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On Theoretical Plasticity and Crack Propagation

by

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ON THEORETICAL PLASTICITY AND CRACK PROPAGATION

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E. H. Mansfield, Sc.D., F.R.Ae.S., A.F.I.A.S.

SUMMARY

The partial differential equations appropriate to certain approximate theories of plasticity are derived and discussed with particular reference to the determination of the stress distribution due to a transverse crack in a sheet under tension. Even the simplest of theories results in a differential equation whose solution will present great difficulties.

A Griffith-type analysis is also given to predict critical crack lengths under static load in terms of material properties.

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1 INTRODUCTION

The classical theory of plasticity (with work hardening) is given in detail in Ref.1. A feature of the theory is that the principal axes of the plastic strain increment (and not the total strain) coincide with the axes of principal stress. The classical theory is thus an "incremental theory" and solutions generally can only be obtained by a step-by-step integration process.

A simplification of the classical theory is possible if there is no work hardening, i.e. the material is "perfectly plastic"[2]. Solutions to a variety of simple problems are possible[3] if the elastic component of the strains can be neglected. For the crack problem this is not so, and one of the ensuing difficulties in this problem is the determination of the boundary between elastic and plastic states. The shape of the boundary is, of course, a function of the applied load.

The difficulties encountered due to the different behaviour of the elastic and plastic regions may be overcome by assuming a smooth stress strain relation for the material[4]. The difficulties inherent in an "incremental theory" are avoided if a simple deformation theory is used[5]. A deformation theory is characterised by a one-one correspondence between stress and strain, independent of the loading path. The partial differential equations resulting from various deformation theories are derived in sections 3-6. It is hoped that these will provide a springboard for future work in this field.

Griffith's work on microscopic cracks in glass[10] has recently been applied[11,12] with some success, to cracks of moderate size in steel and light alloy. A similar type of analysis is presented in sections 7-9 where physical and engineering arguments are employed in an attempt to relate the "energy of fracture" with material properties obtained from a simple tensile test. An estimate is also made of the effects of plate buckling on the critical crack length. The analysis of sections 7-9 is undoubtedly much simpler - and possibly more realistic - than the more mathematical approach introduced in sections 3-6, but further experiments are required before the results can be accepted with confidence.

2 LIST OF SYMBOLS

Sections 3-6:

x, y = Cartesian coordinates

E = Young's modulus at low stress level

ν = Poisson's ratio

$\sigma_x, \sigma_y, \tau_{xy}$ = direct and shear stresses

σ_1 = equivalent stress, defined by equation (1)

$\varepsilon_x, \varepsilon_y, \gamma_{xy}$ = direct and shear strains

E_s = secant modulus

H = E/E_s

ϕ, ψ = stress functions defined by equations (5), (17)

v = transverse displacement

κ, n = constants

F, J introduced in equations (11), (20)

X, Y introduced in section 5.2

ω = stress energy density

U_c = complementary energy

U_c' defined by equation (29)

Sections 7-9:

structure properties:

E = Young's modulus

t = thickness of sheet

ℓ = length of crack

w = width of crack

t_1 is introduced in equations (45) and (46)

ℓ_n = length of neck in tensile test (see Fig.4)

γ = angle of neck at failure (see Fig.4)

t_f = reduced thickness at neck after failure (see Fig.4)

stresses:

σ = uniformly applied stress away from crack

σ_{ult} = ultimate nominal stress of material in tensile test

σ_{fr} = nominal stress at fracture of material in tensile test (see Fig.2)

σ_{crit} = critical value of σ at which crack propagates

σ'_{crit} = value of σ_{crit} if buckling is prevented

σ_{buc} = value of σ at onset of buckling elongations

e	=	elongation
$e_{fr,t}$	=	elongation of "neck" at fracture in tensile test
$e_{fr,c}$	=	elongation of "neck" at end of crack

energies

V	=	reduction of strain energy due to crack
V_b	=	reduction of strain energy due to buckling
W	=	energy required to fracture sheet

non-dimensional parameters

K	=	$(1/t)e_{fr,t}$, (see equation (44))
μ	=	$1 - t_f/t$
λ	=	coefficient of buckling (see equation (53))
k	=	coefficient of buckling energy (see equation (57))
Ω	=	$\sigma_{buc}/\sigma'_{crit}$
Γ	=	$\sigma_{crit}/\sigma'_{crit}$
α, β	=	constants introduced prior to equation (45)

3 CRITERION FOR YIELDING

For a material with a well defined yield point in simple tension σ_1 say, the von Mises criterion for yielding under combined stresses states that yielding occurs when

$$\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2 = \sigma_1^2 . \tag{1}$$

For a material with no well defined yield point the secant modulus under combined loading may be identified with the secant modulus in tension under an equivalent stress σ_1 as defined by equation (1).

There are, of course, other yield criteria (such as the Tresca maximum shear criterion) but these require a prior knowledge of the relative magnitudes of the principal stresses; in the crack problem there will be boundaries defining regions in the plate in which the yield criterion assumes different forms, and the difficulties inherent in determining these boundaries will be comparable to those in an "incremental theory".

4 STRESS-STRAIN RELATIONS FOR SIMPLE DEFORMATION THEORY

The stress components are related to the strain components as follows:-

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) + \left(\frac{1}{E_s} - \frac{1}{E}\right)(\sigma_x - \tfrac{1}{2}\sigma_y) \\ \varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) + \left(\frac{1}{E_s} - \frac{1}{E}\right)(\sigma_y - \tfrac{1}{2}\sigma_x) \\ \gamma_{xy} &= \frac{1}{G}\tau_{xy} + 3\left(\frac{1}{E_s} - \frac{1}{E}\right)\tau_{xy} . \end{aligned}\right\} \quad (2)$$

4.1 Simplification to stress-strain relations

For the case of a circular hole in a plate under uniaxial tension[5] it has been shown that the stress distribution is independent of Poisson's ratio.

This would not, of course, have been so if the boundary conditions were mixed[6], but it does suggest a way for simplifying equation (2) for holes of arbitrary shape. The simplifying assumption is that if the boundary conditions are not mixed we may take

$$\nu = \tfrac{1}{2} ,$$

so that equation (2) reduces to:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E_s}(\sigma_x - \tfrac{1}{2}\sigma_y) \\ \varepsilon_y &= \frac{1}{E_s}(\sigma_y - \tfrac{1}{2}\sigma_x) \\ \gamma_{xy} &= \frac{3}{E_s}\tau_{xy} . \end{aligned}\right\} \quad (3)$$

Further simplification of these equations for the crack problem will be discussed later.

4.2 Equilibrium and compatibility

The equations of equilibrium

and

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} &= 0\\ \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} &= 0\end{aligned}\right\} \qquad (4)$$

are satisfied by introducing a stress function ϕ such that the stresses are derived from it by the relations

$$\left.\begin{aligned}\sigma_x &= \frac{\partial^2\phi}{\partial y^2}\\ \sigma_y &= \frac{\partial^2\phi}{\partial x^2}\\ \tau_{xy} &= -\frac{\partial^2\phi}{\partial x\partial y} .\end{aligned}\right\} \qquad (5)$$

The equation of compatibility[7] is

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y} . \qquad (6)$$

5 THE GOVERNING DIFFERENTIAL EQUATION

The governing differential equation for the stress function ϕ is obtained from equations (3), (5) and (6), and may be written in the form

$$2\nabla^2(H\nabla^2\phi) - 3\Diamond^4(H,\phi) = 0 \qquad (7)$$

where $H = E/E_s$ and, for convenience of presentation, we have introduced the invariant "die-operator" $\Diamond^4$ defined by

$$\Diamond^4(H,\phi) \equiv \frac{\partial^2 H}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - 2\frac{\partial^2 H}{\partial x\partial y}\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2 H}{\partial y^2}\frac{\partial^2\phi}{\partial x^2} .$$

Equation (7) is identical with that (Ref.14) for an elastic plate whose thickness varies as E_s.

In deformation theory H is a function of the equivalent stress σ_1, and it is therefore a function of ϕ in virtue of the relation

$$\sigma_1^2 = (\nabla^2\phi)^2 - \frac{3}{2}\,\Diamond^4(\phi,\phi) \tag{8}$$

obtained from equations (1) and (5).

5.1 Relation between secant modulus E_s and equivalent stress σ_1

The E_s, σ_1 relation for a given material cannot generally be expressed (even approximately) by a simple mathematical form. However, the approximate nature of the theory of plasticity, particularly the deformation theory, does not justify great accuracy in the mathematical formulation of this relation.

One of the earliest mathematical relations between E_s and σ_1 is due to Prager[8], who proposed

$$\frac{\sigma_1}{\sigma_{ult}} = \tanh\left(\frac{E\,\sigma_1}{E_s\,\sigma_{ult}}\right)$$

but the resultant differential equation for ϕ is extremely complicated.

A simpler differential equation is obtained by assuming

$$H = E/E_s = \exp\left(\kappa\sigma_1^{2n}\right) \tag{9}$$

where κ and n are arbitrary constants, but probably the simplest relation is that due to Ramberg and Osgood who proposed

$$H = E/E_s = 1 + \kappa\sigma_1^{2n}\,. \tag{10}$$

Typical stress-strain graphs appropriate to equations (9) and (10) are shown in Fig.1, where, for ease of comparison, the constant κ has been chosen so that all the curves pass through the point $\sigma = 1$, $E\varepsilon = 1.2$.

The governing differential equation may now be expressed solely in terms of ϕ in virtue of equations (7), (8) and (9) or (10). In this connection it is convenient to write

$$F = \kappa\left\{(\nabla^2\phi)^2 - \frac{3}{2}\,\Diamond^4(\phi,\phi)\right\}^n \tag{11}$$

whence if equation (9) is adopted we find

$$4(1-F)\nabla^4\phi + 4\nabla^2(F\nabla^2\phi) - 6(1-F)\Diamond^4(F,\phi)$$

$$+ 2\left\{\nabla^2(F^2) - 2F\nabla^2 F\right\}\nabla^2\phi - 3\Diamond^4(F^2,\phi) = 0 , \tag{12}$$

while if equation (10) is adopted:

$$2\nabla^4\phi + 2\nabla^2(F\nabla^2\phi) - 3\Diamond^4(F,\phi) = 0 . \tag{13}$$

The simplest stress function equation for any (inelastic) deformation theory is obtained by taking n = 1 in equations (11) and (13), to give

$$\frac{4}{\kappa}\nabla^4\phi + 2\nabla^2\left[\left\{2(\nabla^2\phi)^2 - 3\Diamond^4(\phi,\phi)\right\}\nabla^2\phi\right] - 3\Diamond^4\left\{\phi,\ 2(\nabla^2\phi)^2 - 3\Diamond^4(\phi,\phi)\right\} = 0 .$$

..... (14)

It need scarcely be added that the difficulties encountered in the solution of this equation (possibly by finite difference representation) are formidable.

5.2 Further simplification of the equation for the stress function

If the applied load acts in one direction only, as in the basic crack problem, a reasonable assumption to make is that the transverse strain may be neglected. If, in addition, the structure and the distribution of applied load possess symmetry about a longitudinal axis the transverse displacement will also be zero. Thus if the load is applied in the direction of the x axis,

$$v = 0 \tag{15}$$

and the equation of compatibility is

$$\frac{\partial \varepsilon_x}{\partial y} - \frac{\partial \gamma_{xy}}{\partial x} = 0 . \tag{16}$$

The single equilibrium equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

is satisfied by introducing a stress function ψ such that the stresses are derived from it by the relations

$$\left.\begin{aligned} \sigma_x &= \frac{\partial \psi}{\partial y} \\ \tau_{xy} &= -\frac{\partial \psi}{\partial x} . \end{aligned}\right\} \tag{17}$$

The stress strain relations may be taken as

$$\left.\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E_s} \\ \gamma_{xy} &= \frac{3\tau_{xy}}{E_s} \end{aligned}\right\} \tag{18}$$

and the "yield" condition may be taken as

$$\sigma_x^2 + 3\tau_{xy}^2 = \sigma_1^2 . \tag{19}$$

Now if the secant modulus is defined by equation (9) or (10) the differential equation for ψ may be determined from equations (16), (17), (18) and (19). In this connection it is convenient to refer the operators ∇^2 and ∇^4 to fresh coordinates X, Y given by

$$X = \frac{x}{\sqrt{3}} ,$$

$$Y = y ,$$

so that, in this context,

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} , \quad \text{etc.}$$

We may now write

$$\kappa\sigma_1^{2n} = J, \text{ say}$$

$$= \kappa\left\{\tfrac{1}{2}\nabla^2(\psi^2) - \psi\nabla^2\psi\right\}^n , \tag{20}$$

whence if equation (9) is adopted we find

$$(2-J)\nabla^2\psi + \nabla^2(J\psi) - \psi\nabla^2 J = 0 , \tag{21}$$

while if equation (10) is adopted:

$$(2+J)\nabla^2\psi + \nabla^2(J\psi) - \psi\nabla^2 J = 0 . \tag{22}$$

If we restrict attention to the simplest forms of these equations, by taking n = 1 in equation (20), we may recast equation (21) in the form

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If we restrict attention to the simplest forms of these equations, by taking n = 1 in equation (20), we may recast equation (21) in the form

$$\frac{1}{\kappa}\nabla^2\psi + \left\{\nabla^2(\psi^2) - 2\psi\nabla^2\psi\right\}\nabla^2\psi - \Diamond^4(\psi^2,\psi) + 2\psi\Diamond^4(\psi,\psi) = 0 , \qquad (23)$$

while equation (22) becomes

$$\frac{1}{\kappa}\nabla^2\psi + \frac{3}{2}\left\{\nabla^2(\psi^2) - 2\psi\nabla^2\psi\right\}\nabla^2\psi - \Diamond^4(\psi^2,\psi) + 2\psi\Diamond^4(\psi,\psi) = 0 . \qquad (24)$$

Although equations (23) and (24) are simpler than equation (14) they will nevertheless present considerable difficulties in their solution.

6 STRAIN ENERGY METHODS

For deformation theories, the variational principles for the stresses involve the stress-energy density, which in the plane stress case can be written as

$$\omega(\sigma_x,\sigma_y,\tau_{xy}) = \int_0^{\sigma_x,\sigma_y,\tau_{xy}} (\varepsilon_x\, d\sigma_x + \varepsilon_y\, d\sigma_y + \gamma_{xy}\, d\tau_{xy}) \qquad (25)$$

where the strains are considered as functions of the stresses; these functions are assumed to be such that the line integral in equation (25) is independent of the path. For a finite two-dimensional domain A having stresses prescribed over its boundaries, the complementary energy is defined as

$$U_c = \int_A \omega \, dA . \qquad (26)$$

The variational principle states that, when U_c is written for the true solution to the boundary-value problem,

$$\delta U_c = 0 \qquad (27)$$

for all stress variations $\delta\sigma_x, \delta\sigma_y$ and $\delta\tau_{xy}$ that satisfy equilibrium and provide no stress resultants on the boundaries.

It is shown in Ref.5 that if the domain A is infinite it is necessary to replace equation (27) by the equation

$$\delta U_c' = 0 \qquad (28)$$

where $U_c' = \int\limits_A (\omega-\omega^o)dA - \lim\limits_{R\to\infty} \oint \left\{u_r\left(\sigma_r-\sigma_r^o\right) + u_\theta\left(\tau_{r\theta}-\tau_{r\theta}^o\right)\right\} ds$ (29)

and the superscript ^{o} indicates the solution of the elastic problem.

Now from equation (1)

$$d(\sigma_1^2) = 2\sigma_x\, d\sigma_x + 2\sigma_y\, d\sigma_y - \sigma_x\, d\sigma_y - \sigma_y\, d\sigma_x + 6\tau_{xy}\, d\tau_{xy}$$

$$= 2E_s\, (\varepsilon_x\, d\sigma_x + \varepsilon_y\, d\sigma_y + \gamma_{xy}\, d\tau_{xy}) \qquad (30)$$

from equation (3).

The stress-energy density can therefore be written in the form:-

$$\left.\begin{aligned} \omega &= \tfrac{1}{2}\int \frac{d(\sigma_1^2)}{E_s} \\ &= \int_o^{\sigma_1} \frac{\sigma_1}{E_s}\, d\sigma_1 . \end{aligned}\right\} \qquad (31)$$

If we adopt expression (9) for the secant modulus and take n = 1, we find

$$\omega = \frac{1}{2E\kappa}\left(e^{\kappa\sigma_1^2} - 1\right) , \qquad (32)$$

while if expression (10) is adopted we find, for any value of n,

$$\omega = \frac{\sigma_1^2}{2E}\left\{1 + \left(\frac{\kappa}{n+1}\right)\sigma_1^{2n}\right\} . \qquad (33)$$

In terms of the stress function we therefore find, from equations (32), (8), (26) and (27):

$$\delta \int\limits_A \exp\, \kappa\left\{(\nabla^2\phi)^2 - \tfrac{3}{2}\,\Diamond^4(\phi,\phi)\right\} dA = 0 \qquad (34)$$

while if expression (10) is adopted:

$$\delta \int_A \left\{(\nabla^2\phi)^2 - \frac{3}{2}\,\Diamond^4(\phi,\phi)\right\}\left[1 + \frac{\kappa}{n+1}\left\{(\nabla^2\phi)^2 - \frac{3}{2}\,\Diamond^4(\phi,\phi)\right\}^n\right] dA = 0 . \qquad \ldots\ldots (35)$$

Approximate solutions may now be obtained by substituting into equation (34) or (35) an expression for the stress function that contains undetermined parameters and by minimising the integrals with respect to these parameters.

If the simplified approach of section 5.2 is used, the variational equations corresponding to equations (34) and (35) are, respectively

$$\delta \int_A \exp \;\kappa\left\{3\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right\} dA = 0 \qquad (36)$$

and

$$\delta \int_A \left\{3\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right\}\left[1 + \frac{\kappa}{n+1}\left\{3\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right\}^n\right] dA = 0. \qquad (37)$$

7 CONDITIONS FOR CRACK PROPAGATION - A GRIFFITH-TYPE ANALYSIS

A further approximate analysis is presented below for relating the critical length of a transverse crack in a wide sheet under tension to the material properties obtained from a simple tensile test.

If an elastic sheet is loaded in tension and the boundaries rigidly held, the introduction of a small transverse crack of length ℓ will cause a reduction in the stored strain energy[13] of an amount

$$V = \frac{\pi t \ell^2 \sigma^2}{4E} . \qquad (38)$$

If the crack increases to a length $\ell + \delta\ell$ the release of strain energy is therefore

$$\delta V = \left(\frac{\pi t \ell \sigma^2}{2E}\right) \delta\ell . \qquad (39)$$

Although equation (39) is based on purely elastic considerations it can be expected to be reasonably valid despite the presence of plastic regions at the ends of the crack. This is because the plastic regions are localised and in part of them the stresses are falling and therefore following the elastic unloading line.

The crack will be self-propagating if

$$\delta V \geq \frac{\partial W}{\partial \ell}\,\delta \ell \tag{40}$$

where ${}^{\partial W}/_{\partial \ell}$ is the energy required per unit length to fracture the sheet.

7.1 Approximate interpretation of ${}^{\partial W}/_{\partial \ell}$

Consider a long strip of the sheet loaded in tension with the ends rigidly held. Failure occurs almost immediately after the onset of necking when the stress in the sheet reaches σ_{ult}. During failure however the localised necking becomes more pronounced and plastic work is required before actual fracture occurs; if the strip is very short the release of elastic energy may not be sufficient to deform the neck plastically and failure will occur at a lower stress associated with a higher strain.

If the relationship between nominal stress and elongation of the neck is as shown in Fig.2, the plastic work required per unit width of strip during failure to cause fracture is the area shown shaded multiplied by the sheet thickness. Now this area can be expressed very simply and with negligible loss of accuracy by representing the σ, e relation by the quadrant of the ellipse shown in Fig.3; the plastic work required per unit width of strip is then equal to

$$\frac{\pi}{4}\, t\sigma_{ult}\, e_{fr,t}\ . \tag{41}$$

If we equate expression (41) to ${}^{\partial W}/_{\partial \ell}$ it is possible to obtain a formula for the critical crack length or for σ_{crit}. Thus

$$\left.\begin{aligned} \frac{\partial W}{\partial \ell} &\approx \frac{\pi}{4}\, t\sigma_{ult}\, e_{fr,t} \\ &= \frac{\pi t \ell \sigma_{crit}^2}{2E} \end{aligned}\right\} \tag{42}$$

from equations (39) and (40), whence

$$\sigma_{crit} \approx \left[\frac{E\ \sigma_{ult}\ e_{fr,t}}{2\ell}\right]^{\frac{1}{2}}\ . \tag{43}$$

7.2 Interpretation of $e_{fr,t}$ and $e_{fr,c}$

Equation (43) cannot be used without some modification. This is because $e_{fr,t}$ is the elongation in a simple tensile specimen where necking can occur more readily than at the end of a crack. That part of the sheet which is adjacent to the edges of the crack and which is comparatively lightly stressed, tends to resist the necking at the ends of the crack, and in doing so introduces tensile stresses across the thickness of the sheet in the necking regions. The effect of these induced tensile stresses is to raise the average level of the hydrostatic component of the stresses and correspondingly lower the deviatoric stress in the necking parts in the direction of the applied loads; the final effect being that the effective rate of work-hardening is increased and the material behaves in a more brittle manner. Let us now attempt to relate $e_{fr,c}$ with $e_{fr,t}$.

For a simple tensile specimen we would expect from geometrical similarity that

$$e_{fr,t} \approx Kt \tag{44}$$

where K is some constant; but from the previous paragraph $e_{fr,c}$ at the ends of a crack is less than $e_{fr,t}$ by virtue of the restraining influence of the adjacent lightly stressed parts of the sheet. For the case of a very fine crack, let us represent the ratio $e_{fr,c} : e_{fr,t}$ by the factor $1/(\alpha+\beta t)$. If we consider the limiting case as t tends to zero, the restraining influence vanishes so that we must take α equal to unity. Furthermore, if when the thickness is t_1 the ratio of $e_{fr,c} : e_{fr,t}$ is a half, we may write

$$\left.\begin{aligned} e_{fr,c} &\approx \frac{e_{fr,t}}{1 + {}^{t}/t_1} \\ &\approx \frac{Kt}{1 + {}^{t}/t_1} \end{aligned}\right\} \tag{45}$$

from equation (44). Equation (45) is, of course, empirical and by making t tend to infinity it allows an alternative interpretation of t_1. Thus

$$t_1 = \left(\frac{1}{K}\right) (e_{fr,c})_{\text{large } t} \; . \tag{46}$$

8 INFLUENCE OF CRACK WIDTH

Artificially made cracks may have a finite width at their ends and the restraining influence of the lightly stressed parts of the sheet in resisting the necking will be reduced. In a simple tensile test necking is confined to a length ℓ_n, say, and a possible modification to equation (45), to take account of the influence of crack width, would be to take

$$e_{fr,c} = \frac{Kt}{1 + (t/t_1)[1 - w/\ell_n]^*} \qquad (47)$$

where it is understood that the asterisked expression in square brackets is zero if w exceeds ℓ_n.

For many materials the shape of the neck after fracture in a tensile test is similar to that shown in Fig.4 and ℓ_n can therefore be expressed in terms of K, t and γ. The fact that the volume of the material does not vary significantly implies that

$$\tfrac{1}{2}\ell_n^2 \sin\gamma = Kt^2$$

whence

$$\ell_n = t\sqrt{\frac{2K}{\sin\gamma}} \qquad (48)$$

and equation (47) becomes

$$e_{fr,c} = \frac{Kt}{1 + (t/t_1)\left[1 - \frac{w}{t}\sqrt{\frac{\sin\gamma}{2K}}\right]^*} . \qquad (49)$$

An alternative scheme is to express K in terms of the fractional reduction in thickness, $1 - t_f/t$, denoted by μ. This gives

$$K = \frac{\mu^2}{2\sin\gamma} \qquad (50)$$

and equation (49) becomes

$$e_{fr,c} = \frac{\left(\frac{\mu^2 t}{2\sin\gamma}\right)}{1 + (t/t_1)\left[1 - \frac{w\sin\gamma}{\mu t}\right]^*} . \qquad (51)$$

Combining equations (43) and (51) gives

$$\sigma_{crit} = \frac{\frac{\mu}{2}\left(\frac{Et\,\sigma_{ult}}{\ell\sin\gamma}\right)^{\frac{1}{2}}}{\left\{1 + (t/t_1)\left[1 - \frac{w\sin\gamma}{\mu t}\right]^*\right\}^{\frac{1}{2}}} . \qquad (52)$$

9 BUCKLING OF THE SHEET

If buckling of the sheet occurs equation (52) predicts a value for σ_{crit} which is too high. In this section a formula is developed for the necessary correction factor.

9.1 Onset of buckling

The elastic stress distribution in the sheet is such that there is a lateral compressive stress σ at the edges of the crack. Buckling will therefore occur before the crack propagates if

$$\sigma_{crit} > \sigma_{buc} \tag{53}$$

where

$$\sigma_{buc} = \lambda E t^2/\ell^2 \tag{54}$$

and λ is a constant which preliminary calculations show to be about 10.

9.2 Effects of buckling

The effects of buckling are twofold and are difficult to estimate. First, the bending of the sheet introduces variations in stress across the thickness of the sheet which, at the ends of the crack would facilitate crack propagation. However, this effect is probably small because it is the lateral stress distribution at the ends of the crack that is primarily affected. We shall note in passing that after buckling the curvature at a point in the sheet will vary with increasing σ approximately as

$$\{\sigma - \sigma_{buc}\}^{\frac{1}{2}} . \tag{55}$$

The second and probably more important effect arises from the fact that in the buckled regions ABC of Fig.5 the middle surface compressive forces vary little from their values at the onset of buckling. In the regions ABC the lateral compressive stresses prior to buckling vary approximately linearly to zero at C. After buckling therefore the stress distribution in the unbuckled region can be regarded as composed of two parts: the first part being that which would exist if buckling did not take place and the second part being that due to the self-equilibrating system shown in Fig.6. The peak values of this linearly varying stress distribution occur at A and B and are of magnitude

$$\sigma - \sigma_{buc} . \tag{56}$$

This self-equilibrating system tends to "open up" the crack, an effect which becomes particularly important if the sheet is of finite width and the crack extends over a fair proportion of the width. (Similar reasoning explains the drop in strength of a sheet with a crack in it, on the removal of a circle of sheet whose diameter encompasses most of the crack length.)

The influence of buckling on σ_{crit} will now be determined by its effect on the release of stored strain energy of the system. The onset of buckling causes a reduction in the stored strain energy (as compared with an unbuckled sheet) of an amount given by

$$V_b = \frac{\pi k t \ell^2}{4E} (\sigma - \sigma_{buc})^2 \qquad (57)$$

where preliminary calculations indicate that k is approximately 0.3.

Referring back to equation (39) it will be seen that if the crack increases to a length $\ell + \delta\ell$ the total release of strain energy is given by

$$\delta(V + V_b) = \frac{\pi t \ell \delta\ell}{2E} \{\sigma^2 + k(\sigma - \sigma_{buc})^2\} . \qquad (58)$$

If we denote by σ'_{crit} the value obtained for σ_{crit} if the effects of buckling are ignored (i.e. V_b is zero) and write

$$\left.\begin{aligned} \Omega &= \frac{\sigma_{buc}}{\sigma'_{crit}} , \text{ necessarily less than unity} \\ &\text{and} \\ \Gamma &= \frac{\sigma_{crit}}{\sigma'_{crit}} \end{aligned}\right\} \qquad (59)$$

we find from equations (42) and (58) that

$$\Gamma = \frac{k\Omega + \sqrt{(1 + k - k\Omega^2)}}{1 + k} . \qquad (60)$$

For example, if

$$\Omega = 1/3$$

$$\Gamma = 0.94 .$$

According to the present theory the greatest reduction in σ_{crit} due to buckling occurs when Ω tends to zero, whence

$$\Gamma \to \frac{1}{\sqrt{(1 + k)}} . \qquad (61)$$

It must be realised however that the effect of buckling will be more marked for a sheet of finite width.

10 GENERAL CONCLUSIONS

Two possible analytical approaches have been presented to the problem of estimating the static strength of a plate containing a transverse crack. In sections 3-6 the effects of plasticity in the plate are included by the adoption of a deformation theory together with appropriate stress-strain relationships. The simplest of the resulting governing differential equations will present considerable difficulties in its solution, and this has not been attempted here. A simpler and more promising approach is presented in sections 7-9 where an attempt is made to relate the strength of a cracked plate to the material properties measured in a simple tensile test. Predictions of strengths from the resulting formula are of the correct order, but further confirmatory experimental results are required.

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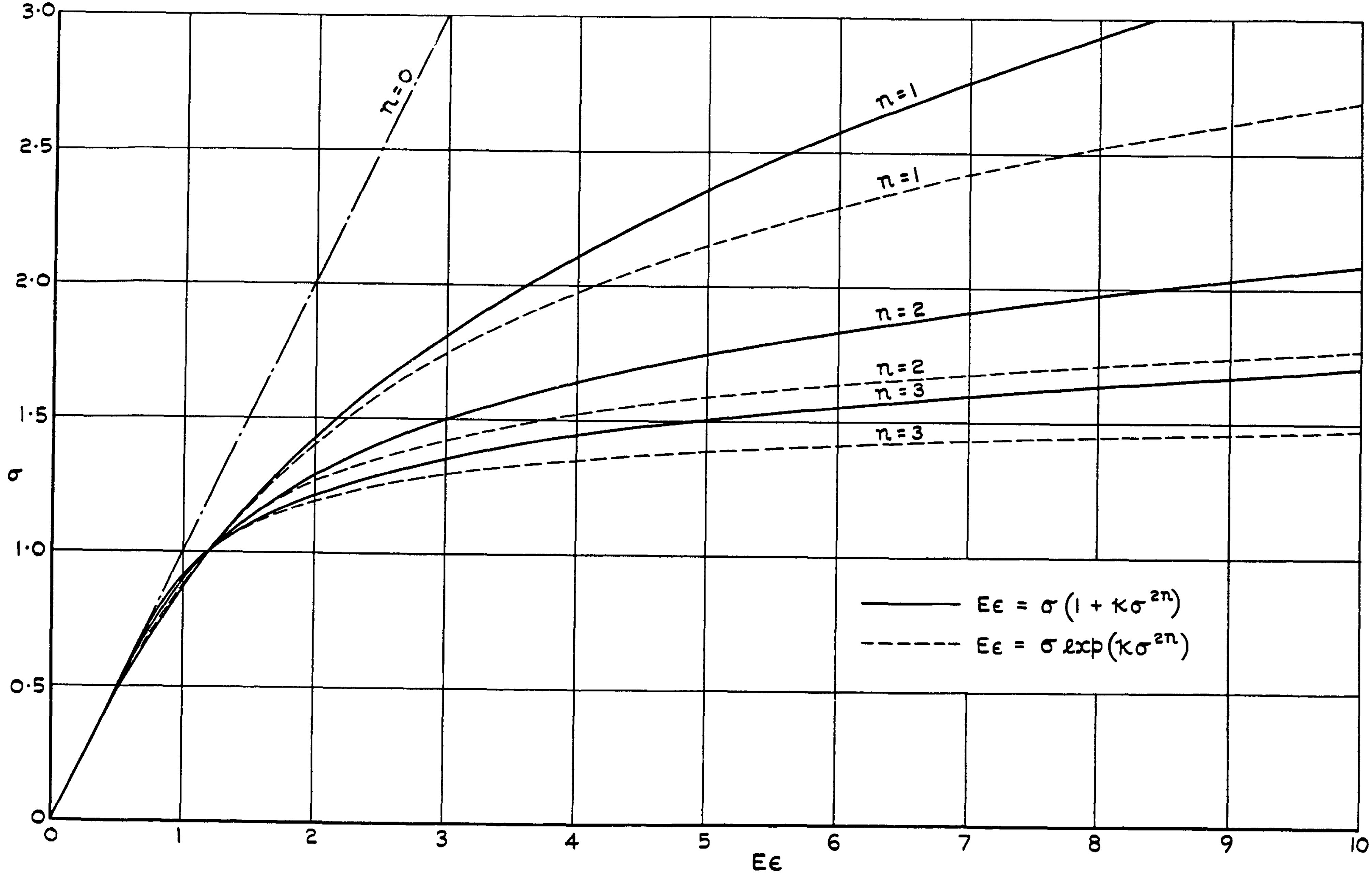

FIG. 1. THEORETICAL STRESS - STRAIN RELATIONS

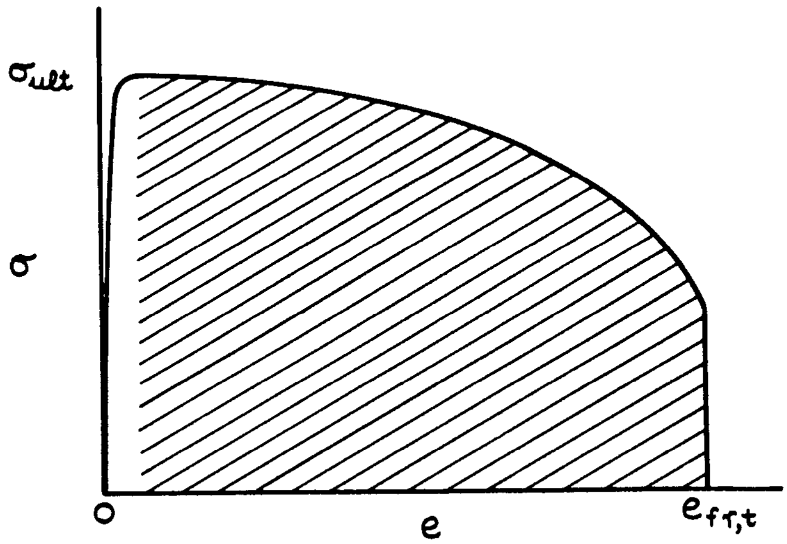

FIG. 2. TYPICAL σ, e RELATION FOR THE NECK.

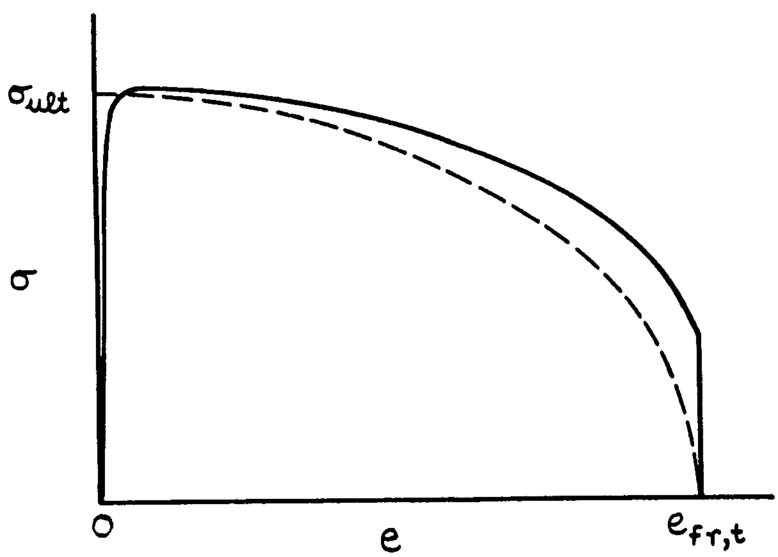

FIG. 3. SIMPLIFIED σ, e RELATION.

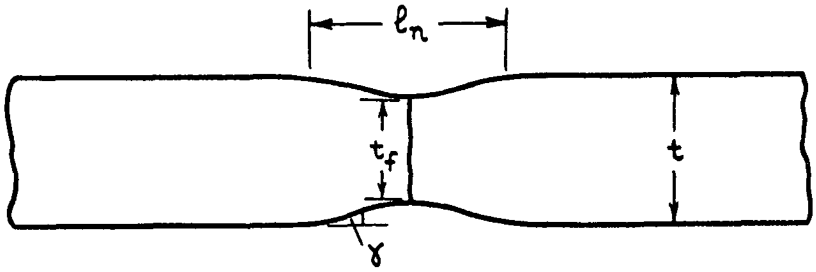

FIG. 4. TYPICAL SHAPE OF NECK.

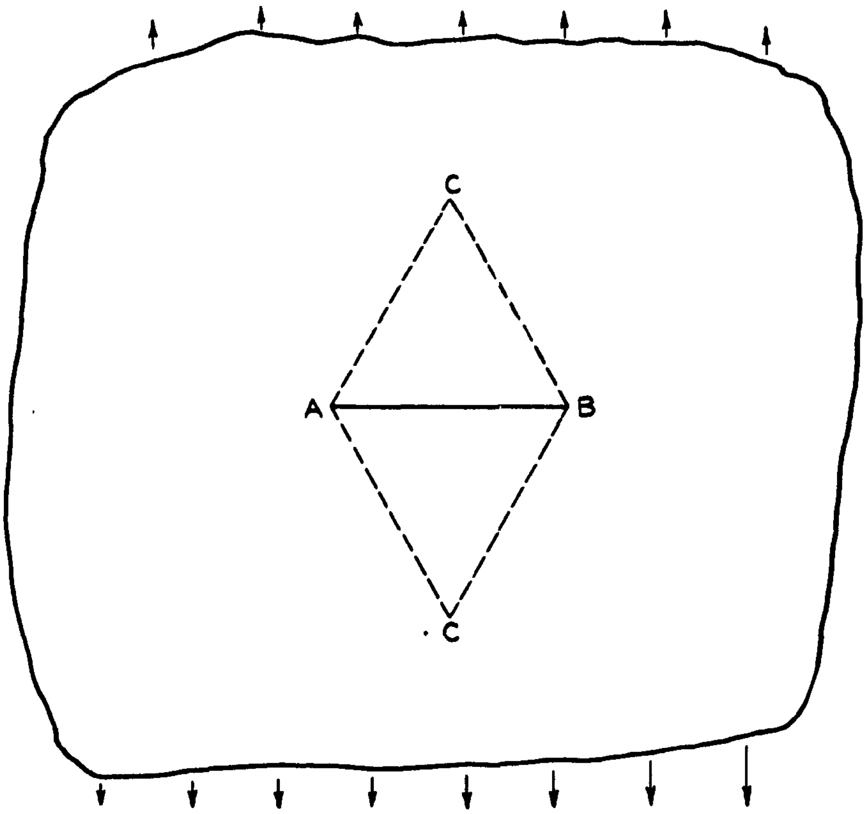

FIG. 5. THE BUCKLED REGIONS IN THE SHEET.

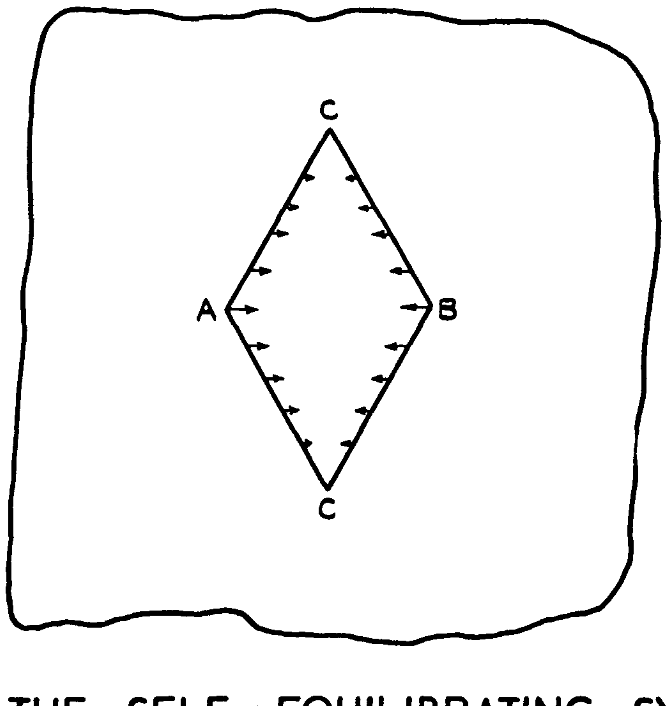

FIG. 6. THE SELF - EQUILIBRATING SYSTEM.

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