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Theoretical Comparison of the Flow over a Flat Delta Wing and a Rectangular Pyramid

by

E. Eminton

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THEORETICAL COMPARISON OF THE FLOW OVER A FLAT DELTA WING AND A RECTANGULAR PYRAMID

by

E. Eminton

SUMMARY

Slender-body theory is used to compare the flow over the bottom surface of a flat delta wing at incidence and the delta-like surfaces of a pyramid of rectangular section, interest being focussed on the centre line and on the flow attachment lines where they exist. Pyramids are found which reproduce closely the flow in the neighbourhood of the delta centre line. Other pyramids are found which have the same position of the attachment lines as a delta wing, the approximation to the flow being good when the attachment lines are near the centre line but deteriorating as they move outboard.

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1 INTRODUCTION

Part of a general programme of research into the aerodynamic properties of lifting slender wings is concerned with the problem of heat transfer to the surfaces, especially near the edges and along the attachment lines. The free-flight technique is particularly suited to make an experimental contribution in this study, especially if it is possible to simulate the essential features of the flow on non-lifting bodies since the complications associated with flying lifting models can then be avoided. The purpose of this note is to investigate whether non-lifting body shapes can be found with flows which are similar to those typically found with slender lifting wings. The investigation is restricted to lifting wings of delta planform and hence to non-lifting conical bodies with triangular surfaces.

An obvious shape of this kind is the pyramid with rectangular section flying point first which is shown in Fig.1. All four surfaces in this are forward-facing like the bottom surface of a flat delta at incidence. To obtain a rearward-facing surface like the top surface of a delta wing we should need the tetrahedron also shown in Fig.1 flying edge first. The top and bottom surfaces here are like the top of a flat delta at incidence; the sides are deltas flying backwards!

One way of investigating whether the flow over these bodies retains the features typical of the flow over a flat delta at incidence is to make some wind tunnel models and compare oil flow patterns in the surface. D.A. Treadgold has made such tests with four models - one delta wing, two pyramids and one tetrahedron - at Mach numbers 1.57 and 4.4. His results will be incorporated in the report on the free-flight work that his tests were designed to help. A few flow patterns are reproduced in Fig.5.

The present note is concerned with theoretical flows obtained by applying slender-body theory to a lifting delta wing (a solution which is, of course, well known) and to a slender rectangular pyramid. The type of flow considered is allowed to have, in general, infinite velocities along the edges, i.e. vortex sheets from the edges are not considered. The flow over the top surface of a delta wing is not therefore properly represented but it will be seen that some deductions can be made about the similarity between the flows over the bottom surface and over the smaller sides of a pyramid. Because of the non-slenderness of the tetrahedron near its leading edge, this shape is not considered.

2 SURFACE VELOCITIES CALCULATED BY SLENDER-BODY THEORY

In small-perturbation theory the equation for the perturbation velocity potential ϕ becomes

$$(1-M^2)\,\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad . \tag{1}$$

The slender body theory of Munk and Jones assumes in addition that $\phi_{xx} \ll \phi_{yy}, \phi_{zz}$. Consequently the potential equation becomes

$$\phi_{yy} + \phi_{zz} = 0 \tag{2}$$

and the potential is of the form

$$\phi(x,y,z) = \phi_1(y,z;x) + \phi_2(x) \tag{3}$$

where ϕ_1 is the solution of this two-dimensional Laplace equation in the transverse plane x = constant. The solution of the equation depends on the conditions which the flow is required to satisfy at its boundaries. Usually, as in our problem, the disturbance created by the body must die out at infinity, and on the body the velocity must be tangential to the surface.

Let us introduce complex variables and denote a point in the transverse plane x = constant by $t = y + iz$. Since the body is slender we can use the boundary condition on the surface to give explicitly V_n, the velocity in the transverse plane normal to and away from the body cross-section. If we know a relation which transforms the region outside this cross-section in the t-plane into the outside of a circle in some other plane where $t' = y' + iz'$, the normal velocity in this new plane will be

$$V_n' = \left|\frac{dt}{dt'}\right| V_n \tag{4}$$

and we can appeal to a standard result, recalled in Appendix 1, to give the corresponding velocities tangential to the cross-section, V_t' and V_t. Finally since the body is slender we can show that the other component of velocity in the surface, V_s, is to first order equal to U, the velocity of the undisturbed stream.

2.1 Surface velocities on a flat delta wing at incidence

We consider a flat delta wing inclined at an angle α to a uniform stream of speed U. The velocity distribution on its surface is well known but for completeness is derived here by the same method as will be applied in the next section to a pyramid. The x-axis lies in the stream direction and at the point $x = \ell$ bisects the trailing edge, which is parallel to the y-axis and of length $2a\ell$. The nose lies on the z-axis at the point $z = b\ell$ so that $b = \tan\alpha$. Fig.2 shows the notation.

The cross-section in the transverse plane x = constant is the line

$$z = b(\ell - x), \qquad -ax \leqslant y \leqslant ax \quad . \tag{5}$$

Introducing complex variables $t = y + iz$, we can relate the t-plane outside this line to the t'-plane outside a circle of radius ax/2 by the transformation

$$t - ib(\ell - x) = t' + \left(\frac{ax}{2}\right)^2 \frac{1}{t'} , \tag{6}$$

whence

$$\frac{dt}{dt'} = 1 - \left(\frac{ax}{2}\right)^2 \frac{1}{t'^2} \quad . \tag{7}$$

On the circle, let

$$t' = \frac{ax}{2} e^{i\theta}, \qquad 0 \leqslant \theta \leqslant 2\pi \quad . \tag{8}$$

Then

$$t = ax \cos\theta + ib(\ell - x) \tag{9}$$

$$\frac{dt}{dt'} = 1 - e^{-2i\theta} \tag{10}$$

and

$$\left|\frac{dt}{dt'}\right| = 2\,|\sin\theta| \quad . \tag{11}$$

If we assume that a and b are both small - $O(\varepsilon)$ where $\varepsilon \ll 1$ - then according to Ward[1] ϕ_x/U is even smaller - $O(\varepsilon^2 \log \varepsilon)$. To satisfy the boundary condition that requires the component of velocity normal to the surface to be zero we must have

$$(U + \phi_x) \sin\alpha + \phi_z \cos\alpha = 0 \tag{12}$$

whence

$$\phi_z/U = -b + O(\varepsilon^2) \quad . \tag{13}$$

The velocity V_n has the same magnitude as ϕ_z but is directed away from the surface so that

$$\begin{aligned} V_n/U &= -b + O(\varepsilon^2), \text{ above the wing} \\ &= \ \ b + O(\varepsilon^2), \text{ below the wing.} \end{aligned} \tag{14}$$

In the t'-plane therefore, by (4) and (11),

$$V_n'/U = -2b \sin\theta + O(\varepsilon^2), \qquad 0 \leq \theta \leq 2\pi \quad . \tag{15}$$

We can now apply the result of Appendix 1 to give V_t as follows: since V_n' is an odd function of θ we use equation (40) which becomes

$$\begin{aligned} \frac{1}{r}\phi_\theta/U &= -\frac{2b}{\pi}\int_0^\pi \frac{\sin^2\zeta \, d\zeta}{\cos\zeta - \cos\theta} + O(\varepsilon^2), \qquad 0 \leq \theta \leq 2\pi, \\ &= 2b\cos\theta + O(\varepsilon^2), \qquad 0 \leq \theta \leq 2\pi \quad . \end{aligned} \tag{16}$$

If the corresponding velocity in the t-plane is V_t, taken anti-clockwise around the cross-section, we have by (4) and (11)

$$V_t/U = \frac{b \cos \theta}{|\sin \theta|} + O(\varepsilon^2), \qquad 0 \leqslant \theta \leqslant 2\pi \quad . \tag{17}$$

In this case V_t is always parallel to the y-axis, so that equation (17) can also be written

$$V_y/U = - b \cot \theta + O(\varepsilon^2), \qquad 0 \leqslant \theta \leqslant 2\pi \quad . \tag{18}$$

The other component of velocity tangential to the surface, V_s, is given by

$$\begin{aligned} V_s &= (U + \phi_x) \cos \alpha - \phi_z \sin \alpha \\ &= (U + \phi_x) \sec \alpha \end{aligned} \tag{19}$$

using the boundary condition (12) to eliminate ϕ_z, whence

$$V_s/U = 1 + O(\varepsilon) \quad . \tag{20}$$

2.2 Surface velocities on a rectangular pyramid

We consider a pyramid of rectangular section pointing into a uniform stream of speed U. The origin of co-ordinates is at the apex of the pyramid. The x-axis lies in the stream direction and coincides with the axis of the pyramid, which is of length ℓ. The edges of its base are parallel to the y and z axes and are of length $2a\ell$ and $2b\ell$ respectively. Fig.2 shows the notation.

The cross-section in the transverse plane x = constant is the rectangle with sides

$$\begin{aligned} z &= \pm bx, \qquad - ax \leqslant y \leqslant ax \\ y &= \pm ax, \qquad - bx \leqslant z \leqslant bx \quad . \end{aligned} \tag{21}$$

Introducing complex variables $t = y + iz$, we can relate the t-plane outside this rectangle to the t'-plane outside a circle of radius r by the transformation*

$$\frac{dt}{dt'} = \sqrt{\left(1 - \frac{2r^2}{t'^2} \cos 2\gamma + \frac{r^4}{t'^4}\right)} \quad . \tag{22}$$

The corners of the rectangle $t = \pm ax \pm ibx$ correspond to the four points $t' = \pm re^{\pm i\gamma}$.

*This is a particular case of a transformation given by Jeffreys and Jeffreys[2].

It can be shown that r is a function of a, b and x but γ depends only on the ratio a/b. We must find this dependence in order to work specific examples; details are given in Appendix 2 and Fig.10. On the circle, let

$$t' = re^{i\theta}, \quad 0 \leq \theta \leq 2\pi \quad . \tag{23}$$

Then

$$\frac{dt}{dt'} = 2e^{-i\theta} \sqrt{(\cos^2 \theta - \cos^2 \gamma)} \tag{24}$$

$$\left|\frac{dt}{dt'}\right| = 2 \sqrt{|\cos^2 \theta - \cos^2 \gamma|} \tag{25}$$

and

$$t = 2ir \int \sqrt{(\cos^2 \theta - \cos^2 \gamma)} \, d\theta \quad . \tag{26}$$

If we assume, as in section 2.1, that a and b are both $O(\varepsilon)$ where $\varepsilon \ll 1$ then ϕ_x/U is $O(\varepsilon^2 \log \varepsilon)$ and we can obtain V_n from the boundary condition that requires the component of velocity normal to the surface to be zero. The top and bottom faces are inclined at an angle α to the horizontal where $\tan \alpha = b$, and V_n on these faces has the same magnitude as ϕ_z but is directed away from the surface. To satisfy the boundary condition on these faces we must have

$$(U + \phi_x) \sin \alpha = V_n \cos \alpha \tag{27}$$

whence

$$V_n/U = b + O(\varepsilon^2) \quad .$$

Similarly on the side faces

$$V_n/U = a + O(\varepsilon^2) \quad . \tag{28}$$

In the t'-plane, therefore, by (4) and (25),

$$\begin{aligned} V_n'/U &= 2a \sqrt{(\cos^2 \theta - \cos^2 \gamma)} + O(\varepsilon^2), \quad 0 \leq \theta \leq \gamma \\ &= 2b \sqrt{(\cos^2 \gamma - \cos^2 \theta)} + O(\varepsilon^2), \quad \gamma \leq \theta \leq \pi/2 \end{aligned} \tag{29}$$

and by symmetry

$$V_n'(\theta) = V_n'(-\theta) = V_n'(\pi - \theta) \quad . \tag{30}$$

We can now apply the result of Appendix 1 to give V_t as follows: since V'_n is an even function of θ we use equation (39) which becomes

$$\frac{1}{r}\phi_\theta/U = \frac{2}{\pi}\sin\theta\left\{ a\int_0^\gamma \frac{\sqrt{(\cos^2\zeta - \cos^2\gamma)}}{\cos\zeta - \cos\theta}\,d\zeta + a\int_{\pi-\gamma}^{\pi} \frac{\sqrt{(\cos^2\zeta - \cos^2\gamma)}}{\cos\zeta - \cos\theta}\,d\zeta\right.$$

$$\left. + b\int_\gamma^{\pi-\gamma} \frac{\sqrt{(\cos^2\gamma - \cos^2\zeta)}}{\cos\zeta - \cos\theta}\,d\zeta + O(\varepsilon^2)\right\},\quad 0 \leqslant \theta \leqslant 2\pi$$

$$= \frac{2}{\pi}\sin 2\theta\left\{ a\int_0^\gamma \frac{\sqrt{(\cos^2\zeta - \cos^2\gamma)}}{\cos^2\zeta - \cos^2\theta}\,d\zeta + b\int_\gamma^{\pi/2} \frac{\sqrt{(\cos^2\gamma - \cos^2\zeta)}}{\cos^2\zeta - \cos^2\theta}\,d\zeta\right.$$

$$\left. + O(\varepsilon^2)\right\},\quad 0 \leqslant \theta \leqslant 2\pi \quad . \tag{31}$$

If the corresponding velocity in the t-plane is V_t taken anti-clockwise around the cross-section, we have by (4) and (25)

$$V_t/U = \frac{\sin 2\theta}{\pi\sqrt{|\cos^2\theta - \cos^2\gamma|}}\left\{ a\int_0^\gamma \frac{\sqrt{(\cos^2\zeta - \cos^2\gamma)}}{\cos^2\zeta - \cos^2\theta}\,d\zeta\right.$$

$$+ b\int_\gamma^{\pi/2} \frac{\sqrt{(\cos^2\gamma - \cos^2\zeta)}}{\cos^2\zeta - \cos^2\theta}\,d\zeta$$

$$\left. + O(\varepsilon^2)\right\},\quad 0 \leqslant \theta \leqslant 2\pi \quad . \tag{32}$$

Equations (26) and (32) for t and the corresponding velocity V_t involve elliptic integrals. They can, with a little manipulation, be expressed in terms of elliptic integrals of first and second kind. This is an important step in evaluating specific examples, but the details have been confined to Appendix 2.

The other component of velocity tangential to the surface, V_s, is given on the top and bottom faces by

$$V_s = (U + \phi_x) \cos \alpha + V_n \sin \alpha$$

$$= (U + \phi_x) \sec \alpha \tag{33}$$

using the boundary condition (27) to eliminate V_n.

Hence

$$V_s/U = 1 + O(\varepsilon) \tag{34}$$

an expression that holds true on the side faces as well.

3 RESULTS

To compare the bottom surface of a flat delta at incidence with any one surface of a rectangular pyramid let us take new axes in the surface: the origin at the apex, $O\xi$ along the centre line and $O\eta$ perpendicular to it. The components of velocity in these two directions are V_s and V_t, derived in section 2 by slender body theory. The form of the results confirms, of course, that the flow is conical since V_s is constant and both V_t/U and η/ξ are functions of a single variable θ. Figs.3 and 4 show the variation of V_t/Ua with $\eta/\xi a$ on the surface defined by $\eta = \pm \xi a$ for different values of b/a. At first sight there appears a marked similarity between the two sets of curves but notice that among the pyramid curves only those for $b/a > 1$ resemble the delta family. This means that it is the two smaller faces of a pyramid that correspond to the bottom surface of a delta.

For these results to be valid they must be consistent with the slender-body assumptions which imply that $V_s/U = 1 + O(\varepsilon)$ and $V_t/U = O(\varepsilon)$ where $\varepsilon \ll 1$. The condition on V_t is apparently violated near the leading edge since $V_t/Ua \to \infty$ as $\eta \to \xi a$, both on the bottom surface of a delta and on the smaller faces of a pyramid. Experimental evidence is known to show that viscous flow cannot negotiate these corners without separating. The oil flow patterns reproduced in Fig.5 from Treadgold's photographs suggest the separations sketched in Fig.6. To take proper account of this we should have to introduce some model of the separated flow such as that devised by K.W. Mangler and J.H.B. Smith[3]. This would alter appreciably the flow over the faces on which the vortices lie but on the faces that we are comparing our results should still be significant away from the leading edge. The broken curves in Fig.3 that support this assertion are from unpublished calculations by Smith.

An important part of the free-flight programme is an investigation of the heating effects of the flow. Of particular interest are the heating rates along the flow attachment lines. These lines are not easy to define in general but if the flow is conical they must be straight lines through the apex which form a parting from which the surface streamlines diverge.

In all our results the ξ-axis is just such a line and of interest as a line of symmetry. Fig.7 therefore takes each of the relevant curves from Fig.4 and pairs it with that curve of the delta family which best fits it near the origin. This relates each pyramid with $b/a \geq 1$ to a delta with $b/a \geq 0.43$.

In general the flow is directed along a straight line through the apex wherever

$$V_t/V_s = \eta/\xi \tag{35}$$

or, since $V_s/U = 1 + O(\varepsilon)$, wherever

$$V_t/Ua = \eta/\xi a \quad . \tag{36}$$

Fig.3 and 4 confirm that this is always true along the centre line where $\eta = V_t = 0$ and show that for limited ranges of b/a there is a second pair of lines on both delta wing and pyramid where equation (36) is satisfied, and from which the streamlines diverge. On a delta wing the range of b/a is $0 \leq b/a \leq 1$; on a pyramid it is $1 \leq b/a \leq 1.84$. In each case the lines move inwards from leading edge to centre line as b/a increases from one limit to the other. Fig.8 shows a selection of results between these limits paired according to the position of these attachment lines. Fig.9 shows how the position varies with b/a. Unfortunately none of Treadgold's models allow us to check the existence of these lines since for his values of b/a the theory either predicts no attachment lines at all or places them too close to the leading edge for the theory to be reliable.

Summarising - for a delta wing of given b/a, we can find a pyramid which gives quite a good approximation to the flow in the vicinity of the centre line whenever b/a is greater than about 0.43. For values of b/a less than 1 there is another pair of attachment lines and we can find a pyramid which exactly reproduces the position of these lines; but the smaller b/a becomes the further the lines move from the centre line and the worse the approximation becomes over the rest of the flow.

LIST OF SYMBOLS

a, b	dimensions of models defined in Fig.2
E, F, K	standard elliptic integrals (defined in Appendix 2)
ℓ	dimension of models defined in Fig.2
r	r, θ - polar co-ordinates in the t'-plane
$t = y + iz$	the t-plane is the transverse plane x = constant
$t' = y' + iz' = re^{i\theta}$	the t'-plane is a transformation of the t-plane
U	free stream velocity
V_n, V_n'	velocities normal to the cross-section in t- and t'-planes
V_s	velocity tangential to the surface and normal to V_t
V_t, V_t'	velocities tangential to the cross-section in t- and t'-planes
V_y	velocity in the y-direction

LIST OF SYMBOLS (Cont'd)

x, y, z	Cartesian co-ordinates in space
y', z'	Cartesian co-ordinates in the t'-plane
Z	standard elliptic integral (defined in Appendix 2)
α	inclination of model surface defined in Fig.2 (also used in Appendix 2 in another sense)
γ	parameter related to model dimensions
ε	measure of the magnitude of a, b and α
ξ, η	Cartesian co-ordinates in the surface
θ	r, θ - polar co-ordinates in the t'-plane
ϕ	perturbation velocity potential (also used in Appendix 2 in another sense)
Π, Λ_0	standard elliptic integrals (defined in Appendix 2).

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APPENDIX 1

THE TANGENTIAL VELOCITY ASSOCIATED WITH A GIVEN RADIAL VELOCITY DISTRIBUTION AROUND A CIRCLE

The solution of Laplace's equation in two dimensions which produces a given radial velocity around a circle can be constructed by distributing elementary sources around the circumference and at the centre*.

A source of strength m at the centre of a circle of radius r produces a radial velocity m/r at the circumference. A source distribution of intensity $q(\theta)$, where $\int_0^{2\pi} q(\theta)\, d\theta = 0$, around the circle produces a radial velocity $\pi\, q(\theta)$ at the circumference. A radial velocity $V_n(\theta)$ can therefore be produced by a source distribution of intensity $\frac{1}{\pi}(V_n(\theta) - \bar{V}_n)$ where $\bar{V}_n = \frac{1}{2\pi}\int_0^{2\pi} V_n(\theta)\, d\theta$, around the circle and a source of strength $r\bar{V}_n$ at the centre.

The potential at the circumference is

$$\phi(re^{i\theta}) = r\bar{V}_n \log r + \int_0^{2\pi} \frac{1}{\pi}(V_n(\zeta) - \bar{V}_n)\, r\, d\zeta \log 2r\left|\sin\frac{\theta-\zeta}{2}\right| . \tag{37}$$

The corresponding tangential velocity is therefore

$$\frac{1}{r}\frac{\partial\phi(re^{i\theta})}{\partial\theta} = \frac{1}{2\pi}\int_0^{2\pi} V_n(\zeta)\cot\frac{\theta-\zeta}{2}\, d\zeta + \frac{\bar{V}_n}{2\pi}\int_0^{2\pi} \cot\frac{\theta-\zeta}{2}\, d\zeta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} V_n(\zeta)\cot\frac{\theta-\zeta}{2}\, d\zeta \quad . \tag{38}$$

If V_n is an even function of θ so that $V_n(\theta) = V_n(-\theta)$ then equation (38) reduces to

$$\frac{1}{r}\frac{\partial\phi(re^{i\theta})}{\partial\theta} = \frac{1}{\pi}\int_0^{\pi} \frac{\sin\theta\, V_n(\zeta)}{\cos\zeta - \cos\theta}\, d\zeta \tag{39}$$

and if V_n is an odd function of θ so that $V_n(\theta) = -V_n(-\theta)$ then it reduces to

$$\frac{1}{r}\frac{\partial\phi(re^{i\theta})}{\partial\theta} = \frac{1}{\pi}\int_0^{\pi} \frac{\sin\zeta\, V_n(\zeta)}{\cos\zeta - \cos\theta}\, d\zeta \tag{40}$$

---

*This technique appears in a paper by J. Weber[5] who attributes it to A. Betz.

APPENDIX 2

EVALUATION OF CERTAIN ELLIPTIC INTEGRALS IN TERMS OF THE FUNCTIONS $E(\phi,k)$ and $F(\phi,k)$

Since notations vary so much from book to book we shall first define the elliptic integrals that appear in this Appendix. They are: the elliptic integral of the first kind

$$F(\phi,k) \equiv \int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}, \qquad K(k) \equiv F(\pi/2,k);$$

the elliptic integral of the second kind

$$E(\phi,k) \equiv \int_0^{\phi} \sqrt{1-k^2 \sin^2 \phi}\, d\phi, \qquad E(k) \equiv E(\pi/2,k);$$

and the complete elliptic integral of the third kind

$$\Pi(\alpha^2,k) \equiv \int_0^{\pi/2} \frac{d\phi}{(1-\alpha^2 \sin^2 \phi)\sqrt{1-k^2 \sin^2 \phi}} .$$

Also appearing are $K(k)Z(\phi,k)$ and $\Lambda_0(\phi,k)$ which are defined as they arise. All relations quoted here can be found in the introduction to Ref.6.

1 Evaluation of equation (26)

$$t = 2ir \int \sqrt{(\cos^2 \theta - \cos^2 \gamma)}\, d\theta .$$

When $0 \leqslant \theta \leqslant \gamma$ we substitute ϕ for θ where $\sin \phi = \sin \theta / \sin \gamma$ and write $\sin \gamma = k$ so that

$$t(\theta) - t(0) = 2ir \int_0^{\theta} \sqrt{(\cos^2 \theta - \cos^2 \gamma)}\, d\theta$$

$$= 2ir \int_0^{\phi} \frac{k^2 \cos^2 \phi}{\sqrt{1-k^2 \sin^2 \phi}}\, d\phi$$

$$= 2ir \{E(\phi,k) - (1-k^2)\, F(\phi,k)\} .$$

When $\gamma \leqslant \theta \leqslant \pi/2$ we substitute ϕ' for θ where $\sin \phi' = \cos \theta / \cos \gamma$ and write $\cos \gamma = k'$ so that

$$t(\theta) - t(\pi/2) = 2r \int_{\theta}^{\pi/2} \sqrt{(\cos^2 \gamma - \cos^2 \theta)}\, d\theta$$

$$= 2r \int_{0}^{\phi'} \frac{k'^2 \cos^2 \phi'}{\sqrt{1-k'^2 \sin^2 \phi'}}\, d\phi'$$

$$= 2r \{E(\phi', k') - (1-k'^2)\, F(\phi', k')\} \quad .$$

In particular, when $\theta = \gamma$ we deduce

$$ax/2r = E(k') - k^2\, K(k')$$

$$bx/2r = E(k) - k'^2\, K(k) \quad .$$

The ratio a/b is therefore a function of γ only and, given this ratio, r is directly proportional to ax and bx which are the dimensions of the cross-section at x. Fig.10 shows the relation between a/b and γ.

2 Evaluation of equation (32)

$$V_t/U = \frac{\sin 2\theta}{\pi\sqrt{|\cos^2 \theta - \cos^2 \gamma|}} \left\{ a \int_0^{\gamma} \frac{\sqrt{(\cos^2 \zeta - \cos^2 \gamma)}}{\cos^2 \zeta - \cos^2 \theta}\, d\zeta \right.$$

$$\left. + \, b \int_{\gamma}^{\pi/2} \frac{\sqrt{(\cos^2 \gamma - \cos^2 \zeta)}}{\cos^2 \zeta - \cos^2 \theta}\, d\zeta + O(\varepsilon^2) \right\} .$$

Of these two integrals, the first is singular when $0 \leqslant \theta \leqslant \gamma$ and the second when $\gamma \leqslant \theta \leqslant \pi/2$. In the first integral we substitute ξ for ζ where $\sin \xi = \sin \zeta / \sin \gamma$ and write $\sin \gamma = k$, $\sin \gamma / \sin \theta = \alpha$ so that

$$\int_0^{\gamma} \frac{\sqrt{(\cos^2 \zeta - \cos^2 \gamma)}}{\cos^2 \zeta - \cos^2 \theta}\, d\zeta = \int_0^{\pi/2} \frac{\alpha^2 \cos^2 \xi}{1-\alpha^2 \sin^2 \xi} \frac{d\xi}{\sqrt{1-k^2 \sin^2 \xi}}$$

$$= K(k) - (1-\alpha^2)\, \Pi\, (\alpha^2, k) \quad .$$

If $0 \leqslant \theta \leqslant \gamma$ then $\alpha \geqslant 1$ and

$$(1-\alpha^2)\ \Pi\ (\alpha^2,k) \quad = \quad \alpha \sqrt{\frac{\alpha^2-1}{\alpha^2-k^2}}\ K(k)\ Z\ (\phi,k)$$

$$= \frac{\sqrt{(\cos^2\theta - \cos^2\gamma)}}{\sin\theta\ \cos\theta}\ [K(k)\ E(\phi,k) - E(k)\ F(\phi,k)]$$

where

$$\sin\phi \quad = \quad 1/\alpha \quad = \quad \sin\theta/\sin\gamma \quad .$$

But if $\gamma \leqslant \theta \leqslant \pi/2$ then $k \leqslant \alpha \leqslant 1$ and

$$(1-\alpha^2)\ \Pi\ (\alpha^2,k) \quad = \quad \alpha \sqrt{\frac{1-\alpha^2}{\alpha^2-k^2}}\ \frac{\pi}{2}\ \Lambda_o\ (\phi',k)$$

$$= \frac{\sqrt{(\cos^2\gamma - \cos^2\theta)}}{\sin\theta\ \cos\theta}\ [E(k)\ F(\phi',k') + K(k)\ E(\phi',k') - K(k)\ F(\phi',k')]$$

where

$$\sin\phi' \quad = \quad \frac{1}{\alpha}\sqrt{\frac{\alpha^2-k^2}{1-k^2}} \quad = \quad \cos\theta/\cos\gamma \quad .$$

In the second integral we substitute ξ for ζ where $\sin\xi = \cos\zeta/\cos\gamma$ and write $\cos\gamma = k'$, $\cos\gamma/\cos\theta = \alpha'$ so that

$$\int_{\gamma}^{\pi/2} \frac{\sqrt{(\cos^2\gamma - \cos^2\zeta)}}{\cos^2\zeta - \cos^2\theta}\, d\zeta \quad = \quad \int_{0}^{\pi/2} \frac{\alpha'^2 \cos^2\xi}{\alpha'^2 \sin^2\xi - 1}\, \frac{d\xi}{\sqrt{1-k'^2\sin^2\xi}}$$

$$= \quad (1-\alpha'^2)\ \Pi\ (\alpha'^2,k') - K(k') \quad .$$

If $0 \leqslant \theta \leqslant \gamma$ then $k' \leqslant \alpha' \leqslant 1$ and

$$(1-\alpha'^2)\,\Pi\,(\alpha'^2,k') \;=\; \alpha'\sqrt{\frac{1-\alpha'^2}{\alpha'^2-k'^2}}\,\frac{\pi}{2}\,\Lambda_o(\phi,k')$$

$$= \frac{\sqrt{(\cos^2\theta - \cos^2\gamma)}}{\sin\theta\cos\theta}\,[E(k')\,F(\phi,k) + K(k')\,E(\phi,k) - K(k')\,F(\phi,k)]$$

where

$$\sin\phi \;=\; \frac{1}{\alpha'}\sqrt{\frac{\alpha'^2-k'^2}{1-k'^2}} \;=\; \sin\theta/\sin\gamma \quad .$$

But if $\gamma \leqslant \theta \leqslant \pi/2$ then $\alpha' \geqslant 1$ and

$$(1-\alpha'^2)\,\Pi\,(\alpha'^2,k') \;=\; \alpha'\sqrt{\frac{\alpha'^2-1}{\alpha'^2-k'^2}}\,K(k')\,Z(\phi',k')$$

$$= \frac{\sqrt{(\cos^2\gamma - \cos^2\theta)}}{\sin\theta\cos\theta}\,[K(k')\,E(\phi',k') - E(k')\,F(\phi',k')]$$

where

$$\sin\phi' \;=\; 1/\alpha' \;=\; \cos\theta/\cos\gamma \quad .$$

Collecting these results together we have when $0 \leqslant \theta \leqslant \gamma$

$$V_t/U \;=\; [aK(k) - bK(k')]\left[\frac{\sin 2\theta}{\pi\sqrt{(\cos^2\theta - \cos^2\gamma)}} - \frac{2}{\pi}\,E(\phi,k)\right]$$

$$+\,\frac{2}{\pi}\,[aE(k) + bE(k') - bK(k')]\,F(\phi,k)$$

and when $\gamma \leqslant \theta \leqslant \pi/2$

$$V_t/U \;=\; [aK(k) - bK(k')]\left[\frac{\sin 2\theta}{\pi\sqrt{(\cos^2\gamma - \cos^2\theta)}} - \frac{2}{\pi}\,E(\phi',k')\right]$$

$$-\,\frac{2}{\pi}\,[aE(k) + bE(k') - aK(k)]\,F(\phi',k') \quad .$$

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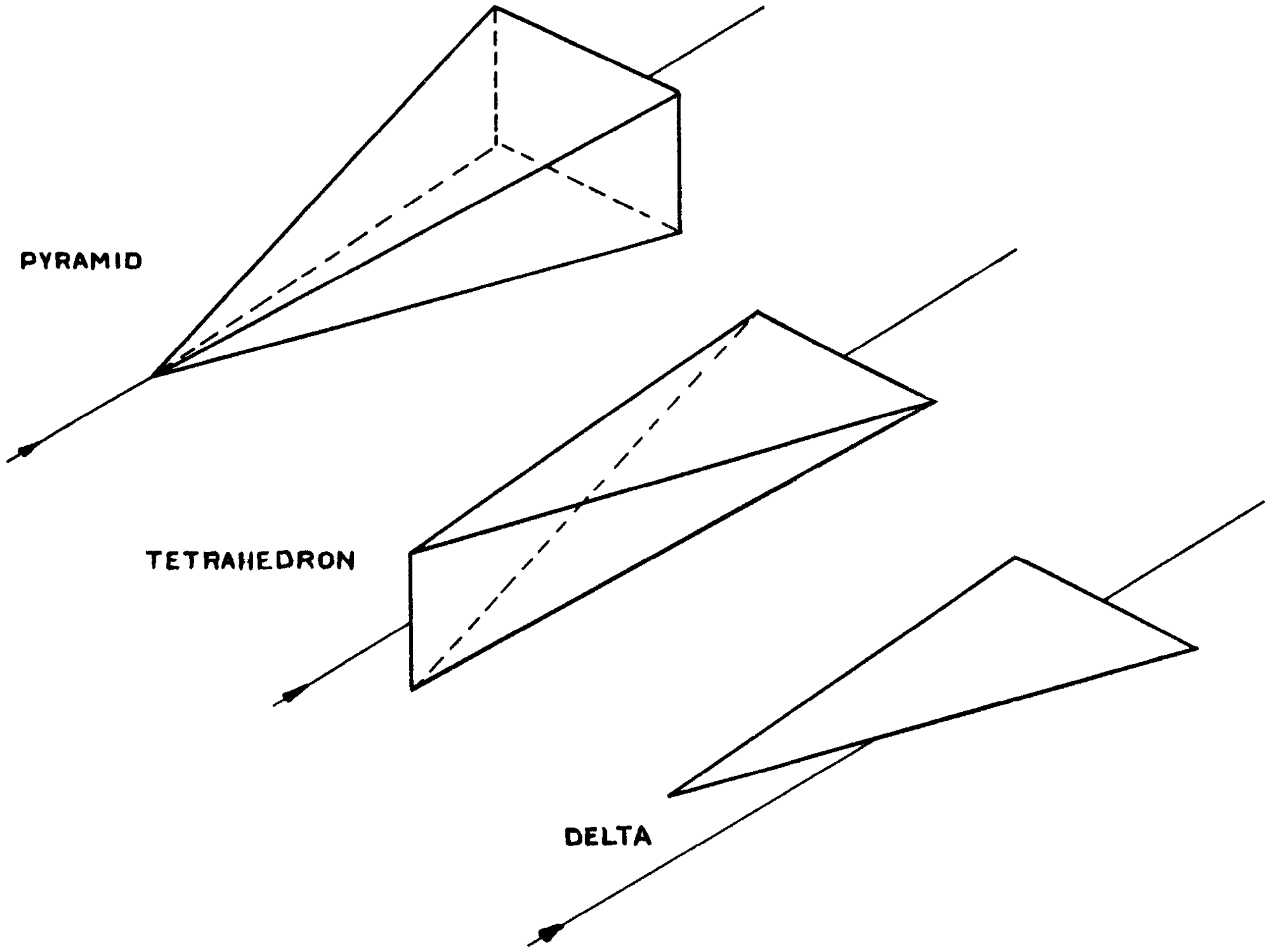


FIG. 1. MODELS WITH DELTA-LIKE SURFACES.

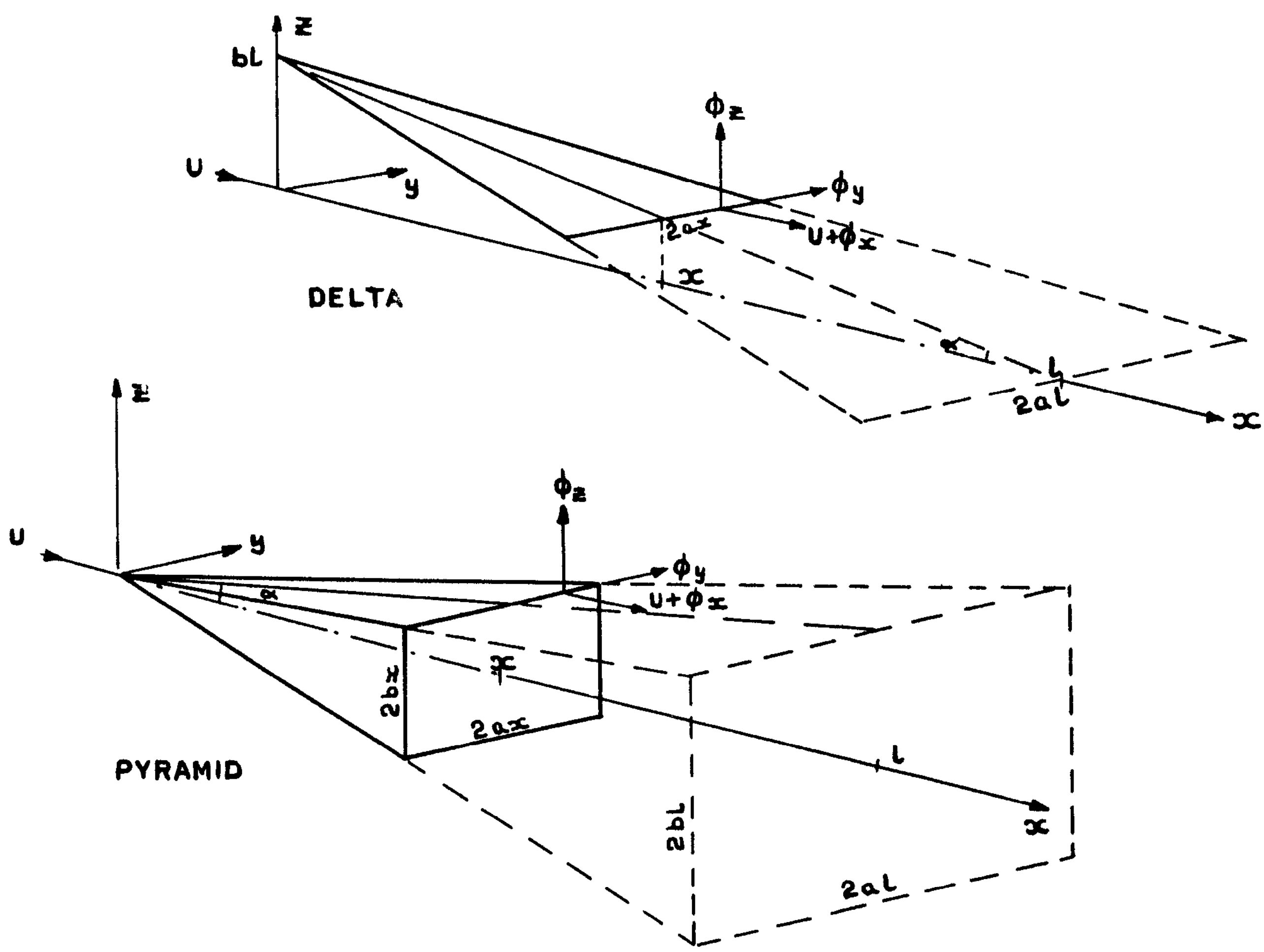


FIG. 2. NOTATION.

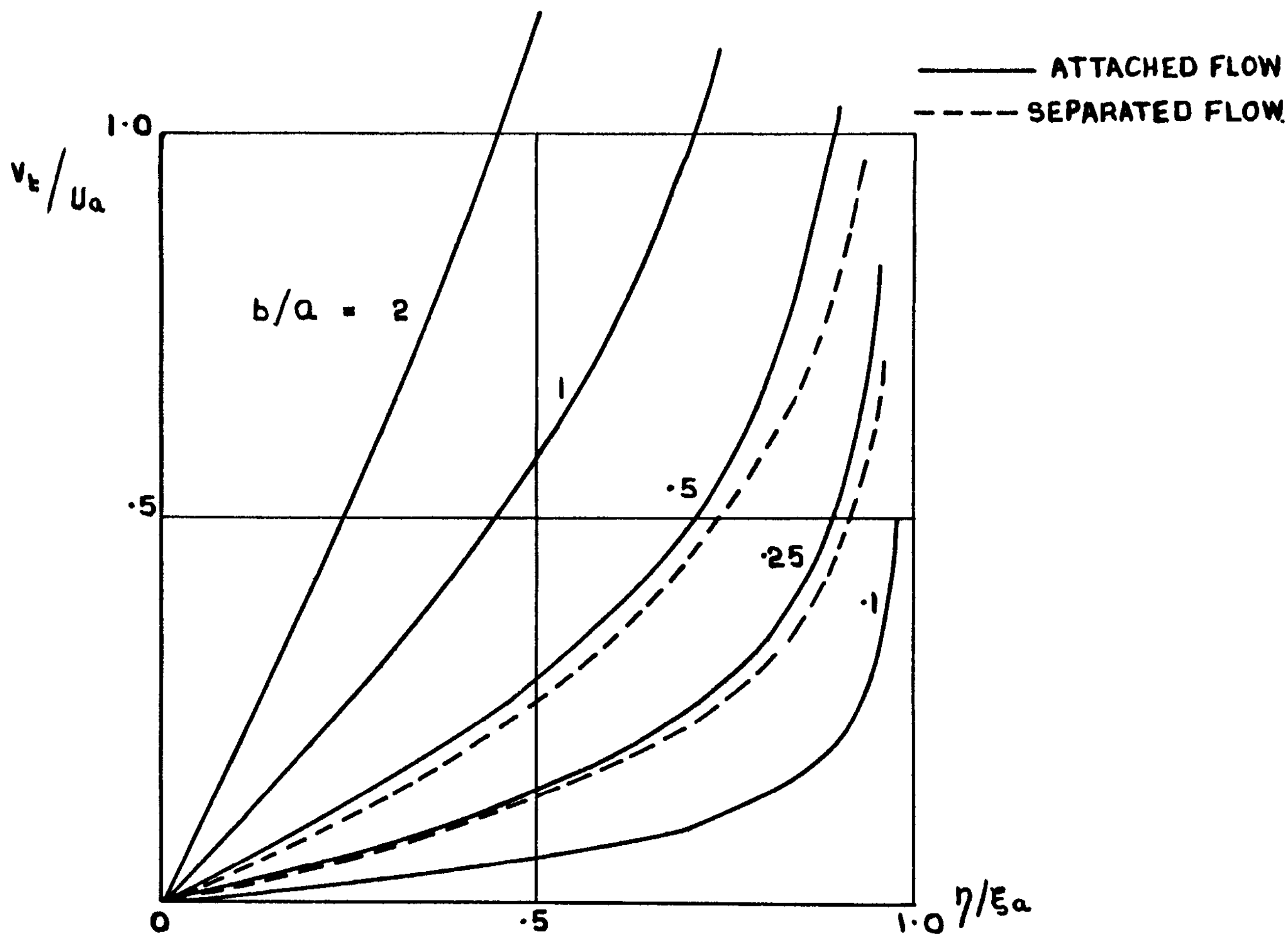


FIG. 3. SURFACE VELOCITIES ON THE BOTTOM SURFACE OF A DELTA.

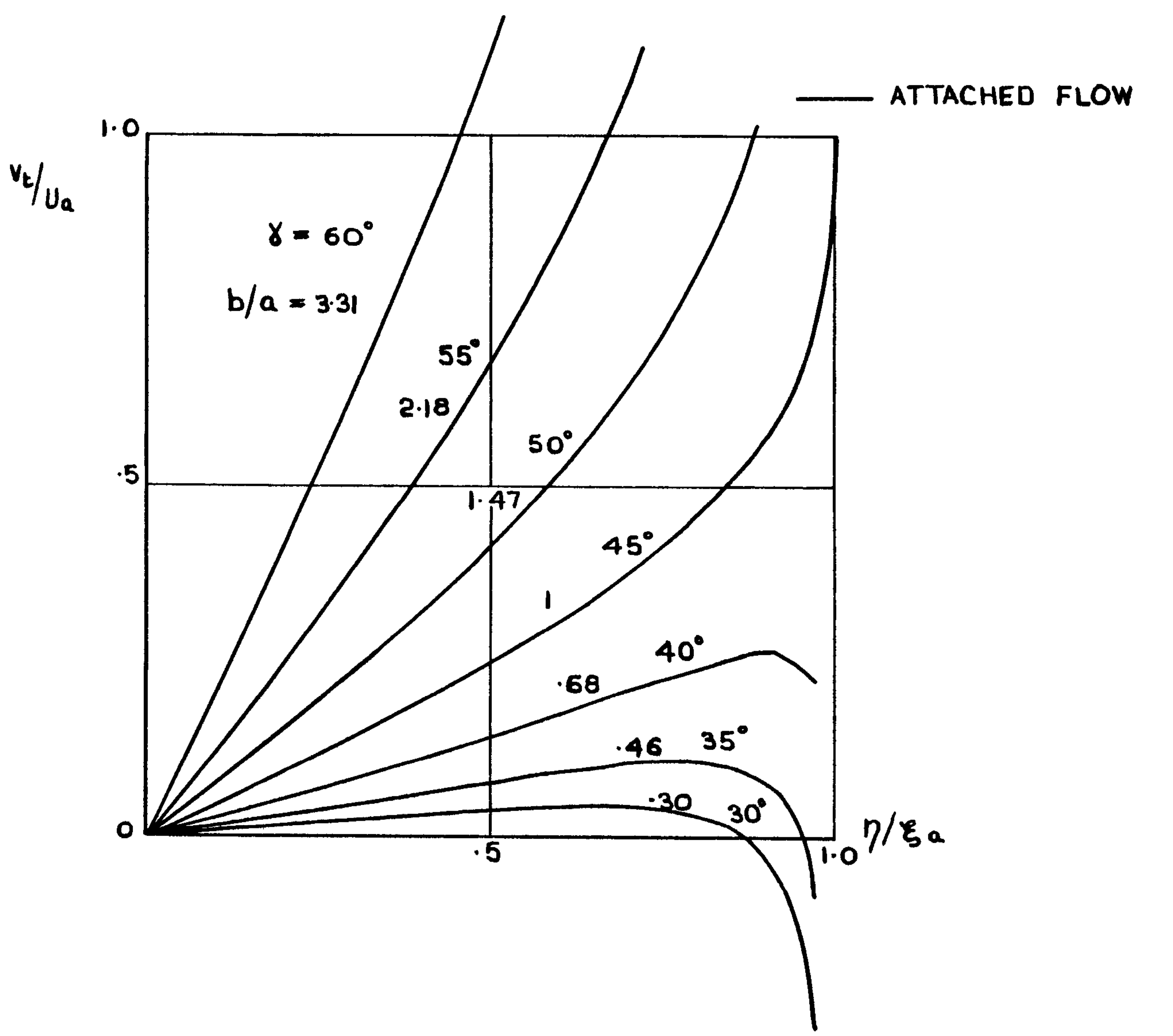


FIG. 4. SURFACE VELOCITIES ON A RECTANGULAR PYRAMID.

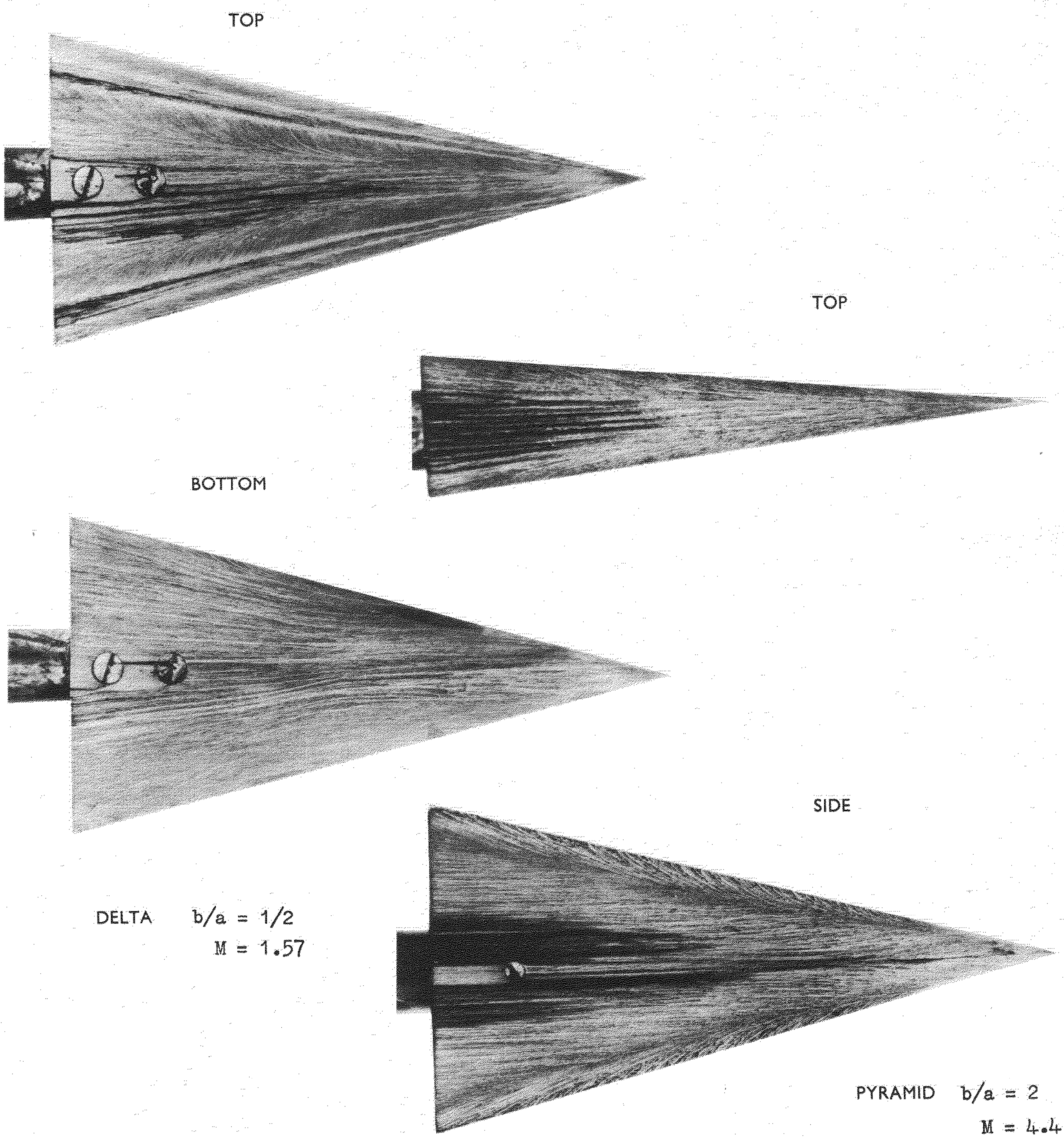
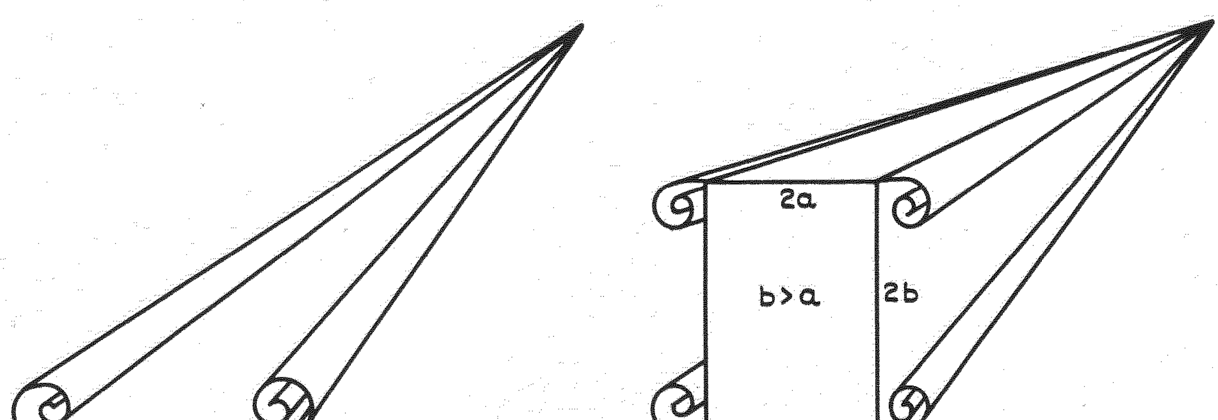


FIG.5. A FEW OF TREADGOLD'S OIL FLOW PATTERNS



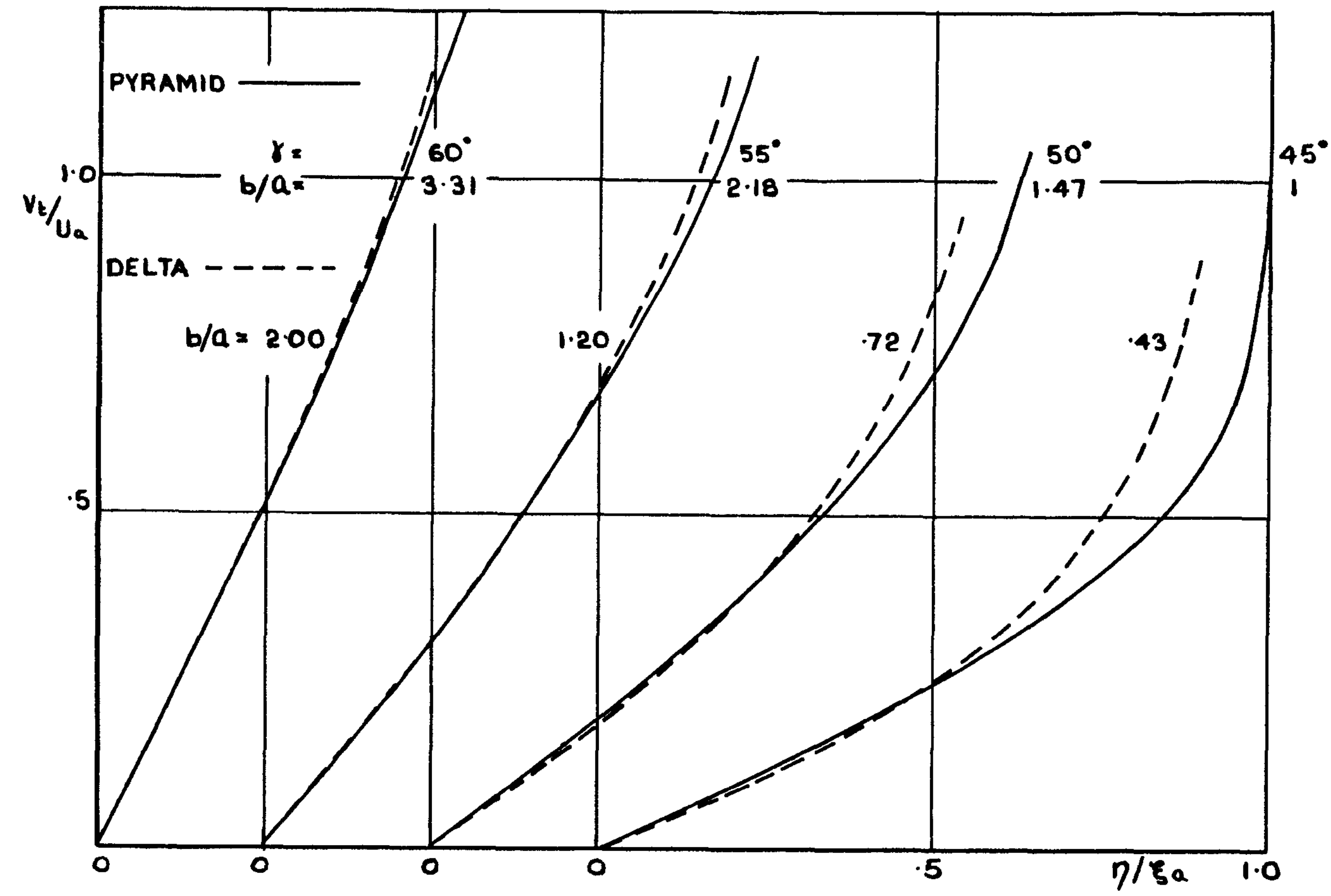


FIG. 7. SURFACE VELOCITIES MATCHED NEAR THE CENTRE-LINE.

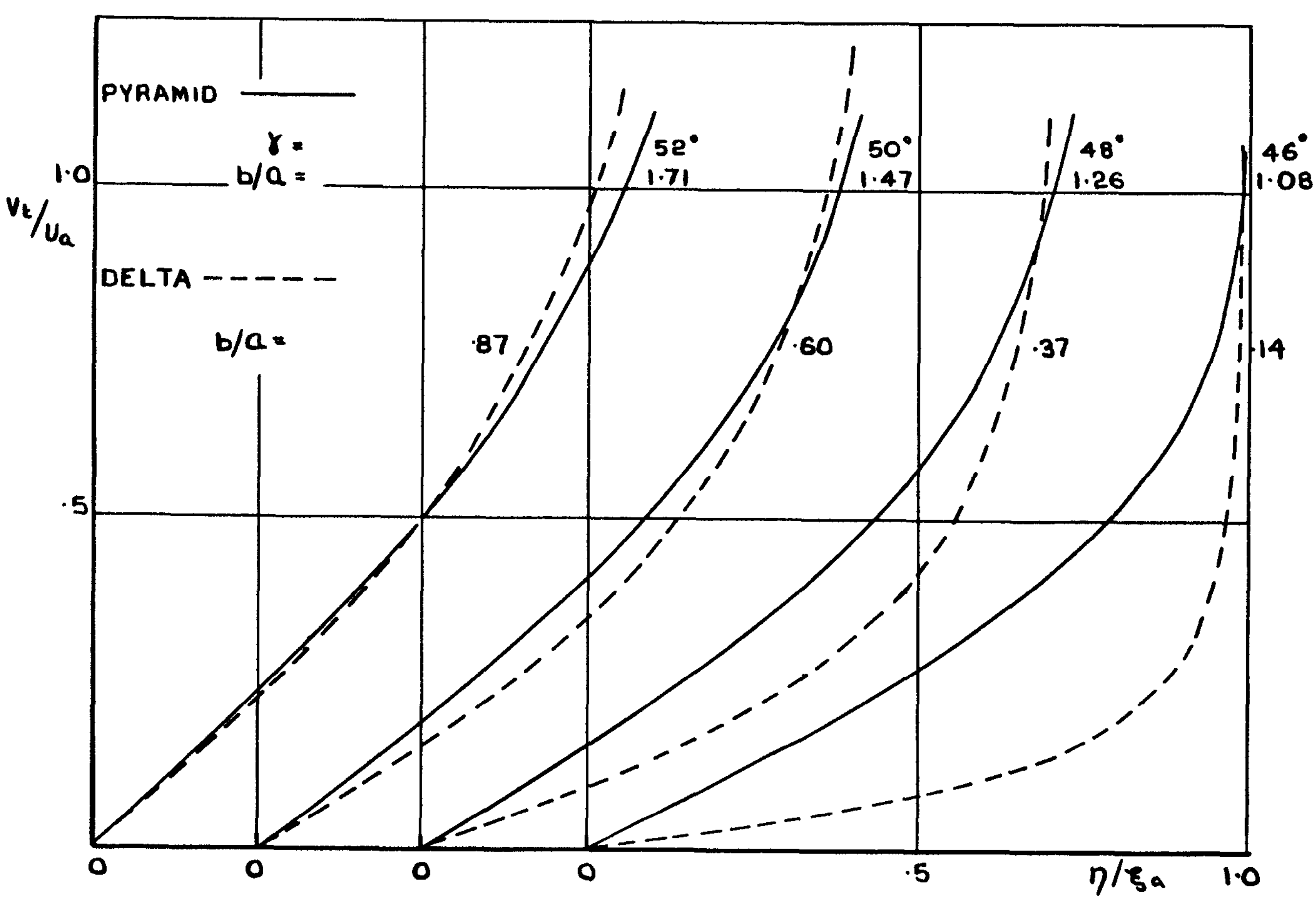


FIG. 8 SURFACE VELOCITIES MATCHED AT THE ATTACHMENT LINE.

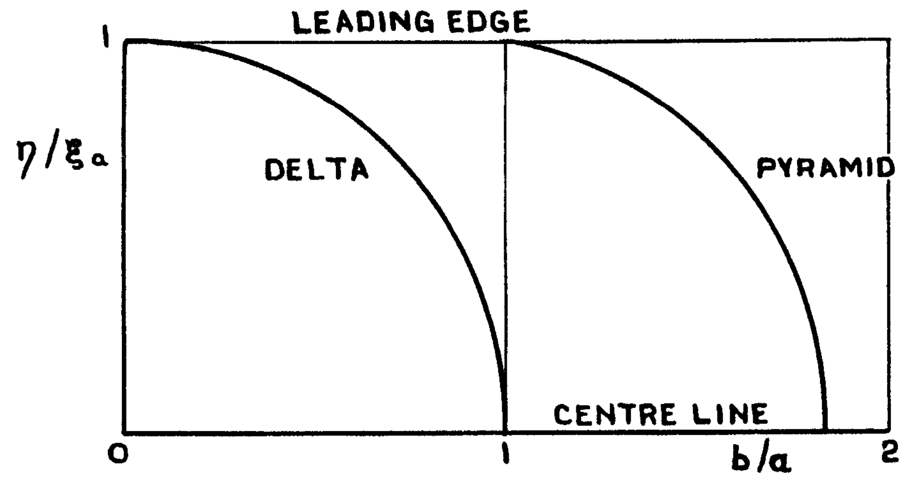


FIG. 9. VARIATION OF ATTACHMENT LINE POSITION WITH b/a.

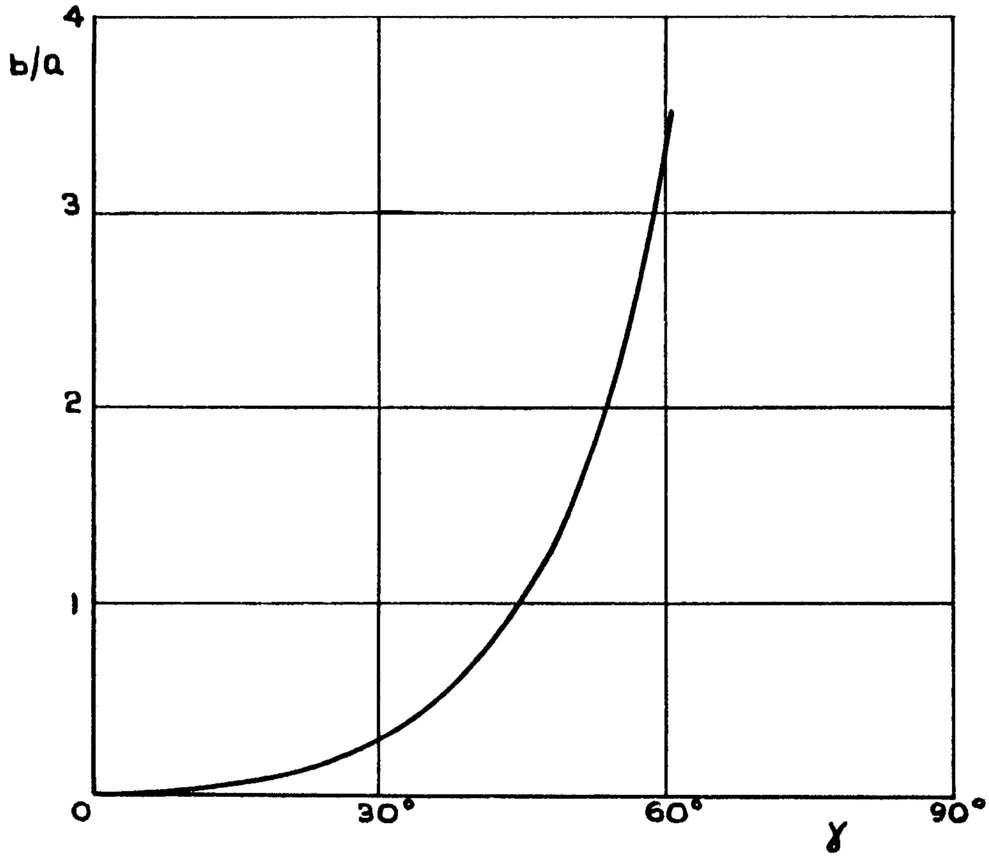


FIG. 10. VARIATION OF b/a WITH γ.

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THEORETICAL COMPARISON OF THE FLOW OVER A FLAT DELTA WING AND A RECTANGULAR PYRAMID. Eminton, E. November, 1961.

Slender-body theory is used to compare the flow over the bottom surface of a flat delta wing at incidence and the delta-like surfaces of a pyramid of rectangular section, interest being focussed on the centre line and on the flow attachment lines where they exist. Pyramids are found which reproduce closely the flow in the neighbourhood of the delta centre line. Othe pyramids are found which have the same position of the attachment lines as a delta wing, the approximation to the flow being good when the attachment lines are near the centre line but deteriorating as they move outboard.

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