C.P. No. 68
(5804)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL

CURRENT PAPERS

Approximate Two-dimensional Aerofoil Theory.

Part I. Velocity Distributions for Symmetrical Aerofoils

By

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LONDON: HER MAJESTY'S STATIONERY OFFICE

1952

Price 11s 0d net.

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PREFACE

This paper constitutes Part I of an approximate theory of two-dimensional aerofoils. Parts I and II deal with the "direct" problem of the calculation of the velocity distribution over the surface of a given aerofoil, Part I for symmetrical sections and Part II for cambered sections. Parts III and IV are concerned with the "inverse" problem of designing an aerofoil with a specified velocity distribution, Part III for symmetrical sections and Part IV for the camber-line. Part V discusses the chordwise position of the peak velocity, and also the theoretical C_L-range of low-drag aerofoils. Aerofoils with hinged flaps form the subject of Part VI.

These papers are being published, without amendment, in the form in which they were written several years ago and with the dates when they were received by the Council. They will carry the following consecutive numbers:— Current Papers Nos. 68, 69, 70, 71, 72 and 73.

J. L. N.

December, 1951

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14th May, 1942.

1. Introduction

In these notes we shall be concerned with irrotational two-dimensional flow past aerofoils, with the establishment of simple methods and formulae for finding useful information about a given aerofoil, and with the inverse problem of establishing methods and formulae useful in designing an aerofoil with given properties.

The usual theory of thin aerofoils as presented, for example, by Glauert[1], is in fact a theory of aerofoils of zero thickness and small camber at small lift coefficients. In these notes the approximations applied require only that the thickness and camber shall not be too large; the errors with various thicknesses, cambers and lift coefficients will be illustrated by comparisons with numerically accurate results derived from an exact theory.

Apart from the reports on the exact theory by Theodorsen[2] and by Theodorsen and Garrick[3] the only papers in which thickness is considered appear to be a paper by Jeffreys[4] and a recent note by Squire[5]; reference to these papers will be made later.

The results in these notes may be obtained by several methods; we here deduce them all from the results of the exact theory (See References 2 and 3).

In this first part we are concerned with the velocity distribution (and therefore, because of Bernoulli's equation, with the pressure distribution) at the surface of a given symmetrical aerofoil. Later notes will deal with cambered aerofoils, with the inverse problem of the design of aerofoils for given velocity distributions, with calculations of moments, and with numerical comparisons of this theory with accurate results from the exact theory.

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When applied to find the properties of a given aerofoil, the theory is specially designed to be applied to aerofoils with ordinates given by algebraical formulae, or which can be fitted to such formulae. It may also be applied if the ordinates are given numerically only (though the fitting of algebraical formulae, if possible, is recommended). It is not meant to be applied to aerofoils derived by simple conformal transformations from a circle.

In this first part the requisite formulae are obtained in three different forms. We have, first, Fourier series conjugate to Fourier series for given functions. These Fourier series are summed and the sums expressed generally by means of integrals. Finally the sums are evaluated in finite terms, with the occurrence of simple functions only, for a wide variety of cases in which the aerofoil ordinate is given by algebraical formulae; these evaluations in finite terms may, for example, be carried out if, with the aerofoil chord divided into any number of segments, in each segment the ordinate y is a polynomial of any degree in $x^{\frac{1}{2}}$ (including, of course, a polynomial in x) or in $(1-x)^{\frac{1}{2}}$, where x is the distance in fractions of the chord from the leading edge; in addition an elliptic nose may be fitted, if desired, up to any distance back from the leading edge, and the trailing edge may be rounded off by a circle, an ellipse or a hyperbola up to any distance from the trailing edge. Moreover the same mathematical methods will avail to provide results, if required, for certain other types of formulae. In this way results in finite terms may be obtained for practically every aerofoil for which formulae have ever been proposed. For a number of definite aerofoils (NACA 0012, NACA 16-012, Clark Y fairing, EQH 1260, EQH 1250, EQH 1240) these formulae in finite terms are set out as examples. For one aerofoil (EQH 1260), all the computations are carried through, and the results compared with numerically accurate results from the exact theory, with a wholly satisfactory measure of agreement.

Finally, it should be remarked that the results of the approximate theory of these notes may be used as a kind of first approximation in exact calculations of numerically accurate results.

2. Exact Theory and Nomenclature

As usual we denote the lift coefficient by C_L and the geometrical incidence by α. We denote by $-\beta$ the theoretical no-lift angle, by U the velocity of the undisturbed stream, by q the fluid velocity at a point on the aerofoil surface, by x the distance in fractions of the chord from the leading edge (measured along the chord), by y the ordinate in fractions of the chord.

According to the exact theory, if*

$$C_L = a_0 \sin(\alpha + \beta), \qquad \ldots (1)$$

then

$$\frac{q}{U} = F(\theta)\left| \left(1 - \frac{C_L^2}{a_0^2}\right)^{\frac{1}{2}} \sin(\theta + \varepsilon - \beta) + \frac{C_L}{a_0}\cos(\theta + \varepsilon - \beta) + \frac{C_L}{8\pi a_0}[\Psi]\right. \qquad \ldots (2)$$

where/

*If the experimental no-lift angle is not equal to its theoretical value, or if in any other respect (1) does not hold, we must substitute $\sin(\alpha + \beta)$ for C_L/a_0 in the first two terms of equation (2).

where

$$\left.\begin{aligned} x &= 2a \cosh \psi_L - 2a \cosh \psi \cos \theta \\ y &= 2a \sinh \psi \sin \theta \end{aligned}\right\} \quad \ldots\ldots(3)$$

$$\varepsilon(\theta) = -\frac{1}{2\pi} P \int_{-\pi}^{\pi} \psi(t) \{1 + \varepsilon'(t)\} \cot \tfrac{1}{2}[t + \varepsilon(t) - \theta - \varepsilon(\theta)]dt, \quad \ldots\ldots(4)$$

P denotes that the "principal value" of the integral is to be taken, and the dash denotes a derivative,

$$F(\theta) = \frac{e^{[\Psi]}(1 + \varepsilon'(\theta))}{[1 + (\psi'(\theta))^2]^{\frac{1}{2}}[\sinh^2 \psi + \sin^2 \theta]^{\frac{1}{2}}}, \quad \ldots\ldots(5)$$

and

$$[\Psi] = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\psi(t) - \varepsilon(t)\psi'(t)]dt. \quad \ldots\ldots(6)$$

θ is positive on the upper surface and negative on the lower surface; it is zero at the leading edge and $\pm\pi$ at the trailing edge. When the Kutta-Joukowski condition is satisfied at a sharp trailing edge, or the trailing edge, if rounded, is a stagnation point, the theoretical value of a_0 is $8\pi a e^{[\Psi]}$.

ψ_L is the value of ψ at the leading edge, $\theta = 0$. Equations (3) define ψ and θ at any point of the aerofoil, and define ψ as a function of θ. Equation (4) is an integral equation for ε. The value of β is the value of ε at $\theta = \pm\pi$.

The choice of a and ψ_L is to some extent arbitrary, but they are always chosen so that when second powers of the thickness are neglected we may always take $\cosh \psi_L = 1$, $4a = 1$.

3. First Approximation for an Aerofoil of Small Thickness.

To begin with neglect second powers of the thickness, and therefore second and higher powers and products of ψ, ψ', ε, ε'. Then

$$x = \tfrac{1}{2}(1 - \cos \theta), \qquad y = \tfrac{1}{2}\psi \sin \theta \quad \ldots\ldots(7)$$

$$[\Psi] = C_0 \text{ (say)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(t)dt, \quad \ldots\ldots(8)$$

$$\varepsilon(\theta) = -\frac{1}{2\pi} P \int_{-\pi}^{\pi} \psi(t) \cot \tfrac{1}{2}(t - \theta)dt. \quad \ldots\ldots(9)$$

Also

$$F(\theta) = \frac{1 + C_0 + \varepsilon'(\theta)}{|\sin \theta|}, \quad \ldots\ldots(10)$$

and if we also neglect $(C_L/a_0)^2$ and the product of C_L/a_0 with ε, ε', or C_0, we have

q/U /

$$\frac{q}{U} = \frac{1 + C_0 + \varepsilon'(\theta)}{|\sin\theta|}\left|\sin\theta + (\varepsilon - \beta)\cos\theta + \frac{C_L}{a_0}\cos\theta + \frac{C_L}{2\pi}\right|$$

$$= 1 + C_0 + \varepsilon'(\theta) + (\varepsilon - \beta)\cot\theta + \frac{C_L}{a_0}\cot\theta + \frac{C_L}{2\pi}\operatorname{cosec}\theta. \quad (11)$$

The theoretical value of a_0 is approximately $2\pi e^{C_0}$.

4. Symmetrical Aerofoil at Zero Incidence.

For a symmetrical aerofoil at zero incidence

$$q/U = 1 + g, \qquad \ldots\ldots (12)$$

where

$$g = C_0 + \varepsilon'(\theta) + \varepsilon\cot\theta. \qquad \ldots\ldots (13)$$

For a symmetrical aerofoil ψ is even in θ and ε is odd, and if

$$\psi = \sum_{n=0}^{\infty} C_n \cos n\theta, \qquad \ldots\ldots (14)$$

then (cf. Appendix, Lemma (5)).

$$\varepsilon = \sum_{n=1}^{\infty} C_n \sin n\theta. \qquad \ldots\ldots (15)$$

Hence (Appendix, Lemma (1))

$$2y = \psi\sin\theta = (C_0 - \tfrac{1}{2}C_2)\sin\theta + \sum_{n=2}^{\infty}\tfrac{1}{2}(C_{n-1} - C_{n+1})\sin n\theta. \quad (16)$$

Since $\psi\sin\theta$ is zero at 0 and π its Fourier sine series may be differentiated term-by-term, so if we write

$$y(x) = f(\theta), \qquad \ldots\ldots (17)$$

then

$$2f'(\theta) = (C_0 - \tfrac{1}{2}C_2)\cos\theta + \sum_{n=2}^{\infty}\tfrac{1}{2}n(C_{n-1} - C_{n+1})\cos n\theta. \quad (18)$$

Now by Lemma (3) (Appendix),

$$\varepsilon\sin\theta = \tfrac{1}{2}C_1 + \tfrac{1}{2}C_2\cos\theta - \sum_{n=2}^{\infty}\tfrac{1}{2}(C_{n-1} - C_{n+1})\cos n\theta, \quad (19)$$

and a Fourier cosine series may be differentiated term-by-term, so

$$g\sin\theta = C_0\sin\theta + \frac{d}{d\theta}(\varepsilon\sin\theta)$$

$$= (C_0 - \tfrac{1}{2}C_2)\sin\theta + \sum_{n=2}^{\infty}\tfrac{1}{2}n(C_{n-1} - C_{n+1})\sin n\theta, \quad (20)$$

and/

and $g \sin\theta$ has a Fourier series conjugate to that for $2f'(\theta)$. Hence (Appendix, Lemma (6)),

$$g \sin\theta = -\frac{2\sin\theta}{\pi} P\int_0^\pi \frac{f'(t)dt}{\cos\theta - \cos t}, \qquad \ldots\ldots (21)$$

i.e.,*

$$g = -\frac{2}{\pi} P\int_0^\pi \frac{f'(t)dt}{\cos\theta - \cos t}$$

$$= -\frac{1}{\pi} P\int_0^1 \frac{y'(\xi)d\xi}{\xi - x}. \qquad \ldots\ldots (22)$$

The same result may also be obtained as follows. By Lemma (5), Appendix,

$$\varepsilon = -\frac{1}{\pi} P\int_0^\pi \frac{\psi(t)\sin\theta}{\cos\theta - \cos t}dt, \quad \varepsilon'(\theta) = -\frac{1}{\pi} P\int_0^\pi \frac{\psi'(t)\sin t}{\cos\theta - \cos t}dt.$$

$$\ldots\ldots (23)$$

Also

$$C_0 = \frac{1}{\pi}\int_0^\pi \psi(t)dt. \qquad \ldots\ldots (24)$$

Hence

$$g = C_0 + \varepsilon' + \varepsilon\cot\theta = -\frac{1}{\pi} P\int_0^\pi \frac{dt}{\cos\theta - \cos t}\{-\psi(t)\cos\theta$$

$$+ \psi(t)\cos t + \psi'(t)\sin t + \psi(t)\cos\theta\}$$

$$= -\frac{1}{\pi} P\int_0^\pi \frac{dt}{\cos\theta - \cos t}\frac{d}{dt}[\psi(t)\sin t] = -\frac{2}{\pi} P\int_0^\pi \frac{f'(t)dt}{\cos\theta - \cos t}.$$

$$\ldots\ldots (22\text{ bis})$$

5. <u>A Better Approximation for a Symmetrical Aerofoil at Zero Incidence.</u>

It will be noticed that in approximating to $F(\theta)$ in (10)

$$\frac{e^{[\Psi]}}{[\sinh^2\psi + \sin^2\theta]^{\frac{1}{2}}}$$

was/

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*The formula $q/U = 1 + g$, where

$$g = -\frac{1}{\pi} P\int_0^1 \frac{y'(\xi)d\xi}{\xi - x}$$

for the velocity distribution due to a symmetrical aerofoil at zero incidence has been obtained independently by Squire[5] by the method of sources. From this formula he deduces the conjugate relation between the Fourier series for $g \sin\theta$ and $2f'(\theta)$ in (18) and (20), and proposes the use of the series in calculations to find the values of $f(\theta)$ for a given g.

was replaced by

$$\frac{1 + C_0}{|\sin\theta|}.$$

When $\sin\theta$ is small this is not a good approximation; a much better approximation is obtained by replacing the above expression by

$$\frac{1 + C_0 + \frac{1}{2}C_0^2}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}},$$

which is equivalent to multiplying our previous expression for q/U, namely, $1 + g$, by

$$\frac{(1 + \frac{1}{2}C_0^2)|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}. \qquad \ldots\ldots (25)$$

Our approximate formula for q/U is now

$$\frac{q}{U} = \frac{(1 + \frac{1}{2}C_0^2)|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}(1 + g) \qquad \ldots\ldots (26)$$

where g is given by (22), ψ is found as $2y$ cosec θ (see (7)), and C_0 is given by (24).

C_0 may be found by numerical integration of the values of ψ. In many cases, however, an analytical evaluation of the integral in (24) is possible, and in preparation we note that

$$C_0 = \frac{1}{\pi}\int_0^\pi \psi(t)dt = \frac{2}{\pi}\int_0^\pi \frac{f(t)}{\sin t}dt = \frac{1}{\pi}\int_0^1 \frac{y(\xi)d\xi}{\xi(1-\xi)}. \qquad (27)$$

6. Symmetrical Aerofoil at Non-Zero Lifts.

If C_L/a_0 is small, and we neglect its square and its products with ε, ε' and C_0, then from (11) it follows that

$$q/U = 1 + g_S + g_L \qquad \ldots\ldots (28)$$

where

$$g_S = C_0 + \varepsilon'(\theta) + \varepsilon\cot\theta, \qquad \ldots\ldots (29)$$

and is the g of §4, being given by (22), and

$$g_L = \frac{C_L}{a_0}\cot\theta + \frac{C_L}{2\pi}\operatorname{cosec}\theta = \frac{C_L}{2\pi}\cot\tfrac{1}{2}\theta + C_L\left(\frac{1}{a_0} - \frac{1}{2\pi}\right)\cot\theta. \qquad \ldots\ldots (30)$$

A better approximation is again obtained by multiplying the answers so found by (25), and then our approximate formula for q/U becomes

q/U /

$$\frac{q}{U} = \frac{(1 + \frac{1}{2}C_0{}^2)|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}(1 + g_S + g_L)$$

$$= \frac{(1 + \frac{1}{2}C_0{}^2)|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}(1 + g_S) \pm \frac{(1 + \frac{1}{2}C_0{}^2)C_L}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}\left\{\frac{1}{2\pi} + \frac{\cos\theta}{a_0}\right\}, \quad (31)$$

the positive sign being taken on the upper surface and the negative sign on the lower surface. (Note that, θ being positive on the upper surface and negative on the lower surface, ε is odd, ε' even, g_S even and g_L odd.)

7. A Closer Approximation at High Lifts.

The assumption that C_L/a_0 is small is quite distinct from the assumption of small thickness; for large values of C_L for which the approximations of §6 are not satisfactory we may revert to the original expressions (2) and (5), simplifying them to the formula*

$$\frac{q}{U} = \frac{e^{C_0}(1 + \varepsilon')}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}\left|\left(1 - \frac{C_L{}^2}{a_0{}^2}\right)^{\frac{1}{2}}\sin(\theta + \varepsilon) + \frac{C_L}{a_0}\cos(\theta + \varepsilon) + \frac{C_L e^{-C_0}}{2\pi}\right|, \quad \ldots\ldots (32)$$

θ being negative on the lower surface, with ψ even, ε odd and ε' even.

To use (32) we require ε and ε' separately; we shall calculate them for the upper surface (θ positive). Note that if we intend to use (32) and calculate ε, ε' we do not calculate g.

Since $y(x) = f(\theta) = \frac{1}{2}\psi(\theta)\sin\theta$, it follows from (23) that

$$\varepsilon = -\frac{2\sin\theta}{\pi} P\int_0^{\pi} \frac{f(t)dt}{\sin t(\cos\theta - \cos t)}$$

$$= -\frac{\sin\theta}{2\pi} P\int_0^1 \frac{y(\xi)d\xi}{\xi(1 - \xi)(\xi - x)} \quad \ldots\ldots (33)$$

and/

---

*Jeffreys[4] obtained a formula for the fluid velocity at a point on the surface of any aerofoil which is equivalent to

$$\frac{q}{U} = \frac{(1 + C_0)(1 + \varepsilon')}{(1 + \psi'^2)^{\frac{1}{2}}(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}\left\{\sin(\alpha + \theta + \varepsilon) + \frac{C_L}{8\pi a(1 + C_0)}\right\},$$

(though it is not quite in this form), where ψ is $y/(2a\sin\theta)$, and if

$$\psi = \sum_{n=0}^{\infty}(C_n\cos n\theta + D_n\sin n\theta)$$

then

$$\varepsilon = \sum_{n=1}^{\infty}(C_n\sin n\theta - D_n\cos n\theta).$$

The Fourier series for ε was not summed.

and

$$\sin\theta = 2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}, \qquad \ldots\ldots (34)$$

so

$$\varepsilon = -\frac{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}{\pi} P\int_0^1 \frac{y(\xi)d\xi}{\xi(1-\xi)(\xi-x)}. \qquad \ldots\ldots (35)$$

8. Remarks on the Formulae.

In the following sections we shall show how to evaluate g, C_0, ε, ε' in finite terms when, the chord being divided into any number of segments, in each segment the ordinate y is given by an algebraic formula of very general type. Before we become immersed in the details of these calculations, however, we may make some remarks on the formulae already obtained.

We have three different approximate formulae for q/U, namely (28), (31) and (32). Formula (31) is more accurate but harder to use than (28), and (32) more accurate but harder to use than (31). To use (28) we require g only; to use (31) we require (besides ψ, which is easily calculated as $2y \operatorname{cosec}\theta$) g and C_0; to use (32) we require ε, ε' and C_0, but not g. Thus, although in the following sections we shall exhibit formulae for all the three functions g, ε and ε', it should be remembered that we shall never need to calculate all three; we shall calculate either g only or ε and ε'.

In computing g or ε or ε' three methods are apparently open to us; we may use the Fourier series, the integral, or the formulae of the following sections. In all cases it is thoroughly recommended that the formulae of the following sections be used wherever possible.

The Fourier series look charmingly simple, but their use in computation is definitely not recommended unless they terminate, which is the case if, and only if, y is equal to $x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$ multiplied by a polynomial in x. In any other case to evaluate for example, we should have to sum a Fourier series with coefficients order $1/n^2$, and the summation would have to be carried out with considerable accuracy for small values of $\sin\theta$, since we have to divide the sum by $\sin\theta$, or at any rate by $(\psi^2 + \sin^2\theta)^{\frac{1}{2}}$, where ψ^2 is small.

The integrals are improper integrals, but may be changed into proper integrals by the use of Lemma (7), Appendix, and then used for computation. For example, from (22) and Lemma (7),

$$g_S = \frac{2}{\pi}\int_0^\pi \frac{f'(\theta) - f'(t)}{\cos\theta - \cos t}\,dt, \qquad \ldots\ldots (36)$$

where y is $f(\theta)$. If y is given numerically only, we must find numerically a smooth set of values of $f'(\theta)$ which integrate to y; it is, in fact, necessary to find $f''(\theta)$ also, since at $t = \theta$ the integrand is given by

$$\lim_{t\to\theta} \frac{f'(\theta) - f'(t)}{\cos\theta - \cos t} = -\frac{f''(\theta)}{\sin\theta}.$$

The/

The use of the integrals for computation is not recommended if suitable algebraic formulae can be fitted to the ordinate, and use made of the results of the following sections.

9. Results for Aerofoils with Ordinates Given by Algebraical Formulae.

In this section we shall exhibit formulae in closed form for g_S, C_0, ε and ε' when the aerofoil ordinates are given by algebraical formulae of considerable generality. We consider five different algebraical representations of the ordinates, which among them include as special cases practically every formula so far proposed for aerofoil ordinates, and we then explain briefly how results relative to many other algebraical representations may be written down at sight.

To exhibit the algebraical representations of the ordinates and the resulting formulae for g_S, C_0, ε and ε' we introduce four algebraical functions of x : $y_1(x)$, a polynomial in $x^{\frac{1}{2}}$ without a constant term; $y_2(x)$, a polynomial in $x^{\frac{1}{2}}$ with a constant term; $y_3(x)$, the ordinate of a hyperbola through $x = 1$; $y_4(x)$, the ordinate of an ellipse through $x = 0$. Specifically,

$$y_1(x) = \sum_{n=1}^{2M} a_n x^{n/2}$$

$$y_2(x) = \sum_{n=0}^{2m} b_n x^{n/2}$$

$$y_3(x) = \{C(1 - x) + D(1 - x)^2\}^{\frac{1}{2}}$$

$$y_4(x) = (Ax - Bx^2)^{\frac{1}{2}} \quad (0 \leqslant x \leqslant A/B)$$

M and m are any positive integers whatever, A, B, C, D and a_1 are any positive numbers, and the b_n and remaining a_n are any real numbers. (If the term of highest degree in y_1 or y_2 is an odd power of $x^{\frac{1}{2}}$, we simply put $a_{2M} = 0$ or $b_{2m} = 0$.) If $A/B < 1$, we also write

$$y_4(x) = (Bx^2 - Ax)^{\frac{1}{2}} \quad (A/B \leqslant x \leqslant 1).$$

We define the symbols I, j, J by the equations

$$I_1\ (x;\ X_1,\ X_2) = P\int_{X_1}^{X_2} \frac{y_1'(\xi)d\xi}{\xi - x}$$

$$j_1\ (X_1,\ X_2) = \int_{X_1}^{X_2} \frac{y_1(\xi)d\xi}{\xi(1 - \xi)}$$

J_1/

$$J_1\ (x;\ X_1,\ X_2) \ = \ x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\ P\int_{X_1}^{X_2} \frac{y_1(\xi)d\xi}{\xi(1-\xi)(\xi-x)}$$

(with $0 \leqslant X_1 \leqslant 1$, $0 \leqslant X_2 \leqslant 1$), and similarly for I_2, I_3, I_4, j_2, j_3, j_4, J_2, J_3, J_4.

Case I.

Let the aerofoil ordinate be given by

$$y \ = \ y_1(x) \qquad (0 \leqslant x \leqslant 1).$$

Then from (22), (27) and (35)

$$g_S \ = \ -\frac{1}{\pi}\ I_1\ (x;\ 0,\ 1)$$

$$C_0 \ = \ \frac{1}{\pi}\ j_1\ (0,\ 1)$$

$$\varepsilon \ = \ -\frac{1}{\pi}\ J_1\ (x;\ 0,\ 1).$$

Case II.

Let the aerofoil ordinate be given by

$$y \ = \ y_1(x) \qquad (0 \leqslant x \leqslant X_1)$$

$$y \ = \ y_2(x) \qquad (X_1 \leqslant x \leqslant 1).$$

Then

$$g_S \ = \ -\frac{1}{\pi}\ \{I_1\ (x;\ 0,\ X_1) + I_2\ (x;\ X_1,\ 1)\}$$

$$C_0 \ = \ \frac{1}{\pi}\ \{j_1\ (0,\ X_1) + j_2\ (X_1,\ 1)\}$$

$$\varepsilon \ = \ -\frac{1}{\pi}\ \{J_1\ (x;\ 0,\ X_1) + J_2\ (x;\ X_1,\ 1)\}.$$

Case III.

Let

$$y \ = \ y_1(x) \qquad (0 \leqslant x \leqslant X_1)$$

$$y \ = \ y_2(x) \qquad (X_1 \leqslant x \leqslant X_2)$$

$$y \ = \ y_3(x) \qquad (X_2 \leqslant x \leqslant 1).$$

Then

$$g_S \ = \ -\frac{1}{\pi}\ \{I_1\ (x;\ 0,\ X_1) + I_2\ (x;\ X_1,\ X_2) + I_3\ (x;\ X_2,\ 1)\}$$

C_0/

$$C_0 = \frac{1}{\pi}\{j_1(0, X_1) + j_2(X_1, X_2) + j_3(X_2, 1)\}$$

$$\varepsilon = -\frac{1}{\pi}\{J_1(x; 0, X_1) + J_2(x; X_1, X_2) + J_3(x; X_2, 1)\}.$$

Case IV.

$$\text{Let} \quad y = y_4(x) \qquad 0 \leqslant x \leqslant X_1 \quad (\leqslant A/B)$$
$$y = y_2(x) \qquad X_1 \leqslant x \leqslant 1.$$

Then

$$g_S = -\frac{1}{\pi}\{I_4(x; 0, X_1) + I_2(x; X_1, 1)\}$$

$$C_0 = \frac{1}{\pi}\{j_4(0, X_1) + j_2(X_1, 1)\}$$

$$\varepsilon = -\frac{1}{\pi}\{J_4(x; 0, X_1) + J_2(x; X_1, 1)\}.$$

Case V.

$$\text{Let} \quad y = y_4(x) \qquad 0 \leqslant x \leqslant X_1 \quad (\leqslant A/B)$$
$$= y_2(x) \qquad X_1 \leqslant x \leqslant X_2$$
$$= y_3(x) \qquad X_2 \leqslant x \leqslant 1.$$

Then

$$g_S = -\frac{1}{\pi}\{I_4(x; 0, X_1) + I_2(x; X_1, X_2) + I_3(x; X_2, 1)\}$$

$$C_0 = \frac{1}{\pi}\{j_4(0, X_1) + j_2(X_1, X_2) + j_3(X_2, 1)\}$$

$$\varepsilon = -\frac{1}{\pi}\{J_4(x; 0, X_1) + J_2(x; X_1, X_2) + J_3(x; X_2, 1)\}.$$

Thus to obtain formulae for these five general cases, we require formulae for the I, j, and J; since we wish also to be able to calculate $\varepsilon'(\theta)$, we shall also exhibit formulae for $dJ/d\theta$. The necessary mathematics is contained in the lemmas in the Appendix, and we have merely to make the correct substitutions in the formulae contained therein.

It is convenient to enlarge our notation slightly by writing

ψ_1/

$$\left.\begin{aligned}\psi_1 &= 2y_1 \operatorname{cosec}\theta = \frac{y_1}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}},\\ \psi_2 &= \frac{y_2}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}},\\ \psi_3 &= \frac{y_3}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}},\\ \psi_4 &= \frac{y_4}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}.\end{aligned}\right\} \quad \ldots\ldots (37)$$

Then

$$\psi_1'(\theta) = y_1'(x) - \frac{1-2x}{2x(1-x)}\, y_1(x), \qquad \ldots\ldots (38)$$

and similarly for $\psi_2'(\theta)$, $\psi_3'(\theta)$, $\psi_4'(\theta)$.

We may note that in case II, for example,

$$\psi = \psi_1, \quad \psi' = \psi_1' \qquad (0 \leqslant x \leqslant X_1)$$

$$\psi = \psi_2, \quad \psi' = \psi_2' \qquad (X_1 \leqslant x \leqslant 1)$$

and similarly in other cases.

In considering formulae for the I, j, J and $dJ/d\theta$ it is convenient to commence with the formulae for $I_2(x;\, X_1, X_2)$, $j_2(X_1, X_2)$, etc. Since

$$y_2'(x) = \sum_{n=-1}^{2m-2} (\tfrac{1}{2}n + 1)\, b_{n+2}\, x^{\frac{1}{2}n},$$

by substituting $M = m - 1$, $c_n = (\frac{1}{2}n + 1)b_{n+2}$ in Lemma 12 we find that

$$\begin{aligned} I_2(x;\, X_1, X_2) = {} & y_2'(x) \log_e \left|\frac{x - X_2}{x - X_1}\right| \\ & - \left(\sum_{n=0}^{m-1} (\tfrac{1}{2}n + 1)\, b_{2n+1}\, x^{n-\frac{1}{2}}\right) \log_e \frac{X_2^{\frac{1}{2}} + x^{\frac{1}{2}}}{X_1^{\frac{1}{2}} + x^{\frac{1}{2}}} \\ & + \sum_{n=0}^{m-2} x^n \sum_{r=1}^{2m-2n-2} \frac{r + 2n + 2}{r}\, b_{r+2n+2}\, (X_2^{\frac{1}{2}r} - X_1^{\frac{1}{2}r}). \end{aligned}$$

Similarly, on putting $c_{-1} = 0$, $c_n = b_n$ for $n \geqslant 0$, $M = m$ in Lemmas 12 and 14, we find that

j_2/

$$j_2(X_1, X_2) = -y_2(1)\log_e \frac{1-X_2}{1-X_1} + 2\left(\sum_{n=0}^{m-1} b_{2n+1}\right)\log_e \frac{1+X_2^{\frac{1}{2}}}{1+X_1^{\frac{1}{2}}}$$

$$+ b_0 \log_e \frac{X_2}{X_1} - \sum_{r=1}^{m-1} \frac{X_2^r - X_1^r}{r} \sum_{s=r+1}^{m} b_{2s}$$

$$- \sum_{r=1}^{m-1} \frac{X_2^{r-\frac{1}{2}} - X_1^{r-\frac{1}{2}}}{r-\frac{1}{2}} \sum_{s=r+1}^{m} b_{2s-1},$$

$$J_2(x; X_1, X_2) = \psi_2(\theta)\log_e \left|\frac{x-X_2}{x-X_1}\right|$$

$$-\frac{2}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}\left(\sum_{n=0}^{m-1} b_{2n+1}x^{n+\frac{1}{2}}\right)\log_e \frac{X_2^{\frac{1}{2}}+x^{\frac{1}{2}}}{X_1^{\frac{1}{2}}+x^{\frac{1}{2}}}$$

$$+\left(\frac{x}{1-x}\right)^{\frac{1}{2}}\left\{2\left(\sum_{n=0}^{m-1} b_{2n+1}\right)\log_e \frac{1+X_2^{\frac{1}{2}}}{1+X_1^{\frac{1}{2}}} - y_2(1)\log_e \frac{1-X_2}{1-X_1}\right\}$$

$$-\left(\frac{1-x}{x}\right)^{\frac{1}{2}} b_0 \log_e \frac{X_2}{X_1} - x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} \sum_{n=0}^{m-3} B_n x^n$$

where

$$B_n = \sum_{r=1}^{m-n-2} \frac{X_2^r - X_1^r}{r} \sum_{s=r+n+2}^{m} b_{2s} + \sum_{r=1}^{m-n-2} \frac{X_2^{r-\frac{1}{2}} - X_1^{r-\frac{1}{2}}}{r-\frac{1}{2}} \sum_{s=r+n+2}^{m} b_{2s-1},$$

and*

$$\frac{d}{d\theta} J_2(x; X_1, X_2) = \psi_2'(\theta)\log_e\left|\frac{x-X_2}{x-X_1}\right| - 2\left\{\sum_{n=0}^{m-1}(n+\tfrac{1}{2})b_{2n+1}x^{n-\frac{1}{2}}\right.$$

$$\left. - \frac{1-2x}{2x(1-x)}\sum_{n=0}^{m-1} b_{2n+1}x^{n+\frac{1}{2}}\right\}\log_e \frac{X_2^{\frac{1}{2}}+x^{\frac{1}{2}}}{X_1^{\frac{1}{2}}+x^{\frac{1}{2}}}$$

$$+\frac{1}{2(1-x)}\left\{2\left(\sum_{n=0}^{m-1} b_{2n+1}\right)\log_e \frac{1+X_2^{\frac{1}{2}}}{1+X_1^{\frac{1}{2}}} - y_2(1)\log_e \frac{1-X_2}{1-X_1}\right\}$$

$$+\frac{b_0}{2x}\log_e \frac{X_2}{X_1} + \sum_{n=0}^{m-2} \beta_n x^n + \underline{\frac{y_2(X_2)}{x-X_2} - \frac{y_2(X_1)}{x-X_1}}$$

where

$$\beta_n = \sum_{r=1}^{2m-2n-2} \frac{r+2n}{r} b_{r+2n+2}\left(X_2^{\frac{1}{2}r} - X_1^{\frac{1}{2}r}\right) - \tfrac{1}{2}B_n,$$

with B_n as above for $0 \leqslant n \leqslant m-3$ and $B_{m-2} = 0$.

The/

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*Terms underlined in formulae for $dJ/d\theta$ may be omitted, since, y being continuous, they cancel in the calculation of ε'.

The values of $I_2(x; X_1, 1)$, etc. are now found from the above formulae by putting $X_2 = 1$; these values are required only in Cases II and IV, in which cases, the ordinate vanishing at the trailing edge, we shall have $y_2(1) = 0$. With $y_2(1) = 0$ we find that

$$I_2(x;X_1,1) = y_2'(x)\log_e\frac{1-x}{|x-X_1|} - \Big(\sum_{n=0}^{m-1}(2n+1)b_{2n+1}x^{n-\frac{1}{2}}\Big)\log_e\frac{1+x^{\frac{1}{2}}}{X_1^{\frac{1}{2}}+x^{\frac{1}{2}}}$$

$$+\sum_{n=0}^{m-2}x^n\sum_{r=1}^{2m-2n-2}\frac{r+2n+2}{r}b_{r+2n+2}\,(1-X_1^{\frac{1}{2}r}),$$

$$j_2(X_1,1) = 2\Big(\sum_{n=0}^{m-1}b_{2n+1}\Big)\log_e\frac{2}{1+X_1^{\frac{1}{2}}} - b_0\log_e X_1 - \sum_{r=1}^{m-1}\frac{1-X_1^{r}}{r}\sum_{s=r+1}^{m}b_{2s}$$

$$-\sum_{r=1}^{m-1}\frac{1-X_1^{r-\frac{1}{2}}}{r-\frac{1}{2}}\sum_{s=r+1}^{m}b_{2s-1},$$

$$J_2(x;X_1,1) = \psi_2(\theta)\log_e\frac{1-x}{|x-X_1|} - \frac{2}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}\Big(\sum_{n=0}^{m-1}b_{2n+1}x^{n+\frac{1}{2}}\Big)\log_e\frac{1+x^{\frac{1}{2}}}{X_1^{\frac{1}{2}}+x^{\frac{1}{2}}}$$

$$+2\left(\frac{x}{1-x}\right)^{\frac{1}{2}}\Big(\sum_{n=0}^{m-1}b_{2n+1}\Big)\log_e\frac{2}{1+X_1^{\frac{1}{2}}} + \left(\frac{1-x}{x}\right)^{\frac{1}{2}}b_0\log_e X_1$$

$$-x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\sum_{n=0}^{m-3}B_n x^n,$$

where $$B_n = \sum_{r=1}^{m-n-2}\frac{1-X_1^{r}}{r}\sum_{s=r+n+2}^{m}b_{2s} + \sum_{r=1}^{m-n-2}\frac{1-X_1^{r-\frac{1}{2}}}{r-\frac{1}{2}}\sum_{s=r+n+2}^{m}b_{2s-1},$$

and

$$\frac{d}{d\theta}J_2(x;X_1,1) = \psi_2'(\theta)\log_e\frac{1-x}{|x-X_1|} - 2\Big\{\sum_{n=0}^{m-1}(n+\tfrac{1}{2})b_{2n+1}x^{n-\frac{1}{2}}$$

$$-\frac{1-2x}{2x(1-x)}\sum_{n=0}^{m-1}b_{2n+1}x^{n+\frac{1}{2}}\Big\}\log_e\frac{1+x^{\frac{1}{2}}}{X_1^{\frac{1}{2}}+x^{\frac{1}{2}}}$$

$$+\frac{1}{1-x}\Big(\sum_{n=0}^{m-1}b_{2n+1}\Big)\log_e\frac{2}{1+X_1^{\frac{1}{2}}} - \frac{b_0}{2x}\log_e X_1$$

$$+\sum_{n=0}^{m-2}\beta_n x^n - \underline{\frac{y_2(X_1)}{x-X_1}}$$

where $$\beta_n = \sum_{r=1}^{2m-2n-2}\frac{r+2n}{r}b_{r+2n+2}\,(1-X_1^{\frac{1}{2}r}) - \tfrac{1}{2}B_n,$$

with the values of B_n as in $J_2(x; X_1, 1)$ for $0 \leqslant n \leqslant m-3$ and $B_{m-2} = 0$.

In/

In Cases II and IV, with $y_2(1) = 0$, the aerofoil will have a sharp trailing edge. The expression for $y_2(x)$ will contain $1 - x^{\frac{1}{2}}$ as a factor; it may have $1 - x$ as a factor, and if so, $\sum_{n=0}^{m-1} b_{2n+1}$ will be zero, and the terms in $j_2(X_1, 1)$, $J_2(x; X_1, 1)$ and $dJ_2(x; X_1, 1)/d\theta$ containing this expression will be zero.

Before we pass on to consider I_1, etc. we may also note that if in the expression for $y_2(x)$ we have $b_{2n} = 0$, then

$$y_2(x) = \sum_{n=0}^{m-1} b_{2n+1} x^{n+\frac{1}{2}}, \qquad y_2'(x) = \tfrac{1}{2} \sum_{n=0}^{m-1} (2n+1) b_{2n+1} x^{n-\frac{1}{2}},$$

and the first two terms in $I_2\,(x; X_1, X_2)$, for example, become

$$y_2'(x) \left\{ \log_e \left| \frac{x - X_2}{x - X_1} \right| - 2 \log_e \frac{X_2^{\frac{1}{2}} + x^{\frac{1}{2}}}{X_1^{\frac{1}{2}} + x^{\frac{1}{2}}} \right\}$$

$$= y_2'(x) \left\{ \log_e \frac{X_1^{\frac{1}{2}} + x^{\frac{1}{2}}}{|x^{\frac{1}{2}} - X_1^{\frac{1}{2}}|} - \log_e \frac{X_2^{\frac{1}{2}} + x^{\frac{1}{2}}}{|X_2^{\frac{1}{2}} - x^{\frac{1}{2}}|} \right\},$$

with corresponding simplifications in other cases.

We now proceed to find $I_1\,(x; 0, X_1)$, etc. by substituting $m = M$, $b_0 = 0$, $b_n = a_n$ for $n \geqslant 1$, $X_1 = 0$ and replacing X_2 by X_1 in $I_2\,(x; X_1, X_2)$ etc. In this way we find that

$$I_1(x;0,X_1) = y_1'(x) \log_e \frac{|x-X_1|}{x} - \left(\sum_{n=0}^{M-1} (2n+1) a_{2n+1} x^{n-\frac{1}{2}}\right) \log_e \frac{X_1^{\frac{1}{2}} + x^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

$$+ \sum_{n=0}^{M-2} x^n \cdot \sum_{r=1}^{2M-2n-2} \frac{r + 2n + 2}{r} a_{r+2n+2} X_1^{\frac{1}{2}r},$$

$$j_1(0,X_1) = -y_1(1) \log_e (1-X_1) + 2\left(\sum_{n=0}^{M-1} a_{2n+1}\right) \log_e (1 + X_1^{\frac{1}{2}})$$

$$- \sum_{r=1}^{M-1} \frac{X_1^{r}}{r} \sum_{s=r+1}^{M} a_{2s} - \sum_{r=1}^{M-1} \frac{X_1^{r-\frac{1}{2}}}{r - \frac{1}{2}} \sum_{s=r+1}^{M} a_{2s-1},$$

$$J_1(x;0,X_1) = \psi_1(\theta) \log_e \frac{|x-X_1|}{x} - \frac{2}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} \left(\sum_{n=0}^{M-1} a_{2n+1} x^{n+\frac{1}{2}}\right) \log_e \frac{X_1^{\frac{1}{2}} + x^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

$$+ \left(\frac{x}{1-x}\right)^{\frac{1}{2}} \left\{ 2\left(\sum_{n=0}^{M-1} a_{2n+1}\right) \log_e (1+X_1^{\frac{1}{2}}) - y_1(1) \log_e (1-X_1) \right\}$$

$$- x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} \sum_{n=0}^{M-3} A_n x^n,$$

where $$A_n = \sum_{r=1}^{M-n-2} \frac{X_1^{r}}{r} \sum_{s=r+n+2}^{M} a_{2s} + \sum_{r=1}^{M-n-2} \frac{X_1^{r-\frac{1}{2}}}{r - \frac{1}{2}} \sum_{s=r+n+2}^{M} a_{2s-1},$$

and/

and

$$\frac{d}{d\theta} J_1(x;0,X_1) = \psi_1'(\theta)\log_e \frac{|x-X_1|}{x} - 2\left\{\sum_{n=0}^{M-1} (n+\tfrac{1}{2})a_{2n+1}x^{n-\frac{1}{2}} - \frac{1-2x}{2x(1-x)}\sum_{n=0}^{M-1} a_{2n+1}x^{n+\frac{1}{2}}\right\}\log_e \frac{X_1^{\frac{1}{2}}+x^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$
$$+ \frac{1}{2(1-x)}\left\{2(\sum_{n=0}^{M-1} a_{2n+1})\log_e(1+X_1^{\frac{1}{2}}) - y_1(1)\log_e(1-X_1)\right\} + \sum_{n=0}^{M-2} \alpha_n x^n + \frac{y_1(X_1)}{x - X_1}$$

where $\alpha_n = \sum_{r=1}^{2M-2n-2} \frac{r+2n}{r} a_{r+2n+2} X_1^{\frac{1}{2}r} - \frac{1}{2}A_n,$

with A_n as above for $0 \leqslant n \leqslant M-3$ and $A_{M-2} = 0.$

The values of $I_1(x;0,1)$, etc. are now found by putting $X_1 = 1$ in the immediately preceding formulae; these values are required in Case I only, in which case, the ordinate vanishing at the trailing edge, we have $y_1(1) = 0$. With $y_1(1) = 0$ we find that

$$I_1(x;0,1) = y_1'(x)\log_e \frac{1-x}{x} - (\sum_{n=0}^{M-1} (2n+1)a_{2n+1}x^{n-\frac{1}{2}})\log_e \frac{1+x^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \sum_{n=0}^{M-2} x^n \sum_{r=1}^{2M-2n-2} \frac{r+2n+2}{r} a_{r+2n+2},$$

$$j_1(0,1) = 2(\sum_{n=0}^{M-1} a_{2n+1})\log_e 2 - \sum_{r=1}^{M-1} \frac{1}{r}\sum_{s=r+1}^{M} a_{2s} - \sum_{r=1}^{M-1}\frac{1}{r-\frac{1}{2}}\sum_{s=r+1}^{M} a_{2s-1}$$

$$J_1(x;0,1) = \psi_1(\theta)\log_e \frac{1-x}{x} - \frac{2}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}(\sum_{n=0}^{M-1} a_{2n+1}x^{n+\frac{1}{2}})\log_e \frac{1+x^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$
$$+ 2\left(\frac{x}{1-x}\right)^{\frac{1}{2}}(\sum_{n=0}^{M-1} a_{2n+1})\log_e 2 - x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\sum_{n=0}^{M-3} A_n x^n,$$

where $A_n = \sum_{r=1}^{M-n-2} \frac{1}{r}\sum_{s=r+n+2}^{M} a_{2s} + \sum_{r=1}^{M-n-2}\frac{1}{r-\frac{1}{2}}\sum_{s=r+n+2}^{M} a_{2s-1},$

and

d/dθ /

$$\frac{d}{d\theta} J_1(x;0,1) = \psi_1'(\theta) \log_e \frac{1-x}{x} - 2\left\{\sum_{n=0}^{M-1} (n+\tfrac{1}{2}) a_{2n+1} x^{n-\frac{1}{2}} - \frac{1-2x}{2x(1-x)} \sum_{n=0}^{M-1} a_{2n+1} x^{n+\frac{1}{2}}\right\} \log_e \frac{1+x^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

$$+ \frac{1}{1-x}\left(\sum_{n=0}^{M-1} a_{2n+1}\right) \log_e 2 + \sum_{n=0}^{M-2} \alpha_n x^n,$$

where $\alpha_n = \sum_{r=1}^{2M-2n-2} \frac{r' + 2n}{r} a_{r+2n+2} - \frac{1}{2}A_n$

and A_n has the same values as in $J_1\ (x;\ 0,\ 1)$ for $0 \leqslant n \leqslant m-3$, with $A_{m-2} = 0$.

In Case I, with $y_1(1) = 0$, the aerofoil will have a sharp trailing edge. If the expression for $y_1(x)$ contains $1-x$ as a factor, $\sum_{n=0}^{M-1} a_{2n+1}$ will be zero, and therefore so will the terms containing this expression in $j_1\ (0,\ 1)$, $J_1(x;\ 0,\ 1)$ and $dJ_1(x;\ 0,\ 1)/d\theta$.

If $a_{2n} = 0$ in the expression for $y_1(x)$, then

$$y_1(x) = \sum_{n=0}^{m-1} a_{2n+1} x^{n+\frac{1}{2}}, \qquad y_1'(x) = \tfrac{1}{2} \sum_{n=0}^{m-1} (2n+1) a_{2n+1} x^{n-\frac{1}{2}}$$

and the first two terms in $I_1\ (x;\ 0,\ X_1)$, for example, become

$$y_1'(x)\left\{\log_e \frac{|x - X_1|}{x} - 2 \log_e \frac{X_1^{\frac{1}{2}} + x^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right\}$$

$$= y_1'(x)\{\log_e |x - X_1| - 2 \log_e (X_1^{\frac{1}{2}} + x^{\frac{1}{2}})\},$$

with corresponding simplifications in other cases.

Turning next to $I_3\ (x;\ X_2,\ 1)$, etc., we see from Lemma (21) that

$$I_3(x;X_2,1) = 2\sqrt{D} \sinh^{-1} [D(1 - X_2)/C]^{\frac{1}{2}} + y_3'(x)\log_e \frac{(1-x)[1+2D(1-X_2)/C]+1-X_2+2y_3(X_2)y_3(x)/C}{|x - X_2|}$$

$$j_3(X_2,1) = -2\sqrt{D} \sinh^{-1} [D(1 - X_2)/C]^{\frac{1}{2}} + (C+D)^{\frac{1}{2}}\log_e \frac{2-X_2+2D(1-X_2)/C+2y_3(X_2)(C+D)^{\frac{1}{2}}/C}{X_2}$$

J_3/

$$J_3(x,X_2,1) = -\left(\frac{1-x}{x}\right)^{\frac{1}{2}} (C+D)^{\frac{1}{2}} \log_e \frac{2-X_2+2D(1-X_2)/C + 2y_3(X_2)(C+D)^{\frac{1}{2}}/C}{X_2}$$

$$+ \psi_3(\theta)\log_e \frac{(1-x)[1+2D(1-X_2)/C]+1-X_2+2y_3(X_2)y_3(x)/C}{|x-X_2|}$$

and

$$\frac{d}{d\theta} J_3(x;X_2,1) = \frac{(C+D)^{\frac{1}{2}}}{2x} \log_e \frac{2-X_2+2D(1-X_2)/C+2y_3(X_2)(C+D)^{\frac{1}{2}}/C}{X_2}$$

$$+ \psi_3'(\theta)\log_e \frac{(1-x)[1+2D(1-X_2)/C]+1-X_2+2y_3(X_2)y_3(x)/C}{|x-X_2|}$$

$$- \frac{y_3(X_2)}{x-X_2} .$$

Finally, from Lemma (16) it follows that

$$I_4(x;0,X_1) = -2\sqrt{B}\sin^{-1}(BX_1/A)^{\frac{1}{2}} - y_4'(x)\log_e \frac{x(1-2BX_1/A)+X_1+2y_4(X_1)y_4(x)/A}{|x-X_1|} \quad \text{for } x < A/B$$

$$= -2\sqrt{B}\sin^{-1}(BX_1/A)^{\frac{1}{2}} + \frac{BX_1}{y_4(X_1)} \quad \text{for } x = A/B$$

$$= -2\sqrt{B}\sin^{-1}(BX_1/A)^{\frac{1}{2}} + 2y_4'(x)\tan^{-1}\left(\frac{BX_1x-AX_1}{Ax-BX_1x}\right)^{\frac{1}{2}} \quad \text{for } x > A/B$$

where
$$y_4'(x) = \frac{A-2Bx}{2y_4(x)} \quad \text{for } x < A/B$$

$$= \frac{2Bx-A}{2y_4(x)} \quad \text{for } x > A/B ,$$

and from Lemmas (17), (18) and (19) it follows that

$$j_4(0,X_1) = 2\sqrt{B}\sin^{-1}(BX_1/A)^{\frac{1}{2}} + (A-B)^{\frac{1}{2}}\log_e \frac{1+X_1-2BX_1/A+2y_4(X_1)(A-B)^{\frac{1}{2}}/A}{1-X_1} \quad \text{for } A > B$$

$$= 2\sqrt{B}\sin^{-1}(BX_1/A)^{\frac{1}{2}} - 2(B-A)^{\frac{1}{2}}\tan^{-1}\left(\frac{BX_1-AX_1}{A-BX_1}\right)^{\frac{1}{2}} \quad \text{for } A < B$$

J_4/

$$J_4(x;0,X_1) = \left(\frac{x}{1-x}\right)^{\frac{1}{2}} (A-B)^{\frac{1}{2}} \log_e \frac{1+X_1-2BX_1/A+2y_4(X_1)(A-B)^{\frac{1}{2}}/A}{1-X_1}$$

$$- \psi_4(\theta)\log_e \frac{x(1-2BX_1/A)+X_1+2y_4(X_1)y_4(x)/A}{|x-X_1|}$$

for $A \geqslant B$, $x < A/B$,

$$= -2\left(\frac{x}{1-x}\right)^{\frac{1}{2}} (B-A)^{\frac{1}{2}} \tan^{-1}\left(\frac{BX_1-AX_1}{A-BX_1}\right)^{\frac{1}{2}}$$

$$- \psi_4(\theta)\log_e \frac{x(1-2BX_1/A)+X_1+2y_4(X_1)y_4(x)/A}{|x-X_1|}$$

for $A < B$, $x < A/B$,

$$= -2\left(\frac{x}{1-x}\right)^{\frac{1}{2}} (B-A)^{\frac{1}{2}} \tan^{-1}\left(\frac{BX_1-AX_1}{A-BX_1}\right)^{\frac{1}{2}}$$

$$+ 2\psi_4(\theta)\tan^{-1}\left(\frac{BX_1x-AX_1}{Ax-BX_1x}\right) \quad \text{for } A < B, \; x > A/B,$$

and

$$\frac{d}{d\theta} J_4(x;\theta,X_1) = \frac{(A-B)^{\frac{1}{2}}}{2(1-x)} \log_e \frac{1+X_1-2BX_1/A+2y_4(X_1)(A-B)^{\frac{1}{2}}/A}{1-X_1}$$

$$- \psi_4'(\theta) \log_e \frac{x(1-2BX_1/A)+X_1+2y_4(X_1)y_4(x)/A}{|x-X_1|} + \underline{\frac{y_4(X_1)}{x-X_1}}$$

for $A > B$, $x < A/B$,

$$= -\frac{(B-A)^{\frac{1}{2}}}{1-x} \tan^{-1}\left(\frac{BX_1-AX_1}{A-BX_1}\right)^{\frac{1}{2}}$$

$$- \psi_4'(\theta) \log_e \frac{x(1-2BX_1/A)+X_1+2y_4(X_1)y_4(x)/A}{|x-X_1|} + \underline{\frac{y_4(X_1)}{x-X_1}}$$

for $A < B$, $x < A/B$

$$= -\frac{(B-A)^{\frac{1}{2}}}{1-x} \tan^{-1}\left(\frac{BX_1-AX_1}{A-BX_1}\right)^{\frac{1}{2}} + \frac{BX_1}{y_4(X_1)} + \underline{\frac{y_4(X_1)}{x-X_1}}$$

for $A < B$, $x = A/B$

$$= -\frac{(B-A)^{\frac{1}{2}}}{1-x} \tan^{-1}\left(\frac{BX_1-AX_1}{A-BX_1}\right)^{\frac{1}{2}} + 2\psi_4'(\theta)\tan^{-1}\left(\frac{BX_1x-AX_1}{Ax-BX_1x}\right)^{\frac{1}{2}}$$

$$+ \underline{\frac{y_4(X_1)}{x-X_1}} \quad \text{for } A < B, \; x > A/B.$$

The/

The formulae for the I, j, J and $dJ/d\theta$ required for the five general cases previously listed have now all been set out. These five general cases include practically all aerofoils for whose ordinates formulae have been proposed, but the mathematical results already obtained enable us to write down, if we wish, formulae for the I, etc. relative to other cases. There is no absolute need, for example, in proposing formulae for the ordinates, to restrict the number of segments of the chord to two or three. In Case III for example, instead of writing

$$y = y_2(x) \qquad X_1 \leqslant x \leqslant X_2$$

$$y = y_3(x) \qquad X_2 \leqslant x \leqslant 1,$$

we may write

$$y = y_2(x) \qquad X_1 \leqslant x \leqslant X_2,$$

$$y = y_5(x) = \sum_{n=0}^{2m_1} c_n x^{n/2} \qquad X_2 \leqslant x \leqslant X_3,$$

$$y = y_3(x) \qquad X_3 \leqslant x \leqslant 1.$$

Then in the expression for g_θ in Case III, we change $I_3(x;\, X_2,\, 1)$ into $I_3(x;\, X_3,\, 1)$ and add $-I_5(x;\, X_2,\, X_3)/\pi$, where the formula for $I_5(x;\, X_2,\, X_3)$ is obtained from that for $I_2(x;\, X_1,\, X_2)$ by changing m into m_1, b_n into c_n, X_1 into X_2 and X_2 into X_3; and similar remarks apply to the expressions for C_0, ε and ε'. There is, in fact, no theoretical difficulty in increasing the number of segments in each of which the ordinate is represented by a different algebraical expression, but the labour of computing definite numerical results increases with the number of segments.

Another change that may easily be made is the use of a polynomial in $(1-x)^{\frac{1}{2}}$ instead of the polynomial $y_2(x)$ in $x^{\frac{1}{2}}$ in Cases II, III, IV and V. For simplicity let us suppose that y_2 is replaced by

$$y_6(x) = \sum_{n=0}^{2m} b_n (1-x)^{n/2},$$

the symbols m and b_n being kept unchanged from those used in $y_2(x)$. Then we must replace $I_2(x;\, X_1,\, X_2)$, $j_2(X_1,\, X_2)$, $J_2(x;\, X_1,\, X_2)$ by, say, $I_6(x;\, X_1,\, X_2)$, $j_6(X_1,\, X_2)$, $J_6(x;\, X_1,\, X_2)$ respectively, and it follows from Lemma (20), Appendix, that

$$I_6(x;\, X_1,\, X_2) = I_2(1-x;\, 1-X_2,\, 1-X_1)$$

$$j_6(X_1,\, X_2) = j_2(1-X_2,\, 1-X_1)$$

$$J_6(x;\, X_1,\, X_2) = -J_2(1-x;\, 1-X_2,\, 1-X_1)$$

and $dJ_6(x;\, X_1,\, X_2)/d\theta$ is obtained from $dJ_2(x;\, X_1,\, X_2)/d\theta$ by changing x into $1-x$, X_1 into $1-X_2$, X_2 into $1-X_1$ and leaving the sign unchanged.

In the same way we may suppose the trailing edge rounded off by an elliptic or a circular arc instead of a hyperbolic arc, i.e., $y_3(x)$ replaced by

$$y_7(x) = \{E(1-x) - F(1-x)^2\}^{\frac{1}{2}}.$$

Then/

Then $I_3\ (x; X_2, 1)$, $j_3(X_2, 1)$, $J_3\ (x; X_2, 1)$ must be replaced by $I_7\ (x; X_2, 1)$, $j_7\ (X_2, 1)$, $J_7\ (x; X_2, 1)$, where from Lemma (20) it follows that we obtain $I_7\ (x; X_2, 1)$, $j_7\ (X_2, 1)$, $J_7\ (x; X_2, 1)$ and $dJ_7\ (x; X_2, 1)/d\theta$ from $I_4\ (x; 0, X_1)$, $j_4\ (0, X_1)$, $J_4\ (x; 0, X_1)$ and $dJ_4\ (x; 0, X_1)/d\theta$, respectively, by replacing A and B by E and F, x by $1 - x$ and X_1 by $1 - X_2$, and changing the sign in the case of J, leaving it unchanged for the other quantities.

The formulae of this section therefore cover a very wide variety of cases. It is exactly on account of this very considerable generality that the formulae probably appear quite repulsively complicated to many who are not mathematicians by nature or training, and definite, limited formulae for a number of special aerofoils have therefore been set out in the next section.

Before we leave the formulae of this section, however, a word or two should be said about the occurrence of infinities.

Clearly infinities will not occur except possibly at 0, 1, X_1, X_2.

Since y and y' must be supposed continuous at X_1 and X_2 in all cases, there are no infinities in g, ε or ε' at these points; moreover in the $dJ/d\theta$ the terms underlined cancel and may therefore be omitted.

At $x = 0$ and $x = 1$ the J vanish in all cases, i.e., $\varepsilon = 0$ at the leading and trailing edges, as it should since it is an odd periodic function of θ.

On the other hand we find that the approximate methods of this note make g and ε' logarithmically infinite when $x \to 0$ unless the coefficient of x in the expansion of y in powers of $x^{\frac{1}{2}}$ near the nose vanishes. If this coefficient is a_2, then $g \to \infty$ like $(a_2/\pi) \log_e x$ and $\varepsilon' \to \infty$ like $(a_2/2\pi)\log_e x$. Similarly if the coefficient of $1 - x$ in the expansion of y in powers of $(1 - x)^{\frac{1}{2}}$ near the tail is a_2', then when $x \to 1$, $g \to \infty$ like $(a_2'/\pi) \log_e (1 - x)$ and $\varepsilon' \to \infty$ like $(a_2'/2\pi) \log_e (1 - x)$.*

10. <u>Special Examples.</u>

For the ordinates of symmetrical aerofoils the following formulae have been proposed.

The N.A.C.A. have published two types of formulae; one, of which the N.A.C.A.0012 is an example, is of the form[6]

$$y = a_1 x^{\frac{1}{2}} + a_2 x + a_4 x^2 + a_6 x^3 + a_8 x^4 \qquad (0 \leqslant x \leqslant 1);$$

and the other, of which the N.A.C.A.0012-63 is an example, is of the form[7]

$$y = a_1 x^{\frac{1}{2}} + a_2 x + a_4 x^2 + a_6 x^3 \qquad (0 \leqslant x \leqslant X_1)$$

$$= b_0 + b_2 x + b_4 x^2 + b_6 x^3 \qquad (X_1 \leqslant x \leqslant 1).$$

I have fitted a formula of the latter type to the N.A.C.A. 16 series †

R. C. Pankhurst has fitted formulae, hitherto unpublished, to the Clark Y series. The formula for the thickness distribution is of the form

y/

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*These logarithmic infinities arise from discontinuities in ψ', for example if $a_2 \neq 0$, ψ' changes abruptly at the leading edge from $-\frac{1}{2}a_2$ on the lower surface to $+\frac{1}{2}a_2$ on the upper surface.

† Now R. & M. 2130

$$y = a_1x^{\frac{1}{2}} + a_3x^{3/2} + a_5x^{5/2} + a_7x^{7/2} \qquad (0 \leqslant x \leqslant X_1)$$

$$= b_0 + b_2x + b_4x^2 + b_6x^3 \qquad (X_1 \leqslant x \leqslant 1).$$

Formulae used in this country for "laminar-flow" aerofoils have been of the form

$$y = (Ax - Bx^2)^{\frac{1}{2}} \qquad (0 \leqslant x \leqslant X_1)$$

$$= b_0 + b_2x + b_4x^2 + b_6x^3 + b_8x^4 \qquad (X_1 \leqslant x \leqslant 1)$$

(sometimes with $b_8 = 0$), or, if the trailing edge has been rounded off in such a way as to leave the curvature everywhere continuous,

$$y = (Ax - Bx^2)^{\frac{1}{2}} \qquad (0 \leqslant x \leqslant X_1)$$

$$= b_0 + b_2x + b_4x^2 + b_6x^3 + b_8x^4 \qquad (X_1 \leqslant x \leqslant X_2)$$

$$= [C(1 - x) + D(1 - x)^2]^{\frac{1}{2}} \qquad (X_2 \leqslant x \leqslant 1).$$

In addition Lock and Preston[9] have proposed a formula

$$y = kx^{\frac{1}{2}}(1 - x)(1 + b - bx)^{\frac{1}{2}}$$

as an approximation to Prof. Piercy's first series of aerofoils, and Duncan[10] has proposed the general formula

$$y = x^n(1 - x)f(x)$$

where $f(x)$ is a polynomial in x.

All these examples except the last two are covered explicitly by the general formulae of §9. For the formulae of Lock and Preston the necessary integrations can be carried out and the results expressed in terms of simple functions by using methods similar to those used in Lemma (16), Appendix; until the formulae are required in practice it is not considered necessary to set them out, especially as the answers may also be obtained as approximations to results obtained by simple conformal transformations. In the case of Duncan's aerofoils the leading edge radius is finite only when $n = \frac{1}{2}$, and Duncan's formula is then a special example of Case I of §9. (The necessary integrals may be expressed in terms of simple functions for certain other special values of n, but not for a general value.)

In the cases of the N.A.C.A. aerofoils and the Clark Y series, the formulae as set out leave a finite thickness at the trailing edge. In manufacture the trailing edge may be rounded off or "radiused"; strictly speaking this rounding off should be taken into account in the calculations, particularly of ε and ε'; but it is surely sufficiently accurate simply to omit, without considering the details of the rounding off, all terms containing as a factor the ordinate at the trailing edge; this procedure will be adopted.

We shall now set out numerical formulae for certain special aerofoils in order to exhibit the nature of such formula. With the approximations of the methods used in this note, $-g$, ψ, $\bar{\varepsilon}$, ε' are all proportional to the thickness; the numbers in the following examples are all for aerofoils 12 per cent thick.

Actual numerical computations of g, or of ε and ε', are much facilitated by using the specially prepared results tabulated in Table 1.

Example/

Example 1. N.A.C.A.0012.

The equation to the aerofoil is

$$y = 0{\cdot}17814x^{\frac{1}{2}} - 0{\cdot}07560x - 0{\cdot}21096x^2 + 0{\cdot}17058x^3 - 0{\cdot}06090x^4$$

so

$$y'(x) = 0{,}08907x^{-\frac{1}{2}} - 0{\cdot}07560 - 0{\cdot}42192x + 0{\cdot}51174x^2 - 0{,}24360x^3$$

This aerofoil is a special example of Case I of the preceding section, and

$$g_S = -\frac{1}{\pi} I_1\ (x;\ 0,\ 1), \qquad C_0 = \frac{1}{\pi} j_1\ (0,\ 1), \qquad \varepsilon = -\frac{1}{\pi} J_1\ (x;\ 0,\ 1)$$

where

$$I_1(x;0,1) = y'(x) \log_e \frac{1-x}{x} - \frac{a_1}{x^{\frac{1}{2}}} \log_e \frac{1+x^{\frac{1}{2}}}{x^{\frac{1}{2}}} + 2a_4 + \frac{3}{2} a_6 + \frac{4}{3} a_8$$

$$+ (3a_6 + 2a_8)x + 4a_8x^2$$

with $a_1 = 0{\cdot}17814$, $a_4 = -0{\cdot}21096$, $a_6 = 0{\cdot}17058$, $a_8 = -0{\cdot}06090$.

The formulae for $j_1\ (0,\ 1)$, $J_1\ (x;\ 0,\ 1)$ and $dJ_1\ (x;\ 0,\ 1)/d\theta$ may be similarly written out in full, and numerical values inserted. In this way we find that

$$g_S = -\frac{y'(x)}{\pi} \log_e \frac{1-x}{x} + \frac{0{\cdot}17814}{\pi x^{\frac{1}{2}}} \log_e \frac{1+x^{\frac{1}{2}}}{x^{\frac{1}{2}}} + 0{\cdot}0787021 - 0{\cdot}1241218x$$

$$+ 0{\cdot}0775403x^2,$$

$$C_0 = 0{\cdot}09985,$$

$$\varepsilon = -\frac{\psi(\theta)}{\pi} \log_e \frac{1-x}{x} + \frac{0{\cdot}35628}{\pi(1-x)^{\frac{1}{2}}} \log_e \frac{1+x^{\frac{1}{2}}}{x^{\frac{1}{2}}} - 0{\cdot}0786081 \left(\frac{x}{1-x}\right)^{\frac{1}{2}}$$

$$+ x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}(0{\cdot}0252197 - 0{\cdot}0193851x),$$

$$\varepsilon' = -\frac{\psi'(\theta)}{\pi} \log_e \frac{1-x}{x} + \frac{0{\cdot}17814x^{\frac{1}{2}}}{\pi(1-x)} \log_e \frac{1+x^{\frac{1}{2}}}{x^{\frac{1}{2}}} - \frac{0{\cdot}0393040}{1-x} + 0{\cdot}0446483$$

$$- 0{\cdot}0892095x + 0{\cdot}0581552x^2,$$

where, as always,

$$\psi(\theta) = \frac{2y}{\sin\theta} = \frac{y(x)}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}$$

$$\psi'(\theta) = y'(x) - \frac{2\cos\theta}{\sin^2\theta} y(x) = y'(x) - \frac{1-2x}{2x(1-x)} y(x).$$

Example/

Example 2. N.A.C.A. 16–012.

The ordinates of this aerofoil appear to be well fitted by the formulae

$$y = y_1(x) = 0{\cdot}11876x^{\frac{1}{2}} - 0{\cdot}02871x - 0{\cdot}00492x^2 - 0{\cdot}067128x^3 \qquad (0 \leqslant x \leqslant 0{\cdot}5)$$

$$\left.\begin{aligned} = y_2(x) &= 0{\cdot}06 - 0{\cdot}1476(x - 0{\cdot}5)^2 - 0{\cdot}1752(x - 0{\cdot}5)^3 \\ &= 0.0450 + 0{\cdot}0162x + 0{\cdot}1152x^2 - 0{\cdot}1752x^3 \end{aligned}\right\}(0{\cdot}5 \leqslant x \leqslant 1).$$

Hence

$$y_1'(x) = 0{\cdot}05938x^{-\frac{1}{2}} - 0{\cdot}02871 - 0{\cdot}00984x - 0{\cdot}201384x^2$$

$$y_2'(x) = 0{\cdot}0162 + 0{\cdot}2304x - 0{\cdot}5256x^2.$$

This aerofoil is a special example of Case II of the preceding section, so

$$g_S = -\frac{1}{\pi}\{I_1\ (x;\ 0,\ X_1) + I_2\ (x;\ X_1,\ 1)\},$$

$$C_0 = \frac{1}{\pi}\{j_1\ (0,\ X_1) + j_2\ (X_1,\ 1)\},$$

$$\varepsilon = -\frac{1}{\pi}\{J_1\ (x;\ 0,\ X_1) + J_2\ (x;\ X_1,\ 1)\},$$

with the following numerical values:

$M = 3$, $a_1 = 0{\cdot}11876$, $a_3 = a_5 = 0$, $a_2 = -0{\cdot}02871$,

$a_4 = -0{\cdot}00492$, $a_6 = -0{\cdot}067128$,

$m = 3$, $b_{2n+1} = 0$, $b_0 = 0{\cdot}0450$, $b_2 = 0{\cdot}0162$,

$b_4 = 0{\cdot}1152$, $b_6 = -0{\cdot}1752$,

$$X_1 = 0{\cdot}5.$$

When we write out the general formulae in full, as in Example 1, and insert numerical values (see Table 1 for $\frac{1}{\pi}\log_e X_1$, $\frac{1}{\pi}\log_e(1 + X_1^{\frac{1}{2}})$ etc.), we find that

$$g_S = -\frac{y_1'(x)}{\pi}\log_e\frac{|x-0{\cdot}5|}{x} + \frac{0{\cdot}11876}{\pi x^{\frac{1}{2}}}\log_e\frac{x^{\frac{1}{2}}+0{\cdot}7071068}{x^{\frac{1}{2}}}$$

$$-\frac{y_2'(x)}{\pi}\log_e\frac{1-x}{|x-0{\cdot}5|} + 0{\cdot}0356485 + 0{\cdot}1157031x,$$

$$C_0 = 0{\cdot}09893,$$

ε/

$$\varepsilon = -\frac{\psi_1(\theta)}{\pi}\log_e\frac{|x-0\cdot5|}{x} + \frac{0\cdot23752}{\pi(1-x)^{\frac{1}{2}}}\log_e\frac{x^{\frac{1}{2}}+0\cdot7071068}{x^{\frac{1}{2}}}$$
$$-\frac{\psi_2(\theta)}{\pi}\log_e\frac{1-x}{|x-0\cdot5|} - 0\cdot0444054\left(\frac{x}{1-x}\right)^{\frac{1}{2}} + 0\cdot0099286\left(\frac{1-x}{x}\right)^{\frac{1}{2}}$$
$$-0\cdot0385677x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$$

$$\varepsilon' = -\frac{\psi_1'(\theta)}{\pi}\log_e\frac{|x-0\cdot5|}{x} + \frac{0\cdot11876x^{\frac{1}{2}}}{\pi(1-x)}\log_e\frac{x^{\frac{1}{2}}+0\cdot7071068}{x^{\frac{1}{2}}}$$
$$-\frac{\psi_2'(\theta)}{\pi}\log_e\frac{1-x}{|x-0\cdot5|} - \frac{0\cdot0222027}{1-x} - \frac{0\cdot0049643}{x}$$
$$+ 0.0103323 + 0\cdot0771354x,$$

where $\psi_1(\theta)$, $\psi_1'(\theta)$ are connected with $y_1(x)$, $y_1'(x)$, and $\psi_2(\theta)$, $\psi_2'(\theta)$ with $y_2(x)$, $y_2'(x)$ in the same way as $\psi(\theta)$, $\psi'(\theta)$ were connected with $y(x)$, $y'(x)$ in the preceding example.

Example 3. Clark Y Fairing, 12 per cent thick.

For a 12 per cent thick fairing of a Clark Y aerofoil the formulae suggested by Pankhurst become

$$y = y_1(x) = x^{\frac{1}{2}}(0\cdot1621728 - 0\cdot1348548x - 0\cdot2947560x^2 + 0\cdot5252112x^3)$$
$$0 \leqslant x \leqslant 0\cdot3317$$
$$= y_2(x) = 0\cdot0379188 + 0\cdot1426932x - 0\cdot2583216x^2 + 0\cdot0868980x^3$$
$$0.3317 \leqslant x \leqslant 1.$$

Thus

$$y_1'(x) = x^{-\frac{1}{2}}(0\cdot0810864 - 0\cdot2022822x - 0\cdot7368900x^2 + 1\cdot8382392x^3)$$
$$y_2'(x) = 0\cdot1426932 - 0\cdot5166432x + 0\cdot2606940x^2.$$

This symmetrical aerofoil is another special example of Case II of the preceding section, with the additional simplification that in $y_1(x)$ we have $a_{2n} = 0$, so

$$\sum_{n=0}^{3} a_{2n+1}x^{n+\frac{1}{2}} = y_1(x), \qquad \sum_{n=0}^{3}(n+\tfrac{1}{2})a_{2n+1}x^{n-\frac{1}{2}} = y_1'(x),$$

$$\sum_{n=0}^{3} a_{2n+1} = y_1(1).$$

The numerical values to be inserted in the formulae of Case II are

M/

$$M = 4,\quad a_{2n} = 0,\quad a_1 = 0{\cdot}1621728,\quad a_3 = -0{\cdot}1348548,$$
$$a_5 = -0{\cdot}2947560,\quad a_7 = 0{\cdot}5252112,$$
$$m = 3,\quad b_{2n+1} = 0,\quad b_0 = 0{\cdot}0379188,\quad b_2 = 0{\cdot}1426932,$$
$$b_4 = -0{\cdot}2583216,\quad b_6 = 0{\cdot}0868980,$$
$$X_1 = 0{\cdot}3317,\quad X_1^{\frac{1}{2}} = 0{\cdot}5759340,\quad y_1(1) = 0{\cdot}2577732,$$

from which values we find that

$$g_S = -\frac{y_1'(x)}{\pi}\{\log_e|x-0{\cdot}3317| - 2\log_e(x^{\frac{1}{2}}+0{\cdot}5759340)\}$$
$$-\frac{y_2'(x)}{\pi}\log_e\frac{1-x}{|x-0{\cdot}3317|} + 0{\cdot}1621868+0{\cdot}1402039x-0{\cdot}6739922x^2$$

$$C_0 = 0{\cdot}09656$$

$$\varepsilon = -\frac{\psi_1(\theta)}{\pi}\{\log_e|x-0{\cdot}3317| - 2\log_e(x^{\frac{1}{2}}+0{\cdot}5759340)\}$$
$$-\frac{\psi_2(\theta)}{\pi}\log_e\frac{1-x}{|x-0{\cdot}3317|} - 0{\cdot}1077105\left(\frac{x}{1-x}\right)^{\frac{1}{2}} + 0{\cdot}0133195\left(\frac{1-x}{x}\right)^{\frac{1}{2}}$$
$$+ x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\{0{\cdot}1242738 + 0{\cdot}1925692x\},$$

$$\varepsilon' = -\frac{\psi_1'(\theta)}{\pi}\{\log_e|x-0{\cdot}3317| - 2\log_e(x^{\frac{1}{2}}+0{\cdot}5759340)\}$$
$$-\frac{\psi_2'(\theta)}{\pi}\log_e\frac{1-x}{|x-0{\cdot}3317|} - \frac{0{\cdot}0538553}{1-x} - \frac{0{\cdot}0066597}{x}$$
$$+ 0{\cdot}1245241 + 0{\cdot}1681931x - 0{\cdot}4814230x^2.$$

<u>Example 4.</u> <u>EQH 1260.</u>

For this aerofoil

$$y = y_4(x) = (0{\cdot}012x-0{\cdot}010x^2)^{\frac{1}{2}} \qquad 0 \leqslant x \leqslant 0{\cdot}6$$
$$= y_2(x) = 0{\cdot}06-0{\cdot}08\dot{3}(x-0{\cdot}6)^2-1{\cdot}4252\dot{7}(x-0{\cdot}6)^3+1{\cdot}787152\dot{7}(x-0{\cdot}6)^4$$
$$= 0{\cdot}569475-2{\cdot}9834x+6{\cdot}34241\dot{6}x^2-5{\cdot}71\dot{4}x^3+1{\cdot}737152\dot{7}x^4$$
$$0{\cdot}6 \leqslant x \leqslant 0{\cdot}9760155$$
$$= y_3(x) = \{0{\cdot}000873882(1-x)+0{\cdot}079607107(1-x)^2\}^{\frac{1}{2}}$$
$$0{\cdot}9760155 \leqslant x \leqslant 1.$$

Thus

$$y_4'(x) = \frac{0{\cdot}006 - 0{\cdot}01x}{y_4(x)}$$

$y_2'(x)$/

$$y_2'(x) = -2 \cdot 9834 + 12 \cdot 6848\dot{3}x - 17 \cdot 14\dot{3}x^2 + 7 \cdot 1486\dot{1}x^3$$

$$y_3'(x) = -\frac{0 \cdot 000436941 + 0 \cdot 079607107(1-x)}{y_3(x)}.$$

This aerofoil is a special example of Case V of the preceding section, with $A/B > 1$, so that $x < A/B$ for all x.

The numerical values required are

$X_1 = 0 \cdot 6$, $X_2 = 0 \cdot 9760155$, $A = 0 \cdot 012$, $B = 0 \cdot 010$, $A - 2BX_1 = 0$,

$2\sin^{-1}(BX_1/A)^{\frac{1}{2}} = \frac{1}{2}\pi$, $(A-B)^{\frac{1}{2}} = 0 \cdot 044721$, $y_4(X_1) = 0 \cdot 06$, $2y_4(X_1)/A = 10$,

$m = 4$, $b_{2n+1} = 0$, $b_0 = 0 \cdot 569475$, $b_2 = -2.9834$, $b_4 = 6 \cdot 34241\dot{6}$,

$b_6 = -5 \cdot 71\dot{4}$, $b_8 = 1 \cdot 787152\dot{7}$

$C = 0 \cdot 000873882$, $D = 0 \cdot 079607107$, $\sqrt{D} = 0 \cdot 2821473$,

$$\frac{D(1-X_2)}{C} = 2 \cdot 1848907, \quad \sinh^{-1}\left[\frac{D(1-X_2)}{C}\right]^{\frac{1}{2}} = 1 \cdot 1825745,$$

$$2\sqrt{D}\sinh^{-1}\left[\frac{D(1-X_2)}{C}\right]^{\frac{1}{2}} = 0 \cdot 6673204, \quad y_3(X_2) = 0 \cdot 008170319,$$

$$\frac{2y_3(X_2)}{C} = 18 \cdot 698907, \quad y_3(0) = (C+D)^{\frac{1}{2}} = 0 \cdot 2836917.$$

With these numerical values the formulae of Case V become, with

$$f = 0 \cdot 0239845 + 5 \cdot 3697814(1-x) + 18 \cdot 698907y_3(x)$$

and $X_2 = 0 \cdot 9760155$ as above,

$$g_S = \frac{y_4'(x)}{\pi}\log_e\frac{10y_4+0 \cdot 6}{|x-0 \cdot 6|} - \frac{y_2'(x)}{\pi}\log_e\left|\frac{x-X_2}{x-0 \cdot 6}\right| - \frac{y_3'(x)}{\pi}\log_e\frac{f}{|x-X_2|}$$

$$- 0 \cdot 6051406 + 1 \cdot 3776462x - 0 \cdot 8556133x^2$$

$C_0 = 0 \cdot 10277$,

$$\varepsilon = \frac{\psi_4(\theta)}{\pi}\log_e\frac{10y_4+0 \cdot 6}{|x-0 \cdot 6|} - \frac{\psi_2(\theta)}{\pi}\log_e\left|\frac{x-X_2}{x-0 \cdot 6}\right| - \frac{\psi_3(\theta)}{\pi}\log_e\frac{f}{|x-X_2|}$$

$$- 0 \cdot 0147752\left(\frac{x}{1-x}\right)^{\frac{1}{2}} + 0 \cdot 3041134\left(\frac{1-x}{x}\right)^{\frac{1}{2}}$$

$$- x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}(0 \cdot 3014979 - 0 \cdot 2139033x),$$

$$\varepsilon' = \frac{\psi_4'(\theta)}{\pi}\log_e\frac{10y_4+0 \cdot 6}{|x-0 \cdot 6|} - \frac{\psi_2'(\theta)}{\pi}\log_e\left|\frac{x-X_2}{x-0 \cdot 6}\right| - \frac{\psi_3'(\theta)}{\pi}\log_e\frac{f}{|x-X_2|}$$

$$- \frac{0 \cdot 0073876}{1-x} - \frac{0 \cdot 1522067}{x} - 0 \cdot 2379744 + 0 \cdot 9691966x - 0 \cdot 6417100x^2.$$

Example/

Example 5. EQH 1250.

For this aerofoil

$$y = y_1(x) = 0{\cdot}12(x-x^2)^{\frac{1}{2}} \qquad 0 \leqslant x \leqslant 0{\cdot}5$$

$$\left.\begin{aligned} &= y_2(x) = 0{\cdot}06-0{\cdot}12(x-0{\cdot}5)^2-0{\cdot}535(x-0{\cdot}5)^3+0{\cdot}609(x-0{\cdot}5)^4 \\ &= 0{\cdot}1349375-0{\cdot}58575x+1{\cdot}596x^2-1{\cdot}753x^3+0{\cdot}609x^4 \end{aligned}\right\}$$

$$0{\cdot}5 \leqslant x \leqslant 0{\cdot}9653726$$

$$= y_3(x) = \{0{\cdot}0006260362(1-x)+0{\cdot}044389956(1-x)^2\}^{\frac{1}{2}}$$

$$0{\cdot}9653726 \leqslant x \leqslant 1.$$

This aerofoil is another special example of Case V of the preceding section, but we now have $A = B = 0{\cdot}0144$. In consequence

$$y_1 = 0{\cdot}06 \sin\theta, \qquad \psi_1 = 0{\cdot}12, \qquad \psi_1' = 0,$$

and

$$y_1'(x) = 0{\cdot}06 \frac{1-2x}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} = 0{\cdot}12 \cot\theta.$$

Also

$$y_2'(x) = -0{\cdot}58575 + 3{\cdot}192x - 5{\cdot}259x^2 + 2{\cdot}436x^3,$$

$$y_3'(x) = -\frac{0{\cdot}0003130181 + 0{\cdot}044389956(1-x)}{y_3(x)}.$$

Other numerical values required are

$X_1 = 0{\cdot}5$, $X_2 = 0{\cdot}9653726$, $2\sin^{-1}(BX_1/A)^{\frac{1}{2}} = \frac{1}{2}\pi$, $A-2BX_1 = 0$,

$y_1(X_1) = 0{\cdot}06$, $2y_1(X_1)/A = 8{\cdot}\dot{3}$,

$m = 4$, $b_{2n+1} = 0$, $b_0 = 0{\cdot}1349375$, $b_2 = -0{\cdot}58575$, $b_4 = 1{\cdot}596$,

$b_6 = -1{\cdot}753$, $b_8 = 0{\cdot}609$,

$C = 0{\cdot}0006260362$, $D = 0{\cdot}044389956$, $\sqrt{D} = 0{\cdot}2106892$,

$$\frac{D(1-X_2)}{C} = 2{\cdot}4553036, \quad \sinh^{-1}\left[\frac{D(1-X_2)}{C}\right]^{\frac{1}{2}} = 1{\cdot}2313306,$$

$$2\sqrt{D}\sinh^{-1}\left[\frac{D(1-X_2)}{C}\right]^{\frac{1}{2}} = 0{\cdot}5188563,$$

$y_3(X_2) = 0{\cdot}0086547156$, $2y_3(X_2)/C = 27{\cdot}649250$, $y_3(0) = (C+D)^{\frac{1}{2}} = 0{\cdot}21216$

If/

If we now write

$$f = 0{\cdot}0346274 + 5{\cdot}9106072(1 - x) + 27{\cdot}649250y_3(x)$$

and $X_2 = 0{\cdot}9653726$ as above, we find that for this aerofoil

$$g_S = \frac{y_4'(x)}{\pi}\log_e \frac{8{\cdot}\dot{3}y_4+0{\cdot}5}{|x-0{\cdot}5|} - \frac{y_2'(x)}{\pi}\log_e\left|\frac{x-X_2}{x-0{\cdot}5}\right|$$

$$-\frac{y_3'(x)}{\pi}\log_e\frac{f}{|x-X_2|} - 0{\cdot}2074402+0{\cdot}5146391x-0{\cdot}3608512x^2,$$

$$C_0 = 0{\cdot}10039,$$

$$\varepsilon = \frac{0{\cdot}12}{\pi}\log_e\frac{8{\cdot}\dot{3}y_4+0{\cdot}5}{|x-0{\cdot}5|} - \frac{\psi_2(\theta)}{\pi}\log_e\left|\frac{x-X_2}{x-0{\cdot}5}\right|$$

$$-\frac{\psi_3(\theta)}{\pi}\log_e\frac{f}{|x-X_2|} - 0{\cdot}0010092\left(\frac{x}{1-x}\right)^{\frac{1}{2}} + 0{\cdot}1973894\left(\frac{1-x}{x}\right)^{\frac{1}{2}}$$

$$-x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}(0{\cdot}1033661 - 0{\cdot}0902128x),$$

$$\varepsilon' = -\frac{\psi_2'(\theta)}{\pi}\log_e\left|\frac{x-X_2}{x-0{\cdot}5}\right| - \frac{\psi_3'(\theta)}{\pi}\log_e\frac{f}{|x-X_2|} - \frac{0{\cdot}0005046}{1-x}$$

$$-\frac{0{\cdot}0986947}{x} - 0{\cdot}0577508+0{\cdot}3661666x-0{\cdot}2706384x^2,$$

where $8.\dot{3}y_4(x) = 0{\cdot}5\sin\theta$.

<u>Example 6.</u> <u>EQH 1240.</u>

For this aerofoil

$$y = y_4(x) = (0{\cdot}018x-0{\cdot}0225x^2)^{\frac{1}{2}} \qquad 0 \leqslant x \leqslant 0{\cdot}4$$

$$\left.\begin{aligned} = y_2(x) &= 0{\cdot}06-0{\cdot}1875(x-0{\cdot}4)^2+0{\cdot}02083(x-0{\cdot}4)^3+0{\cdot}03240(x-0{\cdot}4)^4 \\ &= 0{\cdot}02949632+0{\cdot}151704x-0{\cdot}181392x^2-0{\cdot}03161x^3+0{\cdot}03240x^4 \end{aligned}\right\}$$

$$0{\cdot}4 \leqslant x \leqslant 0{\cdot}9640731$$

$$= y_3(x) = \{0{\cdot}000536980(1-x)+0{\cdot}027022625(1-x)^2\}^{\frac{1}{2}}$$

$$0.9640731 \leqslant x \leqslant 1.$$

This aerofoil is a third special example of Case V of the preceding section, but in this example $A/B = 0{\cdot}8$, and we have somewhat different formulae according as $x \lessgtr 0{\cdot}8$. (The limiting values as $x \to 0{\cdot}8$ are the same for the two sets of formulae.)

The form above defines $y_4(x)$ only for $0 \leqslant x \leqslant 0{\cdot}8$. For $0{\cdot}8 < x \leqslant 1$ we define $y_4(x)$ by

$$y_4(x) = (0{\cdot}0225x^2 - 0{\cdot}018x)^{\frac{1}{2}}.$$

Then/

Then

$$y_4'(x) = \frac{0\cdot009 - 0\cdot0225x}{y_4(x)} \qquad 0 < x < 0\cdot8$$

$$= \frac{0\cdot0225x - 0\cdot009}{y_4(x)} \qquad 0\cdot8 < x \leqslant 1,$$

$$y_2'(x) = 0\cdot151704 - 0\cdot362784x - 0\cdot09303x^2 + 0\cdot12960x^3$$

$$y_3'(x) = -\frac{0\cdot000268490 + 0\cdot027022625(1 - x)}{y_4(x)},$$

$X_1 = 0\cdot4$, $X_2 = 0\cdot9640731$, $A = 0\cdot018$, $B = 0\cdot0225$, $2\sin^{-1}(BX_1/A)^{\frac{1}{2}} - \frac{1}{2}\pi$,
$(B-A)^{\frac{1}{2}} = 0\cdot06708204$, $A - 2BX_1 = 0$, $y_4(X_1) = 0\cdot06$, $2y_4(X_1)/A = 6\cdot\dot{6}$,
$m = 4$, $b_{2n+1} = 0$, $b_0 = 0\cdot02949632$, $b_2 = 0\cdot151704$, $b_4 = -0\cdot181392$,
$b_6 = -0\cdot03101$, $b_8 = 0\cdot03240$,

$C = 0\cdot000536980$, $D = 0\cdot027022625$, $\sqrt{D} = 0\cdot1643856$,

$$\frac{D(1-X_2)}{C} = 1.8079615, \quad \sinh^{-1}\left[\frac{D(1-X_2)}{C}\right]^{\frac{1}{2}} = 1\cdot1053569,$$

$$2\sqrt{D}\,\sinh^{-1}\left[\frac{D(1-X_2)}{C}\right]^{\frac{1}{2}} = 0\cdot3634095,$$

$y_3(X_2) = 0\cdot0073601$, $2y_3(X_2)/C = 27\cdot412939$, $y_3(0) = (C+D)^{\frac{1}{2}} = 0\cdot1660109$.

Hence, with

$$f = 0\cdot0359269 + 4\cdot6159230(1 - x) + 27\cdot412939y_3(x)$$

and $X_2 = 0\cdot9640731$ as above, we find that for $x \leqslant 0\cdot8$,

$$g_S = \frac{y_4'(x)}{\pi}\log_e\frac{6\cdot\dot{6}y_4+0\cdot4}{|x-0\cdot4|} - \frac{y_2'(x)}{\pi}\log_e\left|\frac{x-X_2}{x-0\cdot4}\right| - \frac{y_3'(x)}{\pi}\log_e\frac{f}{|x-X_2|}$$
$$+ 0\cdot0244118 + 0\cdot0008327x - 0\cdot0232696x^2,$$

$$C_0 = 0\cdot09920,$$

$$\varepsilon = \frac{\psi_4(\theta)}{\pi}\log_e\frac{6\cdot\dot{6}y_4+0\cdot4}{|x-0\cdot4|} - \frac{\psi_2(\theta)}{\pi}\log_e\left|\frac{x-X_2}{x-0\cdot4}\right| - \frac{\psi_3(\theta)}{\pi}\log_e\frac{f}{|x-X_2|},$$
$$+ 0\cdot0168853\left(\frac{x}{1-x}\right)^{\frac{1}{2}} + 0\cdot1274775\left(\frac{1-x}{x}\right)^{\frac{1}{2}}$$
$$+ x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}[0\cdot0042173 + 0\cdot0058174x],$$

ε'/

$$\varepsilon' = \frac{\psi_4'(\theta)}{\pi}\log_e\frac{6\cdot\dot{6}y_4+0\cdot4}{|x-0\cdot4|} - \frac{\psi_2'(\theta)}{\pi}\log_e\left|\frac{x-X_2}{x-0\cdot4}\right| - \frac{\psi_3'(\theta)}{\pi}\log_e\frac{f}{|x-X_2|}$$

$$+\frac{0\cdot0084427}{1-x} - \frac{0\cdot0637388}{x} + 0\cdot0336914 + 0\cdot0021413x - 0\cdot0174523x^2.$$

For $x \geqslant 0\cdot8$,

$$\log_e\frac{6\cdot\dot{6}y_4 + 0\cdot4}{|x-0\cdot4|}$$

must be replaced by

$$-2\tan^{-1}\left(1-\frac{0\cdot8}{x}\right)^{\frac{1}{2}},$$

and as $x \to 0\cdot8$

$$y_4'(x)\log_e\frac{6\cdot\dot{6}y_4 + 0\cdot4}{|x-0\cdot4|} \to -0\cdot15$$

$$\psi_4(\theta)\ \log_e\frac{6\cdot\dot{6}y_4 + 0\cdot4}{|x-0\cdot4|} \to 0$$

$$\psi_4'(\theta)\log_e\frac{6\cdot\dot{6}y_4 + 0\cdot4}{|x-0\cdot4|} \to -0\cdot15.$$

11. Calculated Results and Comparisons for EQH 1260.

As an illustration the computations have been carried out by all three approximate methods for EQH 1260, and the results compared with each other and with numerically accurate results obtained from the exact theory. Other results and comparisons will be set out in later reports.

Table 2 contains some numerically accurate results for this aerofoil*; Table 3 shows the computation of g_S, and of $g_S + g_L$ for $C_L = 0.4$, $C_L/\sin\alpha = 4.4$; Table 4 the values of q/U from the formula (31); Table 5 the computation of ε; Table 6 the computation of ε'; and Table 7 the values of q/U from the formula (32). An attempt to exhibit graphically comparisons of these three sets of approximate values of q/U with the accurate values has been made in Figs. 1, 2 and 3, but in some ways the comparison is clearer from the tables. The results appear wholly satisfactory. Probably the only noteworthy discrepancy is that at $C_L = 0$ the approximate values are slightly lower than the accurate values round about the bump in the curve near $x = 0.7$; it is interesting to note that in the same region the values measured by Fage and Walker[11] are lower than the calculated values, and there is some reason to believe that, at any rate for aerofoils of this thickness or less, the regions in which

values/

*Values of q/U at $C_L = 0$ in Table 2 differ somewhat from values published by Fage and Walker[11]; this latter set of values was the result of some preliminary calculations only.

values calculated by the approximate method will differ perceptibly from accurate values will be included in the regions in which experimental values might be expected to differ perceptibly from theoretical values.

12. Summary.

Three formulae, of increasing accuracy and complexity, are given for finding the velocity at the surface of a symmetrical aerofoil.

Computations required in using any one of these formulae may be carried out by any one of three methods: Fourier series, integration, or the use of the analytical results of §9. These analytical results apply when, with the aerofoil chord divided into any number of segments, in each segment the ordinate y is a polynomial of any degree in $x^{\frac{1}{2}}$ (including, of course, a polynomial in x) or in $(1 - x)^{\frac{1}{2}}$, where x is the distance in fractions of the chord from the leading edge; in addition an elliptic nose may be fitted up to any distance back from the leading edge, and the trailing edge may be rounded off by a circle, ellipse or hyperbola up to any distance from the trailing edge. The Fourier series should be used when y is $x^{\frac{1}{2}}(1 - x)^{\frac{1}{2}}$ multiplied by a polynomial in x, since the Fourier series then terminate; in all other cases (except those derived by simple conformal transformations, to which the present theory should not be applied at all) it is recommended that formulae be fitted to y of the type used in §9, if possible, and use made of the analytical results.

Definite numerical formulae are set out, as examples, for N.A.C.A.0012, N.A.C.A.16-012, a Clark Y fairing, EQH 1260, EQH 1250 and EQH 1240. The computations are carried out for EQH 1260 and the results compared with accurate results from the exact theory; the agreement is wholly satisfactory.

Appendix.

Lemma 1.

$$\sin\theta \sum_{n=0}^{\infty} A_n \cos n\theta = (A_0 - \tfrac{1}{2}A_2)\sin\theta + \sum_{n=2}^{\infty} \tfrac{1}{2}(A_{n-1} - A_{n+1}) \sin n\theta.$$

Lemma 2.

$$\cos\theta \sum_{n=0}^{\infty} A_n \cos n\theta = \tfrac{1}{2}A_1 + (A_0 + \tfrac{1}{2}A_2)\cos\theta + \sum_{n=2}^{\infty} \tfrac{1}{2}(A_{n-1} + A_{n+1})\cos n\theta$$

Lemma 3.

$$\sin\theta \sum_{n=1}^{\infty} B_n \sin n\theta = \tfrac{1}{2}B_1 + \tfrac{1}{2}B_2\cos\theta - \sum_{n=2}^{\infty} \tfrac{1}{2}(B_{n-1} - B_{n+1}) \cos n\theta.$$

Lemma 4.

$$\cos\theta \sum_{n=1}^{\infty} B_n \sin n\theta = \tfrac{1}{2}B_2 \sin\theta + \sum_{n=2}^{\infty} \tfrac{1}{2}(B_{n-1} + B_{n+1}) \sin n\theta.$$

Lemma/

Lemma 5. If

$$f(\theta) = \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)$$

and

$$g(\theta) = \sum_{n=1}^{\infty} (A_n \sin n\theta - B_n \cos n\theta),$$

then[12]

$$f(\theta) = \frac{1}{2\pi} \int_0^{\pi} \{g(\theta + t) - g(\theta - t)\} \cot \tfrac{1}{2}t\,dt$$

$$= \frac{1}{2\pi} P \int_{-\pi}^{\pi} g(t) \cot \tfrac{1}{2}(t - \theta)dt,$$

and

$$g(\theta) = -\frac{1}{2\pi} \int_0^{\pi} \{f(\theta + t) - f(\theta - t)\} \cot \tfrac{1}{2}t\,dt$$

$$= -\frac{1}{2\pi} P \int_{-\pi}^{\pi} f(t) \cot \tfrac{1}{2}(t - \theta)dt,$$

where P denotes that the principal values of the integral is to be taken.

Lemma 6. If in Lemma (5) $f(\theta)$ is even and $g(\theta)$ is odd, so that $B_n = 0$ and

$$f(\theta) = \sum_{n=1}^{\infty} A_n \cos n\theta, \qquad g(\theta) = \sum_{n=1}^{\infty} A_n \sin n\theta,$$

then

$$f(\theta) = \frac{1}{2\pi} P \int_{-\pi}^{\pi} g(t)\cot \tfrac{1}{2}(t-\theta)dt = \frac{1}{2\pi} P \int_{-\pi}^{\pi} g(-t)\cot(-\tfrac{1}{2}t-\tfrac{1}{2}\theta)dt$$

$$= \frac{1}{2\pi} P \int_{-\pi}^{\pi} g(t) \cot \tfrac{1}{2}(t + \theta)dt$$

$$= \frac{1}{4\pi} P \int_{-\pi}^{\pi} g(t)\{\cot \tfrac{1}{2}(t - \theta) + \cot \tfrac{1}{2}(t + \theta)\}dt$$

$$= \frac{1}{2\pi} P \int_{-\pi}^{\pi} \frac{g(t) \sin t}{\cos \theta - \cos t} dt$$

$$= \frac{1}{\pi} P \int_{0}^{\pi} \frac{g(t) \sin t}{\cos \theta - \cos t} dt.$$

Similarly

$$g(\theta) = -\frac{\sin \theta}{\pi} P \int_0^{\pi} \frac{f(t)}{\cos \theta - \cos t} dt.$$

Lemma/

Lemma 7.

$$P\int_0^{\theta_1}\frac{dt}{\cos\theta-\cos t}=\frac{1}{\sin\theta}\log_e\left|\frac{\sin\frac{1}{2}(\theta-\theta_1)}{\sin\frac{1}{2}(\theta+\theta_1)}\right|$$

$$P\int_0^{\pi}\frac{dt}{\cos\theta-\cos t}=0.$$

(See Reference 1, pp. 92, 93).

Lemma 8. For any q_r and a_r

$$\sum_{n=0}^{m-1}x^n\sum_{r=1}^{m-n}a_{r+n}q_r-x\sum_{r=1}^{m}q_r\sum_{s=r}^{m}a_s-(1-x)\sum_{r=1}^{m}a_rq_r$$

$$=-x(1-x)\sum_{n=0}^{m-3}A_nx^n,$$

where $A_n=\sum_{r=1}^{m-n-2}q_r\sum_{s=r+n+2}^{m}a_s \qquad (0\leqslant n\leqslant m-3),$

and if $A_{m-2}=0,$

$$A_{n-1}-A_n=\sum_{r=1}^{m-n-1}a_{r+n+1}q_r \qquad (1\leqslant n\leqslant m-2).$$

Lemma 9. If $f(\xi)=\sum_{n=0}^{m}a_n\xi^n,$

then

$$\frac{f(\xi)-f(x)}{\xi-x}=\sum_{n=0}^{m-1}x^n\sum_{r=0}^{m-n-1}a_{r+n+1}\xi^r=\sum_{r=0}^{m-1}\xi^r\sum_{n=0}^{m-r-1}a_{r+n+1}x^n,$$

whence

$$P\int_{x_1}^{x_2}\frac{f(\xi)d\xi}{\xi-x}=f(x)\log_e\left|\frac{x-x_2}{x-x_1}\right|+\sum_{n=0}^{m-1}x^n\sum_{r=1}^{m-n}a_{r+n}\frac{x_2^{\,r}-x_1^{\,r}}{r}$$

and

$$\int_{x_1}^{x_2}\frac{f(\xi)d\xi}{1-\xi}=-f(1)\log_e\frac{1-x_2}{1-x_1}-\sum_{r=1}^{m}\frac{x_2^{\,r}-x_1^{\,r}}{r}\sum_{s=r}^{m}a_s.$$

Also

$$\int_{x_1}^{x_2}\frac{f(\xi)d\xi}{\xi}=a_0\log_e\frac{x_2}{x_1}+\sum_{r=1}^{m}a_r\frac{x_2^{\,r}-x_1^{\,r}}{r}.$$

Hence/

Hence

$$x(1-x)\ P\int_{x_1}^{x_2} \frac{f(\xi)d\xi}{\xi(1-\xi)(\xi-x)} = P\int_{x_1}^{x_2}\left\{\frac{1}{\xi-x} + \frac{x}{1-\xi} - \frac{1-x}{\xi}\right\} f(\xi)d\xi$$

$$= f(x)\log_e\left|\frac{x-x_2}{x-x_1}\right| - xf(1)\log_e\frac{1-x_2}{1-x_1} - (1-x)a_0\log_e\frac{x_2}{x_1}$$

$$+ \sum_{n=0}^{m-1} x^n \sum_{r=1}^{m-n} a_{r+n}\frac{x_2{}^r - x_1{}^r}{r} - x\sum_{r=1}^{m}\frac{x_2{}^r - x_1{}^r}{r}\sum_{s=r}^{m} a_s$$

$$- (1-x)\sum_{r=1}^{m} a_r\frac{x_2{}^r - x_1{}^r}{r}$$

$$= f(x)\log_e\left|\frac{x-x_2}{x-x_1}\right| - xf(1)\log_e\frac{1-x_2}{1-x_1} - (1-x)a_0\log_e\frac{x_2}{x_1}$$

$$- x(1-x)\sum_{n=0}^{m-3} A_n x^n,$$

(by Lemma (8)), where

$$A_n = \sum_{r=1}^{m-n-2}\frac{x_2{}^r - x_1{}^r}{r}\sum_{s=r+n+2}^{m} a_s \qquad (0 \leqslant n \leqslant m-3),$$

and if $A_{m-2} = 0$ then

$$A_{n-1} - A_n = \sum_{r=1}^{m-n-1} a_{r+n+1}\frac{x_2{}^r - x_1{}^r}{r} \qquad (1 \leqslant n \leqslant m-2).$$

Lemma 10.

$$\int\frac{d\xi}{\xi^{\frac{1}{2}}(\xi - x)} = \frac{1}{x^{\frac{1}{2}}}\{\log_e(\xi - x) - 2\log_e(\xi^{\frac{1}{2}} + x^{\frac{1}{2}})\}$$

so

$$P\int_{x_1}^{x_2}\frac{d\xi}{\xi^{\frac{1}{2}}(\xi - x)} = \frac{1}{x^{\frac{1}{2}}}\left\{\log_e\left|\frac{x-x_2}{x-x_1}\right| - 2\log_e\frac{x_2^{\frac{1}{2}} + x^{\frac{1}{2}}}{x_1^{\frac{1}{2}} + x^{\frac{1}{2}}}\right\},$$

and

$$P\int_{x_1}^{x_2}\frac{\xi^{\frac{1}{2}}d\xi}{\xi - x} = P\int_{x_1}^{x_2}\left\{\frac{1}{\xi^{\frac{1}{2}}} + \frac{x}{\xi^{\frac{1}{2}}(\xi-x)}\right\}d\xi$$

$$= 2(x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}}) + x^{\frac{1}{2}}\left\{\log_e\left|\frac{x-x_2}{x-x_1}\right| - 2\log_e\frac{x_2^{\frac{1}{2}}+x^{\frac{1}{2}}}{x_1^{\frac{1}{2}}+x^{\frac{1}{2}}}\right\}.$$

Lemma/

Lemma 11.

If $$F(\xi) = \sum_{n=0}^{m} b_n \xi^{n-\frac{1}{2}},$$

then $$\frac{F(\xi)}{\xi-x} = \frac{\xi^{-\frac{1}{2}}}{\xi-x} \sum_{n=0}^{m} b_n x^n + \sum_{n=0}^{m-1} x^n \sum_{r=0}^{m-n-1} b_{r+n+1} \xi^{r-\frac{1}{2}} \quad \text{(by Lemma (9))},$$

so

$$P \int_{x_1}^{x_2} \frac{F(\xi)d\xi}{\xi - x} = F(x) \left\{ \log_e \left|\frac{x-x_2}{x-x_1}\right| - 2 \log_e \frac{x_2^{\frac{1}{2}}+x^{\frac{1}{2}}}{x_1^{\frac{1}{2}}+x^{\frac{1}{2}}} \right\} + \sum_{n=0}^{m-1} x^n \sum_{r=1}^{m-n} b_{r+n} \frac{x_2^{r-\frac{1}{2}}-x_1^{r-\frac{1}{2}}}{r - \frac{1}{2}} \quad \text{(by Lemma (10))},$$

and

$$\int_{x_1}^{x_2} \frac{F(\xi)d\xi}{1 - \xi} = - F(1) \left\{ \log_e \frac{1-x_2}{1-x_1} - 2 \log_e \frac{1+x_2^{\frac{1}{2}}}{1+x_1^{\frac{1}{2}}} \right\} - \sum_{r=1}^{m} \frac{x_2^{r-\frac{1}{2}}-x_1^{r-\frac{1}{2}}}{r - \frac{1}{2}} \sum_{s=r}^{m} b_s$$

$$\int_{x_1}^{x_2} \frac{F(\xi)d\xi}{\xi} = \sum_{r=0}^{m} b_r \frac{x_2^{r-\frac{1}{2}} - x_1^{r-\frac{1}{2}}}{r - \frac{1}{2}}.$$

Hence

$$\begin{aligned} x(1-x)\, P \int_{x_1}^{x_2} \frac{F(\xi)d\xi}{\xi(1-\xi)(\xi-x)} &= F(x) \left\{ \log_e \left|\frac{x-x_2}{x-x_1}\right| - 2 \log_e \frac{x_2^{\frac{1}{2}}+x^{\frac{1}{2}}}{x_1^{\frac{1}{2}}+x^{\frac{1}{2}}} \right\} \\ &\quad - xF(1) \left\{ \log_e \frac{1-x_2}{1-x_1} - 2 \log_e \frac{1+x_2^{\frac{1}{2}}}{1+x_1^{\frac{1}{2}}} \right\} \\ &\quad - 2(1-x) b_0 (x_1^{-\frac{1}{2}} - x^{-\frac{1}{2}}) \\ &\quad + \sum_{n=0}^{m-1} x^n \sum_{r=1}^{m-n} b_{r+n} \frac{x_2^{r-\frac{1}{2}}-x_1^{r-\frac{1}{2}}}{r - \frac{1}{2}} - x \sum_{r=1}^{m} \frac{x_2^{r-\frac{1}{2}}-x_1^{r-\frac{1}{2}}}{r - \frac{1}{2}} \sum_{s=r}^{m} b_s \\ &\quad - (1 - x) \sum_{r=1}^{m} b_r \frac{x_2^{r-\frac{1}{2}} - x_1^{r-\frac{1}{2}}}{r - \frac{1}{2}} \\ &= F(x) \left\{ \log_e \left|\frac{x-x_2}{x-x_1}\right| - 2 \log_e \frac{x_2^{\frac{1}{2}}+x^{\frac{1}{2}}}{x_1^{\frac{1}{2}}+x^{\frac{1}{2}}} \right\} \\ &\quad - xF(1) \left\{ \log_e \frac{1-x_2}{1-x_1} - 2 \log_e \frac{1+x_2^{\frac{1}{2}}}{1+x_1^{\frac{1}{2}}} \right\} - 2(1-x) b_0 (x_1^{-\frac{1}{2}} - x_2^{-\frac{1}{2}}) \\ &\quad - x(1 - x) \sum_{n=0}^{m-3} B_n x^n \quad \text{(by Lemma (8))}, \end{aligned}$$

where/

where

$$B_n = \sum_{r=1}^{m-n-2} \frac{x_2^{r-\frac{1}{2}} - x_1^{r-\frac{1}{2}}}{r-\frac{1}{2}} \sum_{s=r+n+2}^{m} b_s \qquad (0 \leqslant n \leqslant m-3)$$

and if $B_{m-2} = 0$ then

$$B_{n-1} - B_n = \sum_{r=1}^{m-n-1} b_{r+n+1} \frac{x_2^{r-\frac{1}{2}} - x_1^{r-\frac{1}{2}}}{r-\frac{1}{2}} \qquad (1 \leqslant n \leqslant m-2).$$

Lemma 12.

If $$g(\xi) = \sum_{n=-1}^{2M} c_n \xi^{n/2},$$

then by Lemmas (9) and (11),

$$P\int_{x_1}^{x_2} \frac{g(\xi)d\xi}{\xi - x} = g(x) \log_e \left|\frac{x-x_2}{x-x_1}\right| - 2\left(\sum_{n=0}^{M} c_{2n-1} x^{n-\frac{1}{2}}\right) \log_e \frac{x_2^{\frac{1}{2}}+x^{\frac{1}{2}}}{x_1^{\frac{1}{2}}+x^{\frac{1}{2}}}$$
$$+ \sum_{n=0}^{M-1}\left(x^n \sum_{r=1}^{2M-2n} c_{r+2n} \frac{x_2^{\frac{1}{2}r} - x_1^{\frac{1}{2}r}}{\frac{1}{2}r}\right)$$

and

$$x(1-x)\, P\int_{x_1}^{x_2} \frac{g(\xi)d\xi}{\xi(1-\xi)(\xi-x)} = g(x) \log_e \left|\frac{x-x_2}{x-x_1}\right|$$
$$- 2\left(\sum_{n=0}^{M} c_{2n-1} x^{n-\frac{1}{2}}\right) \log_e \frac{x_2^{\frac{1}{2}}+x^{\frac{1}{2}}}{x_1^{\frac{1}{2}}+x^{\frac{1}{2}}}$$
$$+ x\left\{2\left(\sum_{n=0}^{M} c_{2n-1}\right) \log_e \frac{1+x_2^{\frac{1}{2}}}{1+x_1^{\frac{1}{2}}} - g(1) \log_e \frac{1-x_2}{1-x_1}\right\}$$
$$- (1-x)\left\{2c_{-1}(x_1^{-\frac{1}{2}} - x_2^{-\frac{1}{2}}) + c_0 \log_e \frac{x_2}{x_1}\right\} - x(1-x) \sum_{n=0}^{M-3} C_n x^n,$$

where

$$C_n = \sum_{r=1}^{M-n-2} \frac{x_2^r - x_1^r}{r} \sum_{s=r+n+2}^{M} c_{2s} + \sum_{r=1}^{M-n-2} \frac{x_2^{r-\frac{1}{2}} - x_1^{r-\frac{1}{2}}}{r-\frac{1}{2}} \sum_{s=r+n+2}^{M} c_{2s-1}$$
$$(0 \leqslant n \leqslant M-3)$$

and, if $C_{M-2} = 0$, then

$$C_{n-1} - C_n = \sum_{r=1}^{2M-2n-2} c_{r+2n+2} \frac{x_2^{\frac{1}{2}r} - x_1^{\frac{1}{2}r}}{\frac{1}{2}r} \qquad (1 \leqslant n \leqslant M-2).$$

Also/

Also

$$\int_{x_1}^{x_2} \frac{g(\xi)d\xi}{\xi(1-\xi)} = \int_{x_1}^{x_2} g(\xi)\left\{\frac{1}{\xi} + \frac{1}{1-\xi}\right\} d\xi$$

$$= - g(1) \log_e \frac{1-x_2}{1-x_1} + 2(\sum_{n=0}^{M} c_{2n-1})\log_e \frac{1+x_2^{\frac{1}{2}}}{1+x_1^{\frac{1}{2}}} + c_0 \log_e \frac{x_2}{x_1}$$

$$+ 2c_{-1}(x_1^{-\frac{1}{2}} - x_2^{-\frac{1}{2}})$$

$$- \sum_{r=1}^{M-1} \frac{x_2^r - x_1^r}{r} \sum_{s=r+1}^{M} c_{2s} - \sum_{r=1}^{M-1} \frac{x_2^{r-\frac{1}{2}} - x_1^{r-\frac{1}{2}}}{r-\frac{1}{2}} \sum_{s=r+1}^{M} c_{2s-1}.$$

{Note that if $g(\xi) = \sum_{n=-1}^{2M-1} c_n \xi^{n/2}$, we have the same formulae with $c_{2M} = 0$.}

Lemma 13.

If $$g(x) = \sum_{n=-1}^{2M} c_n x^{n/2}$$

(as in Lemma (12)), then

$$\frac{g(x)}{x-x_1} - \frac{\sum_{n=0}^{M} c_{2n-1} x^{n-1}}{x_1^{\frac{1}{2}} + x^{\frac{1}{2}}} = \frac{1}{x-x_1}\{g(x) - (x^{\frac{1}{2}} - x_1^{\frac{1}{2}}) \sum_{n=0}^{M} c_{2n-1} x^{n-1}\}$$

$$= \frac{c_{-1} x_1^{\frac{1}{2}}}{x(x-x_1)} + \frac{x_1^{\frac{1}{2}}}{x-x_1} \sum_{n=0}^{M-1} c_{2n+1} x^n + \frac{1}{x-x_1} \sum_{n=0}^{M} c_{2n} x^n$$

$$= \frac{g(x_1)}{x-x_1} - \frac{c_{-1} x_1^{-\frac{1}{2}}}{x} + \sum_{n=0}^{M-1} x^n \sum_{r=0}^{2M-2n-2} c_{r+2n+2}\, x_1^{\frac{1}{2}r}$$

(by the use of the first formula of Lemma (9)).

Lemma 14.

If $x = \frac{1}{2}(1 - \cos\theta) = \sin^2 \frac{1}{2}\theta$,

so that $$\frac{d}{d\theta} = x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} \frac{d}{dx},$$

and if $$J = x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\, P\int_{x_1}^{x_2} \frac{g(\xi)d\xi}{\xi(1-\xi)(\xi - x)},$$

where $$g(\xi) = \sum_{n=-1}^{2M} c_n \xi^{n/2},$$

as/

as in Lemmas (12) and (13), then if we use the result of Lemma (12), differentiate straightforwardly, and use Lemma (13) and the value of $C_{n-1} - C_n$ from Lemma (12), we find after some rearrangement that

$$\frac{dJ}{d\theta} = \left\{g'(x) - \frac{1-2x}{2x(1-x)}\,g(x)\right\} \log_e \left|\frac{x-x_2}{x-x_1}\right|$$

$$- 2\left\{\sum_{n=0}^{M} (n-\tfrac{1}{2})c_{2n-1}x^{n-3/2} - \frac{1-2x}{2x(1-x)}\sum_{n=0}^{M} c_{2n-1}x^{n-\frac{1}{2}}\right\} \log_e \frac{x_2^{\frac{1}{2}}+x^{\frac{1}{2}}}{x_1^{\frac{1}{2}}+x^{\frac{1}{2}}}$$

$$+ \frac{1}{2(1-x)}\left\{2\Big(\sum_{n=0}^{M} c_{2n-1}\Big)\log_e \frac{1+x_2^{\frac{1}{2}}}{1+x_1^{\frac{1}{2}}} - g(1)\log_e \frac{1-x_2}{1-x_1}\right\}$$

$$+ \frac{1}{x}\left\{\frac{c_0}{2}\log_e \frac{x_2}{x_1} + 2c_{-1}(x_1^{-\frac{1}{2}} - x_2^{-\frac{1}{2}})\right\} + \sum_{n=0}^{M-2} \gamma_n x^n$$

$$+ \frac{g(x_2)}{x-x_2} - \frac{g(x_1)}{x-x_1},$$

where

$$\gamma_n = \sum_{r=1}^{2M-2n-2} \frac{r+2n}{r}\, c_{r+2n+2}\,(x_2^{\frac{1}{2}r} - x_1^{\frac{1}{2}r}) - \tfrac{1}{2}C_n \qquad (0 \leqslant n \leqslant m-2)$$

and C_n has the value given in Lemma (12), with $C_{m-2} = 0$.

Lemma 15.

Let
$$\tau = \left(\frac{B\xi}{A - B\xi}\right)^{\frac{1}{2}}$$

and, for $x < A/B$, let

$$X = \left(\frac{Bx}{A - Bx}\right)^{\frac{1}{2}}.$$

Then for $x < A/B$,

$$\int \frac{d\tau}{Bx-(A-Bx)\tau^2} = \frac{1}{\sqrt{B}(Ax-Bx^2)^{\frac{1}{2}}} \tanh^{-1}\frac{\tau}{X}$$

$$= \frac{1}{2\sqrt{B}(Ax-Bx^2)^{\frac{1}{2}}} \log_e \frac{X+\tau}{X-\tau}$$

$$= \frac{1}{2\sqrt{B}(Ax-Bx^2)^{\frac{1}{2}}} \log_e \frac{X^2+\tau^2+2X\tau}{X^2-\tau^2}$$

$$= \frac{1}{2\sqrt{B}(Ax-Bx^2)^{\frac{1}{2}}} \log_e \frac{A(x+\xi)-2Bx\xi+2(Ax-Bx^2)^{\frac{1}{2}}(A\xi-B\xi^2)^{\frac{1}{2}}}{A(x-\xi)}.$$

Similarly/

Similarly

$$\int \frac{d\tau}{Bx - (A-Bx)\tau^2} = \frac{1}{\sqrt{B}(Bx^2 - Ax)^{\frac{1}{2}}} \tan^{-1}\left(\frac{B\xi x - A\xi}{Ax - B\xi x}\right)^{\frac{1}{2}} \quad \text{for} \quad x > A/B$$

$$= \frac{1}{Bx}\left(\frac{B\xi}{A - B\xi}\right)^{\frac{1}{2}} \qquad \text{for} \quad x = A/B.$$

Lemma 16.

$$\text{Let } f(\xi) = (A\xi - B\xi^2)^{\frac{1}{2}} \qquad 0 \leqslant \xi \leqslant A/B$$

$$= (B\xi^2 - A\xi)^{\frac{1}{2}} \qquad \xi \geqslant A/B,$$

and let x_1 be $< A/B$. Then

$$P\int_0^{x_1} \frac{f'(\xi)d\xi}{\xi - x} = P\int_0^{x_1} \frac{A - 2B\xi}{2(A\xi - B\xi^2)^{\frac{1}{2}}} \frac{d\xi}{\xi - x}$$

$$= -\sqrt{B}\, P\int_0^{\tau_1} \left\{\frac{2}{1+\tau^2} + \frac{A - 2Bx}{Bx - (A-Bx)\tau^2}\right\} d\tau,$$

where $\xi = \dfrac{A}{B}\dfrac{\tau^2}{1+\tau^2}$ (i.e., $\xi = \dfrac{A}{B}\sin^2 \tfrac{1}{2}t$ where $\tau = \tan \tfrac{1}{2}t$),

$$\tau = \left(\frac{B\xi}{A - B\xi}\right)^{\frac{1}{2}}; \qquad \tau_1 = \left(\frac{Bx_1}{A - Bx_1}\right)^{\frac{1}{2}}.$$

Since

$$\int_0^{\tau_1} \frac{2d\tau}{1 + \tau^2} = 2\tan^{-1}\tau_1 = 2\sin^{-1}(Bx_1/A)^{\frac{1}{2}},$$

it now follows from Lemma (15) that

$$P\int_0^{x_1} \frac{f'(\xi)d\xi}{\xi - x} = -2\sqrt{B}\sin^{-1}(Bx_1/A)^{\frac{1}{2}} - f'(x)\log_e \frac{x(A-2Bx_1)+Ax_1+2f(x_1)f(x}{A|x - x_1|}$$

$$\text{for} \quad x < A/B,$$

$$= -2\sqrt{B}\sin^{-1}(Bx_1/A)^{\frac{1}{2}} + \frac{Bx_1}{f(x_1)} \qquad \text{for} \quad x = A/B$$

$$= -2\sqrt{B}\sin^{-1}(Bx_1/A)^{\frac{1}{2}} + 2f'(x)\tan^{-1}\left(\frac{Bx_1x - Ax_1}{Ax - Bx_1x}\right)^{\frac{1}{2}}$$

$$\text{for} \quad x > A/B,$$

where

$$f'(x) = \frac{A - 2Bx}{2f(x)} \qquad \text{for} \quad x < A/B$$

$$= \frac{2Bx - A}{2f(x)} \qquad \text{for} \quad x > A/B.$$

Lemma/

Lemma 17.

Let $J = x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} P\int_0^{x_1} \frac{f(\xi)d\xi}{\xi(1-\xi)(\xi-x)}$,

where $f(\xi)$ has the same values as in Lemma (16) and $x_1 < A/B$. Then

$$J = \frac{1}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} P\int_0^{x_1}\left(\frac{1}{\xi-x} + \frac{x}{1-\xi} - \frac{1-x}{\xi}\right) f(\xi)d\xi.$$

By methods similar to those used in Lemma (16) we find that

$$P\int_0^{x_1} \frac{f(\xi)d\xi}{\xi - x} = P\int_0^{x_1} \frac{(A\xi-B\xi^2)^{\frac{1}{2}}}{\xi - x}\, d\xi$$

$$= f(x_1) + \frac{1}{\sqrt{B}}(A-2Bx)\sin^{-1}\left(\frac{Bx_1}{A}\right)^{\frac{1}{2}} - f(x)\log_e \frac{x(A-2Bx_1)+Ax_1+2f(x_1)f(x)}{A|x - x_1|}$$

for $x \leqslant A/B$,

$$= f(x_1) + \frac{1}{\sqrt{B}}(A-2Bx)\sin^{-1}\left(\frac{Bx_1}{A}\right)^{\frac{1}{2}} + 2f(x)\tan^{-1}\left(\frac{Bx_1x-Ax_1}{Ax-Bx_1x}\right)^{\frac{1}{2}}$$

for $x \geqslant A/B$.

Similarly

$$\int_0^{x_1} \frac{f(\xi)d\xi}{1 - \xi} = -f(x_1) - \frac{A-2B}{\sqrt{B}}\sin^{-1}\left(\frac{Bx_1}{A}\right)^{\frac{1}{2}}$$

$$+(A-B)^{\frac{1}{2}}\log_e \frac{A(1+x_1)-2Bx_1+2f(x_1)(A-B)^{\frac{1}{2}}}{A(1 - x_1)} \qquad (A \geqslant B)$$

$$= -f(x_1) - \frac{A-2B}{\sqrt{B}}\sin^{-1}\left(\frac{Bx_1}{A}\right)^{\frac{1}{2}} - 2(B-A)^{\frac{1}{2}}\tan^{-1}\left(\frac{Bx_1-Ax_1}{A - Bx_1}\right)^{\frac{1}{2}} \qquad (A \leqslant B);$$

and

$$\int_0^{x_1} \frac{f(\xi)d\xi}{\xi} = f(x_1) + \frac{A}{\sqrt{B}}\sin^{-1}\left(\frac{Bx_1}{A}\right)^{\frac{1}{2}}.$$

Hence for $A \geqslant B$, $x < A/B$,

$$J = \left(\frac{x}{1-x}\right)^{\frac{1}{2}}(A-B)^{\frac{1}{2}}\log_e \frac{A(1+x_1)-2Bx_1+2f(x_1)(A-B)^{\frac{1}{2}}}{A(1 - x_1)}$$

$$- \frac{f(x)}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}\log_e \frac{x(A-2Bx_1)+Ax_1+2f(x_1)f(x)}{A|x - x_1|},$$

and/

Table 1.

$\frac{2\theta}{\pi}$	$x = \sin^2\frac{1}{2}\theta$	x^2	x^3	x^4	x^5	$x^{\frac{1}{2}} = \sin\frac{\theta}{2}$	$x^{3/2}$	$x^{5/2}$	$x^{7/2}$	$x^{9/2}$	
0	0	0	0	0	0	0	0	0	0	0	20
1	0.00615583	0.00003789	0.00000023	0.00000000	0.00000000	0.07845910	0.00048298	0.00000297	0.00000002	0.00000000	19
2	0.02447174	0.00059887	0.00001465_5	0.00000036	0.00000001	0.15643446_5	0.00382822	0.00009368	0.00000229	0.00000006	18
3	0.05449674	0.00296989	0.00016185	0.00000882	0.00000048	0.23344536	0.01272201	0.00069331	0.00003778	0.00000206	17
4	0.09549150	0.00911863	0.00087075	0.00008315	0.00000794	0.30901699	0.02950850	0.00281781	0.00026908	0.00002569_5	16
5	0.14644661	0.02144661	0.00314078	0.00045995	0.00006736	0.38268343	0.05604269	0.00820726	0.00120193	0.00017602	15
6	0.20610737	0.04248025	0.00875549	0.00180457	0.00037194	0.45399050	0.09357079	0.01928563	0.00397491	0.00081926	14
7	0.27300475	0.07453159	0.02034748	0.00555496	0.00151653	0.52249856	0.14264459	0.03894265	0.01063153	0.00290246	13
8	0.34549150	0.11936438	0.04123938	0.01424785_5	0.00492251	0.58778525	0.20307481	0.07016062	0.02423990	0.00837468	12
9	0.42178277	0.17790070	0.07503545	0.03164866	0.01334886	0.64944805	0.27392599_5	0.11553726	0.04873163	0.02055416	11
10	0.50000000	0.25000000	0.12500000	0.06250000	0.03125000	0.70710678	0.35355339	0.17677670	0.08838835	0.04419417	10
11	0.57821723	0.33433517	0.19331836	0.11178000_5	0.06463312_5	0.76040597	0.43967983	0.25423046	0.14700043	0.08499818	9
12	0.65450850	0.42838137	0.28037925	0.18351060	0.12010925	0.80901699	0.52950850	0.34656781	0.22683158	0.14846319_5	8
13	0.72699525	0.52852209	0.38423305	0.27933560	0.20307566	0.85264016	0.61986535	0.45063916	0.32761253	0.23817275_5	7
14	0.79389263	0.63026550	0.50036313_5	0.39723460	0.31536162	0.89100652	0.70736351	0.56157067	0.44582682	0.35393862	6
15	0.85355339	0.72855339	0.62185922	0.53079004	0.45305764	0.92387953	0.78858051	0.67309557	0.57452300	0.49038606	5
16	0.90450850	0.81813562	0.74001062	0.66934590	0.60542905	0.95105652	0.86023870	0.77809321	0.70379192	0.63658578	4
17	0.94550326	0.89397642	0.84525762	0.79919384	0.75564038	0.97236992	0.91937893	0.86927578	0.82190308_5	0.77711205	3
18	0.97552826	0.95165538	0.92836672	0.90564797	0.88348518	0.98768834	0.96351789	0.93993892_5	0.91693698	0.89449794	2
19	0.99384417	0.98772623_5	0.98164596	0.97560312	0.96959747	0.99691733	0.99078048	0.98468140	0.97861987	0.97259566	1
20	$\frac{1}{1-x}$	$\frac{1}{(1-x)^2}$	$\frac{1}{(1-x)^3}$	$\frac{1}{(1-x)^4}$	$\frac{1}{(1-x)^5}$	$\frac{1}{(1-x)^{\frac{1}{2}}}$	$\frac{1}{(1-x)^{3/2}}$	$\frac{1}{(1-x)^{5/2}}$	$\frac{1}{(1-x)^{7/2}}$	$\frac{1}{(1-x)^{9/2}}$	0
											$\frac{2\theta}{\pi}$

Table 1/

Table 1 (Contd.)

$\frac{20\theta}{\pi}$	$x^{-\frac{1}{2}} = \text{cosec}\frac{1}{2}\theta$	x^{-1}	$x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} = 2\text{cosec}\,\theta$	$\left(\frac{x}{1-x}\right)^{\frac{1}{2}} = \tan\frac{1}{2}\theta$	$\frac{1-2x}{2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} = \cot\theta$	$x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} = \frac{1}{2}\sin\theta$	$x^{3/2}(1-x)^{\frac{1}{2}}$	$\frac{1-2x}{2x(1-x)} = \frac{2\cos\theta}{\sin^2\theta}$	
0	∞	∞	∞	0	∞	0	0	∞	20
1	12.7454948	162.4476389	12.7849064	0.0787017	6.3137515	0.0782172	0.0004815	80.720722	19
2	6.3924532	40.8634582	6.4721360	0.1583844	3.0776835	0.1545085	0.0037811	19.919186	18
3	4.2836576	18.3497222	4.4053785	0.2400788	1.9626105	0.2269952	0.0123705	8.646042	17
4	3.2360680	10.4721360	3.4026032	0.3249197	1.3763819	0.2938926	0.0280642	4.683282	16
5	2.6131259	6.8284271	2.8284271	0.4142136	1.0000000	0.3535534	0.0517767	2.828427	15
6	2.2026893	4.8518400	2.4721360	0.5095254	0.7265425	0.4045085	0.0833722	1.796112	14
7	1.9138809	3.6629399	2.2446525	0.6128008	0.5095254	0.4455033	0.1216245	1.143708	13
8	1.7013016	2.8944272	2.1029244	0.7265425	0.3249197	0.4755283	0.1642910	0.683282	12
9	1.5397690	2.3708887	2.0249303	0.8540807	0.1583844	0.4938442	0.2082950	0.320717	11
10	1.4142136	2.0000000	2	1	0	0.5	0.25	0	10
11	1.3150870	1.7294538	2.0249303	1.1708496	-0.1583844	0.4938442	0.2855492	-0.320717	9
12	1.2360680	1.5278640	2.1029244	1.3763819	-0.3249197	0.4755283	0.3112373	-0.683282	8
13	1.1728277	1.3755248	2.2446525	1.6318517	-0.5095254	0.4455033	0.3238788	-1.143708	7
14	1.1223262	1.2596162	2.4721360	1.9626105	-0.7265425	0.4045085	0.3211363	-1.796112	6
15	1.0823922	1.1715729	2.8284271	2.4142136	-1.0000000	0.3535534	0.3017767	-2.828427	5
16	1.0514622	1.1055728	3.4026032	3.0776835	-1.3763819	0.2938926	0.2658284	-4.683282	4
17	1.0284152	1.0576378	4.4053785	4.1652998	-1.9626105	0.2269952	0.2146247	-8.646042	3
18	1.0124651	1.0250856	6.4721360	6.3137515	-3.0776835	0.1545085	0.1507274	-19.919186	2
19	1.0030922	1.0061940	12.7849064	12.7062047	-6.3137515	0.0782172	0.0777357	-80.720722	1
20	1	1	∞	∞	-∞	0	0	-∞	0
	$[(1-x)^{-\frac{1}{2}} = \sec\frac{1}{2}\theta]$	$(1-x)^{-1}$	$x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}$	$\left(\frac{1-x}{x}\right)^{\frac{1}{2}} = \cot\frac{1}{2}\theta$	$-\frac{1-2x}{2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} = -\cot\theta$	$x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} = \frac{1}{2}\sin\theta$	$x^{\frac{1}{2}}(1-x)^{3/2}$	$-\frac{1-2x}{2x(1-x)} = -\frac{2\cos\theta}{\sin^2\theta}$	$\frac{20\theta}{\pi}$

Table 1/

<u>Table 1 (Contd.)</u>

$\frac{20\theta}{\pi}$	$\frac{x^{\frac{1}{2}}}{1-x}$	$\sin\theta = 2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$	$\sin^2\theta = 4x(1-x)$	$\frac{1}{2\pi}\cot\frac{1}{2}\theta$	$\frac{1}{\pi}\log_e x^{\frac{1}{2}}$	$\frac{1}{\pi}\log_e x$	$\frac{1}{\pi}\log_e\frac{1-x}{x}$	
0	0	0	0	∞	$-\infty$	$-\infty$	$+\infty$	20
1	0.0789451	0.1564345	0.0244717	2.0222552	-0.8101553	-1.6203106	1.6183451	19
2	0.1603587	0.3090170	0.0954915	1.0048647	-0.5905025	-1.1810049	1.1731184	18
3	0.2469006	0.4539905	0.2061074	0.6629280	-0.4630795	-0.9261590	0.9083215	17
4	0.3416408	0.5877853	0.3454915	0.4898285	-0.3738101	-0.7476202	0.7156735	16
5	0.4483415	0.7071068	0.5	0.3842340	-0.3057516	-0.6115033	0.5610998	15
6	0.5718538	0.8090170	0.6545085	0.3123592	-0.2513626	-0.5027252	0.4292570	14
7	0.7187097	0.8910065	0.7938926	0.2597173	-0.2066255	-0.4132509	0.3117624	13
8	0.8980560	0.9510565	0.9045085	0.2190580	-0.1691478	-0.3382957	0.2033735	12
9	1.1231904	0.9876883	0.9755283	0.1863465	-0.1373929	-0.2747857	0.1004137	11
10	1.4142136	1	1	0.1591549	-0.1103178	-0.2206356	0	10
11	1.8028379	0.9876883	0.9755283	0.1359312	-0.0871860	-0.1743720	-0.1004137	9
12	2.3416408	0.9510565	0.9045085	0.1156328	-0.0674611	-0.1349222	-0.2033735	8
13	3.1231697	0.8910065	0.7938926	0.0975303	-0.0507442	-0.1014885	-0.3117624	7
14	4.3230211	0.8090170	0.6545085	0.0810935	-0.0367341	-0.0734682	-0.4292570	6
15	6.3086441	0.7071068	0.5	0.0659241	-0.0252017	-0.0504035	-0.5610998	5
16	9.9595931	0.5877853	0.3454915	0.0517126	-0.0159734	-0.0319467	-0.7156735	4
17	17.8427179	0.4539905	0.2061074	0.0382097	-0.0089187	-0.0178375	-0.9083215	3
18	40.3603612	0.3090170	0.0954915	0.0252077	-0.0039432	-0.0078865	-1.1731184	2
19	161.9468670	0.1564345	0.0244717	0.0125258	-0.0009827	-0.0019655	-1.6183451	1
20	∞	0	0	0	0	0	$-\infty$	0
	$\frac{(1-x)^{\frac{1}{2}}}{x}$	$\sin\theta = 2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$	$\sin^2\theta = 4x(1-x)$	$\frac{1}{2\pi}\tan\frac{1}{2}\theta$	$\frac{1}{\pi}\log_e(1-x)^{\frac{1}{2}}$	$\frac{1}{\pi}\log_e(1-x)$	$\frac{1}{\pi}\log_e\frac{x}{1-x}$	$\frac{20\theta}{\pi}$

Table 1/

Table 1 (Contd.)

$\frac{20\,\theta}{\pi}$	$\frac{1}{\pi}\log_e(1+x^{\frac{1}{2}})$	$\frac{1}{\pi}\log_e\frac{1+x^{\frac{1}{2}}}{x^{\frac{1}{2}}}$	$\frac{1}{\pi}\log_e\lvert x-0.5\rvert$	$\frac{1}{\pi}\log_e\frac{\lvert x-0.5\rvert}{x}$	$\frac{1}{\pi}\log_e(x^{\frac{1}{2}}+0.70710678)$	$\frac{1}{\pi}\log_e\frac{x^{\frac{1}{2}}+0.70710678}{x^{\frac{1}{2}}}$	$\frac{1}{\pi}\log_e\lvert x-0.6\rvert$	
0	0	∞	-0.2206356	∞	-0.1103178	∞	-0.1626008	20
1	0.0240430	0.8341983	-0.2245788	1.3957318	-0.0768244	0.7333309	-0.1658835	19
2	0.0462636	0.6367661	-0.2366090	0.9443959	-0.0467004	0.5438021	-0.1758557	18
3	0.0667850	0.5298645	-0.2573697	0.6687893	-0.0195086	0.4435709	-0.1929106	17
4	0.0857133	0.4595234	-0.2880967	0.4595235	0.0050914	0.3789015	-0.2177783	16
5	0.1031407	0.4088923	-0.3309534	0.2805499	0.0273699	0.3331215	-0.2516693	15
6	0.1191472	0.3705098	-0.3897834	0.1129418	0.0475445	0.2989071	-0.2965619	14
7	0.1338025	0.3404280	-0.4719982	-0.0587473	0.0657926	0.2724181	-0.3558098	13
8	0.1471675	0.3163153	-0.5944457	-0.2561500	0.0822599	0.2514077	-0.4355819	12
9	0.1592952	0.2966881	-0.8111380	-0.5363523	0.0970680	0.2344609	-0.5490056	11
10	0.1702321	0.2805499	-∞	-∞	0.1103178	0.2206356	-0.7329356	10
11	0.1800184	0.2672044	-0.8111380	-0.6367660	0.1220938	0.2092798	-1.2180561	9
12	0.1886889	0.2561500	-0.5944457	-0.4595235	0.1321668	0.1999279	-0.9260904	8
13	0.1962736	0.2470178	-0.4719982	-0.3705097	0.1414963	0.1922405	-0.6568660	7
14	0.2027981	0.2395322	-0.3897834	-0.3163152	0.1492313	0.1859654	-0.5221717	6
15	0.2082841	0.2334858	-0.3309534	-0.2805499	0.1557124	0.1809141	-0.4367788	5
16	0.2127491	0.2287225	-0.2880967	-0.2561500	0.1609727	0.1769461	-0.3784883	4
17	0.2162075	0.2251262	-0.2573697	-0.2395322	0.1650380	0.1739567	-0.3382848	3
18	0.2186701	0.2226133	-0.2366090	-0.2287225	0.1679282	0.1718714	-0.3117596	2
19	0.2201446	0.2211273	-0.2245788	-0.2226133	0.1696568	0.1706395	-0.2966012	1
20	0.2206356	0.2206356	-0.2206356	-0.2206356	0.1702321	0.1702321	-0.2916644	0
	$\frac{1}{\pi}\log_e\left[1+(1-x)^{\frac{1}{2}}\right]$	$\frac{1}{\pi}\log_e\frac{1+(1-x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}$	$\frac{1}{\pi}\log_e\lvert x-0.5\rvert$	$\frac{1}{\pi}\log_e\frac{\lvert x-0.5\rvert}{1-x}$	$\frac{1}{\pi}\log_e\left[(1-x)^{\frac{1}{2}}+0.70710678\right]$	$\frac{1}{\pi}\log_e\frac{(1-x)^{\frac{1}{2}}+0.70710678}{(1-x)^{\frac{1}{2}}}$	$\frac{1}{\pi}\log_e\lvert x-0.4\rvert$	$\frac{20\,\theta}{\pi}$

Table 2/

Table 3 (Contd.)

EQH 1260. Approximate Theory. Computation of g_S and of $g_S + g_L$.

$\frac{20\theta}{\pi}$	x	$y_4'(x)$	$y_2'(x)$	$y_3'(x)$	(5)	(6)	(7)	(8)	$1 + g_S$	$1+g_S+g_L$ for $C_L=0.4$, $\frac{C_L}{\sin\alpha}=4.4$	
										Upper Surface	Lower Surface
11	0.5782	0.00363	+0.00153	−0.28217	1.27599	0.92463	0.77520	−0.09462	1.1273	1.1774	1.0773
12	0.6545	−0.00912	−0.02063	−0.28218	0.98347	0.56489	0.78028	−0.06999	1.1529	1.1903	1.1155
13	0.7270	−0.02166	−0.07548	−0.28220	0.71127	0.21434	0.78784	−0.05581	1.1673	1.1924	1.1422
14	0.7939	−0.03415	−0.14095	−0.28224	0.57155	−0.01993	0.79977	−0.05070	1.1527	1.1653	1.1401
15	0.8536	−0.04663	−0.20062	−0.28233	0.47954	−0.23166	0.82031	−0.05261	1.1102	1.1093	1.1110
16	0.9045	−0.05890	−0.24538	−0.28256	0.41370	−0.46120	0.86084	−0.05906	1.0466_5	1.0298	1.0635
17	0.9455	−0.07043	−0.27317	−0.28334	0.36587	−0.77250	0.96460	−0.06747	0.9690_5	0.9309	1.0072
18	0.9755	−0.08025	−0.28700	−0.28701	0.33269	−2.11591	2.05568	−0.07546	0.8806	0.8068	0.9543
19	0.9938	−0.08701	−0.29217	−0.31994	0.31293	−0.98522	0.58273	−0.08109	0.7903	0.6232_5	0.9573
20	1	−0.08944	−0.29329	$\frac{-0.014781}{(1-x)^{\frac{1}{2}}}+\ldots = -\infty$	0.30635	−0.89574	$7.336048(1-x)^{\frac{1}{2}}+\ldots = 0$	−0.08311	0.7352	$-\infty$	∞

$$\text{Column (5)} = \frac{1}{\pi}\log_e \frac{10y_4(x) + 0.6}{|x - 0.6|}. \qquad \text{Column (6)} = \frac{1}{\pi}\log_e \left|\frac{x - 0.9760155}{x - 0.6}\right|.$$

$$\text{Column (7)} = \frac{1}{\pi}\log_e \frac{0.0239845 + 5.3697814(1-x) + 18.698907\, y_3(x)}{|x - 0.9760155|}.$$

$$\text{Column (8)} = -0.6051406 + 1.3776462x - 0.8556133x^2.$$

Table 4/

Table 4.

EQH 1260. q/U calculated from the formula

$$q/U = \frac{(1 + \frac{1}{2}C_o^{2})\,|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}\,(1 + g_S \pm g_L)$$

$$= \frac{(1 + \frac{1}{2}C_o^{2})\,|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}\,(1 + g_S) \pm \frac{C_L(1 + \frac{1}{2}C_o^{2})}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}\left(\frac{1}{2\pi} + \frac{\cos\theta}{a_o}\right).$$

$\frac{20\theta}{\pi}$	q/U for $C_L = 0$	q/U for $C_L = 0.4$, $a_o = C_L/\sin\alpha = 4.4$	
		Upper Surface	Lower Surface
0	0	1.4185	(-) 1.4185
1	0.9088	1.7164	0.1012
2	1.0458	1.5060	0.5856
3	1.0788	1.3901	0.7675
4	1.0914	1.3220	0.8608
5	1.0976	1.2773	0.9179
6	1.1015	1.2456	0.9574
7	1.1044	1.2219	0.9870
8	1.1072	1.2035	1.0109
9	1.1105	1.1893	1.0318
10	1.1155	1.1790	1.0519
11	1.1248	1.1748	1.0749
12	1.1490$_5$	1.1863	1.1118
13	1.1619	1.1869	1.1369
14	1.1460	1.1586	1.1334$_5$
15	1.1028	1.1019$_5$	1.1037
16	1.0392	1.0225	1.0559
17	0.9616	0.9237	0.9995
18	0.8721	0.7990$_5$	0.9452
19	0.7731	0.6097	0.9364$_5$
20	0	(-) 0.9266	0.9266

$C_o = 0.10277.$

$1 + \frac{1}{2}C_o^2 = 1.00528.$

Theoretical value of $C_L/\sin\alpha = 2\pi e^{C_o} = 6.9633.$

$\psi = \psi_4$ for $20\theta/\pi = 0,1,2,..,11.$

$= \psi_2$ for $20\theta/\pi = 12,13,..,18.$

$= \psi_3$ for $20\theta/\pi = 19,20.$

(See Table 5.)

Table 5/

Table 5.

EQH 1260. Computation of ε on Approximate Theory.

$\frac{200}{\pi}$	ψ_4	ψ_2	ψ_3	(4)	ε
0	0.109545	$\frac{0.569475}{x^{\frac{1}{2}}}+\ldots=\infty$	$\frac{0.283692}{x^{\frac{1}{2}}}+\ldots=\infty$	$\frac{0.304413}{x^{\frac{1}{2}}}+\ldots=\infty$	0
1	0.10960	7.04894	3.60477	3.84330	0.0001
2	0.10977	3.23724	1.79140	1.87387_5	0.0002
3	0.11007	1.87148	1.18230	1.19863	0.0003
4	0.11050	1.14869	0.87361	0.84948	0.0005
5	0.11110	0.71125	0.68553	0.63328	0.0007
6	0.11189	0.43805	0.55756	0.48579	0.0010
7	0.11292	0.27240	0.46389	0.37940	0.0015
8	0.11426	0.18000	0.39159	0.30003	0.0023
9	0.11601	0.13612	0.33347	0.23946_5	0.0034
10	0.11832	0.12154	0.28523	0.19236_5	0.0052
11	0.12142	0.12145	0.24409	0.15488	0.0083
12	0.12565	0.12520	0.20822	0.12404	0.0144
13	0.13163	0.12615	0.17634	0.09739	0.0250_5
14	0.14037	0.12114	0.14754	0.07284	0.0375
15	0.15381	0.10973	0.12117	0.04838	0.0482
16	0.17591	0.09321_5	0.09680	0.02169	0.0541
17	0.21610	0.07372_5	0.07425	-0.01099	0.0534
18	0.30286_5	0.05378_5	0.05378_5	-0.05942	0.0446
19	0.57870	0.03838	0.03704	-0.17073	0.0266
20	$\frac{0.044721}{(1-x)^{\frac{1}{2}}}+\ldots=\infty$	$\frac{0.0012}{(1-x)^{\frac{1}{2}}}+\ldots=\infty$	0.02956	$-\frac{0.014775}{(1-x)^{\frac{1}{2}}}+\ldots=-\infty$	0

$$\text{Column (4)} = -0.0147752\left(\frac{x}{1-x}\right)^{\frac{1}{2}} + 0.3044134\left(\frac{1-x}{x}\right)^{\frac{1}{2}} - x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}(0.3014979-0.2139033x).$$

Values of $\frac{1}{\pi}\log_e\frac{10y_4(x)+0.6}{|x-0.6|}$, etc., as in Table 3.

Table 6/

Table 6.

EQH 1260. Computation of ε' on Approximate Theory.

$\frac{20\theta}{\pi}$	ψ_4'	ψ_2'	ψ_3'	(4)	ε'
0	0	$\frac{-0.284738}{x} - 1.206962 + .. = -\infty$	$\frac{-0.141846}{x} + 0.001540 + .. = -\infty$	$\frac{-0.152207}{x} - 0.245362 + .. = -\infty$	0.0000
1	0.00072	-47.41123	-23.04175	-24.96508_5	0.00055_5
2	0.00148	-12.64634	-5.79552	-6.44191	0.0007
3	0.00231	-6.01486	-2.60202	-2.93783	0.0009
4	0.00325	-3.50325	-1.48458	-1.75337	0.0012_5
5	0.00437	-2.18221	-0.96768	-1.15779	0.0018
6	0.00574	-1.35289	-0.68725	-0.81326	0.0026
7	0.00746	-0.79144	-0.51852	-0.53889	0.0038_5
8	0.00971	-0.41088	-0.40939	-0.43156	0.0058
9	0.01273	-0.16813	-0.33498	-0.31699	0.0090
10	0.01690	-0.03324	-0.28216	-0.23299	0.0146
11	0.02286	+0.02076	-0.24351	-0.17286	0.0259
12	0.03170	+0.02005	-0.21453	-0.13246	0.0548
13	0.04541	-0.01121	-0.19235	-0.10895_5	0.0773
14	0.06784	-0.05294	-0.17505	-0.10055	0.0772
15	0.10718	-0.09089	-0.16116	-0.10700	0.0555
16	0.18322	-0.11708	-0.14933	-0.13197	0.0184
17	0.35369	-0.12847	-0.13762	-0.19181	-0.0289
18	0.85187	-0.12147	-0.12147	-0.36109	-0.0850
19	3.56677	-0.04983	-0.08605	-1.26183	-0.1446
20	$\frac{0.022361}{1-x} - 0.067082 + .. = \infty$	$\frac{0.0006}{1-x} - 0.147244 + .. = \infty$	0	$\frac{-0.007388}{1-x} - 0.062695 + .. = -\infty$	-0.1838

$$\text{Column (4)} = -\frac{0.0073876}{1-x} - \frac{0.1522067}{x} - 0.2379744 + 0.9691966x - 0.6417100x^2$$

Values of $\frac{1}{\pi}\log_e \frac{10y_4(x) + 0.6}{|x - 0.6|}$, etc., as in Table 3.

- 55 -

Table 7/

Table 7.

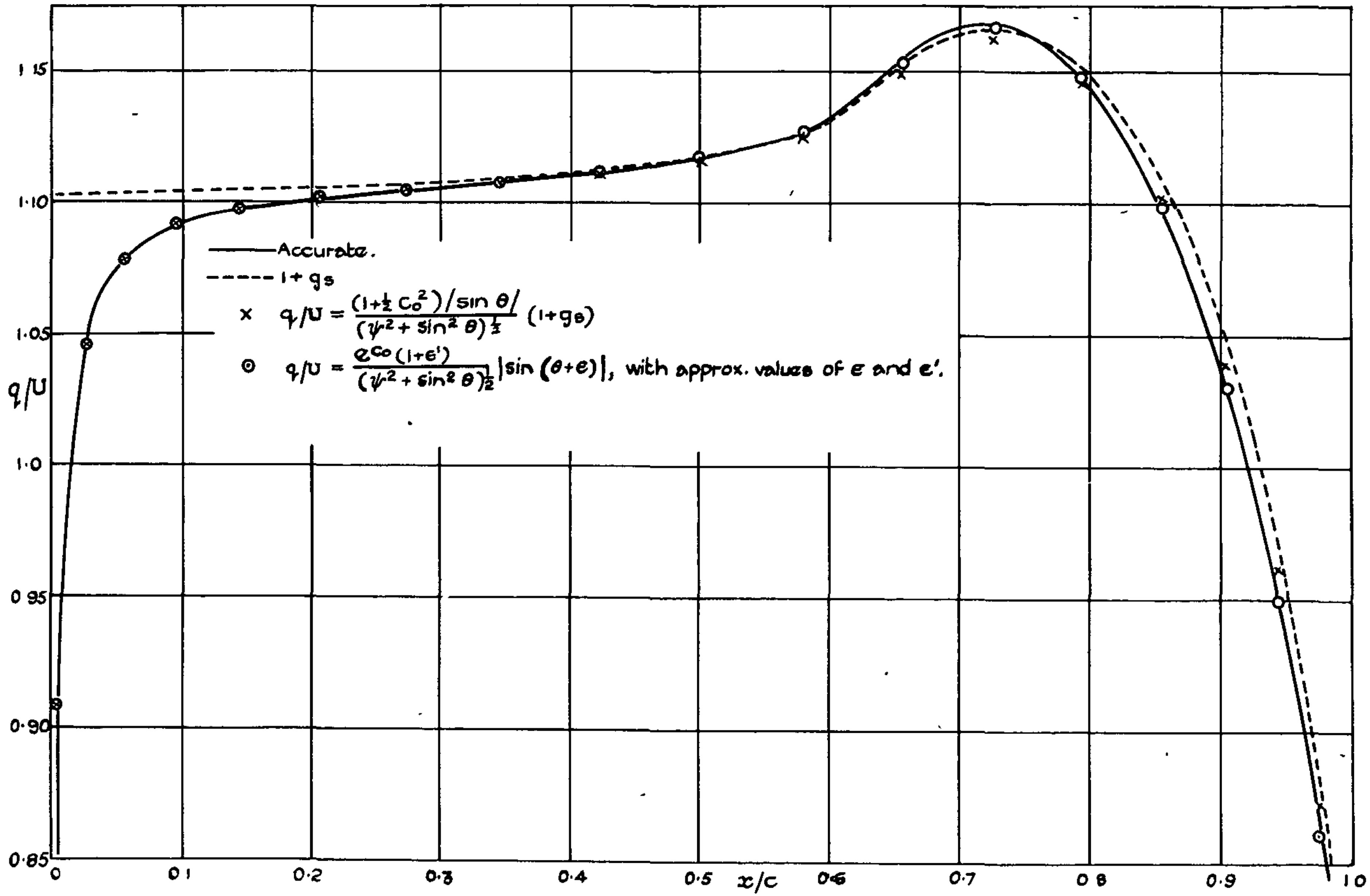
EQH 1260. q/U calculated from the formula

$$q/U = \frac{e^{C_0}(1+\varepsilon')}{(\psi^2+\sin^2\theta)^{\frac{1}{2}}} \left| \left(1-\frac{C_L^2}{a_0^2}\right)^{\frac{1}{2}} \sin(\theta+\varepsilon) + \frac{C_L}{a_0}\cos(\theta+\varepsilon) + \frac{C_L e^{-C_0}}{2\pi} \right| ,$$

with ε, ε' from Tables 5, 6. $e^{C_0} = 1.10824$.

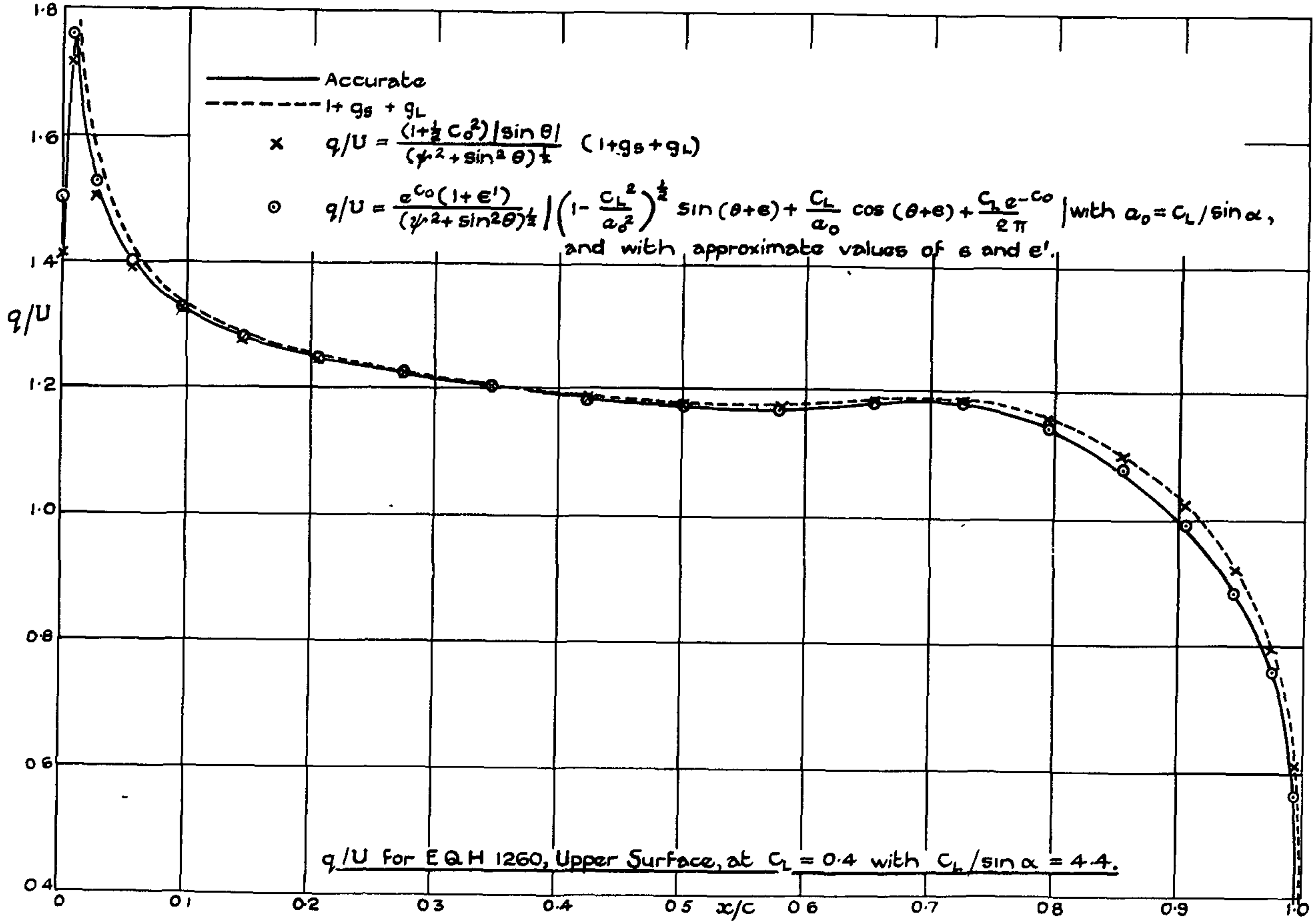
$\frac{2\theta}{\pi}$	q/U for $C_L = 0$	q/U for $C_L = 0.4$, $a_0 = C_L/\sin\alpha = 4.4$	
		Upper Surface	Lower Surface
0	0	1.5009	(-) -1.5009
1	0.9085	1.7595	0.0500_5
2	1.0456	1.5279	0.5547
3	1.0787	1.4030	0.7456
4	1.0913	1.3297	0.8438
5	1.0975	1.2817	0.9043
6	1.1014	1.2476	0.9461
7	1.1045	1.2221	0.9778
8	1.1076	1.2023	1.0037
9	1.1111	1.1868	1.0263
10	1.1166	1.1756	1.0484
11	1.1269_5	1.1708	1.0738
12	1.1534	1.1829	1.1144
13	1.1667	1.1806	1.1431
14	1.1477	1.1449	1.1411
15	1.0989	1.0782	1.1105
16	1.0299	0.9899	1.0615
17	0.9493	0.8853	1.0054
18	0.8612	0.7601	0.9553
19	0.7672	0.5712	0.9568
20	0	(-) 1.0242	1.0242

JR.



q/U for EQH 1260 at $C_L = 0$

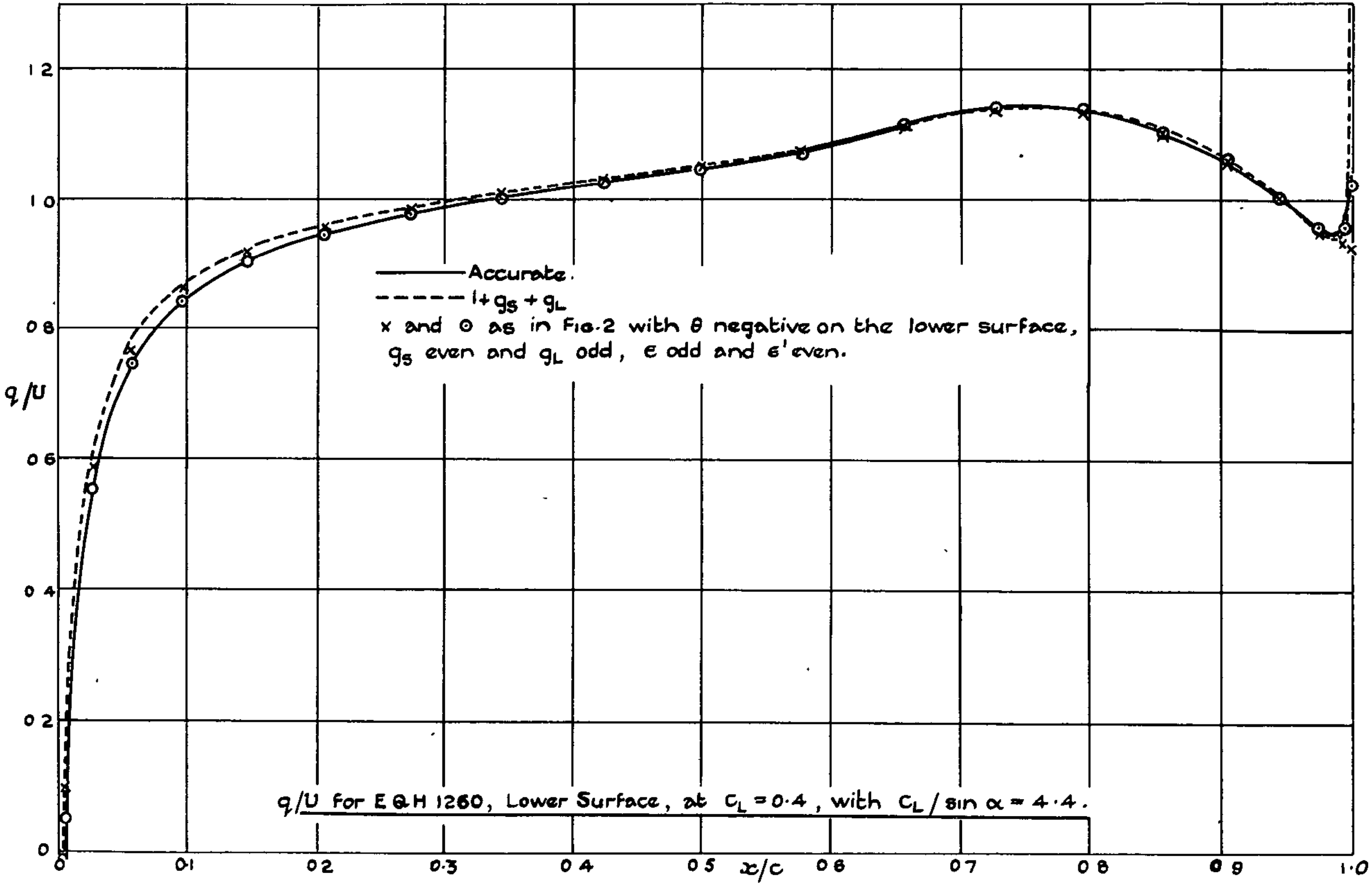
FIG. 1.



q/U for EQH 1260, Upper Surface, at $C_L = 0{\cdot}4$ with $C_L/\sin\alpha = 4{\cdot}4$.

FIG. 2.

V.S.



1 2
1 0
0 8
0 6
0 4
0 2
0
q/U
Accurate.
1+gS+gL
x and ⊙ as in Fig.2 with θ negative on the lower surface,
gS even and gL odd, ε odd and ε' even.
q/U for E Q H 1260, Lower Surface, at CL = 0·4, with CL / sin α = 4·4.
0
0·1
0 2
0·3
0 4
0 5
x/c
0 6
0 7
0 8
0 9
1·0

FIG 3.

V.S.

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