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A Digital Computer Programme for the Calculation of Annular or Two-Dimensional Supersonic Potential Flow in a Duct by the Method of Characteristics

By

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A Digital Computer Programme for the Calculation of Annular or Two-dimensional Supersonic Potential Flow in a Duct by the Method of Characteristics

- By -

P. G. Street

May, 1962

SUMMARY

A programme has been written for the Ferranti Mercury computer to analyse supersonic potential flow in a duct with uniform inlet conditions. Input data consist of duct co-ordinates, initial Mach number and flow angle. The programme calculates Mach number and flow angle at the nodal points of the characteristics net. It can be used for axisymmetric or two-dimensional flow, depending on a parameter set in the initial data.

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ILLUSTRATIONS/

ILLUSTRATIONS

1. Introduction

The detailed calculation of supersonic flow fields by hand using approximate numerical or graphical methods is a tedious and lengthy process and the use of such methods to investigate the performance of a range of possible intake or propelling nozzle designs is hardly practical.

The use of an electronic computer is an obvious alternative. The present report describes a method which enables this to be done using the method of characteristics.

Computer programmes exist in Britain for the calculation of nozzle flow and for the calculation of nozzle contours. These have been written for relatively small, slow computers, such as Deuce; they would require major modifications for them to be used for intakes or on other computers. American programmes appear to exist for intakes, but such programmes written for foreign computers are likely to be of limited use in this country. Consequently, the present programme has been written to calculate the flow in any duct (provided that shock waves do not occur) on the Ferranti Mercury or Atlas computers. Mercury is about 20 times as fast as Deuce, and Atlas, when it comes into service, will be about 30 times faster than Mercury.

The numerical application of a characteristics solution to the calculation of plain and axisymmetric supersonic flows was proposed by Guderly[1] as a development of an approximate graphical method derived by Prandtl and Busemann[2]. The method used in the present programme has been developed by Kantrowitz[3] from Guderly's method. The main virtue, in both conception and execution, of Kantrowitz's technique is its simplicity. He derives compatibility equations along the characteristics in terms of Prandtl-Meyer angle and flow angle and shows that characteristics are Mach waves. Details of Kantrowitz's development of the equations for axisymmetric flow are given in Appendix II. The programme is described in terms of axisymmetric flow, since only a small modification to the data is required to use it for two-dimensional flow. An intake is converted into a duct by extending the cowl forward as a pipe of constant diameter.

The method of characteristics is not valid across shock waves. However, provision has been made to start the calculation behind a bow shock, if one is known to exist.

The general method of calculation is described at the beginning of Section 2 and is followed by a more detailed technical description of the programme in Sections 2.1 to 2.5.

2./

2. The Method

The method of characteristics is used in the following manner. Firstly, the flow conditions must be known at a set of initial points which, for the purposes of description will be assumed to be on a line perpendicular to the axis through the origin. Calculation is started by finding the intersection with the outer surface of the left running characteristic (the characteristic running left relative to the direction of flow) through the initial point of greatest radius. The characteristic is at an angle of $(\theta + \alpha)$, where θ is the flow direction relative to the axis of symmetry and α is the Mach angle, and it passes through the initial point whose co-ordinates are known. Hence the equation of the line can be determined. This equation is solved with the equation representing the required position of the outer surface, thus giving the co-ordinates of a new nodal point. The flow direction at the new point can be found from the differential of the surface equation.

The function "b" $\left[= (\nu + \theta) \right.$, where ν is the Prandtl-Meyer angle of the flow$\left. \right]$ is known at the initial point from the flow conditions, also "δb" can be found from the compatibility relation along the characteristic (see Appendix II):-

$$\delta b = \frac{\sin \theta}{r \sin (\theta + \alpha)} \frac{dr}{M}$$

where "r" is the radius and "M" the Mach number. Hence "b" can be found at the new point and since "θ" is already known from the surface slope, "ν" can be found and all the flow properties can be calculated.

The process is then repeated twice using mean values of the flow conditions at the initial and surface points, in order to improve the solution. This successive approximation technique is also applied to all subsequent calculations.

The next step is to determine a general point at the intersection of a pair of characteristics - with slopes of $(\theta + \alpha)$ and $(\theta - \alpha)$ - through the two initial points of greatest radii. The equations of the characteristics are known as before and can be solved as a pair of simultaneous equations to give the co-ordinates of a new point.

Along the right running characteristic:-

$$\delta a = \frac{\sin \theta}{r \sin (\theta - \alpha)} \frac{dr}{M}$$

and along the left running characteristic:-

$$\delta b = \frac{\sin \theta}{r \sin (\theta + \alpha)} \frac{dr}{M}$$

(see Appendix II). Consequently "a" $\left[= (\nu - \theta) \right]$ and "b" $\left[= (\nu + \theta) \right]$ can be found at the new point, so that "ν" and "θ" can be determined, thus giving the complete flow conditions. The solution can then be improved as before.

The third new point is found by repeating the process for the general point working from the last point found and the point found on the outer surface, as data points. Finally, the characteristic is solved with the outer surface as before, to complete the line. Characteristics are constructed from each of the remaining initial points in this manner to complete the first part of the net.

When/

When the characteristic from the innermost initial point is completed, a right running characteristic is constructed from this initial point to the inner surface using an exactly similar process to that used for the outer surface, except that the function "a" is used in place of "b" - the appropriate compatibility relation now being:-

$$\delta a = \frac{\sin \theta}{r \sin (\theta - \alpha)} \frac{dr}{M}$$

A characteristic is constructed from the new point on the inner surface up to the outer surface. Then a right running characteristic is constructed to the inner surface from the point on this last characteristic next to the inner surface point so that a new line can be started.

The computation can thus be continued constructing a series of characteristics from the inner surface to the outer surface. The calculations may continue either until the required flow field has been covered or until two characteristics of the same family intersect. The intersection, or focussing of two characteristics of the same family means that a shock wave will form in the flow and the compatibility relations will cease to be true.

When starting behind a conical or oblique shock wave, the initial points are treated as though they lay on a characteristic. Thus the first step is to construct an inner surface point from the innermost initial point. A characteristic is then constructed from this new point out to the outer surface, using the appropriate initial point and the last point found as data points for each new point. The procedure is then the same as before.

The layout of the programme to carry out these calculations is described below, the programme itself being presented, with explanatory notes, in Appendix III. The programme is divided into five chapters, Chapter 0 coming last on the tape since the machine automatically starts to work at the beginning of this chapter.

2.1 Chapter 0

Chapter 0 reads in the type of duct, the overall length, and the wall surface points, which are stored on the drum. The coefficients for the polynomial relating Prandtl-Meyer angle and Mach number are stored in HO-H8 for future reference. A cubic expression:-

$$Y = Y1 + (X-X1)Y1' + (X-X1)^2 \; (PX+Q)$$

is used to describe the contours between successive data points, P and Q being found from the conditions that the curve must pass through the point (X0,Y0) with the slope (Y0). It will be seen that the expression satisfies the conditions:-

$$Y = Y1 \text{ and } Y' = Y1' \text{ at } X = X1$$

Co-ordinates and slopes for 32 equally spaced intervals between 0 and X' are calculated from the appropriate cubic expressions and tabulated.

The programme then reads in a value for T. If T = 1, it is assumed that the initial points lie near a line vertical to the axis but if $T \neq 1$, it is assumed that the initial points lie near an oblique line in the first quadrant, generally through the origin as when starting behind a bow shock. Thus control can be transferred to the appropriate section of Chapter 1 after the number of initial points is read into P.

2.2 Chapter 1

Chapter 1 can best be considered in four sections. The part between labels (1) and (2) is concerned with starting the calculation on a first family characteristic or behind a conical shock wave. The co-ordinates and flow propterties for the P initial points are read in and stored. Finally control is transferred to the second part of the chapter which is the main control loop consisting of the instructions between label (2) and the next "Jump 2". This section primes the control parameters and calls in Chapters 2 and 4 as required in order to calculate the mesh points.

When each characteristic is completed, it is shifted back through 21 stores and the previous contents of these stores are also shifted back so that the last two characteristics calculated can always be found in L = 2 to L = 22 and L = 23 to L = 43. The stores L = 44 to L = 64 are cleared ready to receive the results for the next characteristic.

The section from label (6) to the instruction after label (7) is concerned with starting the calculations near a vertical line relative to the axis. As each of the P initial co-ordinates and flow properties are read in, the characteristic is calculated from this location up to the outer surface.

When the last data point has been read in the appropriate characteristic has been calculated, control is given to the main control loop at label (2), mentioned above.

The last section starting at label (50) prints out the data points followed by the legend "D.P." when each initial point is read in.

2.3 Chapter 2

Chapter 2 is concerned with more detailed organization depending upon the parameters set in Chapter 1. The locations of the data required to calculate a general point L are placed in J and T and the co-ordinates of L are calculated. The compatibility relations are used to calculate the flow parameters AL (i.e., $\nu + \theta$) and BL (i.e., $\nu - \theta$) and the flow angle and Mach number are then calculated. It should be noted that the parameters AL and BL are constant, along second and first family characteristics respectively, in two-dimensional flow. No formula has been found expressing Mach number in terms of Prandtl-Meyer angle so a polynomial expression was used. This expression was determined by the usual curve fitting techniques and the best result was found to be

$$M = 1{\cdot}00917351 + 4{\cdot}07762967\nu - 20{\cdot}5542273\nu^2 + 100{\cdot}102132\nu^3 - 274{\cdot}010723\nu^4$$
$$+ 439{\cdot}034941\nu^5 - 405{\cdot}469672\nu^6 + 199{\cdot}9438\nu^7 - 40{\cdot}7116009\nu^8$$

where the maximum error of 0·00917 occurs at M = 1. The error level between M = 1.3 and M = 4 is about 0·003 on Mach number.

The whole calculation of the general point is now repeated twice using the mean of conditions at the point itself and at the data points in order to improve the solution. When the iterative steps have been completed values of X, Y, M, θ and P_{stat} are printed out on one line; the columns for X and Y are lined up with those printed out in Chapter 0 for the surface contours, so that the output can be used on a tape-operated graph plotter to plot the contours and the nodal points of the characteristics mesh. If the next point lies on the outer surface, control is transferred to Chapter 3. If not, L is advanced by 1 and the chapter is repeated. Should any of the checks fail, or should the Mach number be found to be less than unity, the Mach number ("FL" for the point "L") is made zero. Then control is returned to Chapter 1 in order to start the next characteristic from the inner surface.

2.4 Chapter 3

Chapter 3 is used to calculate a point on the outer surface by finding the intersections of a left running, or a first family, characteristic with the surface. This involves the solution of a cubic equation by iteration. When the co-ordinates of the intersection point have been found the flow direction, θ, at the surface is found using the differential of the cubic equation. The function BL $(= \nu - \theta)$ is found along the characteristic and hence ν is known. The Mach number is found from ν as in Chapter 2 and the solution repeated twice using improved values. The results are printed out, followed by the legend "O.S.".

Should the iteration for the intersection point fail, the iteration limits are modified and the calculation repeated. A check on axial position is agian included.

2.5 Chapter 4

Chapter 4 deals with a point on the inner surface in an exactly similar manner to Chapter 3 for the outer surface except that a right running or second family characteristic is used and the function AL $(= \nu + \theta)$ is calculated. The printing is followed by the legend "I.S.". If the iteration fails, or the machine is in some way unable to find the point on the inner surface, e.g., because compression fan focussing has occurred (see Fig.4), the the programme returns to the first instruction of Chapter 0 to start a new example, since no more work is possible.

3. Tests and Examples

The characteristics meshes obtained from several trial calculations are shown in Figs.1 to 5.

Fig.1 is a two-dimensional compression fan focussing surface which was calculated trigonometrically from two-dimensional theory. The surface was built up from a series of 2° wedges, the final angle being 40°. The intersection points of the wedges were found from a knowledge of the shock angles[5]. The data given to the computer consisted of the co-ordinates of the mid-points of the wedges, together with the slope. The position of the vertex was found by passing a Mach wave through the focus point. It can be seen that the mesh focusses very closely to the theoretical position, thus establishing that the surfaces calculated from the cubics are a good representation of the intended surface. The Mach numbers and flow angles along the first family characteristics are constant as would be expected and correlate with two-dimensional theory to within 0·001 on Mach number.

Fig.2 represents a symmetrical slender centrebody in a large tube. The maximum diameter of the centrebody is 1 in. and the tube is 6 in. diameter. The mean Mach number at the minimum area is 2·974 according to one-dimensional theory if the initial Mach number is 3·004. The mean Mach number at the nodal points in this region was 2·968 which gives a difference of about 0·006 on a Mach number change of 0·03. The mean Mach number at the nodal points in the final plane, where the centrebody diameter was again zero, was 3·011. This high value may be caused by the weighting effect of the group of high Mach number points at the incipient focus.

Fig.3 shows the flow about a 20° cone at Mach 3. The characteristics calculation was started on the nose shock with conditions on the downstream side of the shock. The results at the last four points are quoted together with the theoretical results obtained from Ref.5. The Mach numbers agree to within 0·008. The flow angles agree to within 0·09°.

Fig.4/

Fig.4 is an example of the results of some slight errors in the data; it also shows what the programme does in the event of one or more focus points being found. Focus point (1) is the start of a downward running shock from the cowl, but, if the characteristics are followed back, it becomes apparent that the seat of the trouble is on the inner surface around $X = 5$. Examination of the contour in this region shows a slight wobble due, it is thought, to an error of about $\frac{1}{2}^{\circ}$ in the slope at a datum point at $X = 6{\cdot}339$. Focus point (2) can also be traced to an error in the data for the inner surface at around $X = 12$ where, once again, the slope is too high. The mean Mach number at $X = 16$ for this example was found from one-dimensional theory to be about 2·838. The mean Mach number for the nodal points between $X = 15$ and $X = 17$, making no allowance for radial distribution, was found to be 2·854.

The example of Fig.5 was the same duct as the previous example calculated using twice the linear mesh size. This gave a mean Mach number between $X = 15$ and $X = 17$ of 2·866, thus showing that an increase in step size increases the error. The mean Mach number at $X = 22{\cdot}5$ from one-dimensional theory was found to be 2·098 whilst that at the nodal points in this region was found to be 2·084. Thus the difference would appear to be about 2% of the change of Mach number when the Mach number change is fairly large.

The general error level, so far as it can be calculated by comparison with one-dimensional theory, would seem to be satisfactory. Viscosity effects such as shock waves and boundary layers are not calculated. Several possible methods of including shock waves are being considered but it is not intended to include boundary layer calculations in the programme.

4. Discussion

The programme provides a picture of the Mach number, flow angle and static pressure distributions through an annular or two-dimensional duct.

The method of characteristics is only valid in regions of supersonic flow; also, since flow discontinuities such as shock waves have been ignored the programme is limited to supersonic flow over isentropic surfaces, although it may start behind a bow shock if one exists, as discussed in Appendix IV. If either subsonic flow or compression fan focussing occur then the programme stops.

The programme has been written mainly for axisymmetric intakes, operating at or near design point. It can be used to calculate the flow in two-dimensional intakes by putting a "0" at the head of the data tape instead of a "1". Spill is precluded by the extension of the cowl. It is hoped to overcome the limitations of no spill and no shocks in a new programme. Such a programme will be very complex and it is felt that much useful work can be accomplished using the present simple programme.

Early versions of the programme fitted eighth order polynomials by the least squares technique to a set of co-ordinates representing the surfaces of the duct. This technique was used in an effort to obtain a fair representation of the surfaces without having to determine extremely accurate data. However, in many examples, if a sufficiently high order polynomial was fitted to obtain a maximum residual of acceptable value, then waves, or wobbles, occurred between the data points. The system also had the disadvantage of losing accuracy as X increased, since the value of Y was computed from the small differences of several large quantities whence rounding errors in the computer were apt to become noticeable.

The present system for representing the surface contours was partly evolved from the realisation that the slope of the surface had at least as much importance as the position, so that it should really be

included/

included in the data. As explained in Section 2.1, cubics are fitted between successive data points having given slopes at these data points. Experience with this method has shown that the surfaces must be very accurately known, and in examples where the contours are non-analytical, great care must be taken in determining the data.

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3	A. R. Kantrowitz	The characteristic method for steady supersonic flow. Lecture notes given at M.I.T. Private communication from L. F. Crabtree, R.A.E.
4	H. W. Liepmann A. E. Puckett	Introduction to aerodynamics of a compressible fluid. p.31, Chapman & Hall Ltd., London. 1947.
5	A.R.C. Compressible Flow Tables Panel	A selection of graphs for use in calculations of compressible airflow. Clarendon Press, Oxford. 1954.

APPENDIX I

List of Symbols

Symbols for Appendix II

A	area
a	value of $(\nu + \theta)$ along second family characteristic
b	value of $(\nu - \theta)$ along first family characteristic
M	Mach number
N	distance normal to streamline
r	radius
s	entropy
W	velocity
α	Mach angle $\left(\text{arc sin} \left(\frac{1}{M} \right) \right)$
θ	flow angle
ν	Prandtl-Meyer angle
ρ	density

Symbols for Appendix III

(Note AL etc., represents A_L etc.)

A	working space
A'	current drum address of inner surface data
AI	value of $(\nu + \theta)$, in radians, at point "I"
B	working space
B'	current drum address of outer surface data
BI	value of $(\nu - \theta)$, in radians, at point "I"
C	slope of second family characteristic. Also used as working space when available
C'	slope of first family characteristic. Also used as working space when available
D	working space
D0-D2	surface data for point next downstream of current guess
E	current value of "Y" in surface iterations
E0-E2	surface data for point next upstream of current guess
F	working space
FI	Mach number at point "I".
G	working space

GI	flow angle, in radians, at point "I"
H	working space
H0-H8	coefficients of polynomial relating Mach number and Prandtl-Meyer angle
I	cycle counter
J	index for datum point on second family characteristic; cycle counter
K	= 0 for two-dimensional flow; = 1 for axisymmetric flow; set to 3 to act as a trigger when Chapter 4 fails
L	index of point being calculated
N	number of points specifying surface
P	number of initial points
Q	reference index for inner surface location
R	cycle counter
S	cycle counter
T	index for datum point on first family characteristic; cycle counter
U	working space; current FJ
U'	working space; current FT
V	working space; current AJ
V'	working space; current AT
W	working space; current BJ
W'	working space; current BT
X	axial co-ordinate
X'	overall length of duct
XI	axial co-ordinate of point "I"
Y	radial co-ordinate
YI	radial co-ordinate of point "I"
Z	working space
Z0	Mach angle, in radians, at datum point on second family characteristic
Z1	Mach angle, in radians, at datum point on first family characteristic
π	3·141592654

APPENDIX II

Derivation of the Compatibility Equations in Axisymmetric Flow - vide Reference 3

Characteristics are mathematically defined as lines, or surfaces along which the partial differential equations of the flow field reduce to ordinary differential equations. Alternatively, these may be defined as lines or surfaces along any one of which known values of the flow are insufficient to determine the flow in any other region.

The compatability relations are equations relating the flow properties which hold along the characteristics of the flow field. They may be derived from the equations of irrotationality and mass continuity as follows:-

(1) Equation of irrotationality

$$\frac{\partial\theta}{\partial S} - \frac{1}{\sqrt{M^2 - 1}} \cdot \frac{\partial\nu}{\partial N} = 0 \qquad \ldots (1)$$

(2) Equation of continuity

From Fig.6(a) the area of the annulus, is

$$A = 2\pi r \cdot \Delta N$$

Hence

$$\frac{dA}{dS} = 2\pi r \cdot \frac{d(\Delta N)}{dS} + 2\pi \cdot \Delta N \cdot \frac{dr}{dS}$$

and

$$\frac{1}{A}\frac{dA}{dS} = \frac{1}{\Delta N} \cdot \frac{d(\Delta N)}{dS} + \frac{1}{r} \cdot \frac{dr}{dS}$$

But from Fig.6(b)

$$\frac{d(\Delta N)}{dS} = \lim_{\Delta S \to 0} \frac{(\Delta N_2 - \Delta N_1)}{\Delta S}$$

$$= \lim_{\Delta S \to 0} \frac{\frac{\partial\theta}{\partial N} \cdot \Delta N \cdot \Delta S}{\Delta S}$$

$$= \frac{\partial\theta}{\partial N} \cdot \Delta N$$

so that

$$\frac{1}{A}\frac{dA}{dS} = \frac{\partial\theta}{\partial N} + \frac{1}{r} \cdot \frac{dr}{dS}$$

i.e.,

$$\frac{dA}{A} = \frac{\partial\theta}{\partial N} \cdot dS + \frac{dr}{r}$$

But

$$dr = dS \cdot \sin\theta$$

and therefore

$$\frac{dA}{A} = \frac{\partial\theta}{\partial N} \cdot dS + \frac{\sin\theta}{r} \cdot dS \qquad \ldots (2)$$

The/

The continuity equation is

$$\frac{dA}{A} + \frac{1}{\rho W}\frac{\partial}{\partial S}(\rho W) \cdot dS \qquad \ldots (3)$$

Substituting Equation (3) in Equation (2),

$$-\frac{1}{\rho W} \cdot \frac{\partial}{\partial S}(\rho W) \cdot dS = \frac{\partial \theta}{\partial N} \cdot dS + \frac{\sin\theta}{r} \cdot dS$$

or

$$\frac{\partial \theta}{\partial N} + \frac{\sin\theta}{r} + \frac{1}{\rho} \cdot \frac{\partial \rho}{\partial S} + \frac{1}{W} \cdot \frac{\partial W}{\partial S} = 0 \qquad \ldots (4)$$

From the assumption of constant energy (see Ref.4)

$$\frac{d\rho}{\rho} = -M^2 \cdot \frac{dW}{W} \qquad \ldots (5)$$

Equations (4) and (5) then give

$$\frac{\partial \theta}{\partial N} - (M^2 - 1) \cdot \frac{1}{W} \cdot \frac{\partial W}{\partial S} = -\frac{\sin\theta}{r}$$

But

$$\frac{dW}{W} = \frac{d\nu}{\sqrt{M^2 - 1}}$$

and so

$$\frac{\partial \theta}{\partial N} - \sqrt{M^2 - 1} \cdot \frac{\partial \nu}{\partial S} = -\frac{\sin\theta}{r} \qquad \ldots (6)$$

Now in Ref.3, it is shown that on a second family characteristic

$$\delta a = \left[-\frac{\partial \nu}{\partial N} \cdot \frac{1}{\sqrt{M^2 - 1}} + \frac{\partial \nu}{\partial S} - \frac{\partial \theta}{\partial N} \cdot \frac{1}{\sqrt{M^2 - 1}} + \frac{\partial \theta}{\partial S}\right] \cdot dS \qquad \ldots (7)$$

Putting Equations (1) and (6) in Equation (7)

$$\delta a = \frac{\sin\theta}{r\sqrt{M^2 - 1}} \cdot dS \text{ on a second family characteristic} \qquad \ldots (8)$$

Similarly

$$\delta b = \frac{\sin\theta}{r\sqrt{M^2 - 1}} \cdot dS \text{ on a first family characteristic} \qquad \ldots (9)$$

Furthermore, from Fig.2(c)

$$\frac{ds}{\cos\alpha} = \frac{dx}{\cos(\theta - \alpha)} = \frac{dr}{\sin(\theta - \alpha)}$$

and

$$\cos\alpha = \frac{\sqrt{M^2 - 1}}{M}$$

so/

so that

$$\frac{dS}{\sqrt{M^2 - 1}} = \frac{dx}{M \cos(\theta - \alpha)} = \frac{dr}{M \sin(\theta - \alpha)} \qquad \ldots (10)$$

Putting Equation (10) into Equation (8)

$$\delta a = \frac{\sin\theta}{r \cos(\theta - \alpha)} \cdot \frac{dx}{M} = \frac{\sin\theta}{r \sin(\theta - \alpha)} \cdot \frac{dr}{M} \qquad \ldots (11)$$

Similarly,

$$\delta b = \frac{\sin\theta}{r \cos(\theta + \alpha)} \cdot \frac{dx}{M} = \frac{\sin\theta}{r \sin(\theta + \alpha)} \cdot \frac{dr}{M} \qquad \ldots (12)$$

Equations (11) and (12) are known as the compatibility relations and they enable the variations of the flow properties to be calculated along second family and first family characteristics respectively.

Simultaneous solution of Equations (11) and (12) along the two characteristics from two known points enables one to find the flow properties at a third point - which is at the intersection of the two characteristics.

APPENDIX III

The Programme and Typical Data

Chapter 1

```
A → 64
B → 64
F → 64
G → 64
X → 64
Y → 64                    Sets main variables
Z → 1
D → 2
E → 2
H → 8
```

```
1)Q  =  1(1)P
L  =  65 - Q
READ(XL)
READ(YL)
READ(AL)
READ(BL)
READ(FL)                  Reads in initial data for oblique initial line
READ(GL)
AL  =  0·01745329AL
BL  =  0·01745329BL
GL  =  0·01745329GL
JUMP DŌWN 50
REPEAT
```

```
Q  =  64 - P
JUMP DŌWN 3               Starts first characteristic
JUMP 4
```

```
2)DŌWN 1/2
                          Main control loop
JUMP DŌWN 3
```

4)DŌWN 1/4/

CHAPTER 1 (cont'd)

```
4)DŌWN 1/4        }
JUMP 5,2 > K      }   Check not tripped
ACRŌSS 1/0        }
5)L = L + 1
JUMP 2
```

```
3)N = 2(21)23
M = N + 20
L = N(1)M
XL = X(L + 21)
YL = Y(L + 21)
AL = A(L + 21)        Relocates last two characteristics
BL = B(L + 21)
FL = F(L + 21)
GL = G(L + 21)
REPEAT
REPEAT
```

```
L = 44(1)64
XL = 0
YL = 0
AL = 0
BL = 0
                      Clears stores for next characteristic
FL = 0
GL = 0
REPEAT
L = Q - 1
RETURN
```

---

6)Q = 1(1)P/

CHAPTER 1 (cont'd)

```
6)Q  =  1(1)P
L  =  65-2Q
READ(XL)
READ(YL)
READ(AL)
READ(BL)
READ(FL)
READ(GL)
AL  =  0·01745329AL
BL  =  0·01745329BL
GL  =  0·01745329GL
JUMP DOWN 50
JUMP 7,L  =  63
L  =  L + 1
DOWN 1/2
8)JUMP DOWN 3
REPEAT
```

Reads in initial data for vertical initial line and transfers control to Chapter 2 to calculate the characteristics net up to the first inner surface point

---

```
Q  =  64 - 2P
L  =  Q - 1
JUMP 4
7)DOWN 1/3
JUMP 8
```

Sets parameters and transfers control for main calculation

```
50)NEWLINE
PRINT(XL)3,3
PRINT(YL)3,3
PRINT(FL)3,3
PRINT(57·3GL)3,3
A  =  ψ LOG(1 + ·2FLFL)
A  =  ψ EXP(-3·5A)
PRINT(A)3,3
```

Prints initial points followed by legend:- "D.P."

CHAPTER 1 (cont'd)

CAPTIŌN

D.P.

RETURN

CLŌSE

CHAPTER 2

VARIABLES 1

```
1)J  =  L - 20
T  =  L - 1
S  =  0
U  =  FJ
U'  =  FT
V  =  AJ
V'  =  AT
W  =  BJ
W'  =  BT
```

Sets indices of data and transfers flow properties to temporary stores to prevent overwriting of data in iterative loop

---

```
9)Z  =  ψ SQRT(UU - 1)
ZO  =  ψ ARCTAN(Z,1)
Z  =  ψ SQRT(U'U' - 1)
Z1  =  ψ ARCTAN(Z,1)
C  =  ψTAN(·5V - ·5W - ZO)
C'  =  ψTAN(·5V' - ·5W' + Z1)
A  =  C - C'
B  =  YT - YJ + CXJ - C'XT
XL  =  B/A
C  =  CC'XJ - CC'XT + CYT - C'YJ
YL  =  C/A
```

Calculates position co-ordinates of general point

```
A  =  ψSIN(·5V - ·5W)
A  =  AXL - AXJ
B  =  ψCŌS(·5V - ·5W - ZO)
B  =  ·5UYJB + ·5UYLB
```

AL = AJ + KA/B/

CHAPTER 2 (cont'd)

```
AL  =  AJ + KA/B
A  =  ψSIN(·5V' - ·5W')
A  =  AXL - AXT
B  =  ψCŌS(·5V' - ·5W' + Z1)
B  =  ·5U'YLB + ·5U'YTB
BL  =  BT + KA/B
GL  =  ·5AL - ·5BL
A  =  ·5AL + ·5BL
FL  =  ψ PŌLY(A)H0,8
```

Calculates flow properties at general point

```
JUMP 4,S  =  2
S  =  S + 1
U  =  ·5FJ + ·5FL
U'  =  ·5FT + ·5FL
V  =  ·5AJ + ·5AL
V'  =  ·5AT + ·5AL
W  =  ·5BJ + ·5BL
W'  =  ·5BT + ·5BL
JUMP 9
```

Modifies data to improve solution

```
4)A  =  ψ LŌG(1 + ·2FLFL)
A  =  ψEXP(-3·5A)
NEWLINE
PRINT(XL)3,3
PRINT(YL)3,3
PRINT(FL)3,3
PRINT(57·3GL)3,3
PRINT(A)3,3
CAPTIŌN
G.P.
```

Prints results of general point calculation followed by legend:- "G.P."

```
JUMP 101,XT > XL
JUMP 101,XJ > XL
```

Checks whether rays have focussed

```
JUMP 100,XL ≥ X'
```

Checks within overall length

CHAPTER 2 (cont'd)

JUMP 5,L = 63	Transers control if next point is outer surface
L = L + 1 JUMP 1	Advances count and repeats for a new general point
5)ACRŌSS 1/3	Transfers control to Chapter 3
101)FL = 0 CAPTIŌN FŌCUS JUMP 102 100)FL = 0 CAPTIŌN END 102)L = 64 UP	Rescue procedure. Prints reason for failure and transfers control to Chapter 1 to start a new characteristic from inner surface
CLŌSE	

CHAPTER 3 VARIABLES 1 1)T = L L = L + 1 S = 0 U' = FT V' = AT W' = BT	Sets indices of data for point on outer surface and transfers flow properties to temporary stores to prevent overwriting of data in iterative loop
2)Z = ψSQRT(U'U' - 1) Z1 = ψARCTAN(Z,1) C' = ψTAN(·5V' - ·5W' + Z1) A = YT - C'XT ψ6(201)B,1 B = B + 1 D = YT J = 0	Calculates equation of characteristic and sets iteration limits

CHAPTER 3 (cont'd)

```
3)E  =  •5B + •5D
J  =  J + 1
JUMP 4,J > 50
F  =  E - A
F  =  F/C'
B'  =  200
JUMP 12,F > X'
13)ψ6(B')D0,3
JUMP 16,D0 ≥ F
B'≈ B' + 3
0̄  =  0(1)2
E0̄  =  D0̄
REPEAT
JUMP 13
16)C  =  E0 - D0                     Solves characteristic with appropriate outer
                                      surface cubic to within an error of 0•001 on Y
G  =  ψDIVIDE(-2E1 + 2D1 + CD2 + CE2,CCC)
H  =  ψDIVIDE(E1 - D1 - CD2 - GE0CC,CC)
U  =  F - D0
W  =  GF + H
F  =  D1 + UD2 + UUW
CHECK(B,D,•001,5)
F  =  F - E
JUMP 6,F > 0
B  =  E
JUMP 3
6)D  =  E
JUMP 3
5)YL  =  E
XL  =  YL/C' - A/C'
```

CHAPTER 3 (cont'd)

```
A  =  ψSIN(·5V' - ·5W')
A  =  AXL - AXT
B  =  ψCŌS(·5V' - ·5W' + Z1)
B  =  ·5U'YTB + ·5U'YLB
BL  =  BT + KA/B
A  =  D2 + UUG + 2UW
GL  =  ψARCTAN(1,A)
AL  =  BL + 2GL
A  =  BL + GL
FL  =  ψPŌLY(A)HO,8
```

Calculates flow properties at outer surface point

```
JUMP 9,S  =  2
S  =  S + 1
U'  =  ·5FT + ·5FL
V'  =  ·5AT + ·5AL
W'  =  ·5BT + ·5BL
JUMP 2
```

Modifies data for outer surface point and transfers control

```
9)A  =  ψLOG(1 + ·2FLFL)
A  =  ψEXP(-3·5A)
NEWLINE
PRINT(XL)3,3
PRINT(YL)3,3
PRINT(FL)3,3
PRINT(57·3GL)3,3
PRINT(A)3,3
CAPTIŌN
Ō.S.
```

Prints results and legend:-"Ō.S."

```
UP
```

Returns control to main loop in Chapter 1

```
4)J  =  0/
```

CHAPTER 3 (cont'd)

4)J = 0 ψ6(201)B,1 B = B + 1 D = Y(T - 1) JUMP 3	Adjusts limits if iteration fails
12)F = X' JUMP 13	Limits solution in length if test fails
100)A = A CAPTION CH. 3,FAULT 33. UP	Rescue procedure in event of data Mach number being subsonic

CLOSE

CHAPTER 4

VARIABLES 1 1)L = L + 1 J = L - 20 S = 0 U = FJ V = AJ W = BJ	Sets indices of data for point on inner surface and transfers flow properties to temporary stores as in previous chapters
2)Z = ψSQRT(UU - 1) ZO = ψARCTAN(Z,1) C = ψTAN(·5V - ·5W - ZO) A = YJ - CXJ B = YJ D = 0 T = 0 R = 0	Calculates equation of characteristics and sets iteration limits

3)E = ·5B + ·5D/

CHAPTER 4 (cont'd)

```
3)E  =  •5B + •5D
T  =  T + 1
JUMP 100,T > 50
F  =  E - A
F  =  F/C
A'  =  0
JUMP 12,F > X'
11)ψ6(A')D0,3
JUMP 16,D0 ≥ F
A' ≈ A' + 3
Ō  =  0(1)2
EŌ  =  DŌ
REPEAT
JUMP 11                        Solves characteristic with appropriate inner
                                surface curve to within an error of 0•001
16)C'  =  E0 - D0               on  Y
G  =  ψDIVIDE(-2E1 + 2D1 + C'D2 + C'E2,C'C'C')
H  =  ψDIVIDE(E1 - D1 - C'D2 - GE0C'C',C'C')
U'  =  F - D0
W'  =  GF + H
F  =  D1 + U'D2 + U'U'W'
CHECK(B,D,•001,8)
F  =  F - E
JUMP 4,F > 0
B  =  E
JUMP 3
4)D  =  E
JUMP 3
8)YL  =  E
XL  =  YL/C - A/C
```

CHAPTER 4 (cont'd)

```
A  =  ψSIN(·5V - ·5W)
A  =  AXL - AXJ
B  =  ψCŌS(·5V - ·5W - Z0)
B  =  ·5UYJB + ·5UYLB
AL  =  AJ + KA/B
A  =  D2 + U'U'G + 2U'W'
GL  =  ψARCTAN(1,A)
BL  =  AL - 2GL
A  =  AL - GL
FL  =  ψPŌLY(A)H0,8
```

Calculates flow properties at inner surface points

```
JUMP 14,S  =  2
S  =  S + 1
U  =  ·5FJ + ·5FL
V  =  ·5AJ + ·5AL
W  =  ·5BJ + ·5BL
JUMP 2
```

Modifies data for inner surface point and transfers control

```
14)A  =  ψLŌG(1 + ·2FLFL)
A  =  ψEXP(-3·5A)
NEWLINE
PRINT(XL)3,3
PRINT(YL)3,3
PRINT(FL)3,3
PRINT(57·3GL)3,3
PRINT(A)3,3
CAPTIŌN
I.S.
```

Prints results and legend:-"I.S."

```
UP
```

Transfers control to main loop in Chapter 1

```
12)F  =  X'
R  =  R + 1
JUMP 100,R > 10
JUMP 11
```

Limits solution in length if test fails

CHAPTER 4 (cont'd)

100)K = 3 620,27 620,9 620,9 620,0 NEWLINE CAPTIŌN END ŌF CASE UP	Sets trigger, prints II and legend:- "END ŌF CASE" and returns control to Chapter 1 in event of failure to find a satisfactory inner surface point

CLŌSE

CHAPTER 0 VARIABLES 1 1)READ(K) READ(X')	Reads in type number and overall length
620,30 I = 1(1)10 620,13 REPEAT I = 0(1)20 620,0 REPEAT	Outputs "NEWLINES", followed by a length of blank tape
READ(C) CAPTIŌN CASE NŌ. PRINT(C)3,0	Reads and prints an identifying number

CHAPTER 0 (cont'd)

```
HO  =  1·00917351
H1  =  4·07762967
H2  =  -20·5542273
H3  =  100·102132
H4  =  -274·010723
H5  =  439·034941
H6  =  -405·469672
H7  =  199·9438
H8  =  -40·7116009
```

Sets up coefficients of polynomial for Mach number in terms of Prandtl-Meyer angle

---

```
P  =  0
A'  =  0
2)READ(N)
```

Reads number of data points

```
P  =  P + 1
I  =  1(1)N
READ(X0)
READ(X1)
READ(A)
A  =  ·01745329A
X2  =  ψTAN(A)
ψ7(A')X0,3
A' ≈ A' + 3
REPEAT
```

Reads in inner surface data points, converts angle to slope and stores on drum in appropriate locations

---

```
A'  =  200
JUMP 2,P  =  1
```

Repeats above section for outer surface data

```
P  =  0
A'  =  0
NEWLINE
CAPTIŌN
   X   Y   SLŌPE
NEWLINE
```

Prints heading for contours

CHAPTER 0 (cont'd)

```
620,27
620,17                         Prints "QQ"
620,17
620,0
```

```
3)P  =  P + 1
L  =  0(1)32
X  =  LX'/32
6)ψ6(A')D0,3
JUMP 4,D0  =  0                Calculates 33 points at equal intervals
                               between 0 and X' on both inner and outer
JUMP 5,D0 > X                  profiles
4)A' ≈ A' + 3
Ō  =  0(1)2
EŌ  =  DŌ
REPEAT
JUMP 6
5)C  =  E0 - D0
G  =  ψDIVIDE(-2E1 + 2D1 + CD2 + CE2,CCC)
H  =  ψDIVIDE(E1 - D1 - CD2 - GE0CC,CC)
U  =  X - D0
W  =  GX + H
Y  =  D1 + UD2 + UUW
A  =  D2 + UUG + 2UW
A  =  ψARCTAN(1,A)             Tabulates contours together with slopes in
                               degrees
NEWLINE
PRINT(X)3,3
PRINT(Y)3,3
PRINT(57·3A)3,3
REPEAT
A'  =  200
JUMP 3,P  =  1
```

A' = 0/

CHAPTER 0 (cont'd)

A' = 0 B' = 200	Sets drum addresses for surface data
620,27 620,9 620,9 620,0	Prints "II"
NEWLINE CAPTIŌN X Y M THETA P.STAT NEWLINE	Prints heading for characteristic points
620,27 620,17 620,17 620,0	Prints "QQ"
READ(T)	Reads T, which is 1 for vertical initial line
READ(P)	Reads number of initial points
JUMP 10,T = 1 ACRŌSS 1/1 10)ACRŌSS 6/1	Transfers control to appropriate point of Chapter 1

CLŌSE

→

```
0    22·5    13
22
 0         0        0
 3·0247     ·0522    2
 5·7793     ·1925    4
 8·1780     ·4003    6
```

10·1573	·6381	8
11·7826	·8945	10
13·2843	1·1853	12
14·6142	1·4905	14
15·7350	1·7890	16
16·6987	2·0828	18
17·5589	2·3781	20
18·3447	2·6793	22
19·0235	2·9658	24
19·6050	3·2366	26
20·1253	3·5008	28
20·5865	3·7562	30
21·0034	4·0060	32
21·3764	4·2482	34
21·7190	4·4878	36
22·0340	4·7250	38
22·3558	5·0136	40
22·5254	5·1832	40
2		
0	8	0
22·6	8	0
1		
8		

Typical data tape for a two-dimensional surface with eight initial points on a vertical line, see Appendix IV and Figure 1.

0	7·5	49·8	49·8	3·0	0
0	6·5	49·8	49·8	3·0	0
0	5·5	49·8	49·8	3·0	0
0	4·5	49·8	49·8	3·0	0
0	3·5	49·8	49·8	3·0	0
0	2·5	49·8	49·8	3·0	0
0	1·5	49·8	49·8	3·0	0
0	0·5	49·8	49·8	3·0	0

→

APPENDIX IV/

APPENDIX IV

Running the Programme

Information about the shape of the duct and the initial flow conditions are given to the computer on a data tape. As the machine completes its calculation of each particular example, it calls for more data, which, when read, enables it to proceed with the next example. In this way, the programme can continue ad-infinitum provided a sufficient supply of data tapes is maintained. If the machine stops for some non-programmed reason, such as a parity stop, going off-scale, etc., then the next example, or the same example if desired, can be started using the manual restart procedure, i.e., switch to single, keys 9 and 1 up, clear tape, I.T.B., switch to continuous. The machine will then call for a complete new set of data.

The layout of the data tape is as follows:-

K	must be 0 for two-dimensional flow, and 1 for axisymmetric flow
X'	overall length of duct, or the axial position at which it is desired to stop the calculation
C	an identifying number which is printed out preceded by "Case No."
N	number of points specifying inner surface
N sets of X ... Y ... θ.(in degrees)	data for inner surface
N	number of points specifying outer surface
N sets of X ... Y ... θ.(in degrees)	data for outer surface
T	= 1 for vertical initial line, $\neq 1$ for oblique initial line
P	number of initial points: $P < 21$ if $T \neq 1$; $P < 11$ if $T = 1$
P sets of XI ... YI ... AI ... BI ... FI ... GI.	co-ordinates and flow properties of initial points (see Appendix I for symbols)

The data tape should be terminated with an arrow.

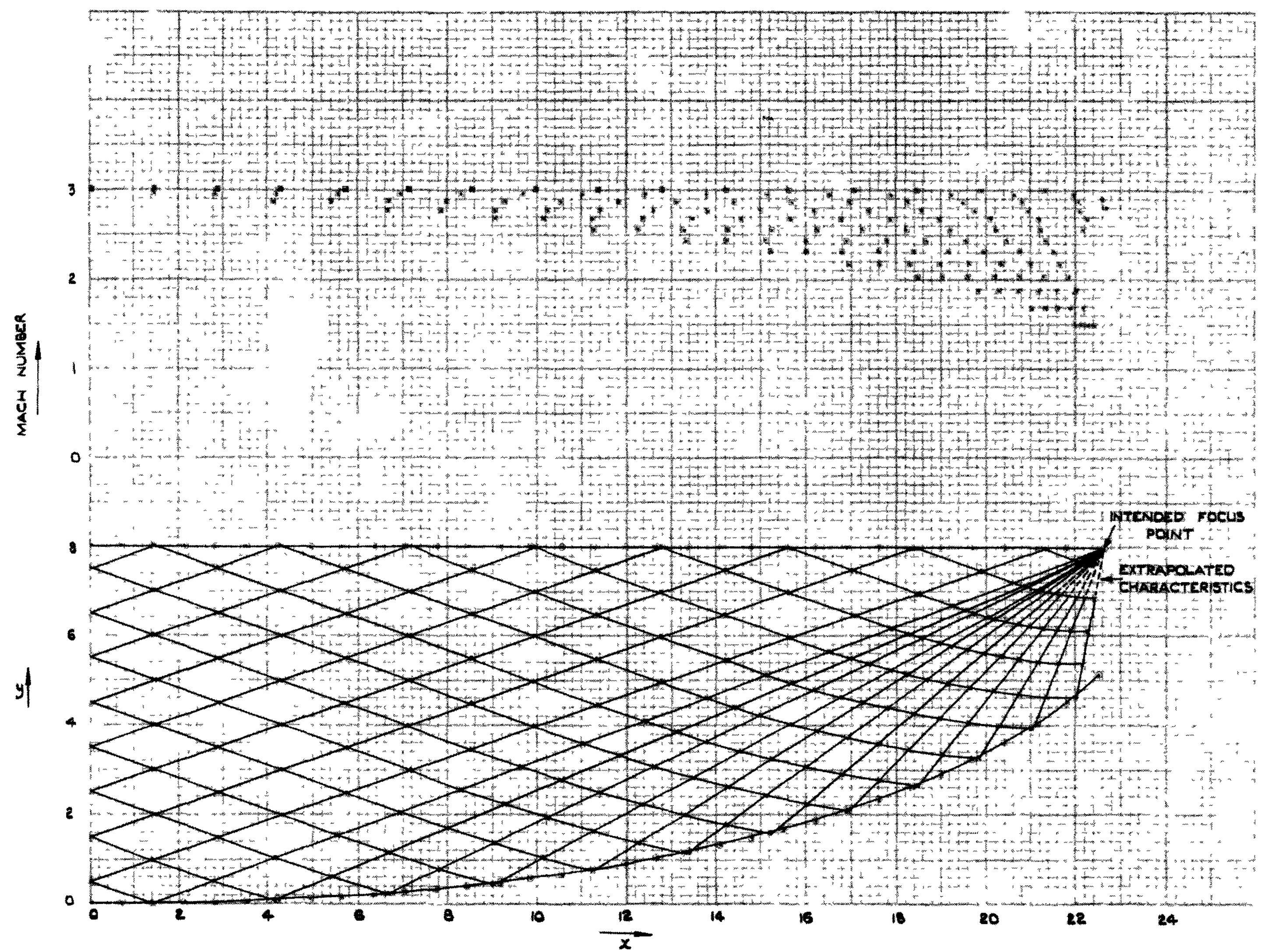
The machine prints out "Case No." followed by the number read into C from the data tape, and then prints out headed columns of co-ordinates and slopes for the surfaces. This is followed by headed columns of X, Y, M, θ and P_{stat}, which are printed out as the computation proceeds.

QQ and II are printed at suitable locations so that the output tape can be used with the R.A.E. tape operated graph plotter.

It is recommended that the scale of the duct should be such that the largest "Y" ordinate is greater than 1 - so that the allowed error of 0·001 in the surface intersection calculations is reasonable - and such that "X" is less than 100, to keep rounding errors in the polynomials small.

DS 74694/1/Wt.59 K.4 9/63 XL & CL

FIG. I



TWO DIMENSIONAL COMPRESSION-FAN FOCUSSING SURFACE

NATIONAL GAS TURBINE ESTABLISHMENT

SK 72300

TITLE

MATERIAL

SCALE

OPEN LIMITS

DEPT

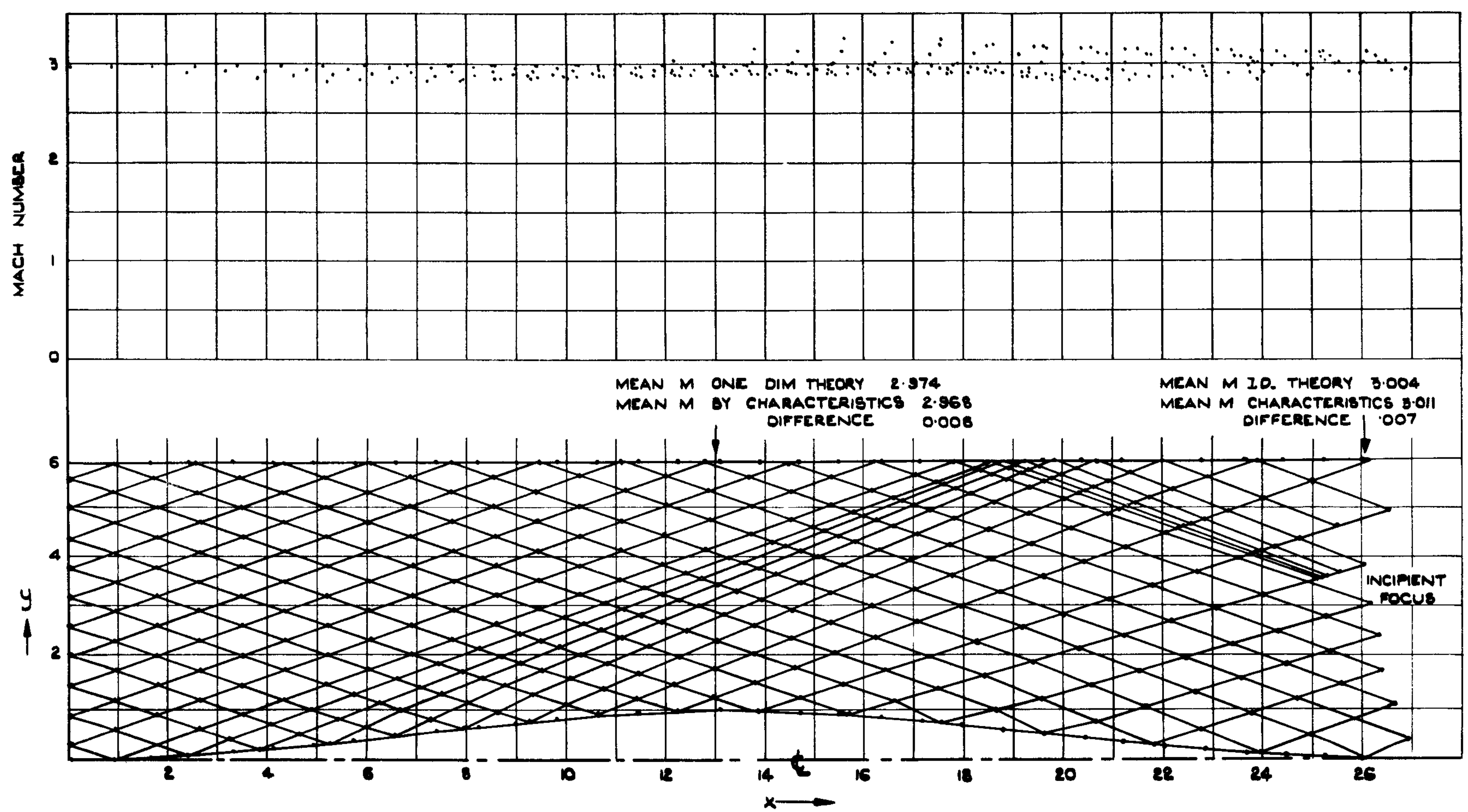
AT WHETSTONE PYESTOCK

ISSUE 1 2 3 4 5 6

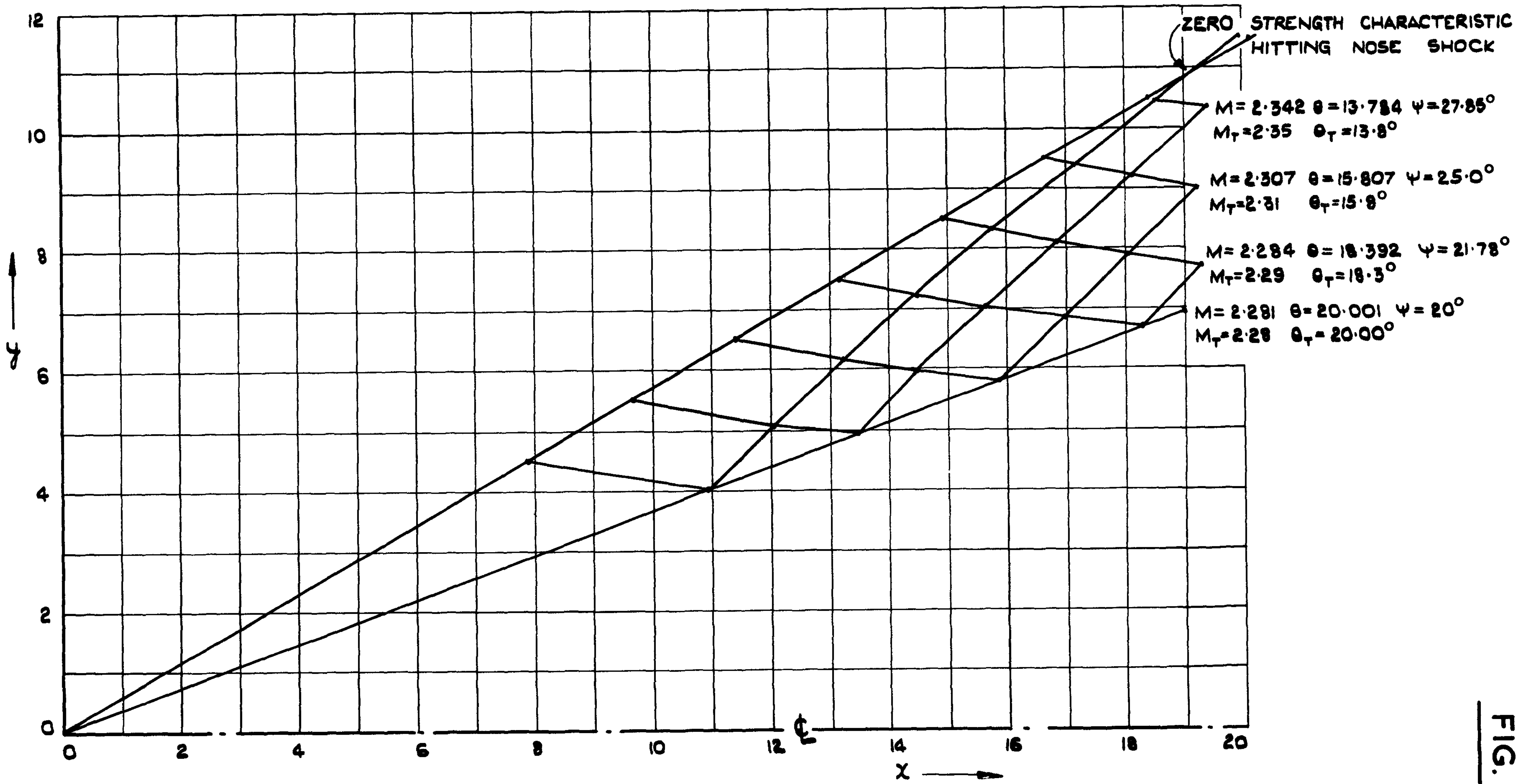
SIG

DATE

FIG. 2

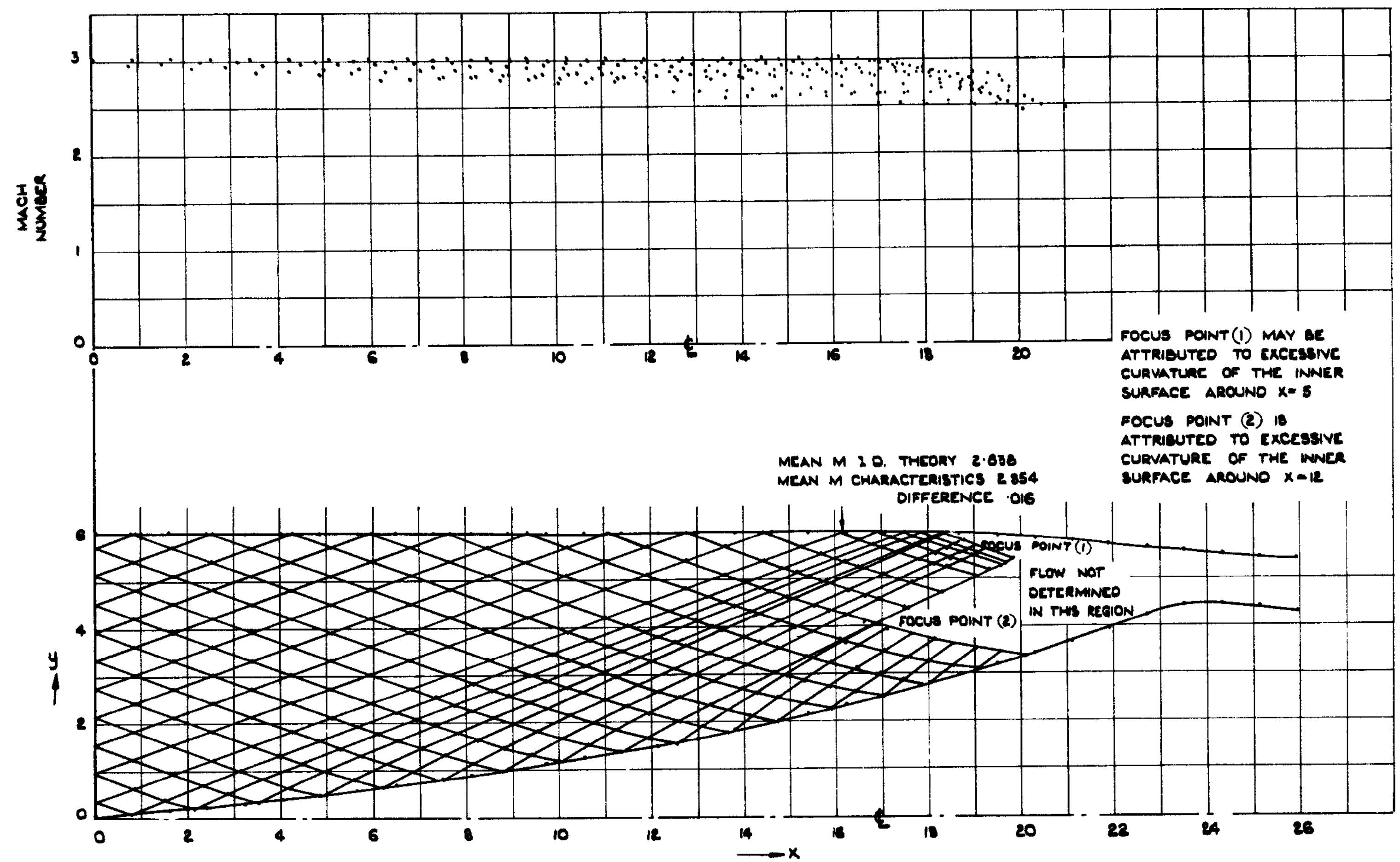


SLENDER SYMMETRICAL CENTRE-BODY IN TUBE



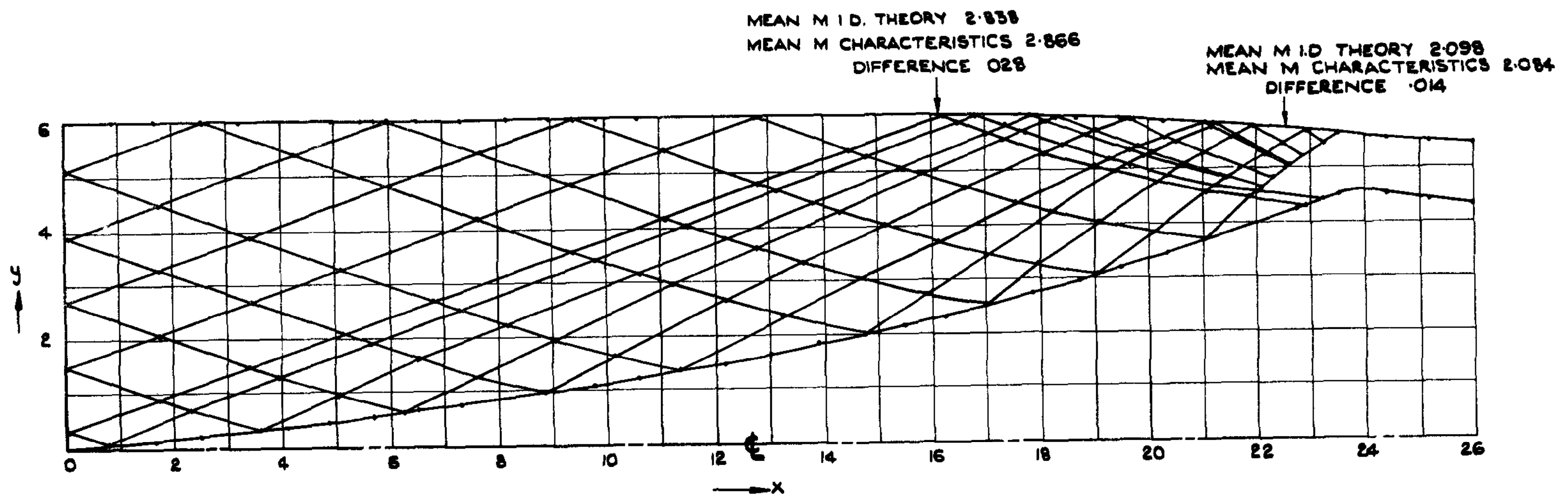
20° CONE AT MACH 3

FIG. 3



AXISYMMETRIC INTAKE WITH TWO FOCUS POINTS

FIG.5



AXISYMMETRIC INTAKE (COARSE MESH)

FIG.6

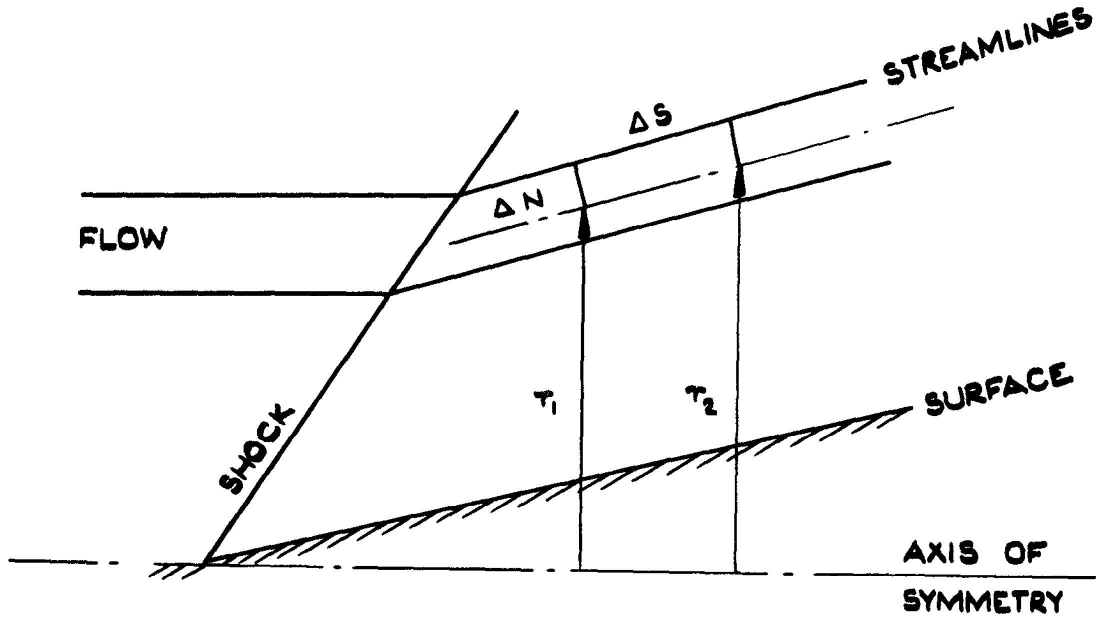


FIG.6a

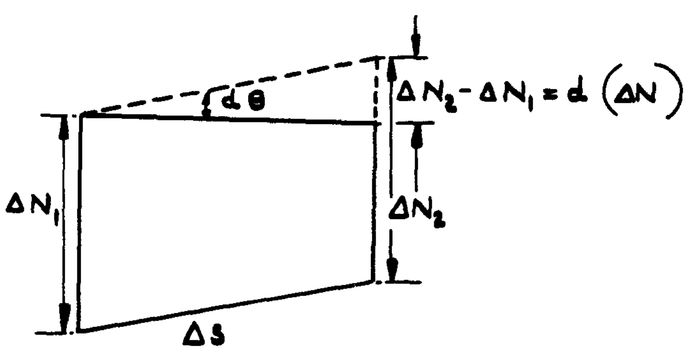


FIG.6b

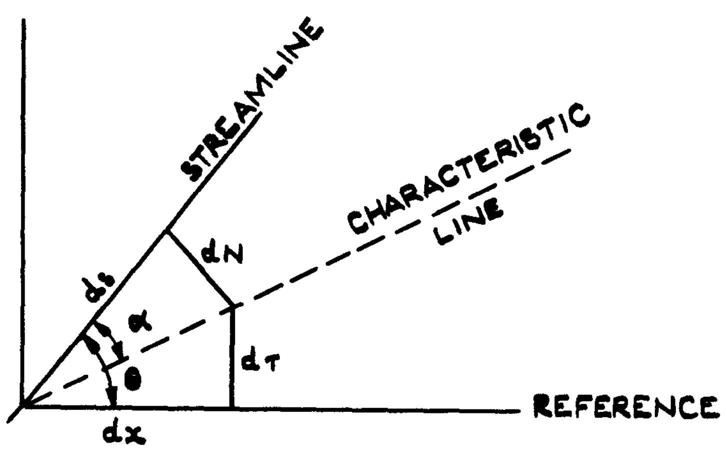


FIG.6c

GEOMETRIC RELATIONS
IN CHARACTERISTICS THEORY

A.R.C. C.P.No.649 518.5:533.6.011.5:
May, 1962 533.697.3
Street, P. G.

A DIGITAL COMPUTER PROGRAMME FOR THE CALCULATION OF ANNULAR OR TWO-DIMENSIONAL SUPERSONIC POTENTIAL FLOW IN A DUCT BY THE METHOD OF CHARACTERISTICS

A programme has been written for the Ferranti Mercury computer to analyse supersonic potential flow in a duct with uniform inlet conditions. Input data consist of duct co-ordinates, initial Mach number and flow angle. The programme calculates Mach number and flow angle at the nodal points of the characteristics net. It can be used for axisymmetric or two-dimensional flow, depending on a parameter set in the initial data.

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