------------------
Xn2=n3-2
```

```
10)v0=MODv102
Xn1=n2-1
→0,0.00000001>v0                  End if  δβ = 0 .
-----------------------------------------------------------------
PRINTv110,1025
PRINTv111,2025
PRINTv99,2005                     Print c-δc, c and δβ.
-----------------------------------------------------------------
v0=v99Xv19
v55=SINv0                                        Calculation of standard
Xv56=COSv0                                       form of blade data,
v57=0.5Xv110                                     stored from location 151.
Xv58=v151

n4=-1
12)n4=n4+1
n5=2Xn4
n5=n5+151
n6=n5+1
v0=vn5-v58
PRINTv0,3045
PRINTvn6,4045      Print blade coordinates with trailing edge at the origin .
-----------------------------------------------------------------
v1=v0Xv56
v2=v0Xv55
v0=vn6Xv55
v0=-v0+v1
vn5=v0+v57
PRINTvn5,4085
v0=vn6Xv56
vn6=v0+v2
PRINTvn6,4045      Print out blade coordinates rotated through
→12,n1>n4                         an angle -δβ .
-----------------------------------------------------------------
n4=n4+1            Store number of given points .
-----------------------------------------------------------------
(→0)
********
Z                  Stop;  to enable blade coordinates to be read in .
-----------------------------------------------------------------
```

D
N
PR.3 T1 - TRANSFORMATICN TO ZETA PLANE

```
J7090                   Initial orders.
J1.0
STOP

n1=TAPE                 Read in input parameter.
→19,n1=1

n27=TAPE
v90=TAPE2
v94=TAPE
v96=TAPE3               Input if calculation is started with this programme.
n4=TAPE
n6=2Xn4
v151=TAPEn6

19)v12=v96Xv97
n10=n27
n12=149
n6=0
n7=1
n1=150                  Constants, i.e. functions of (β-δ β) and γ, and
n2=151                  initial values for the step-by-step calculation
n3=152                  of points in the ζ-plane .
n13=147
n14=148
n15=300
n16=301
n8=0
v18=3.14159265
v19=v18/180
v10=v90Xv19
v14=v94Xv19             θ0 = θT .
v46=SINv14
v47=COSv14
n10=0

v0=1/v91
v1=v91-v0
v28=v1X.5
v2=v91+v0
v29=v2X.5
v26=COSv10
v27=SINv10
v30=v29Xv26
v31=v28Xv27
v32=v29Xv29
v33=v32-1
v0=2Xv32
v34=v0-1
v35=v34Xv34
v0=4Xv97
v36=v0/v18
v37=0.125Xv36
v38=0.5Xv26
v39=-0.5Xv27

v2=v32Xv33
v0=v27Xv27
v0=v32-v0
v1=SQRTv0
v0=v0Xv1
v0=2Xv0
v53=v2/v0               F"(σ0).
```

```
v13=0                    φo = 0 .
13)n8=n8+1
n5=0
v23=v12
n12=n12+2
n1=n1+2
n2=n2+2
n3=n3+2
n13=n13+2
n14=n14+2
n15=n15+2
n16=n16+2
vn13=EXPv13                     Output of e^φ and θ.
PRINTvn13,3065                  Storage locations, e^φ from 151, 153, ........,
vn14=v14                                            θ from 152, 154, ........,
v0=v14/v19                                      cos θ from 303, 305, ........,
PRINTv0,4085                                    sin θ from 304, 306, ........,
vn15=v47
vn16=v46
n7=1
v60=vn2
v61=vn3             z'n+1.
v66=vn2-vn12
v67=vn3-vn1         dz'n .
>0,n8≥n4

12)v46=SINv14
v47=COSv14
v48=v46Xv46
v49=2Xv48
v49=1-v49
v50=v49Xv49

v0=EXPv13                       Calculation of z'(ζ)
v1=1/v0                        using Garrick's equation directly.
v40=v0+v1
v40=.5Xv40
v41=v0-v1
v41=.5Xv41
v42=v40Xv40
v43=v42-1
v44=2Xv42
v44=v44-1
v45=v44Xv44
v45=v45-1

v1=v32+v42
v51=v1-v48
v0=v33+v43
v52=v0+v48

v2=2Xv47
v0=v2Xv29
v0=v0Xv40
v1=v51-v0
v0=v51+v0
v9=v0/v1

v0=v2Xv28
v0=v0Xv41
v1=v52-v0
v0=v52+v0
v8=v0/v1

v1=v28Xv40
v2=v29Xv41
v0=v1-v2
v1=-v1-v2
v7=v46/v0
v6=v46/v1
```

```
v7=ARCTANv7
v6=ARCTANv6
→7,v0>0
→9,v46>0
v7=v7-v18
→8
9)v7=v7+v18
→8
7)→8,0>v1
→6,v46>0
v6=v6+v18
→8
6)v6=v6-v18
8)v9=LOGv9
v9=v9Xv38
v0=v7-v6
v0=v0Xv27
v9=v9+v0
v9=v9Xv37

v8=LOGv8
v8=v8Xv39
v0=v6+v7
v6=v0Xv26
v8=v8+v6
v8=v8Xv37
→10,n6>0
```

As above.

```
16)v24=v60-v9
v25=v61-v8
→11,n5>0
```

$ds'^{(k)}_n$.

```
15)v0=v32-v42
v0=v0-v48
v1=v30Xv47
v0=v1Xv0
v2=v31Xv46
v1=v51Xv2
v2=v34Xv44
v2=v2Xv49
v2=2Xv2

v3=v35+v50
v3=v3+v45
v4=v3-v2
v4=1/v4

v0=v0+v1
v0=v0Xv41
v0=v0Xv36
v20=v0Xv4

v1=v30Xv46
v0=v1Xv52
v1=v42-v32
v1=v1-v48
v2=v31Xv47
v1=v1Xv2
v0=v0+v1
v0=v0Xv40
v0=v0Xv36
v21=v0Xv4

v0=v20Xv20
v1=v21Xv21
v0=v0+v1
v22=1/v0
```

Calculation of $d\sigma^{(k)}_n$ using equations (50) - (53).

```
v0=v20×v24
v1=v21×v25
v0=v0+v1
v1=v20×v25
v2=v21×v24
v1=v1-v2
v0=v0×v22
v1=v1×v22
```

```
v13=v13+v0
v14=v14+v1
```

$\sigma_n^{(k+1)} = \sigma_n^{(k)} + d\sigma_n^{(k)}$.

```
n9=n5+n6
→12,n9≠-1
v2=v63
v0=v13×v13
v1=v14×v14
v0=v0+v1
v63=SQRTv0
→13,v2=•v63
→12
```

Test if $|\sigma_n^{(k+1)}| \simeq |\sigma_n^{(k)}|$.

```
11)n5=n5+1
v2=v23
v0=v24×v24
v1=v25×v25
v0=v0+v1
v23=SQRTv0
→14,v2>v23
n7=2×n7
v66=.5×v66
v67=.5×v67
v60=vn12+v66
v61=vn1+v67
→15,n10=1
→17
14)n11=n10+n5
→17,n11=1
→15,n5=1
→18,n10=0
n7=n7-1
→10,n5≠2
n5=-1
→16

17)v0=v24+v23
v0=v0×v53
v0=SQRTv0
v1=v25×v53
v1=v1/v0
v13=v13+v0
v14=v14+v1
→12
```

Calculation of $d\sigma_\varphi^{(k)}$ for the trailing edge using equations (54) and (55) and the procedure for halving the step used.

```
10)n6=n6+1
v60=v60+v66
v61=v61+v67
→16,n6≠n7
n6=0
n5=-1
→16
```

Procedure for setting up the next sub-step.

```
18)n5=0
n7=1
v23=v12
v60=vn2
v61=vn3
v66=vn2-v9
v67=vn3-v8
n10=1
→16
```

Procedure for setting up the last sub-step to finish with z'_1.

```
(→0)
*********
 +1
```

Data: Input parameter.

```
*******
```

D
N
PR.4 T2 - CHANGE OF ORIGIN

```
J7090                          Initial orders.
J8
XXXXXX
STOP

n10=TAPE                       Read in input parameter .
→28,n10=1
n4=TAPE
n6=n4x2
v149=TAPEn6                    Input if the programme is run separately .
v18=3.14159265
v19=v18/180

n1=148                         Used when programme is run separately .
n7=148+n6
24)n1=n1+2
vn1=vn1xv19
→24,n7>n1

28)v94=v150
n5=n4
n6=2xn5                        Initial constants .
n7=300+n6
n2=300
n3=301
n4=148

→29,n10=1
25)n2=n2+2
n3=n3+2
n4=n4+2                        Calculation of cos θn, sin θn when the
vn2=COSvn4                     programme is run separately .
vn3=SINvn4
→25,n7>n2

29)Xv69=2xv18
v148=v(148+n6)-v69
v(150+n6)=v150+v69             Calculation of ½(θn+1 - θn-1) .

Xn1=146
n2=150
n4=250
n7=250+n5

20)n4=n4+1
n1=n1+2
n2=n2+2
v0=vn2-vn1
vn4=.5xv0
→20,n7>n4
```

```
v3=0
v4=0
v5=0
v6=0
n1=147
n4=250
n2=300
n3=301

21)n4=n4+1
n1=n1+2
n2=n2+2
n3=n3+2
v2=vn1xvn4
vn2=vn1xvn2
vn3=vn1xvn3
v0=v2xvn2
v1=v2xvn3
v11=vn1xvn1
v10=v11xvn4
v3=v3+v2
v4=v4+v0
v5=v5+v1
v6=v6+v10
→21,n7>n4
```

Calculation of $\sum_{n=0}^{2N-1} \frac{1}{2}r_n^2 \sin\theta_n(\theta_{n+1} - \theta_{n-1})$,

$\sum_{n=0}^{2N-1} \frac{1}{2}r_n^2 \cos\theta_n(\theta_{n+1} - \theta_{n-1})$,

$\sum_{n=0}^{2N-1} \frac{1}{2}r_n (\theta_{n+1} - \theta_{n-1})$.

```
v4=v4/v3
v5=v5/v3
v0=v6x0.5
v0=v0/v18
v6=SQRTv0
```

Calculation of $\bar{\zeta}$ and $\bar{r}$.

```
PRINTv4,3065
PRINTv5,4065
PRINTv6,4065
```

Output of $\bar{\zeta}$ and $\bar{r}$.

```
v92=v4
v93=v5
```

Store $\bar{\zeta}$.

```
v0=v4xv4
v1=v5xv5
v0=v0+v1
v0=SQRTv0
v1=v5/v4
v1=ARCTANv1
v1=v1/v19
PRINTv0,4065
PRINTv1,4085
```

Calculation and output of $\bar{\zeta}$ in polar coordinates.

```
Xv7=-100
Xv9=0
n0=199
n1=149
n2=300
n3=301
n7=199+n5                     Transformation of points to the ζ'-plane
                              using T2 .
22)n0=n0+1
n1=n1+1
n2=n2+2
n3=n3+2
v0=vn2-v4
v1=vn3-v5
v2=v0Xv0
v3=v1Xv1
v2=v2+v3
vn0=SQRTv2      Store e^φ' from location 200.
PRINTvn0,3065
v6=v1/v0
v0=ARCTANv6
→23,v6>v7
v9=v9+v18
23)vn1=v0+v9
v7=v6
vn1=vn1/v19     Store θ' from location 150 .
PRINTvn1,4085
→22,n7>n0
----------------------------------------------------------------
Xv95=v150
PRINTv95,1008   Store and print θ'_T .
----------------------------------------------------------------
Xn3=149
Xn7=149+n5                    Output of θ' and e^φ' separately to
26)n3=n3+1                    serve as data for the following
PRINTvn3,3085                 programmes, if required .
→26,n7>n3

Xn0=199
Xn7=199+n5
27)n0=n0+1
PRINTvn0,3065
→27,n7>n0
----------------------------------------------------------------
v10=n5                        Store no. of points.
----------------------------------------------------------------
(→0)
********

+1                            Data: input parameter.
----------------------------------------------------------------
*********
```

D
N
PF.5 LAGRANGIAN INTERPOLATION OF POINTS AT EQUAL ANGLES

```
J7090-1
J512
O7168
```

Initial orders.

```
(300,8X8)•→400
(300,8X8)-(400,8X8)→300
(300,8X8)+(500,8/)→300
(300,8/)X(308,8X1)→60
(60,8/)X(316,8X1)→60
(60,8/)X(324,8X1)→60
(60,8/)X(332,8X1)→60
(60,8/)X(340,8X1)→60
(60,8/)X(348,8X1)→60
(60,8/)X(356,8X1)→60
O7101
```

Coefficients of the interpolation polynomial.

```
I
J320
J7090
J8
```

Initial orders.

```
STOP
```

```
n11=TAPE
n13=TAPE
```

Read in input parameter and number of equally spaced points required.

```
n0=v10
→4,n11=1
```

```
v150=TAPE•
v200=TAPE•
```

Input if programme is run separately.

```
4)n12=n0
```

```
n1=0
14)v(500+n1)=1
n1=n1+1
→14,8>n1
```

Unit vector from location 500.

```
n1=-1
n2=n0-1
n3=n0
n4=0
6)v(150+n1)=v(150+n2)-360
v(150+n3)=v(150+n4)+360
v(200+n1)=v(200+n2)
v(200+n3)=v(200+n4)
n1=n1-1
n2=n2-1
n3=n3+1
n4=n4+1
→6,4>n4
```

Extra end values of ζ' curve (using the periodicity of the functions).

```
v20=v150
v0=n13
v21=360/v0
```

Initial θ' value and θ' interval.

```
n2=0
n3=0
n4=n3
n7=3940
v51=v200
→1
```

Initial constants.

```
12)n0=200
→M7080,300

n8=300
n9=1
n10=0
2)n8=n8+n9

3)vn8=v(26+n9)-v(26+n10)
n8=n8+1
n9=n9+1
→3,8>n9

n10=n10+1
n9=n10+1
→2,7>n10
→M7103,1
```

Calculation of the coefficients of the Lagrangian interpolation polynomial.

```
11)n1=0
v50=1
13)v(40+n1)=v20-v(26+n1)
v50=v50Xv(40+n1)
n1=n1+1
→13,8>n1

v51=0
v52=0
n5=196+n3
n1=0
15)v0=v(40+n1)Xv(60+n1)
v1=1/v0
v0=vn5Xv1
v51=v51+v0
v52=v52+v1
n1=n1+1
n5=n5+1
→15,8>n1

v52=v52Xv50
v51=v51Xv50
```

Calculation of interpolated value.

```
1)PRINTv20,3061
PRINTv51,4028
PRINTn3,4060
vn7=LOGv51
n7=n7+1
```

Output of equally-spaced θ' and ø'. Storage of ø' from location 3940.

```
v20=v20+v21
```

Next value of θ'.

```
n2=n2+1
→18,n2>n13

16)→10,v20>v(150+n3)
→11,n3=n4
n4=n3
```

```
n6=146+n3
n1=0
17)v(26+n1)=vn6
n6=n6+1
n1=n1+1
→17,8>n1
→12

10)n3=n3+1
→16
```

Choice of which given points to use in the interpolation polynomial.

```
18)Xn7=3940
Xn1=n13+3940
19)PRINTvn7,1008
n7=n7+1
→19,n1>n7
```

Output of ø'.

```
(→0)
XXXXXXXXX

+1
```

Data: Input parameter.

```
Z
```

Stop to allow the number of equally-spaced points required to be read in.

```
XXXXXXXXX
```

```
T1600                         Present parameters instruction .
Z                             Stop; to allow the number of interpolated points used to be
                              read in .
D
N
PR.6  T3 - THEODORSENS TRANSFORMATION WITH LAGRANGIAN DIFFN.

J7090-1
J512
O 7168                        Initial orders .

O7060                         Completion of (C) and (D) matrices,
(100,241X1)>100               (C) from location 100,
O7060                         (D) from location 1700 .
(1700,241X1)>1700
O7101

(0,241X1)>3940                Input of ψ' if programme is run separately .
(3940,241X1)>3300
(3300,1X241)X(3300,241X1)>48  ||ψ'|| in location 48 .
O7101

                                        Calculation of (-1)^n ψ'^(n)/n! if n28 = 0,
O7060                                              or      E^(n)/n!  if n28 = 1,
(1700,241X241)X(3300,241X1)>3300        from location (3260 + 40n) .
O7060
(12)X(3340,241X1)>3300
O7101

(100,241X241)X(3940,241X1)>3780
(3780,241X1)>3820                       Calculation of 6 terms of series
(3820,241/)X(3340,241X1)>3940           for Ψ if n28 = 0 ,
(3780,241/)X(3820,241X1)>3820           or ε if n28 = 1 ,
(3820,241/)X(3380,241X1)>3860           from location 3940 .
(3940,241/)+(3860,241/)>3940

(3780,241/)X(3820,241X1)>3820
(3820,241/)X(3420,241X1)>3860
(3940,241/)+(3860,241/)>3940

(3780,241/)X(3820,241X1)>3820
(3820,241/)X(3460,241X1)>3860
(3940,241/)+(3860,241/)>3940

(3780,241/)X(3820,241X1)>3820
(3820,241/)X(3500,241X1)>3860
(3940,241/)+(3860,241/)>3940

(3780,241/)X(3820,241X1)>3820
(3820,241/)X(3540,241X1)>3860
(3940,241/)+(3860,241/)>3940

(3940,241/)+(3300,241/)>3940
(3940,1X241)X(3940,241X1)>49            ||Ψ|| or ||ε|| in location 49 .
O7101

(3940,241X1)>0                          Print Ψ.
(50,1X241)X(3940,241X1)>99
(6)X(99)>99
(99)>0                                  Print Ψ̄.
(3300,241X1)>3380
(100,241X241)X(3940,241X1)>3300
(3300,241X1)>0                          Print E .
(3940,241X1)>260
(3380,241X1)>140
(3300,241X1)>3940
(3940,241X1)>3780
(3340,241/)X(3340,241X1)>100
(50,241/)+(100,241/)>100               1+(ψ'^(1))^2 from location 100 .
(3300,1X241)X(3300,241X1)>48           ||E|| in location 48 .
O7101
```

```
(3940,241X1)→0                              Print ε.
(3940,241X1)→700
(1700,241X241)X(3940,241X1)→180
(180,241/)+(50,241/)→180
O7101

(220,241X1)→0                               Print |dZ/dζ|.
O7101

(3940,241/)→3780
J16

I
J 320
J7090
J8                                          Initial orders.
STOP

v1=3.14159265
n25=TAPE                                    Read in input parameter.
→8,n25=1
n13=TAPE
n27=TAPE                                    Input if programme is run separately.
8)v2=n13
v3=v2/2
n2=v3
n1=v2
v4=v1/v2
v5=v4X2
v6=1/v2

v94=v99                            Re-storing 8β in location 94.
n0=n27

n3=0
4)v(50+n3)=1
n3=n3+1
→4,n1>n3                           Unit vector from location 50.

n30=0
v100=0
v(100+n2)=0
v10=v4
n3=1
v11=-1                             Calculation of first column of (C) matrix.
1)v12=COTv10
n4=n1-n3
→2,v11=-1
v(100+n3)=0
→6
2)v(100+n3)=v12/v3
6)v(100+n4)=-v(100+n3)
v12=v12Xv11
v11=-v11
v10=v10+v4
n3=n3+1
→1,n2>n3

v1700=0
v1701=-540
v1702=+108
v1703=-12
n4=n1-7
n3=0
27)v(1704+n3)=0                    Calculation of first column of (D) matrix.
n3=n3+1                            (Lagrangian differentiation).
→27,n4>n3
v(1704+n3)=+12
v(1705+n3)=-108
v(1706+n3)=+540

v0=v5X720
v0=1/v0
n3=0
28)v(1700+n3)=v(1700+n3)Xv0
n3=n3+1
→28,n1>n3
```

```
n3=n1
n8=1
3)v(100+n3)=v(99+n3)
v(1700+n3)=v(1699+n3)
n29=n3-n1
n31=n3+1
→M7103,1
n8=n8+1
n3=n3+n1
→3,n1>n8
v1700=0
```

Completion of (C) and (D) matrices .

```
n11=6+n25
→M7103,n11

n28=0
```

Set for iteration of ψ .

```
19)n3=1
n29=0
n30=0
n31=40
v11=1
17)→11,n28=0
v12=1/v11
→18
11)v12=-1/v11
18)→M7103,10
n30=n30+40
n31=n31+40
n3=n3+1
v11=v11+1

→17,6>n3
```

Calculation of $(-1)^n\phi'^{(n)}/n!$ if n28 = 0,
$E^{(n)}/n!$ if n28 = 1.

```
5)→15,n28=1
→M7103,15
7)→14,v48≠v49
v48=v49
→5
```

Iteration for ψ , ceasing when inner products are unchanged to n_0 binary significant figures.

```
14)→16,n28=1
→M7103,36
n28=1
→19
```

```
15)→M7103,58
→7
```

Iteration for ε.

```
16)v99=EXPv99
PRINTv99,1028
→M7103,51
n3=0
24)v10=EXPv(140+n3)
v10=v10/v99
v11=SQRTv(100+n3)
v10=v10Xv11
v(220+n3)=v10/v(180+n3)
n3=n3+1
→24,n1>n3
→M7103,56
```

Calculation of |dZ/dζ|.

```
(→0)
**************

+1
***********
```

Data: input parameter.

```
T1600
```

Preset parameters order .

```
Z
```

Stop to allow the number of points used to be read in .

```
D
N
PR.7  POSITION OF SINGULARITY POINTS IN Z-PLANE
```

```
J7090-1
J512
O7168
```

Initial orders.

```
(20)X(600,241X1)→500
(21)X(300,241X1)→540
(500,241X1)+(540,241X1)→540
(24)+(540,241/)→340
(23)X(540,241X1)→500
(22)-(500,241/)→420
(20)X(300,241X1)→500
(21)X(600,241X1)→540
(500,241X1)-(540,241X1)→380
(420,241/),(640,241X1)→460
(340,241/)X(460,241X1)→500
(380,241/)X(460,241X1)→540
(260,1X241)X(500,241X1)→30
(260,1X241)X(540,241X1)→31
O7101
```

Iteration of equations (56) and (57) .

```
(0,241X1)→260
O7101
```

Input of ψ if programme is run separately .

```
I
J320
J7090
J8
```

Initial orders.

```
STOP
```

```
n10=TAPE
```

Read in input parameter.

```
→8,n10=1
n1=TAPE
n27=TAPE
v91=TAPE3
v95=TAPE
v99=TAPE
→M7103,16
```

Input if programme is run separately .

```
8)n2=n1
n0=n27
v88=-v91
v89=v91

Xv12=1/v99
v0=-1
v18=ARCCOSv0
v2=n2
v8=2/v2
v0=v18/v2
v3=2Xv0
v7=2Xv18
v19=180/v18
v0=v95/v19
v27=COSv0
v28=SINv0
```

Calculation of constants .

```
n1=0
v4=0
3)v(600+n1)=COSv4
v(300+n1)=SINv4
n1=n1+1
v4=v3+v4
→3,n2>n1
```

Calculation of cos ϕ, sin ϕ at equal intervals from locations 600 and 300 respectively.

```
v23=2
v24=-1
n1=0
7)v(640+n1)=1
n1=n1+1
→7,n2>n1
```

Scalar matrices in locations 23 and 24.
Unit vector from location 640.

```
n3=0
10)v10=v(88+n3)-v92
v11=-v93
v0=v10Xv27
v1=v11Xv28
v4=v10Xv28
v5=v11Xv27
v10=v0+v1
v11=v5-v4
Xv14=v10
v15=v11
PRINTv10,3025
PRINTv11,4025
```

Position of singular points in the ζ'-plane relative to the radius vector through T.

```
v9=0
n1=0
1)n1=n1+1
v20=v14Xv12
v21=v15Xv12
v0=v20Xv20
v1=v21Xv21
v0=v0+v1
v22=1+v0
→M7103,1

v30=-v8Xv30
v31=v8Xv31
v30=EXPv30
v32=COSv31
v31=SINv31
v0=v10Xv32
v1=v11Xv31
v0=v0+v1
v14=v30Xv0
v0=v10Xv31
v1=v11Xv32
v0=v1-v0
v15=v0Xv30

v0=v14Xv14
v1=v15Xv15
v0=v0+v1
v0=SQRTv0
→2,v0=*v9
v9=v0
→1
```

Iteration of equations (5.6) and (5.7) for $-\beta_1$ or β_2 terminating when $|\beta_1|$ or $|\beta_2|$ is unchanged to n_0 binary significant digits.

```
2)PRINTv14,3028
PRINTv15,4028
```

Print position of singularity point in Z-plane relative to line $\arg(Z) = \theta_T'$.

```
PRINTv0,4028
v(40+n3)=v0

v1=v15/v14
v1=ARCTANv1
→6,v14>0
v1=v1+v18
6)v1=v1Xv19
PRINTv1,4067
PRINTn1,4060
```

Polar coordinates of singularity point in the Z-plane.

```
v(42+n3)=v1

n3=n3+1
→10,n3=1
```

Repeat the calculation for the other singularity point.

```
n3=0
4)v(80+n3)=v(40+n3)
n3=n3+1
→4,4>n3
```

Store polar coordinates of the singularity points from location 80.

```
(→0)
*********

+1
*********
```

Data: input parameter.

```
T1600
```
Present parameter input code.

```
Z
```
Stop to allow number of points used to be read in.

```
D
N
PR.8  INTERPOLATED COORDINATES
```

```
J7090-1
J512
O7168
```
Initial orders.

```
(0,241X1)→140
O7101
```
Input of ψ' at equally spaced θ' if programme is run separately.

```
(700,241X1)→0
(220,241X1)→0
(260,241X1)→0
(300,241X1)→0
(400,241X1)→0
O7101
```
Output of data for the following programme, if required.

```
I
J320
J7090
J8
```
Initial orders.

```
STOP
```

```
n10=TAPE
```
Read in input parameter.

```
v0=-1
v18=ARCCOSv0
v19=v18/180
→18,n10=1
```

```
v90=TAPE9
n2=TAPE
→M7103,1
```
Input if programme is run separately

```
18)v4=-v92
v5=-v93
Xv7=-100
Xv9=0
v2=n2
v2=v2X0.5
v11=v18/v2
v10=v95Xv19
n1=259
n4=139
n3=299
n7=259+n2

22)n1=n1+1
n4=n4+1
n3=n3+1
v0=COSv10
v1=SINv10
v2=EXPvn4
v20=v2Xv0
v21=v2Xv1
v0=v20-v4
v1=v21-v5
v2=v0Xv0
v3=v1Xv1
v2=v2+v3
vn3=SQRTv2
PRINTvn3,3065
v6=v1/v0
v0=ARCTANv6
→23,v6>v7
v9=v9+v18
23)vn1=v0+v9
v7=v6
vn1=vn1/v19
PRINTvn1,4085
v10=v10+v11
→22,n7>n1
```
Transformation of equally spaced interpolated points in the ζ'-plane to the ζ-plane using T_2.

Calculation of constants i.e. functions of β-$\delta\beta$, y $\delta\beta$, c-δc, d, c.

```
v10=v90Xv19
Xv0=1/v91
Xv1=v91-v0
v28=v1X.5
v2=v91+v0
v29=v2X.5
v26=COSv10
v27=SINv10
v30=v29Xv26
v31=v28Xv27
v32=v29Xv29
v33=v32-1
v0=v98Xv97
v57=0.5Xv0
v0=v94Xv19
v34=COSv0
v35=SINv0
v0=v97/v18
v37=0.5Xv0
v38=0.5Xv26
v39=-0.5Xv27
```

Transformation of interpolated points to the z-plane by T_1.

```
n4=0
10)v13=v(300+n4)
v14=v(260+n4)Xv19
v46=SINv14
v47=COSv14
v48=v46Xv46
v49=2Xv48
v49=1-v49
v50=v49Xv49

v0=v13
v1=1/v0
v40=v0+v1
v40=.5Xv40
v41=v0-v1
v41=.5Xv41
v42=v40Xv40
v43=v42-1

v1=v32+v42
v51=v1-v48
v53=v48+v43
v52=v53+v33

v2=2Xv47
v0=v2Xv29
v0=v0Xv40
v1=v51-v0
v0=v51+v0
v9=v0/v1

v0=v2Xv28
v0=v0Xv41
v1=v52-v0
v0=v52+v0
v8=v0/v1

v1=v28Xv40
v2=v29Xv41
v0=v1-v2
v1=-v1-v2
v7=v46/v0
v6=v46/v1
```

```
v7=ARCTANv7
v6=ARCTANv6
→7,v0>0
→9,v46>0
v7=v7-v18
→8
9)v7=v7+v18
→8
7)→8,0>v1
→6,v46>0
v6=v6+v18
→8
6)v6=v6-v18
8)v9=LOGv9
v9=v9Xv38
v0=v7-v6
v0=v0Xv27
v9=v9+v0
v9=v9Xv37
XPv9=v9-v57

v8=LOGv8
v8=v8Xv39
v0=v6+v7
v6=v0Xv26
v8=v8+v6
SPv8=v8Xv37

v0=v9Xv34
v1=v8Xv35
v(400+n4)=v0+v1
v0=v9Xv35
v1=v8Xv34
v(440+n4)=v1-v0
PRINTv(400+n4),3045
PRINTv(440+n4),4045
n4=n4+1
→10,n2>n4
STOP
```

As above.

```
→4,n10=0
Xn4=0
5)PRINTv(80+n4),3108
n4=n4+1
→5,4>n4

4)Xn4=0
Xn11=9+n10
21)PRINTv(90+n4),3108
n4=n4+1
→21,n11>n4

v0=n10
v0=2Xv0
n11=v0
n11=5-n11
→M7103,n11
```

Output of data for the following programme, if required.

```
(→0)
*********
```

Data: input paramter .

```
+1
*************
```

```
T1600
```
Preset parameters input order.

```
Z
```
Stop to allow number of points being used to be read in.

```
D
N
PR.9  VELOCITIES AND BLADE PRESSURE DISTRIBUTION
```

```
J7090-1
J512
07168
```
Initial orders.

```
(0,241X1)>700
(0,241X1)>220
(0,241X1)>260
(0,241X1)>300
(0,241X1)>400
07101
```
Input of programme is run separately.

```
I
J320
J7090
J8
```
Initial orders.

```
STOP
```

```
n10=TAPE
```
Read in input parameter .

```
v14=TAPE2
```
Read in V_1 , α_1 .

```
v0=-1
v18=ARCCOSv0
v19=v18/180
>18,n10=1
```

```
n2=TAPE
v80=TAPE4
v90=TAPE10
>M7103,1
```
Input if programme is run separately.

```
18)v2=n2
v0=2Xv18
v3=v0/v2
n3=0
v4=v95Xv19
n4=700
11)v(100+n3)=v4+vn4
v4=v4+v3
n3=n3+1
n4=n4+1
>11,n2>n3
```
Calculation of $\phi = \theta' + \varepsilon$.

```
n1=0
v82=v82-180
1)v0=v(82+n1)+v95
v(82+n1)=v0Xv19
n1=n1+1
>1,2>n1
v15=v15Xv19
v0=COSv15
v16=v0Xv14
v0=v0Xv97
v27=v0/v18

n12=0
v2=1/v99
v80=v80Xv2
v81=v*81Xv2
v0=1/v80
v1=1/v81
v20=v80-v0
v21=v81-v1
v0=v0+v80
v22=0.5Xv0
v0=v1+v81
v23=0.5Xv0
v24=0.25Xv2
v0=v2Xv2
v28=0.25Xv0
v25=TANv15
```
Calculation of constants and trailing edge initial values.

```
v30=v100-v82
v31=v100-v83
---------------------------------------------------------------
2)v32=SINv30
v33=COSv30
v34=SINv31
v35=COSv31
v2=v22+v33
v3=v23-v35                                     Calculation of W.
v2=1/v2
v3=1/v3

v0=v2Xv32
v1=v3Xv34
v0=v0+v1
v0=2Xv0
v36=v24Xv0            f in location 36 .
------------------------------------------
v0=v20Xv2
v37=-v0Xv24           g in location 37 .
------------------------------------------
v0=v21Xv3
v38=v0Xv24            h in location 38 .
------------------------------------------
→3,n12=1
---------------------------------------------------------------
v0=-v25Xv37
v0=v0+v36
v26=v0/v38            tan α_s in location 26 .
------------------------------------------
v0=ARCTANv26                                   Calculation of α_s and V_s .
Xv0=v0/v19
v1=v26Xv26
v1=1+v1
v1=SQRTv1
v1=v16Xv1             V_s .
------------------------------------------
PRINTv1,3065
PRINTv0,4066          Print V_s, α_s and tan α_s .
PRINTv26,2008
------------------------------------------
Xn12=1
n3=1
---------------------------------------------------------------
v500=v30
v501=v31
v502=v30
v503=v31

v510=v80              Calculation of |d²Ω/dZ²|_T .
v511=-v81
v512=1/v80
v513=-1/v81

n4=0
24)v44=2Xv(500+n4)
v45=COSv(500+n4)
v46=SINv(500+n4)
v47=COSv44
v48=SINv44

v49=1/v(510+n4)
v0=v49+v(510+n4)
v50=0.5Xv0
v0=v50+v45
v0=v0Xv0
v51=v49/v0
v0=v(510+n4)Xv47
v0=v49+v0
v0=0.5Xv0
v58=v45+v0
v0=v(510+n4)Xv48
v0=v0X0.5
v59=v0+v46

v(60+n4)=v58Xv51      A_n.
------------------------------
v(64+n4)=v59Xv51      B_n.
------------------------------
n4=n4+1
→24,4>n4
```

Calculation of $\left|\frac{d^2\Omega}{dZ^2}\right|_T$ (cont.).

```
v0=-v60+v61
v0=v0-v62
v50=v0+v63
v0=v64-v66
v0=-v0Xv25
v1=v65-v67
v1=v1Xv26
v0=v50+v0
v50=v0+v1          P.
```

```
v0=-v64+v65
v0=v0-v66
v51=v0+v67
v0=v60-v62
v0=v0Xv25
v1=v61-v63
v1=-*v1Xv26
v0=v51+v0
v51=v0+v1          Q.
```

```
v0=v50Xv50
v1=v51Xv51
v0=v0+v1
v0=SQRTv0
v0=v0Xv28
v40=v0Xv27
```

Print $\left|\frac{d^2\Omega}{dZ^2}\right|_T$.

```
v0=v100/v19
PRINTv0,3065
PRINTv40,2025
v40=v40/v(219+n3)
→4
```

Calculation of W from equation (31).

```
3)v0=-v37Xv25
v1=-v38Xv26
v0=v0+v1
v40=v0+v36
v40=v40Xv27
PRINTv40,2025
```

$W\times|dZ/d\zeta'|$.

```
4)v(179+n3)=v40/v(219+n3)
PRINTv(179+n3),2025
```

ϕ_1 and ϕ_2.

```
v0=v(100+n3)
n3=n3+1
→14,n2>n3
→15
14)v1=v0/v19
PRINTv1,3065
v30=v0-v82 }
v31=v0-v83 }   φ1 and φ2
→2
```

```
15)v10=v90xv19
Xv0=1/v91
Xv1=v91-v0
v28=v1X.5
v2=v91+v0
v29=v2X.5
v26=COSv10
v27=SINv10
v30=v29Xv26
v31=v28Xv27
v54=v30Xv31
v32=v29Xv29
v33=v32-1
v0=2Xv32
v34=v0-1
v35=v34Xv34
v0=4Xv97
v36=v0/v18
v37=0.125Xv36
v59=2/v36
v38=0.5Xv26
v39=-0.5Xv27

v2=v32Xv33
v0=v27Xv27
v57=v0Xv33
v1=v26Xv26
v58=v32Xv1
```

Functions of $(\beta - \delta\beta)$, γ and d.

```
v0=v32-v0
v1=SQRTv0
v0=v0Xv1
v0=v0/v2
v0=v0Xv37
v61=v0X2
v0=v97Xv96
v62=1/v0
```

Calculation of $\left|\frac{d^2 z'}{d\zeta^2}\right|_T$.

```
n4=0
10)v13=v(300+n4)
v14=v(260+n4)Xv19
v46=SINv14
v47=COSv14
v48=v46Xv46
v49=2Xv48
v49=1-v49
v50=v49Xv49
v0=v46Xv47
v55=2Xv0

v0=v13
v1=1/v0
v40=v0+v1
v40=.5Xv40
v41=v0-v1
v41=.5Xv41
v42=v40Xv40
v43=v42-1
v44=2Xv42
v44=v44-1
v45=v44Xv44
v45=v45-1
v0=v40Xv41
v56=2Xv0

v1=v32+v42
v51=v1-v48
v53=v48+v43
v52=v53+v33
```

Calculation of $\left|\frac{d\zeta}{dz'}\right|$

```
→16,n4=0
v0=v58Xv53
v1=v42-v48
v1=v57Xv1
v2=v54Xv55
v0=v0+v1
v0=v0-v2
v1=v44Xv49
v1=v34-v1
v1=v1Xv1
v2=v56Xv55
v2=v2Xv2
v1=v1+v2
v0=v1/v0
v0=SQRTv0
v0=v0Xv(300+n4)
v60=v59Xv0
→17
16)v60=1/v61
```

```
17)v2=v60Xv(180+n4)                     Calculation of v/V1.
```

```
v1=v(400+n4)Xv62
v1=1+v1
PRINTv1,3065                            Calculation and output of (1 + x/c) and Cp.
v0=v2Xv2
v0=1-v0
PRINTv0,4087
```

```
SPv0=v2
SPv0=v60                                Optional output of v/V1 and |dζ/ds'|.
```

```
n4=n4+1
→10,n2>n4

(→0)
*********

Z
************
```

+17	Specified accuracy.	
+37.5	β .	
+1.0098763	c/d.	Cascade data .
+0.0156	ρ/c .	
+16.0	ε .	
+3.14159265	d^L .	

Z

+1	Input parameter .
+0	Circular arc-camber angle .
+40	Number of given points.

Blade coordinates .

Blade data .

+1.57492474	+0.0
+1.55738553	+0.00513700268
+1.42260484	+0.0462520325
+1.30512035	+0.0817121990
+1.19170238	+0.114693902
+1.02479402	+0.160280410
+0.823077517	+0.209970951
+0.656515987	+0.246172349
+0.464734474	+0.282160026
+0.205922353	+0.320568830
−0.0597352553	+0.346968399
−0.389316024	+0.359104316
−0.572422621	+0.354512283
−0.773142345	+0.338571331
−0.941060388	+0.314989626
−1.14953588	+0.269571148
−1.31718149	+0.215472451
−1.44883128	+0.155355501
−1.48472984	+0.134389826
−1.56604665	+0.0701097394
−1.58312345	+0.0486525124
−1.59417486	+0.0274674713
−1.59821360	+0.00680140778
−1.59380608	−0.0130108841
−1.57877419	−0.0314902570
−1.54957068	−0.0479007055
−1.49974774	−0.0609872117
−1.41507438	−0.0681858518
−1.24565310	−0.0614059729
−1.07310990	−0.0429395536
−0.861599786	−0.0146762290
−0.576222052	+0.0246962586
−0.248940944	+0.0645008134
−0.0115565313	+0.0870590201
+0.358413184	+0.108253224
+0.650360815	+0.110533153
+0.889897786	+0.101185338
+1.08755085	+0.0847222884
+1.31658862	+0.0541017379
+1.47227347	+0.0246978600

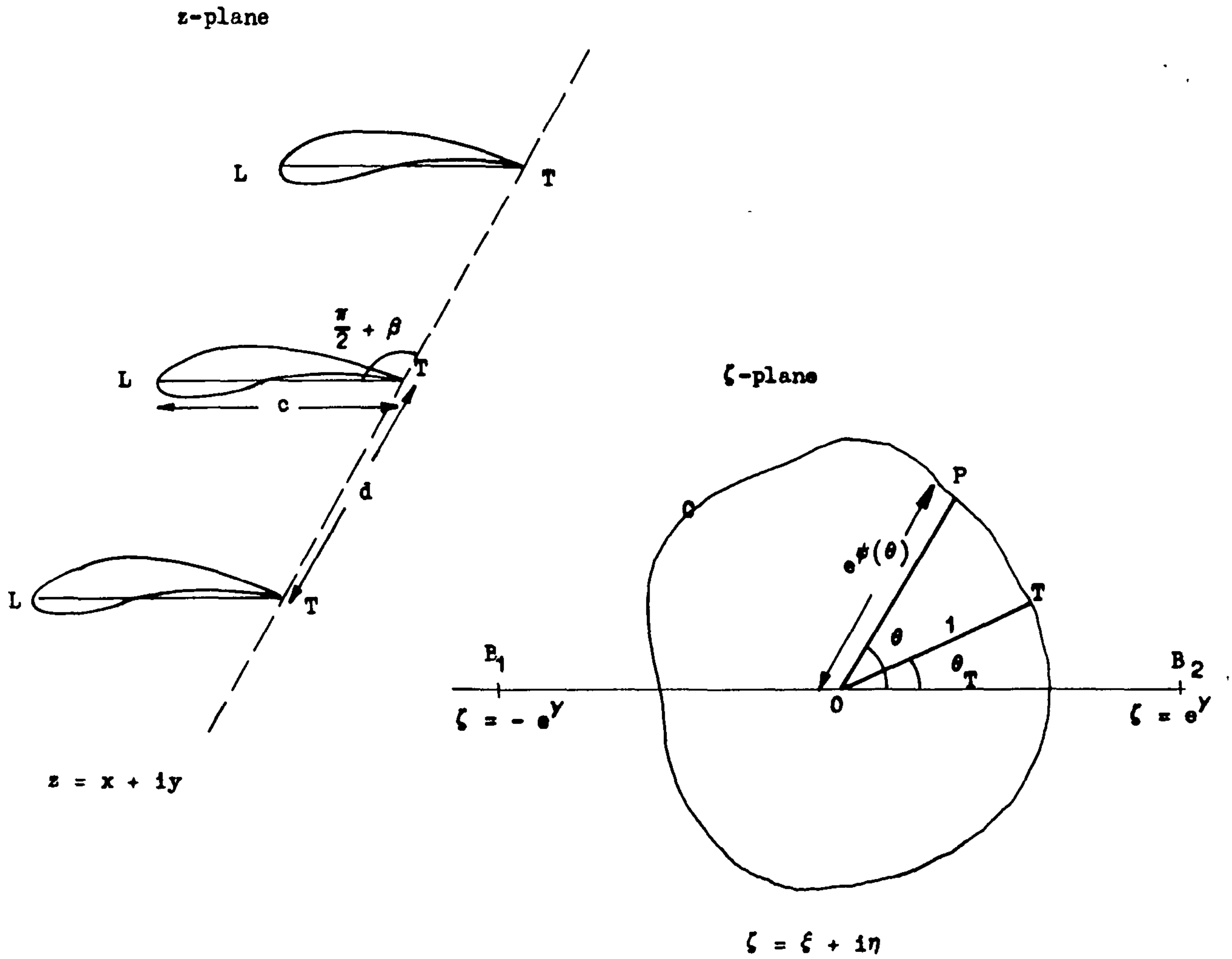
Z

+40 Number of interpolated points.

Z

+1	Input parameter .	
+1.0	V_1 .	Inlet flow data.
+53.5	α_1 .	

Z
****



On $C, \zeta = e^{\phi(\theta)+i\theta} = e^{\sigma}$.

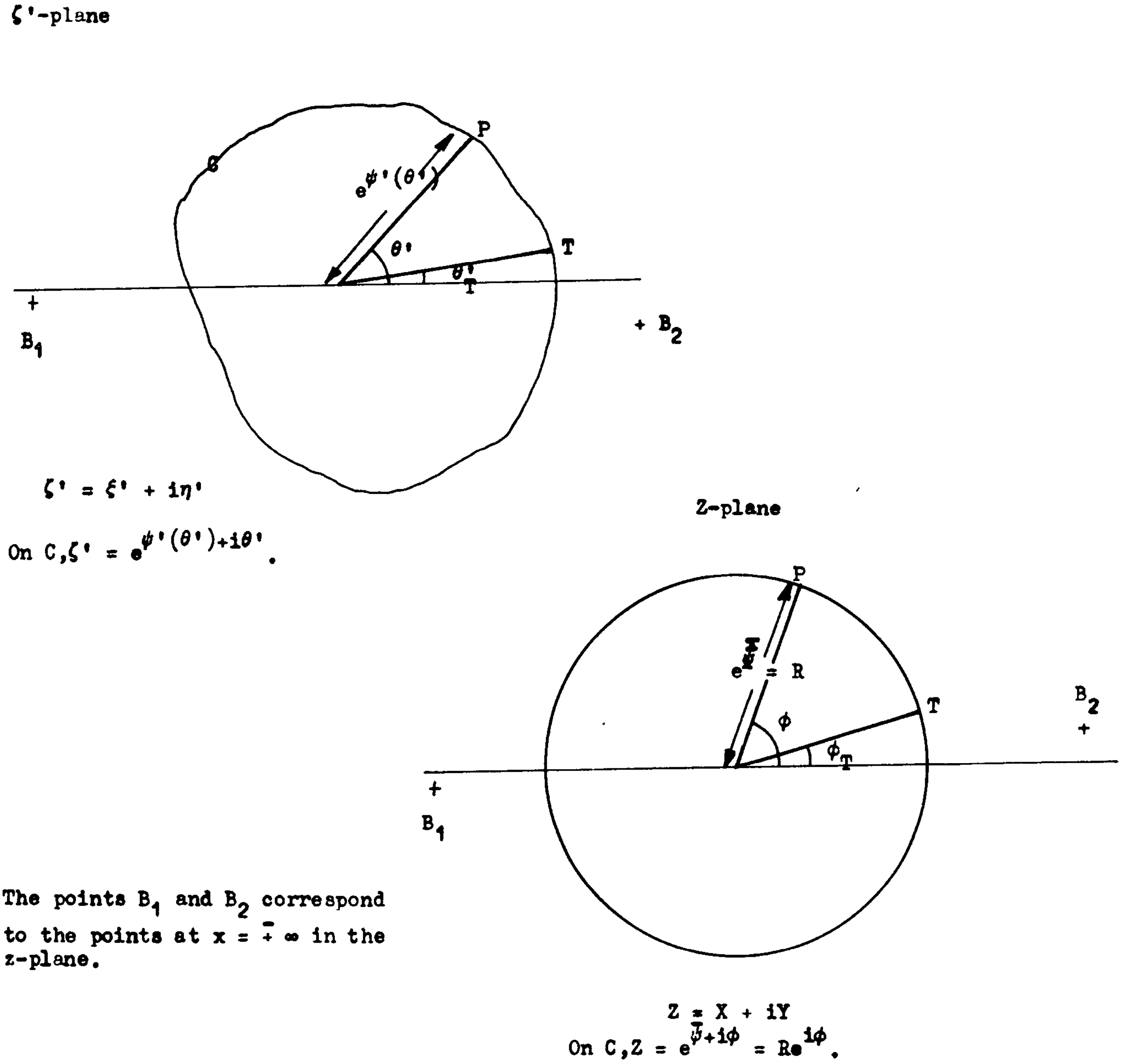
The points B_1 and B_2 correspond to the points at $x = \mp \infty$ in the z-plane.

<u>Transformation</u>

$$T_1 : z = \frac{d}{2\pi}\left[e^{-i\beta}\log\frac{e^{\gamma}+\zeta}{e^{\gamma}-\zeta} + e^{i(\beta-2\delta\beta)}\log\frac{\zeta+e^{-\gamma}}{\zeta-e^{-\gamma}}\right] - \frac{1}{2}(c-\delta_0)e^{-i\delta\beta} \qquad (1)$$

$$= F(\zeta)$$

Figure 1. Some notation and the first transformation of the cascade.



<u>Transformations</u>

$$T_2 : \zeta' = \zeta - \bar{\zeta},\ \bar{\zeta} \text{ constant} \tag{2}$$

$$T_3 : \log(\zeta'/Z) = \sum_{n=1}^{\infty} a_n Z^{-n} \tag{3}$$

Figure 2. Further notation and transformations.

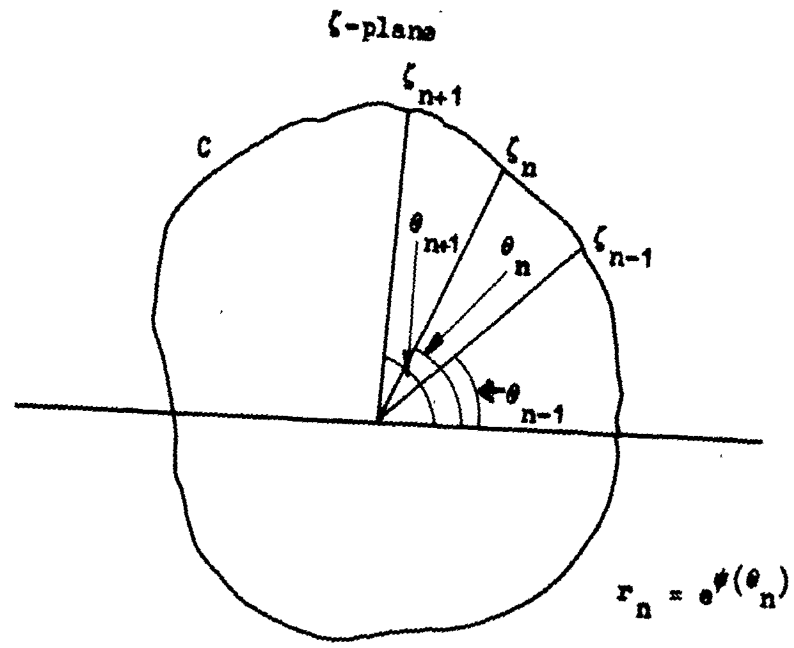
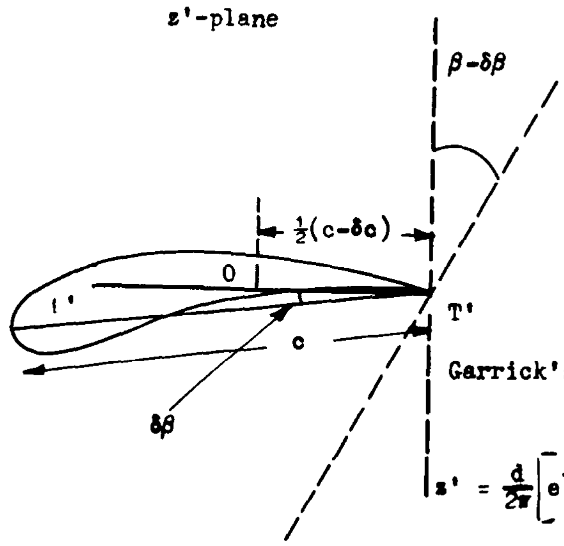


Figure 3. Notation used in the determination of the constant $\bar{\zeta}$ in T_2.

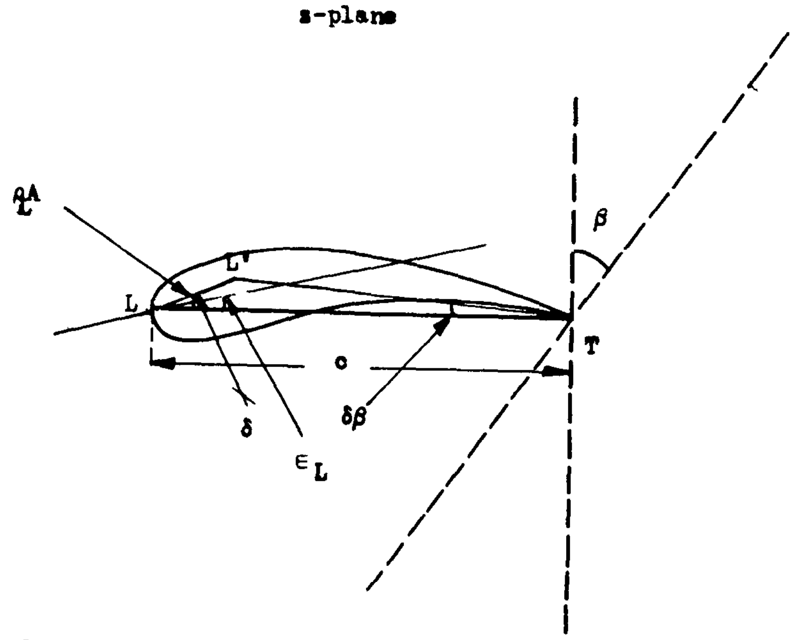
(a)



Garrick's transformation is

$$z' = \frac{d}{2\pi}\left[e^{-i(\beta-\delta\beta)}\log\frac{e^{\gamma}+\zeta}{e^{\gamma}-\zeta} + e^{i(\beta-\delta\beta)}\log\frac{\zeta+e^{-\gamma}}{\zeta-e^{-\gamma}}\right].$$

(b)



The transformation from the z'- to the z-plane is $z = \left\{z' - \frac{1}{2}(c - \delta c)\right\}e^{-i\delta\beta}$; thus we obtain T_1 as

$$z = \frac{d}{2\pi}\left[e^{-i\beta}\log\frac{e^{\gamma}+\zeta}{e^{\gamma}-\zeta} + e^{i(\beta-2\delta\beta)}\log\frac{\zeta+e^{\gamma}}{\zeta-e^{-\gamma}}\right] - \frac{1}{2}(c - \delta c)e^{-i\delta\beta}.$$

Figure 4. Details of transformation T_1, showing its derivation from the standard Garrick equation.

z-plane

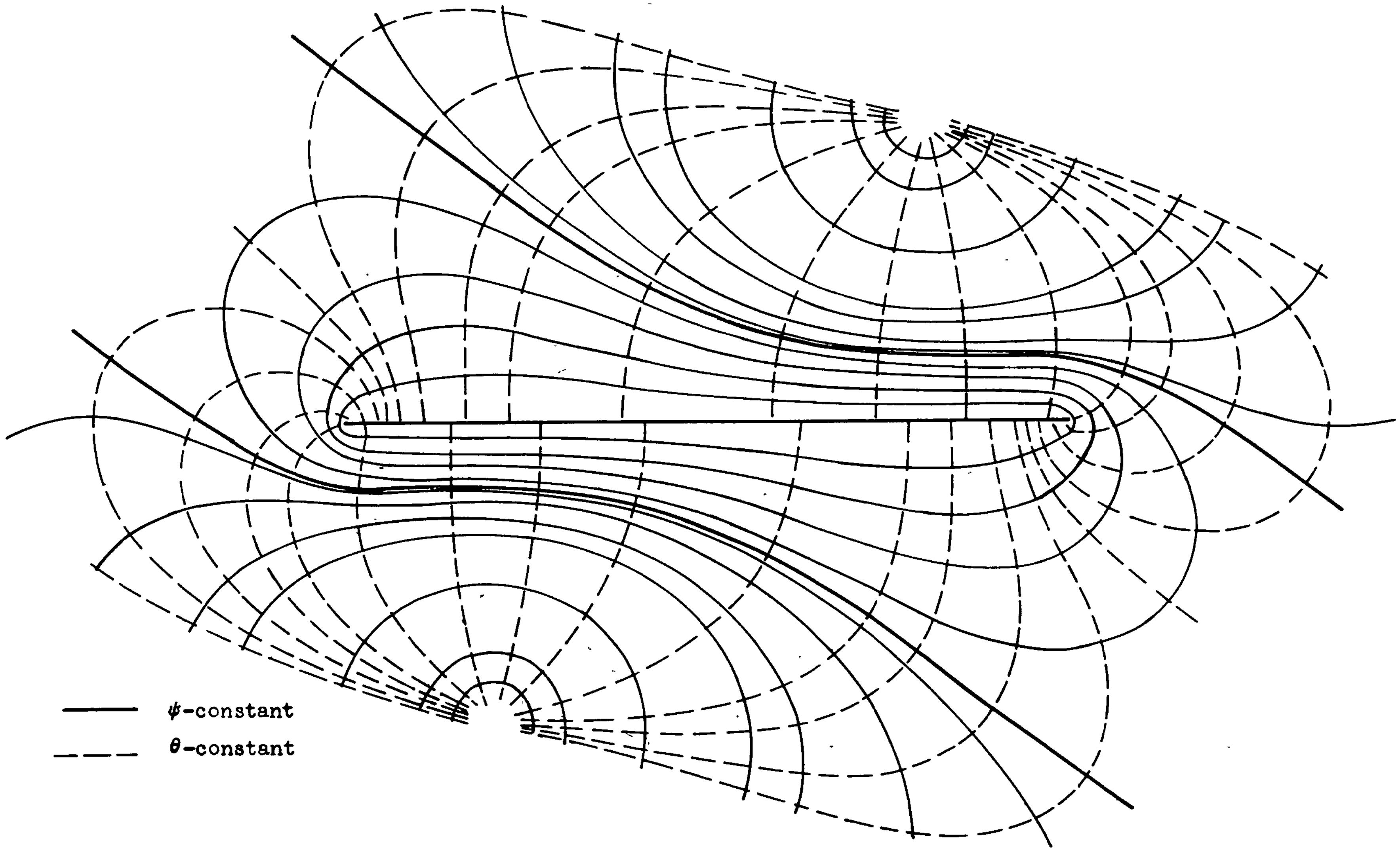


Figure 5. Contours in the z-plane corresponding to ψ = constant and θ = constant in the transformation T_1.

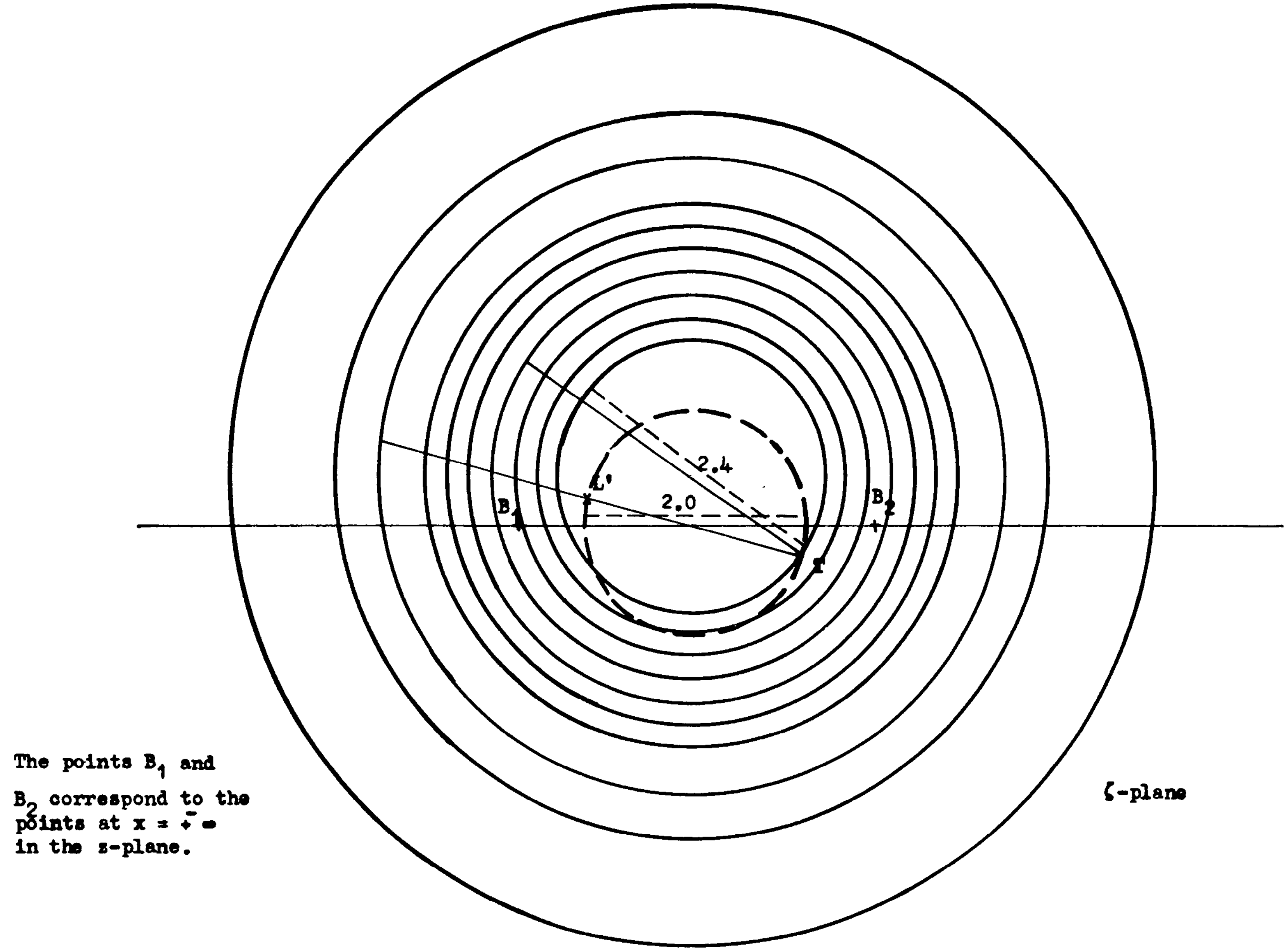


Figure 6. The ζ-plane circles to which the curves in Figure 7 correspond.

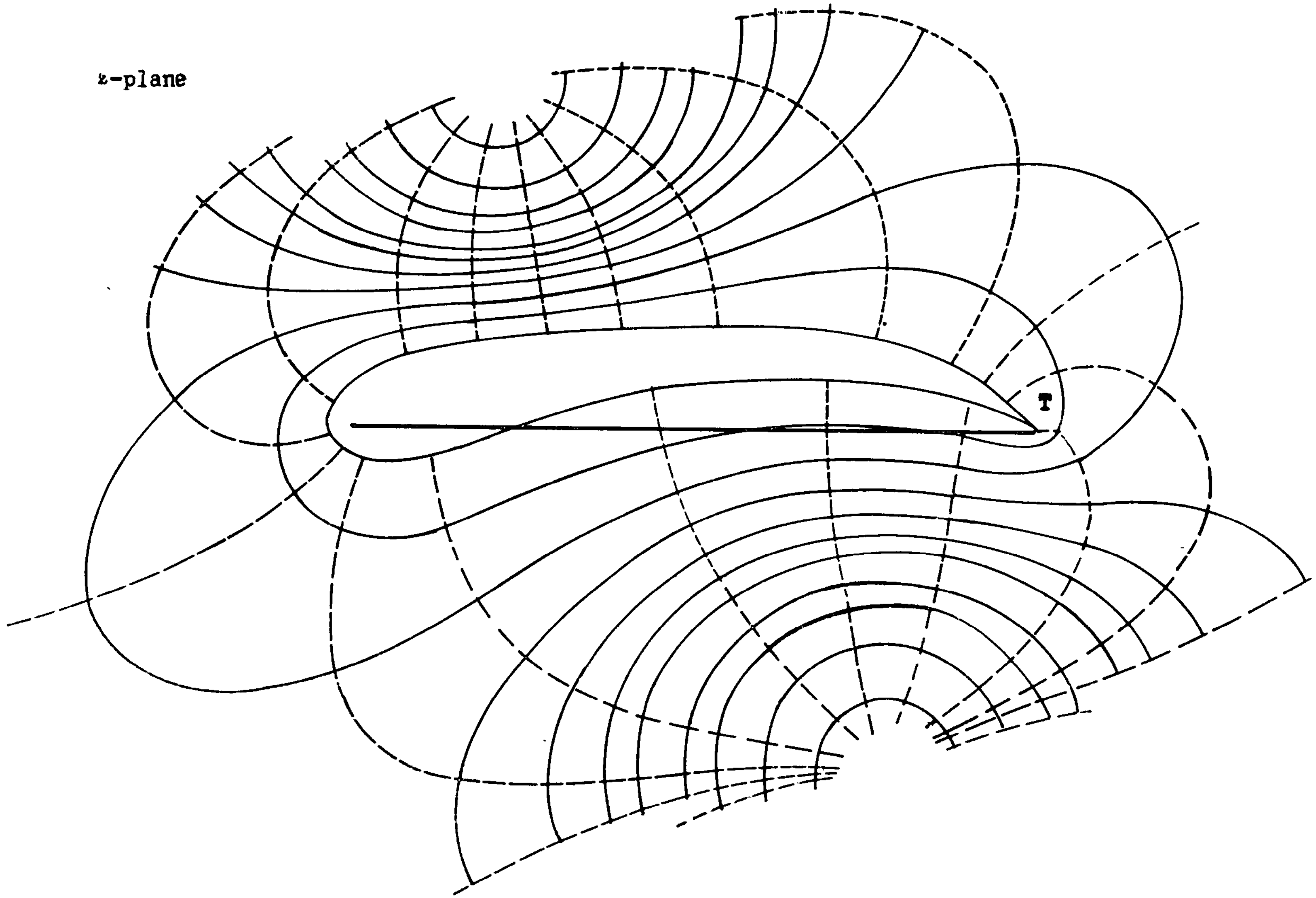
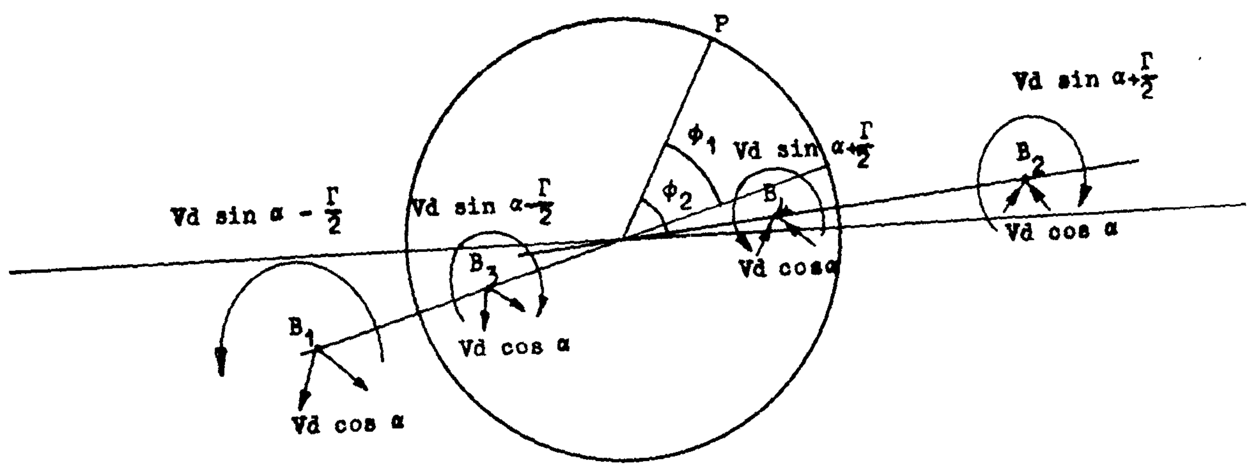


Figure 7. The z-plane curves to which the circles in Figure 6 correspond (full lines) together with the orthogonal curves (broken lines).



The points B_1 and B_2 correspond to the points at $x = \mp \infty$ in the z-plane, B_3 and B_4 are their inverse points in the circle.

At B_1, $Z = -\beta_1$ and $|Z| = m_1 R$.

At B_2, $Z = \beta_2$ and $|Z| = m_2 R$.

At B_3, $Z = -\dfrac{R^2}{\bar{\beta}_1}$ and $|Z| = \dfrac{1}{m_1} R$.

At B_4, $Z = \dfrac{R^2}{\bar{\beta}_2}$ and $|Z| = \dfrac{1}{m_2} R$.

Figure 8. System of singularities for the circle plane.

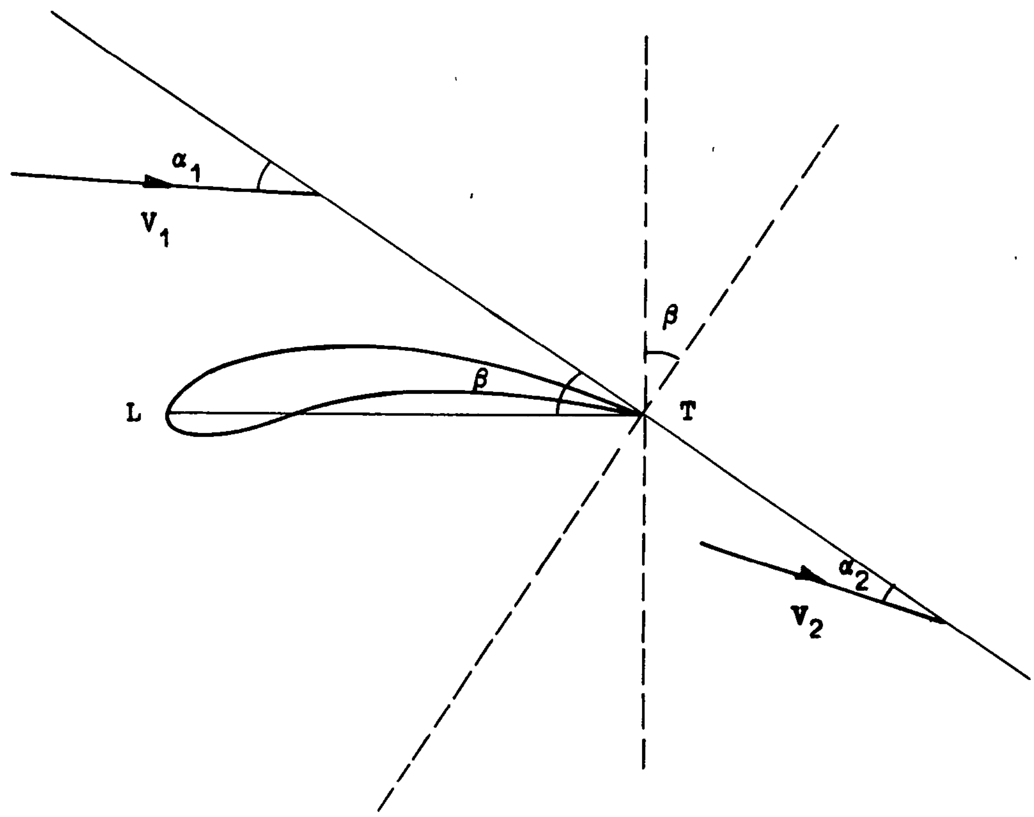


Figure 9. Inlet and outlet velocities.

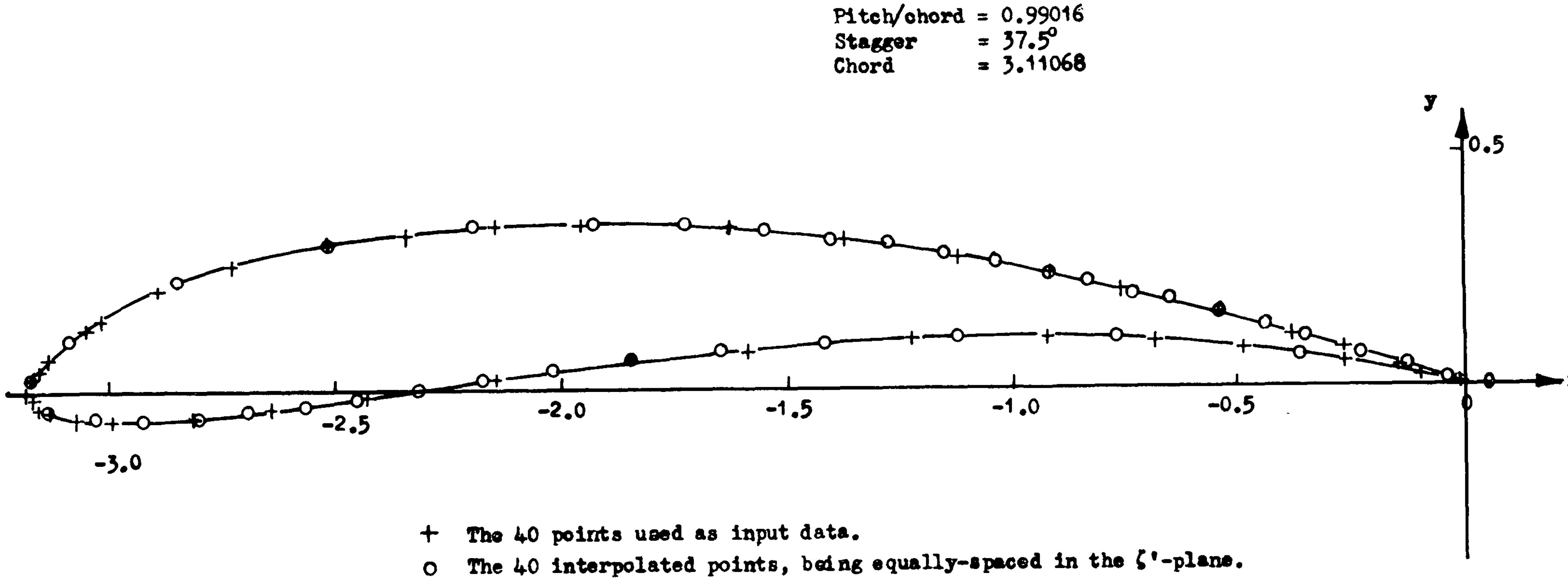
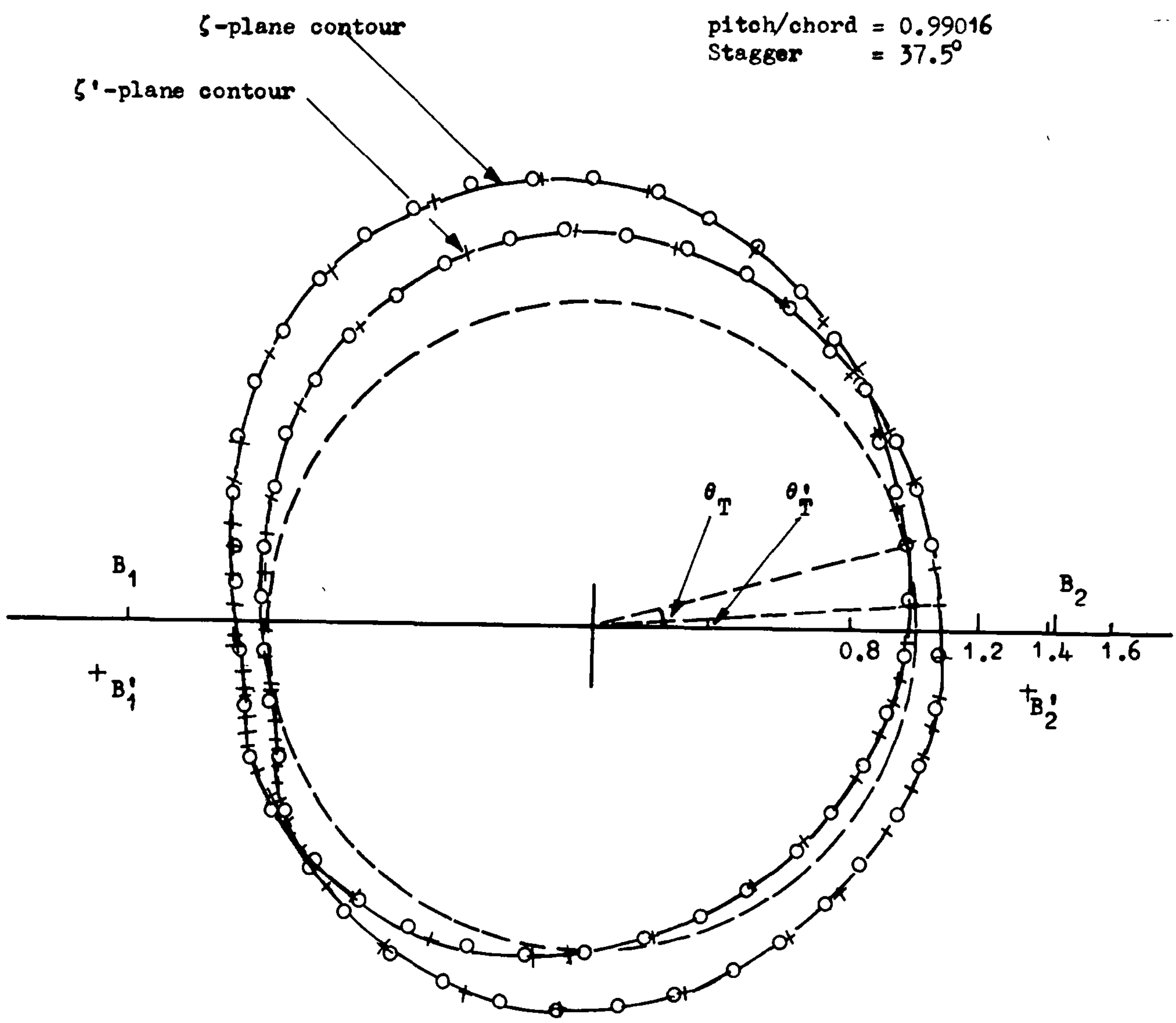


Figure 10. The test blade profile calculated from the method of Merchant and Collar (1941).



+ The 40 given points.

o The points equally spaced in the ζ'-plane.

B_1 and B_2 are the ζ-plane singular points.

B'_1 and B'_2 are the ζ'-plane singular points.

Figure 11. Contours in the ζ- and ζ'-planes for the Merchant and Collar cascade.

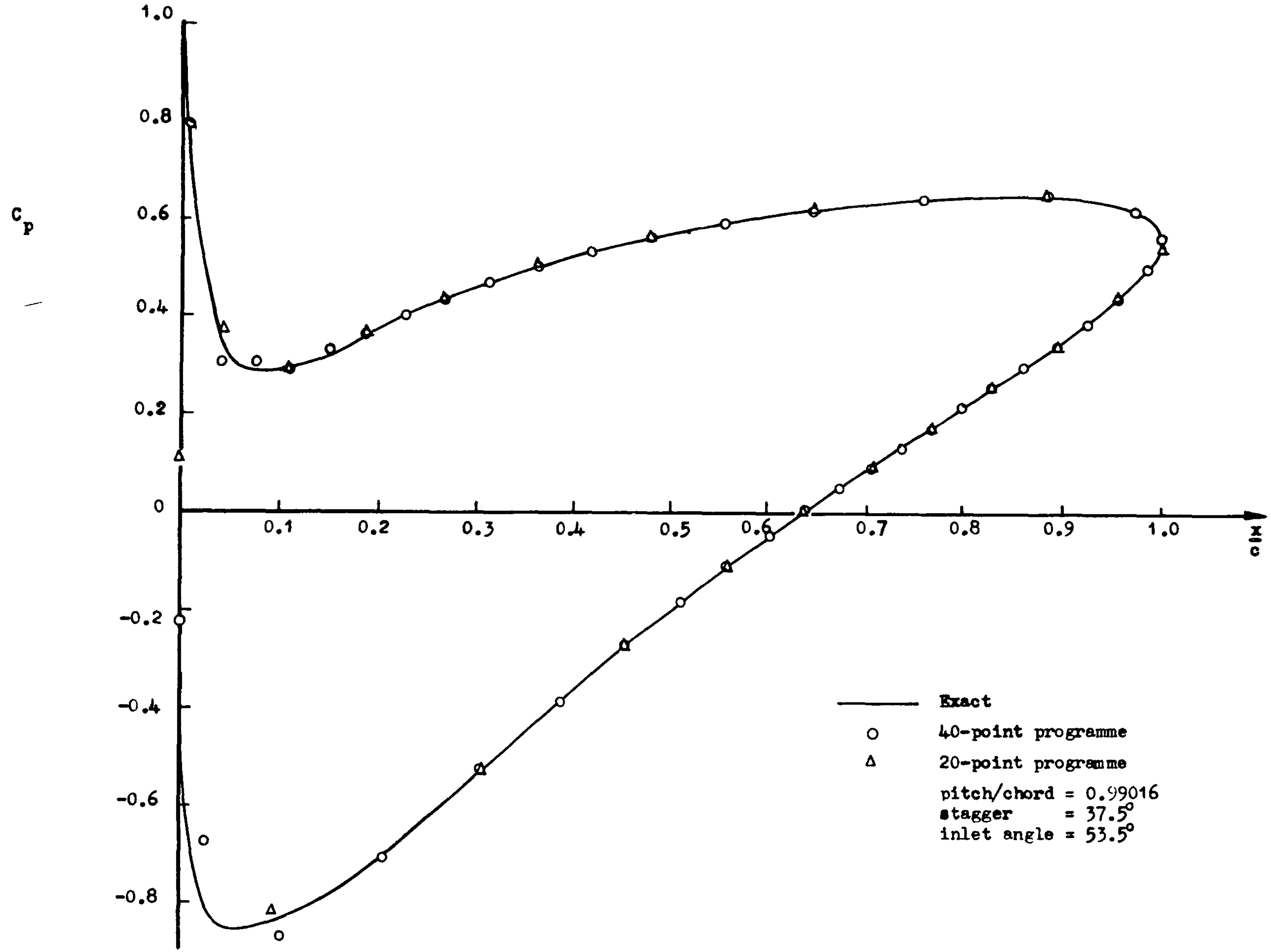


Figure 12. Calculated values of the pressure distribution, compared with exact values, for the Merchant and Collar cascade.

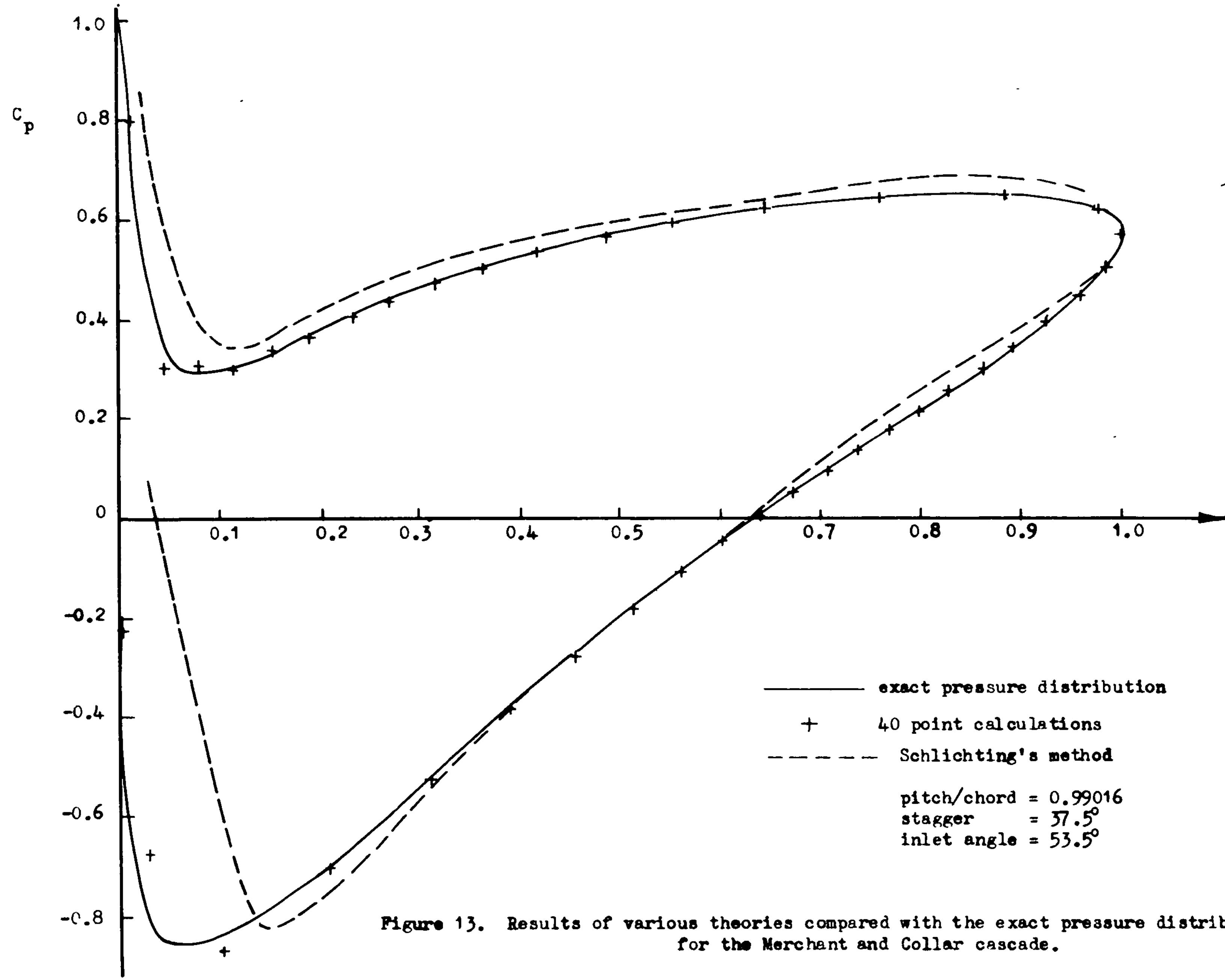
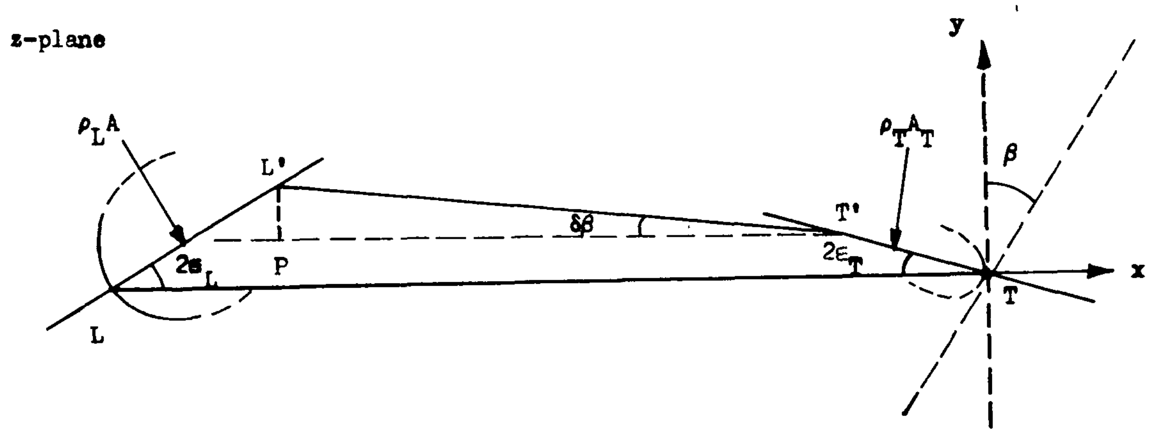


Figure 13. Results of various theories compared with the exact pressure distribution for the Merchant and Collar cascade.



L and T are the leading and trailing edges of a blade.
L' and T' are the corresponding singularities of transformation T_1.

T_1 is now given by

$$z = \frac{d}{2\pi}\left[e^{-i\beta}\log\left(\frac{e^{\gamma}+\zeta}{e^{\gamma}-\zeta}\right) + e^{i(\beta-2\delta\beta)}\log\left(\frac{e^{-\gamma}+\zeta}{\zeta-e^{-\gamma}}\right)\right] - \tfrac{1}{2}(c - \delta c)e^{-i\delta\beta} - \delta z, \qquad (43)$$

where

$$\delta z = \rho_T A_T e^{2i\epsilon_T}$$
$$= \tfrac{1}{2}\rho_T \cos\epsilon_T e^{2i\epsilon_T} \qquad (44)$$

following the procedure of Section 7 to find A_T. Also

$$\delta\beta \approx \frac{L'P}{LT} = (\rho_L A_L \sin 2\epsilon_L - \rho_T A_T \sin 2\epsilon_T)/c$$
$$= \tfrac{1}{2}\left(\frac{\rho_L}{c}\right)\cos\epsilon_L \sin 2\epsilon_L - \tfrac{1}{2}\left(\frac{\rho_T}{c}\right)\cos\epsilon_T \sin 2\epsilon_T, \qquad (45)$$

and

$$\delta c \approx \tfrac{1}{2}\rho_L \cos\epsilon_L \cos 2\epsilon_L + \tfrac{1}{2}\rho_T \cos\epsilon_T \cos 2\epsilon_T. \qquad (46)$$

Figure 14. Calculation of the parameters in transformation T_1 for blades with rounded trailing edges.

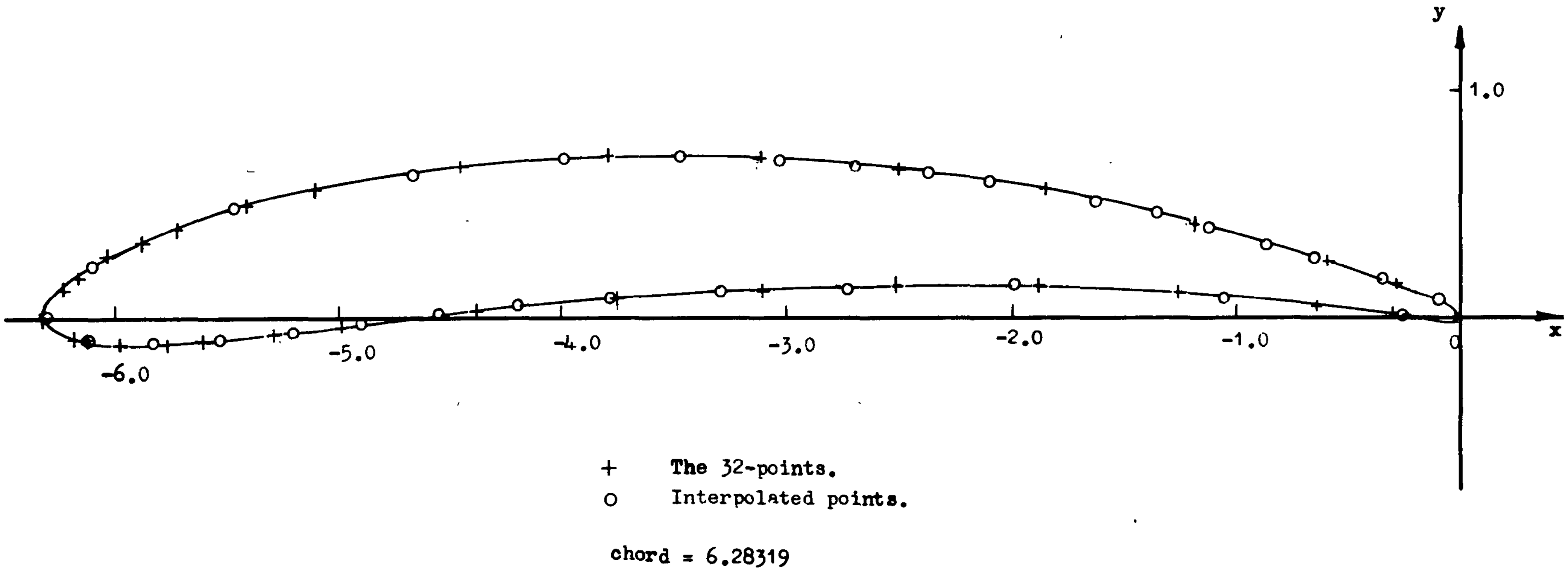
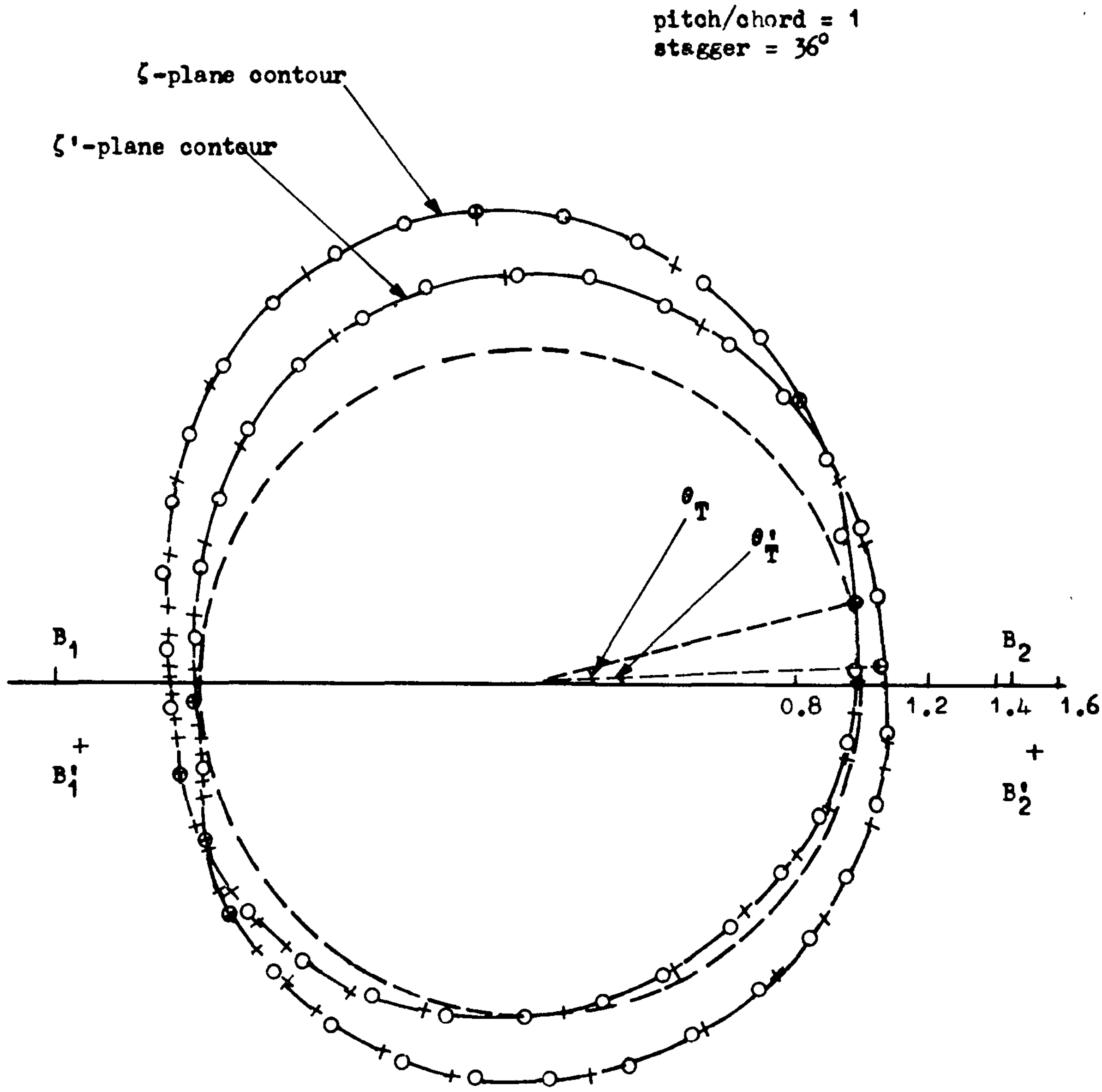


Figure 15. The 10C4/30C50 blade.



+ The 32 given points

o Interpolated points

B_1 and B_2 are the ζ-plane singular points

B'_1 and B'_2 are the ζ'-plane singular points

Figure 16. Contours in the ζ- and ζ'-planes for the cascade of 10C4/30C50 aerofoils.

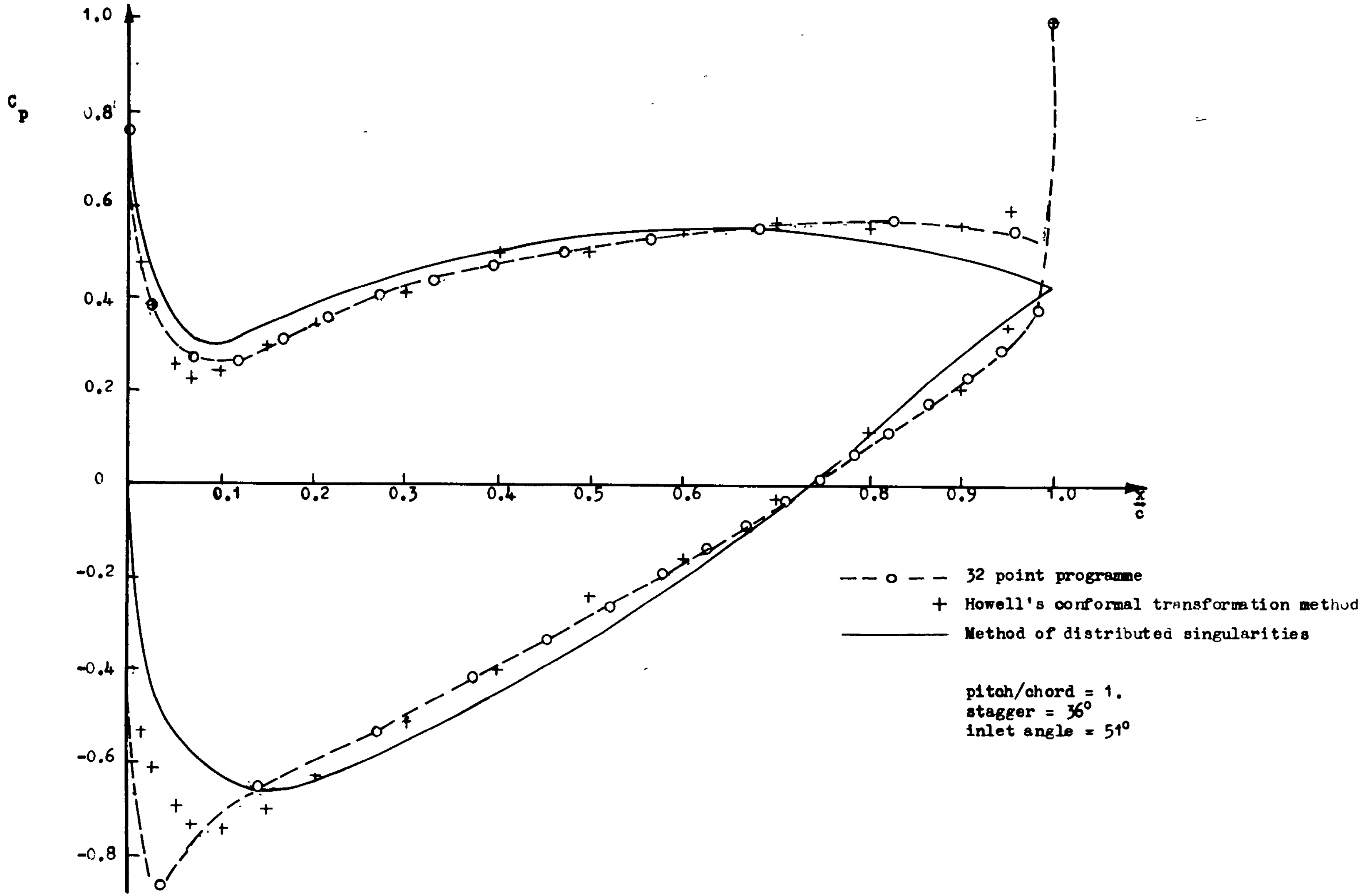
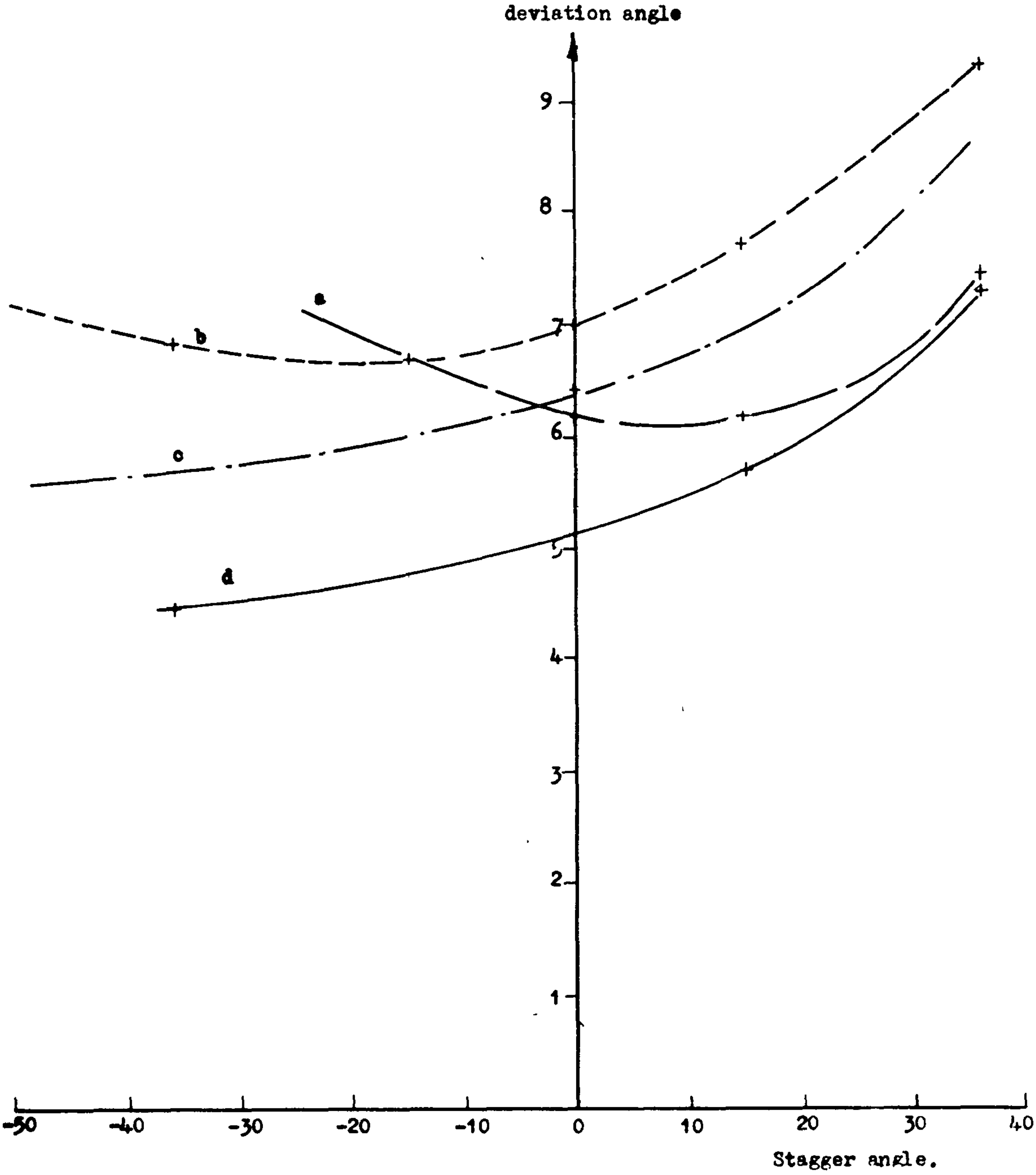


Figure 17. Pressure distribution: Comparative results for the 10C4/30C50 cascade.



Curve a - Howell's conformal transformation method
b - Method of distributed singularities
c - Rule for nominal deviation
d - Present method

} See Pollard and Wordsworth (1962)

Figure 18. Deviation v. stagger: Comparative results for the 10C4/30C50 cascade.

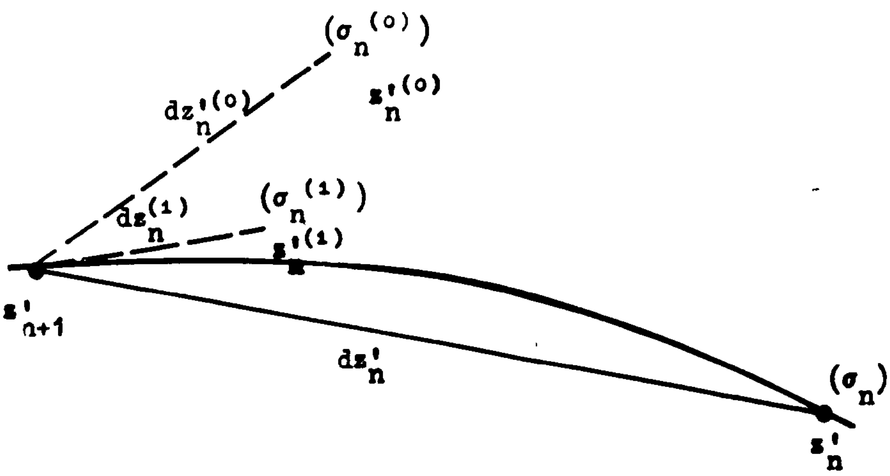


Figure 19. The iterative scheme for calculating ζ-plane points.

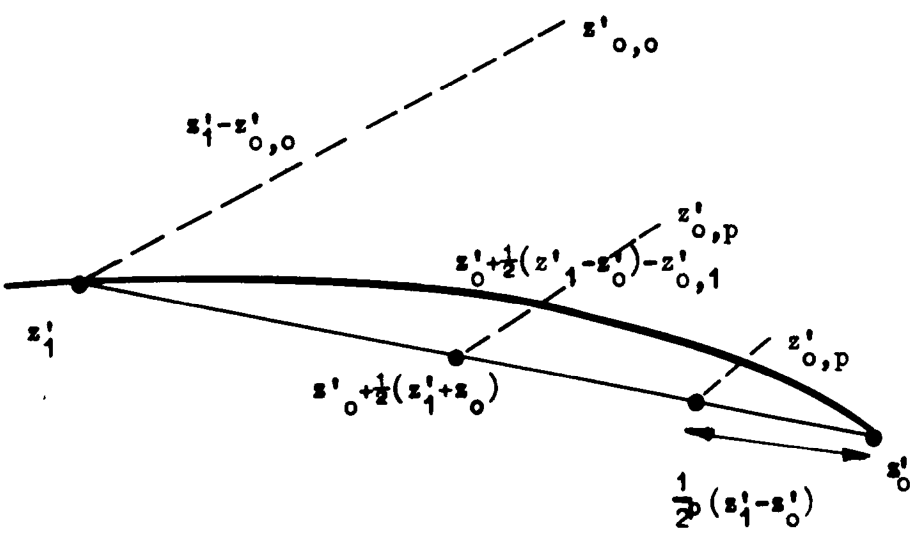


Figure 20. The ζ-plane iterative scheme near the trailing edge.

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The solution is obtained by means of a series of conformal transformations, the most important of which are those due to Garrick and Theodorsen to which, however, essential modifications have been made which allow a practical numerical solution. The mathematical validity is successfully tested against the known surface pressure distribution about a given compressor cascade as calculated by Gostelow.

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