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# A Preliminary Study of Ionic Recombination of Argon in Wind-Tunnel Nozzles

## Part II

*By*

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A Preliminary Study of Ionic Recombination of Argon in Wind-Tunnel Nozzles
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SUMMARY

A study has been made of electron-ion recombination in the flow of a partially ionised Argon plasma through wind-tunnel nozzles. Effects of thermal conduction, convection and two-body recombination processes are neglected, but the temperatures of the ions and electrons are allowed to differ. The equations have been integrated for a variety of stagnation conditions and it has been shown that, for the cases considered, the flow is far removed from thermal equilibrium. Furthermore, as a result of this effect alone, large differences between the temperatures of the heavy particles and the electrons are predicted. The phenomenon of "sudden freezing" characteristic of atomic recombination is not encountered, because of the exponential temperature dependence of the recombination rate which results from the assumed ionic recombination mechanism.

List of Contents



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Replaces A.A.S.U. Report No.185 - A.R.C.23,290.

## 1. Introduction

This paper is a further step in the study of electron-ion recombination in the flow of partially ionised argon through wind-tunnel nozzles. A preliminary study was made by the authors[1] by determining the flow for the two limiting cases in which recombination is either completely frozen or in complete equilibrium, and it is not intended to go into any detail here concerning these results. The main aim of these studies is to obtain a theory which will predict with reasonable accuracy the magnitude of the flow variables for an initially ionised gas at any station in a supersonic nozzle of specified shape. It is not suggested that the results given in this paper represent the completion of these studies; rather, they are an intermediate step between the simple equilibrium and frozen flow cases and more accurate calculations.

The mathematical model used in this intermediate analysis was constructed by making the following basic assumptions:

(i) The flow is one-dimensional, steady and inviscid - a close approximation to this flow may be obtained in the central core of the gas, if the nozzle wall boundary layer does not fill the nozzle completely, and if the rate of change of effective cross-sectional area is small.

(ii) Magneto-fluid-dynamical effects may be neglected - this implied that at any given station in the nozzle there will be a homogeneous' electrically neutral mixture of heavy particles (atoms and ions) and electrons, a condition which is not met if electron diffusion occurs. However this diffusion is strongly resisted by the resulting change separation[2]. Furthermore, it is found in practice that the plasma potential can have widely different values between the arc chamber and the nozzle exit of a plasma-jet wind tunnel, a situation which can give rise to a large drift current, that is, to a flow of the electrons relative to the heavier particles. All currents and body forces on the gas are neglected in the present work.

(iii) Ionisation resulting from the electric field in the arc chamber is also neglected, and it is assumed that the expansion of the hot gases takes place from a large reservoir in which equilibrium conditions prevail.

(iv) Ionisation and recombination on the walls of the nozzle are not considered, as these are boundary-layer phenomena, and outside the scope of this work.

(v) Conductive, radiative and convective heat transfer are neglected - this again is an assumption which may not be justified in practice; conduction in particular may have a very marked effect on the flow[3].

(vi) In general the temperatures of the heavy particles and the electrons will not be equal - a phenomenon observed by several workers[3,4] which is attributed to the fact that atoms and electrons are effectively isolated energetically. In the present work the electron temperature is determined from an energy balance for the electrons. Following Ref.4, this considers only the effects of elastic and inelastic collisions between electrons and ions, effects of conduction and convection within the electron gas are neglected. The magnitude of these neglected terms requires further study. Recombination by radiative capture of an electron is assumed to be a slow process compared with the three-body process involving an ion and two electrons - a condition which cannot be met if the density is very low.

It is recognised by the authors that many factors, which may or may not be important, have been neglected. The reason for these omissions lies in the mathematical complexity of the problem, the aim of this paper is to suggest a crude but workable mathematical model which may be extended in the future. The next stage in this work will be to determine which factors may justifiably be neglected and which must be included, but this is outside the scope of the present paper.

In Section 2 the basic thermodynamic relationship for the ideal ionising monatomic gas are derived, the results being given in terms of dimensionless quantities following the procedure suggested by the authors in Ref.1. Section 3 deals with the quasi-one-dimensional flow equations which are present in a form suitable for integration numerically by a Runge-Kutta process. Section 4 deals with the computational procedure and includes a discussion of the results of the integration.

## 2. The Ideal Monatomic Ionising Gas

The theory of the ideal monatomic ionising gas in equilibrium has been formulated in several papers[1,6,7] and it will be sufficient merely to quote the results here. The equations governing the thermodynamic behaviour of the ideal monatomic ionising gas in equilibrium are:

$$p = \rho T(1 + \alpha) \qquad \ldots(1)$$

$$i = \tfrac{5}{2}T(1 + \alpha) + \alpha \qquad \ldots(2)$$

$$u = \tfrac{3}{2}T(1 + \alpha) + \alpha \qquad \ldots(3)$$

$$\frac{\alpha^2}{1 - \alpha} = \frac{1}{\rho} T^{\frac{3}{2}} e^{-1/T}. \qquad \ldots(4)$$

These equations have been non-dimensionalised by a technique similar to that employed by Lighthill for the ideal dissociating gas[8]. A more detailed account of the method is given in a previous paper by the authors[1]. The values of the characteristic temperature, density, pressure, internal energy and velocity which have been used as units are given for argon in Table 1.

Electronic excitation and multiple ionisation are neglected. Unfortunately, when this theory is extended to the case where equilibrium is not achieved these equations are not adequate. In their derivation it has been assumed that the atom, ion and electron temperatures are all equal, and in general this assumption is not valid in regions where thermal equilibrium is not attained. The remainder of this section will therefore be devoted to a detailed derivation of the equations governing the thermodynamic behaviour of the ideal ionising monatomic gas away from equilibrium.

Table 1/

Table 1

Characteristic quantities

| Quantity | Units | Values for Argon | Units | Values for Argon |
|---|---|---|---|---|
| $p_i'$ | $\frac{lb}{ft^2}$ | $1.1976 \times 10^{11}$ | $\frac{dynes}{cm^2}$ | $5.7362 \times 10^{13}$ |
| $T_i'$ | °K | $1.8210 \times 10^5$ | °K | $1.8210 \times 10^5$ |
| $\rho_i'$ | $\frac{slugs}{ft^3}$ | $2.9326 \times 10^2$ | $\frac{gm}{cc}$ | 145.9 |
| $n_i'$ | $\frac{1}{ft^3}$ | $6.4538 \times 10^{28}$ | $\frac{1}{cc}$ | $2.2791 \times 10^{24}$ |
| $v_i'$ | $\frac{ft}{sec}$ | $2.0208 \times 10^4$ | $\frac{cm}{sec}$ | $6.1594 \times 10^5$ |
| $i_i'$ | $\frac{ft\ lb}{slugs}$ | $4.0836 \times 10^8$ | $\frac{erg}{gm}$ | $3.92990 \times 10^{11}$ |
| $\psi_i'$ | $\frac{slugs}{ft^2\ sec}$ | $5.9262 \times 10^6$ | $\frac{gm}{cm^2\ sec}$ | $8.9859 \times 10^7$ |

It will be assumed throughout the following analysis that the temperatures of the heavy particles (i.e., atoms and ions) are equal at all times but may be different from the electron temperature. The existence of the different temperatures in the atom-ion and electron gases implies that different maxwellian velocity distributions exist in these component gases. This concept is amply justified in the literature. (See for example Refs.4,5.)

The general expression for the equation of state for a mixture of gases as derived from statistical mechanics is:

$$p'V' = k(n_a'T' + n_+'T' + n_e'T_e') \qquad \ldots(5)$$

where $V$ is the volume of the gas under consideration.

But $n_+' = n_e'$ since the gas is electrically neutral

hence $$p'V' = k[(n_a' + n_e')T' + n_e'T_e']$$

also the total mass involved in the system is given by:

$\rho'V'/$

$$\rho'V' = n'_a m_a + n'_+ m_+ + n'_e m_e = (n'_a + n'_e)m_a + n'_e m_e$$

if

$$m_a \simeq m_+ >> m_e$$

then

$$\frac{p'}{\rho'} = \frac{k}{m_a}(T' + \alpha T'_e).$$

If this equation is written in a dimensionless form similar to equation (1) we obtain:

$$p = \rho(T + \alpha T_e). \qquad ...(6)$$

The equation relating enthalpy and temperature may be obtained in a similar manner. In general:

$$E' = n'_a \frac{\Sigma \varepsilon_n^a e^{-\varepsilon_n^a/kT'}}{\Sigma e^{-\varepsilon_n^a/kT'}} + n'_+ \frac{\Sigma(\varepsilon_n^+ + \chi)e^{-(\varepsilon_n^+ - \chi)/kT'}}{\Sigma e^{-(\varepsilon_n^+ - \chi)/kT'}} + n'_e \frac{\Sigma \varepsilon_n^e e^{-\varepsilon_n^e/kT'_e}}{\Sigma e^{-\varepsilon_n^e/kT'_e}} \qquad ...(7)$$

which may be reduced to:

$$E' = [(n'_a + n'_e)T' + n'_e T'_e]k + n'_e \chi$$

so that

$$e = \tfrac{3}{2}(T + \alpha T_e) + \alpha \qquad ...(8)$$

and

$$i = \tfrac{5}{2}(T + \alpha T_e) + \alpha. \qquad ...(9)$$

There are two further equations required to completely specify the thermodynamic quantities, namely an equation giving the rate of production or recombination of electrons, and an equation relating the two temperatures. In a previous paper by the authors[1] a method was outlined by which an approximate rate equation could be obtained by combining equations derived by Petschek and Byron[4] and Bond[5]. A slightly modified and simplified form of this equation has been used in this analysis; it is given by:

$$\frac{d\alpha}{dt} = \Gamma_i - \Gamma_r$$

where

$$\left.\begin{aligned} \Gamma_i &= A\rho\alpha(1 - \alpha)T_e^{\frac{3}{2}}\left(\frac{T_x}{T_e} + 2\right)e^{-T_x/T_e} \\ \Gamma_r &= B\rho^2\alpha^3\left(\frac{T_x}{T_e} + 2\right)e^{1/T_e}\,e^{-T_x/T_e} \end{aligned}\right\} \qquad ...(10)$$

A/

$$A = B = 3.349 \times 10^{16}\,(\text{Sec})^{-1}$$

$$T_x = \frac{T'_x}{T'_e} = 0.7359.$$

This equation assumes that ionisation and recombination occur only through the reaction:

$$A + e \rightleftharpoons A^+ + 2e.$$

It has the same form as the rate equation derived in Ref.1 but two further simplifications have been incorporated. The production and recombination rate coefficients have been assumed to depend on $T_x$ only, whereas in fact they may also depend on $T$ to an unknown extent. A similar assumption was made by Bond[5]. Also, the radiative recombination process discussed in Ref.1 has been completely neglected. Equation (10) is probably a gross over-simplification of the true ionisation and recombination process. However it is hoped that some of the main features of the full rate equation are retained.

The remaining equation relating the two temperatures may be obtained by considering the conservation of energy for electrons. In general heat is transferred to the electron gas by radiation and by conduction and convection within the electron gas, furthermore electrons exchange energy by elastic collisions with heavy particles and lose energy by inelastic collisions which lead to ionisation. However if all these factors are included a second order, second degree differential equation is obtained which makes the solution of the problem very difficult. It has therefore been assumed throughout this paper that convective, conductive and radiative heat transfer terms have negligible effects. No attempt will be made here to justify these assumptions since any one of the terms omitted or all of them may have a considerable effect on the flow; these assumptions represent a certain mathematical model which should be regarded as a tentative step towards a more complete solution.

With these assumptions the equation of conservation of energy for electrons may be written $Q_{EL} = Q_{IN}$ where $Q_{EL}$ is the rate at which electrons gain energy by Coulomb collisions and $Q_{IN}$ is the rate at which electrons lose energy by ionisation. From Ref.4 and equation (10) we have:

$$Q_{EL} = (\text{Constant})\rho^2\alpha^2 T_e^{-\frac{3}{2}}(T - T_e)\ell n\left(\frac{DT_e^3}{\rho\alpha}\right)$$

$$Q_{IN} = \rho^2(\text{Constant})(1 + \tfrac{3}{2}T_e)T_e^{\frac{3}{2}}\alpha\left(\frac{T_x}{T_e} + 2\right)e^{-T_x/T_e}\left[(1 - \alpha) - \rho\alpha^2 T_e^{-\frac{3}{2}} e^{1/T_e}\right]$$

and hence

$$T = T_e + \frac{(\text{Constant})(1 + \tfrac{3}{2}T_e)T_e^3\left(\frac{T_x}{T_e} + 2\right)e^{-T_x/T_e}\left[(1 - \alpha) - \rho\alpha^2 T_e^{\frac{3}{2}} e^{1/T_e}\right]}{\alpha\ell n\left(\frac{DT_e^3}{\rho\alpha}\right)} \qquad \ldots(11)$$

or

$$T = T_e + \phi(T_e, \rho, \alpha).$$

Now/

Now $\phi(T_e,\rho,\alpha)$ is positive when $\Gamma_i > \Gamma_r$ and negative when $\Gamma_i < \Gamma_r$, when $\Gamma_i = \Gamma_r$ $\phi(T_e,\rho,\alpha) = 0$. Physically this implies that:

(a) when the gas is in equilibrium the two temperatures will be equal,

(b) when the nett reaction rate produces ionisation $T > T_e$, this occurs behind a strong shock wave, for example (Ref.4),

(c) when the nett reaction rate produces a recombining flow $T < T_e$, which is the case encountered in a supersonic nozzle.

Using equation (11), the atom temperature has been plotted against the electron temperature for various densities and ionisation fractions, and the results are presented in Figs.1-4.

The four equations (6), (9), (10) and (11) completely specify the thermodynamic behaviour of the ideal ionising monatomic gas away from equilibrium within the initial assumptions.

## 3. Quasi-One-Dimensional Flow Equations

The frictionless adiabatic flow of an ideal ionising monatomic gas through a duct of slowly varying cross-sectional area $A$ is described by the equations of conservation of mass, momentum and energy:

$$\rho v A = \rho^* v^* = \psi \qquad ...(12)$$

$$v\frac{dv}{d\xi} + \frac{1}{\rho}\frac{dp}{d\xi} = 0 \qquad ...(13)$$

$$i + \tfrac{1}{2}v^2 = i_o \qquad ...(14)$$

in which asterisks denote conditions at a sonic throat, $\xi$ is a dimensionless distance defined below, and $A$ is the area ratio $A'/A^{*\prime}$.

The expression specifying the nozzle shape will be the same as that suggested in the previous paper by the authors:

$$A' = A^{*\prime} + K_N^2 (x')^2$$

where $x'$ is the axial distance measured from the throat as datum, and $K_N$ is a constant determining the expansion angle.

$\xi$ the dimensionless distance is defined by:

$$\xi = \frac{K_N x'}{\sqrt{A^{*\prime}}}$$

so that the nozzle shape becomes:

$$A = 1 + \xi^2. \qquad ...(15)$$

Equations/

Equations (12) - (15) together with the thermodynamic relationships, equations (6), (9), (10), (11) and a knowledge of the mass flow $\psi$, and the initial conditions specify the flow.

These equations must now be solved to obtain the flow variables downstream of the throat and this is most easily achieved by reducing them to six simultaneous differential equations in $p$, $\rho$, $T$, $T_e$, $\alpha$ and $i$ with $\xi$ as independent variables. These may then be integrated by a numerical technique.

Equation (10) may be written:

$$\frac{d\alpha}{d\xi} = \Phi_1 \frac{\rho\alpha}{v} T_e^{\frac{3}{2}} \left( \frac{T_x}{T_e} + 2 \right) e^{-T_x/T_e} \left[ (1 - \alpha) - \rho\alpha^2 T_e^{-\frac{3}{2}} e^{1/T_e} \right] \quad \ldots(16)$$

where $\Phi_1 = \frac{A}{K_N} \sqrt{\frac{mA^{*\prime}}{\chi}}$ is the dimensionless rate parameter analagous to that used by Bray[9] for the case of the ideal dissociating gas.

Combination of equations (6) and (9) yields:

$$\frac{5}{2} \frac{dp}{d\xi} = (i - \alpha) \frac{d\rho}{d\xi} + \rho \frac{di}{d\xi} - \rho \frac{d\alpha}{d\xi}$$

and eliminating $d\rho/d\xi$ and $di/d\xi$ with the aid of equations (12) - (15) we obtain:

$$\frac{dp}{d\xi} = \left[ \frac{(i - \alpha)}{v^2} - \frac{3}{2} \right]^{-1} \left[ \rho \frac{d\alpha}{d\xi} + \rho(i - \alpha) \frac{2\xi}{1 + \xi^2} \right] \quad \ldots(17)$$

$$\frac{d\rho}{d\xi} = \frac{1}{v^2} \frac{dp}{d\xi} - \frac{2\rho\xi}{1 + \xi^2} \quad \ldots(18)$$

$$\frac{di}{d\xi} = \frac{1}{\rho} \frac{dp}{d\xi} . \quad \ldots(19)$$

Finally equation (11) may be differentiated yielding:

$$F_1 \frac{dT_e}{d\xi} + F_2 \frac{dT}{d\xi} + F_3 \frac{d\rho}{d\xi} + F_4 \frac{d\alpha}{d\xi} = 0 \quad \ldots(20)$$

where

$$F_1 = Ae^{-T_x/T_e} \left[ (1 - \alpha) - \rho\alpha^2 T_e^{-\frac{3}{2}} e^{1/T_e} \right] \left[ (1 + \tfrac{3}{2} T_e) \left( \frac{T_x}{T_e} + 2 \right) T_e^{\frac{1}{2}} \left( \frac{T_x}{T_e} + 3 \right) \right.$$

$$\left. - (1 + \tfrac{3}{2} T_e) T_x T_e + \frac{3}{2} \left( \frac{T_x}{T_e} + 2 \right) T_e^3 \right] + A\rho\alpha^2 \left( \frac{T_x}{T_e} + 2 \right) (1 + \tfrac{3}{2} T_e)^2 T_e^{1/e} e^{-T_x/T_e} e^{1/T_e}$$

$$+ \alpha \ln \left( \frac{DT_e^3}{\rho\alpha} \right) - \frac{3\alpha}{T_e} (T - T_e)$$

$$F_2 = -\alpha\, \ell n\left(\frac{DT_e^3}{\rho\alpha}\right)$$

$$F_3 = -A(1+\tfrac{3}{2}T_e)\left(\frac{T_x}{T_e}+2\right)T_e^{\frac{3}{2}}\, e^{-T_x/T_e}\, e^{1/T_e}\,\alpha^2 - \frac{\alpha}{\rho}(T-T_e)$$

$$F_4 = A(1+\tfrac{3}{2}T_e)\left(\frac{T_x}{T_e}+2\right)T_e^3\, e^{-T_x/T_e}\left(1+2\rho\alpha\, T_e^{-\frac{3}{2}}\, e^{1/T_e}\right) + (T-T_e)\left[\ell n\left(\frac{DT_e^3}{\rho\alpha}\right)-1\right]$$

and

$$\frac{dT_e}{d\xi} = \left(\alpha - \frac{F_1}{F_2}\right)^{-1}\left\{\frac{F_5}{F_2} - \frac{1}{\rho}\left[\frac{dp}{d\xi} - \rho T_e\frac{d\alpha}{d\xi} - (T+\alpha T_e)\frac{d\rho}{d\xi}\right]\right\} \quad \text{...(21)}$$

$$\frac{dT}{d\xi} = -\left[\frac{F_5}{F_2} + \frac{F_1}{F_2}T_e'\right] \quad \text{...(22)}$$

where

$$F_5 = F_3\frac{d\rho}{d\xi} + F_4\frac{d\alpha}{d\xi}.$$

Equations (16), (17), (18), (19), (21) and (22) together represent a system of six equations with seven unknowns, and before attempting a solution the velocity must be eliminated in terms of the mass flow ($\psi$) which may then be treated as a boundary condition. The remainder of this section will be devoted to a short discussion of the two Mach numbers which may be defined for the flow.

It may be shown from the continuity and momentum equations that:

$$\frac{dp}{d\rho} = \frac{dp/d\xi}{d\rho/d\xi} = a^2 = v^2 - \frac{2\rho v^2\xi}{(1+\xi^2)d\rho/d\xi}. \quad \text{...(23)}$$

This equation shows that the flow velocity is that of sound at the throat, unless $dp/d\xi = 0$.

Furthermore if equations (17) and (18) are combined it may be shown that:

$$M_a^2 = \frac{v^2}{a^2} = 1 - \frac{2\rho v^2\xi}{(1+\xi^2)dp/d\xi} \quad \text{...(24)}$$

and hence irrespective of the mass flow $M_a = 1$ when $\xi = 0$ at the throat unless $dp/d\xi = 0$. This is an important result for the computational procedure that follows. Finally, if the speed of sound is written in terms of the two temperatures we have:

$$a^2 = \frac{5}{3}(T+\alpha T_e) - \frac{2}{3}\frac{\rho}{dp/d\xi}\cdot\frac{d\alpha}{d\xi} \quad \text{...(25)}$$

and if we define a frozen speed of sound (Ref.4) as

$$c^2 = \frac{5}{3}(T+\alpha T_e) \quad \text{...(26)}$$

then/

---

Note: The quantity a defined in equation (23) is not a true speed of sound in a non-equilibrium gas.

then

$$a^2 = c^2 - \frac{2}{3}\rho\,\frac{d\alpha/d\xi}{d\rho/d\xi}. \qquad \ldots(27)$$

A frozen Mach number being defined as

$$M_c = v/c.$$

Now if the flow is supersonic, $d\rho/d\xi$ will always be negative and $d\alpha/d\xi$ will be negative or zero. Therefore:

$$c \geqslant a$$

and

$$M_c \leqslant M_a$$

also equation (17):

$$\frac{d\rho}{d\xi} = \left[\frac{(i - \alpha)}{v^2} - \frac{3}{2}\right]^{-1}\left[\rho\,\frac{d\alpha}{d\xi} + \rho(i - \alpha)\,\frac{2\xi}{1 + \xi^2}\right]$$

which has a singularity at a point where $\frac{i - \alpha}{v^2} = \frac{3}{2}$ i.e., when $v^2 = \frac{2}{3}(i - \alpha) = c^2$. This singular point will therefore occur when $M_c = 1$, that is, downstream of the throat. At the singularity point we must have either $\frac{d\rho}{d\xi} \rightarrow \infty$ or

$$\xi = -\frac{(1 + \xi^2)}{2(i - \alpha)} \cdot \frac{d\alpha}{d\xi}.$$

Clearly, it is the latter condition which represents the physical flow.

Difficulty is experienced in performing the step-by-step integration through this point, and it is therefore useful to compute $M_c$ at intervals during the calculations.

## 4. Computational Procedure

It was decided to integrate the set of equations (16) - (19), (21) and (22) numerically on a Ferranti Pegasus high-speed digital computer using the Runge-Kutta process due to C. Stratchey which is particularly suitable for high-speed computation[11].

There are two difficulties which must be surmounted before the equations may be integrated, these are:

(a) Starting the integration.

(b) Determining the mass flow ($\psi$).

It was decided to commence the integration at $\xi = -\infty$, i.e., in the stagnation region upstream of the throat. To obtain the stagnation conditions it was assumed that the gas started in equilibrium (when $T = T_e$), and then by specifying any two variables the remaining four may be calculated.

Unfortunately,/

Unfortunately, in the stagnation region all the derivatives vanish, and the integration was therefore started by constructing an asymptotic series solution for the flow variables at large area ratios upstream of the throat by expansion about the stagnation conditions at $\xi = -\infty$. This was the method employed by Hall and Russo[12] and a detailed discussion of the technique is included in their paper.

The expansion was carried out for the hyperbolic axisymmetric nozzle specified by:

$$A = 1 + \xi^2$$

in terms of the expansion variable $Z = A^{-\frac{1}{2}}$ assuming that the gas remained in equilibrium.

The leading terms in the series were found to be:

$$\alpha = \alpha_o + \alpha_2 Z^2$$

$$T = T_o + T_2 Z^2$$

$$\rho = \rho_o + \rho_2 Z^2$$

$$p = p_o + p_2 Z^2$$

$$i = i_o + i_2 Z^2$$

$$v = v_o + v_1 Z = v_1 Z$$

where

$$v_1 = \psi/\rho_o$$

$$p_2 = -\tfrac{1}{2}(\psi^2/\rho_o)$$

$$i_2 = -\tfrac{1}{2}(\psi/\rho_o)^2$$

$$T_2 = \frac{\psi^2 T_o[(2-\alpha_o)(1+\alpha_o)T_o + \alpha_o(1-\alpha_o)(1+\frac{3}{2}T_o)]}{\rho_o^2(1-\alpha_o^2)\left[\alpha_o(3T_o+2)(1+\frac{5}{2}T_o) + \dfrac{5(1+\alpha_o)(2-\alpha_o)}{(1-\alpha_o)}T_o^2 + 2\alpha_o T_o\right]}$$

$$\alpha_2 = -\frac{1}{(1+\frac{5}{2}T_o)}\left[\left(\frac{\psi}{\rho_o}\right)^2 + 5(1+\alpha_o)T_2\right]$$

$$\rho_2 = \frac{1}{2}\frac{\psi^2}{p_o}\left[\frac{T_o}{(1+\frac{5}{2}T_o)} - 1\right] + \frac{\rho_o}{T_o}T_2\left[\frac{5}{2}\frac{T_o}{(1+\frac{5}{2}T_o)} - 1\right].$$

The series shows only a very slow departure from stagnation conditions, and at an area ratio of 10 ($\xi = -3.0$) the variables have changed by less than 1% of their initial values.

Having established a suitable starting series, the next stage in the integration is to determine the mass flow yielding supersonic flow downstream of the throat. It will be remembered from the

previous/

previous section that the velocity was eliminated from the equations to be integrated, in favour of the mass flow. The reason for this substitution is that the mass flow is known to lie between quite narrow limits, i.e., those for equilibrium and frozen flow. There is no straightforward analytical method of obtaining the mass flow and hence a method of trial and error was adopted. A value for the mass flow was chosen, and then the integration was carried out from $-\infty$ to the throat, when various parameters were investigated. It was shown in the previous section that the actual Mach number ($M_a$) will be equal to unity at the throat regardless of the value of the mass flow, unless $dp/d\xi = 0$, nevertheless this is a fairly sensitive parameter. If the mass flow chosen is too large then the Mach number reaches unity before the throat is reached. Hence by starting with the largest mass flow (i.e., the frozen mass flow), and then decreasing it gradually a value is reached when the Mach number increases smoothly from zero at $x_i = -\infty$ to unity at the throat. If on integrating downstream of the throat supersonic flow was obtained then it was assumed that the correct mass flow had been determined.

This is a very crude method for finding the correct mass flow, and it is further hindered by the location of a singularity at a point slightly downstream of the throat. Fortunately, it was found that the solution downstream of the throat was not sensitive to the mass flow and an accuracy of three or four significant figures was found to be sufficient; small perturbations of the chosen value were found to induce negligible errors in the solution.

The final difficulty which had to be surmounted was the integration through the critical point where $M_c = 1$. In order to determine the region of influence of the singularity the computer programme was designed to give the value of $dp/d\xi$ at discrete values upstream of the throat. It was found that $d^2p/d\xi^2$ underwent a sharp change as the region of influence of the singularity was entered, and the integration was curtailed at this point. The procedure from here varied slightly depending on the condition of the flow. If the flow was frozen or near frozen considerable difficulty was encountered but plotting $\ell n(\rho)$ or $\rho$ and $T_e$ against $\alpha$ yielded straight lines which facilitated extrapolation. For equilibrium or near equilibrium flow it was found that a graph of $\alpha T_e$ against $\ell n(\rho)$ was a straight line, and then by plotting $\alpha$ and $T_e$ against $\ell n(\rho)$ two further crude extrapolations could be made and the mean of the resulting values taken. These extrapolation techniques were used to estimate values of all the dependent variables at a point downstream of the critical point, which were used as initial values for a step-by-step integration in the supersonic flow region. It is not suggested that this is the best method for getting through the singularity, but an empirical justification was obtained by perturbing the extrapolated values, and noting that only small errors were introduced into the downstream solution.

## 5. Results

In Figs.1 - 4 the variation of atom temperature (T) with the electron temperature ($T_e$) is shown for various ranges of ionisation fractions ($\alpha$) and densities ($\rho$), as described by equation (11). In all cases the graphs may be divided into two halves about the line $T = T_e$ (equilibrium condition). Points above this line ($T > T_e$) represent states in which the ionisation level will increase with time, while points below the line ($T < T_e$) represent states in which recombination will occur.

It/

It may be seen from Figs.1 and 2 that the density dependence is relatively unimportant in the ionising region, large variations in density showing only small changes in temperatures. In the recombining region however changes in density are important, producing large swings in the temperature.

Figs.3 and 4 show the effect of the ionisation fraction on the temperature relationship. The ionising regions of these graphs have the same form as those formulated by Petschek and Byron[4]; there is a slight shift towards the equilibrium line ($T = T_e$) due to the inclusion of the recombining terms in this work. For low densities in the recombining region variations in $\alpha$ are of little importance. There is a wide range of conditions at low density for which $T \simeq T_e$.

Figs.5 - 30 describe the flow conditions downstream of the sonic throat in a near conical nozzle. The equations describing the flow have been solved for a number of stagnation conditions and the results will be described below. Figs.5 - 11 have been obtained for stagnation conditions of $\Phi_1 = 4.0 \times 10^{10}$ and $p_o = 10^{-8}$, $T_o = 0.09$. Fig.11 shows that the ionisation fraction ($\alpha$) changes only slowly through the nozzle, and hence the flow is tending towards a frozen region where $\alpha$ = constant with distance. Conditions are in fact, far removed from equilibrium but the sudden freezing phenomena so characteristic of dissociation in nozzles[1], is not encountered in ionisation. This statement has been verified for a wide range of conditions and may be attributed to the different form of the rate equation which has evolved.

Fig.8 shows the variation of the two temperatures with distance through the nozzle, and it is found in regions where thermal equilibrium does not prevail, that the electron temperature falls less rapidly than the temperature of the heavy particles. Consequently at large area ratios the two temperatures differ by large amounts. For example at an area ratio of 1000 (Fig.5) the difference ($T_e - T$) = 0.01 on a non-dimensional scale; this represents a temperature difference of approximately 1800°K. Large differences between atom and electron temperatures have in fact been measured in plasma-jet wind tunnels[3].

The remaining graphs describe the other flow variables in the nozzle and it may be seen that they lie between the two limiting solutions of equilibrium and frozen flow as is to be expected.

Figs.12 - 18 show the effect of reducing the stagnation temperatures, a value of 0.06 being chosen in this case. The ionisation fraction ($\alpha$) is found to be small at all times (Fig.18) although still not completely frozen. However the other variables are closer to the frozen solutions in this case, and for a given area ratio (A) the difference between the two temperatures is increased. The temperature of the heavy particles is close to its limiting value (zero) at an area ratio of about 1000, and at this point the electron temperature is falling only slowly, and is in fact tending to "freeze out".

In Figs.19 - 25 the stagnation pressure has been reduced to $10^{-9}$ and the stagnation temperature maintained at 0.09. These figures must be compared with Figs.5 - 11.

The flow conditions are seen to be very close to the frozen conditions in this case, and again the graph of the ionisation fraction against distance shows only very small changes. The complete "freeze"

is/

is not accomplished even now however, and $\alpha$ falls gradually from 0.99 to 0.90 at an area ratio of 1000. The atom temperature (Fig.19) lies very close to the frozen value and has almost attained its limiting value at an area ratio of 1000, where the difference between the two temperatures is approximately 2700°K (0.015 on a dimensionless scale).

The remaining curves (Figs.26 - 32) show the effect produced on the flow condition by varying the rate parameter, $\Phi$. The first three solutions described above were all obtained for a value of $\Phi = 4 \times 10^{10}$, whereas this final solution employs a value of $\Phi = 10^{13}$. It is to be noted that if one assumes the rate constant (A of equation (10)) to have a fixed value then variations in $\Phi$ correspond to variations in the nozzle geometry; this is demonstrated in Fig.33, where the rate parameter is plotted against throat diameter for various expansion angles. It may readily be seen from these curves that if the value assumed for A is correct, then $\Phi = 10^{13}$ has no physical significance for nozzles of feasible proportions. However it has been found that under a wide range of starting conditions the flow is not in equilibrium ($\Phi = \infty$) and tends towards the frozen solution ($\Phi = 0$); it was therefore decided to try to obtain a solution which remained near equilibrium at least to some point downstream of the sonic throat. Also, as mentioned in Section 2, the value assumed for the rate constant A may be in error by orders of magnitude. The present set of calculations shows the effect of such an error on the results.

Fig.32 shows that the ionisation fraction falls quite rapidly down the nozzle, but begins to depart appreciably from the true equilibrium solution at an area ratio of about 1.02. The temperature variations (Fig.26) produce an interesting result since the temperature gradients are larger than for equilibrium but the two temperatures remain approximately equal. This perhaps could have been predicted from Fig.1. If the effect of variations in $\alpha$ is assumed to be unimportant in the recombining region of this curve, then it is clearly possible for the two temperatures to remain close to one another, as the density decreases, under certain favourable conditions.

The velocity curve (Fig.27) is particularly interesting, since it is known that conditions substantially depart from equilibrium at an area ratio of 1.02, but the velocity remains at the equilibrium value to an area ratio of about 20. This may be attributed to the form of the governing equations. Since $v = f(i_o - i)$ it follows that the enthalpy also must be close to the equilibrium solution for area ratios less than 20. It may be seen from equation (9) that the enthalpy can remain at the equilibrium value provided $\alpha$, $T$ and $T_e$ vary suitably. In the present case the temperatures are falling more rapidly than at equilibrium, but the ionisation fraction maintains a high level. The enthalpy is therefore subject to two opposing effects and happens to remain at the equilibrium value for some distance downstream of the throat. This phenomenon also occurs during atomic recombination in nozzles as noted by Hall and Russo[12].

## 6. Comments and Conclusions

In conclusion it may be stated that, in the absence of conduction and convection, the atom and ion temperatures may differ considerably from the electron temperature under a wide range of stagnation conditions. This difference in temperatures is predicted entirely from the effects of lack of thermal equilibrium in the nozzle. It is thought that the equations employed show the correct trends in

the/

the flow conditions, but perhaps overestimate the difference in temperatures, which most probably result from a combination of conduction, convection and non-equilibrium effects. Further work is in progress to study the effects of conduction and convection, and also to allow for a two-body recombination process.

It has also been found that reducing the stagnation pressure and temperature tends to remove the solution from the limiting case of thermal equilibrium.

These results are in qualitative agreement with the empirical freezing criterion suggested in Ref.1. However, freezing does not occur nearly as suddenly as in the case of atomic recombination, because of the exponential temperature dependence of the assumed recombination rate, which leads to the prediction of very high rates of recombination at low electron temperatures.

List of Symbols

| | | |
|---|---|---|
| $A$ | area ratio | (dimensionless) |
| $A'$ | area of nozzle at a given station | ($ft^2$) |
| $A^*$ | area of nozzle at sonic throat | ($ft^2$) |
| $c'$ | frozen velocity of sound | (ft/sec) |
| $C_p$ | specific heat at constant pressure | ($slugs^2/ft^4\, sec^2$) |
| $C_v$ | specific heat at constant volume | ($slugs/ft^4\, sec^2$) |
| $e'$ | specific internal energy | (ft lb/slugs) |
| $e'_r$ | radiated energy | (ft lb/slugs) |
| $h$ | Plank's constant $= 4.868 \times 10^{-34}$ | (ft lb/sec) |
| $i'$ | specific enthalpy | (ft lb/slugs) |
| $k$ | Boltzmann's constant $= 1.019 \times 10^{-23}$ | (ft lb/°K mol) |
| $M$ | Mach number | (dimensionless) |
| $M_c$ | frozen Mach number | (dimensionless) |
| $m$ | mass of an atom | (slugs) |
| $m_e$ | mass of an electron | (slugs) |
| $m_+$ | mass of an ion | (slugs) |
| $n'$ | overall number density | ($ft^{-3}$) |
| $n'_a$ | number density of atoms | ($ft^{-3}$) |
| $n'_e$ | number density of electrons | ($ft^{-3}$) |
| $n'_+$ | number density of ions | ($ft^{-3}$) |

p'/

| | | |
|---|---|---|
| $p'$ | pressure | ($lb/ft^2$) |
| $Q = r_i/d\alpha/dt$ | | (dimensionless) |
| $r_i$ | rate of ionisation | ($sec^{-1}$) |
| $r_r$ | rate of recombination | ($sec^{-1}$) |
| $s'$ | specific entropy | (ft lb/°k) |
| $T'$ | plasma or atom temperature | (°k) |
| $T'_e$ | electron temperature | (°k) |
| $T'_{exc}$ | first electronic excitation potential | (°k) |
| $v'$ | velocity | (ft/sec) |
| $x'$ | distance measured from throat as datum | (ft) |
| $\alpha$ | ionisation fraction $= n_e/n_e + n_a$ | (dimensionless) |
| $\gamma$ | ratio of specific heats $= C_p/C_v$ | (dimensionless) |
| $\xi$ | dimensionless distance measured from the throat as datum | |
| $\rho$ | density | ($slugs/ft^3$) |
| $\chi$ | ionisation potential | (ft lb) |

<u>Suffices</u>

| | |
|---|---|
| i | ionisation quantities |
| o | stagnation quantities |
| e | electron quantities |
| eq | equilibrium quantities |
| | primes denote dimensional quantities |
| | superfix * denotes throat conditions |

References/

References


| No. | Author(s) | Title etc. |
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| 12 | J. G. Hall and A. L. Russo | Studies of chemical non-equilibrium in hypersonic nozzle flows. Cornell Aeronautical Laboratory, Inc. November, 1959. |


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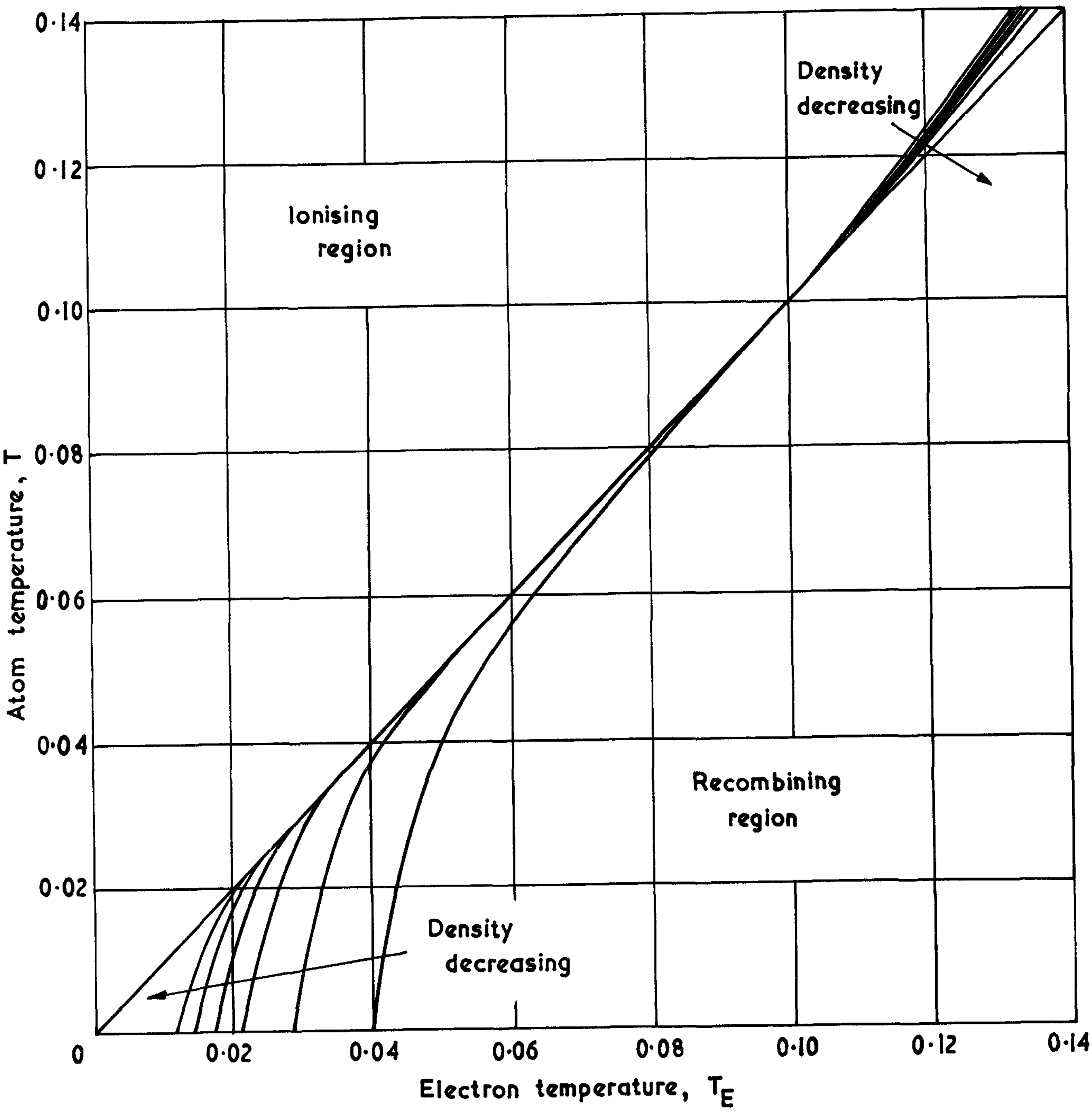
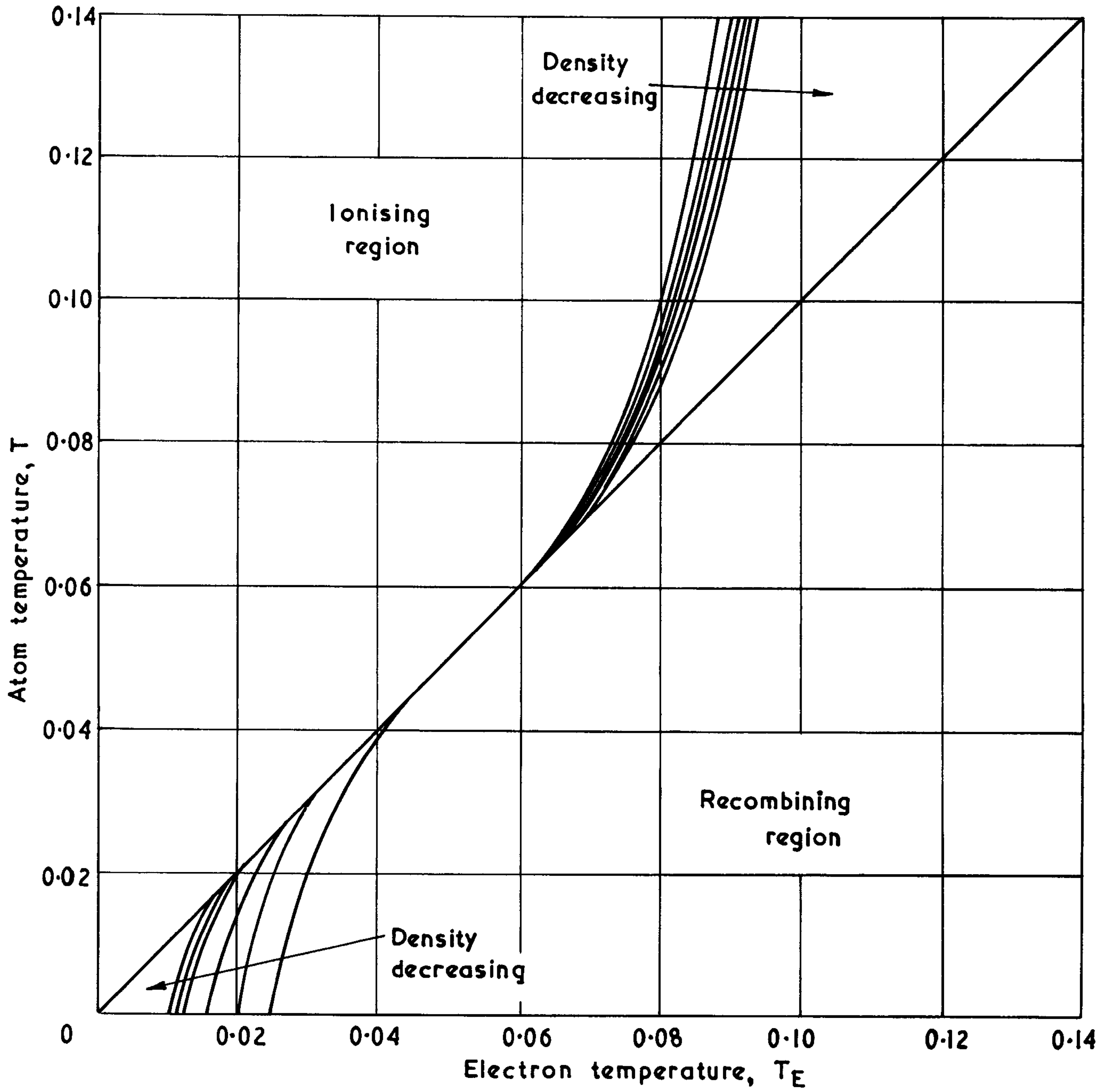
FIG. 1




$\alpha = 0{\cdot}95$

$\rho$ varying between $10^{-7}$ and $10^{-12}$ in steps of $10^{-1}$.

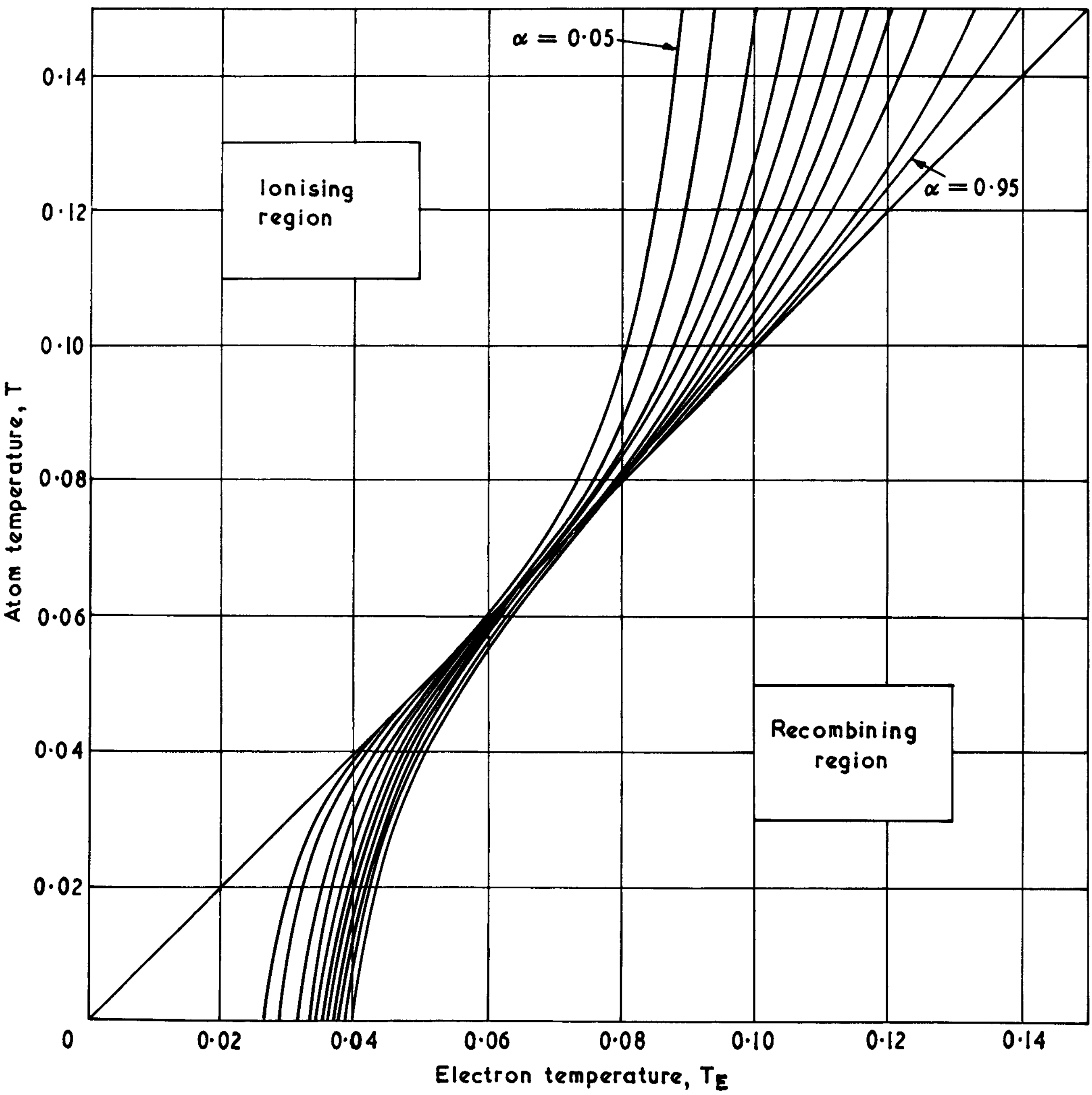
FIG. 2




$\alpha = 0{\cdot}05$

$\rho$ varying between $10^{-7}$ and $10^{-12}$ in steps of $10^{-1}$

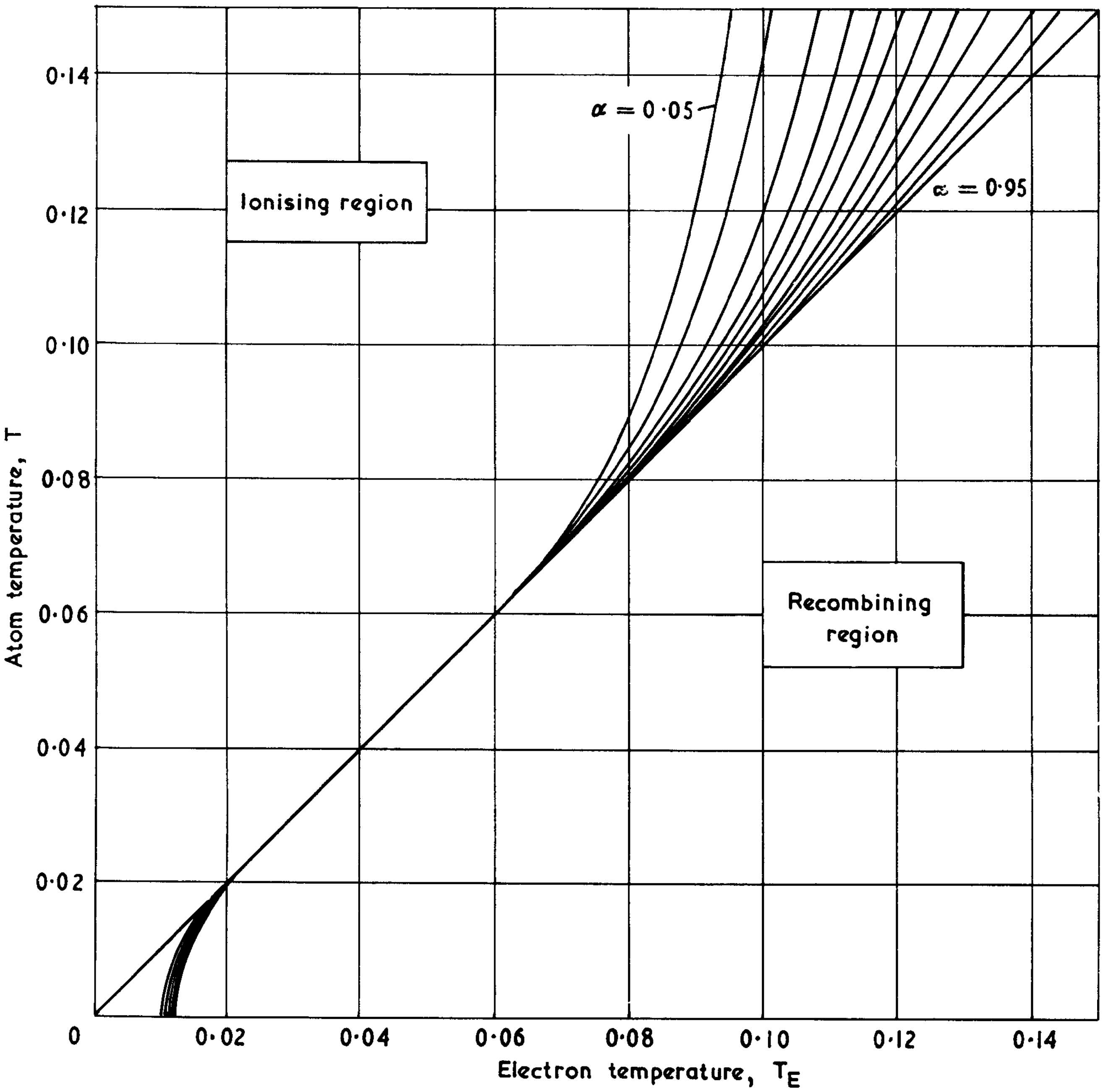
FIG. 3




$\rho = 10^{-7}$

α varying between 0·1 and 0·9 in steps of 0·1

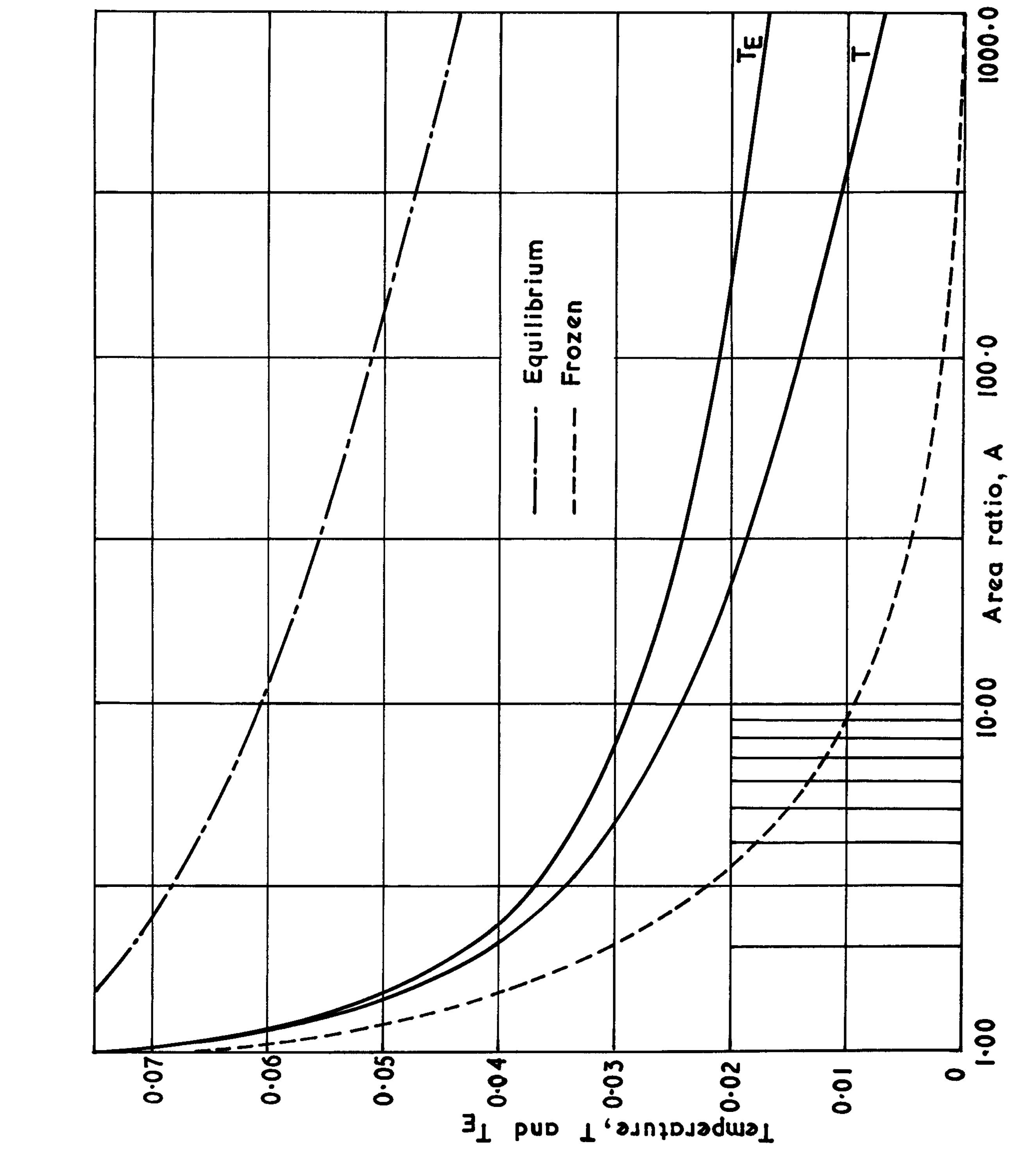
FIG. 4




$\rho = 10^{-12}$

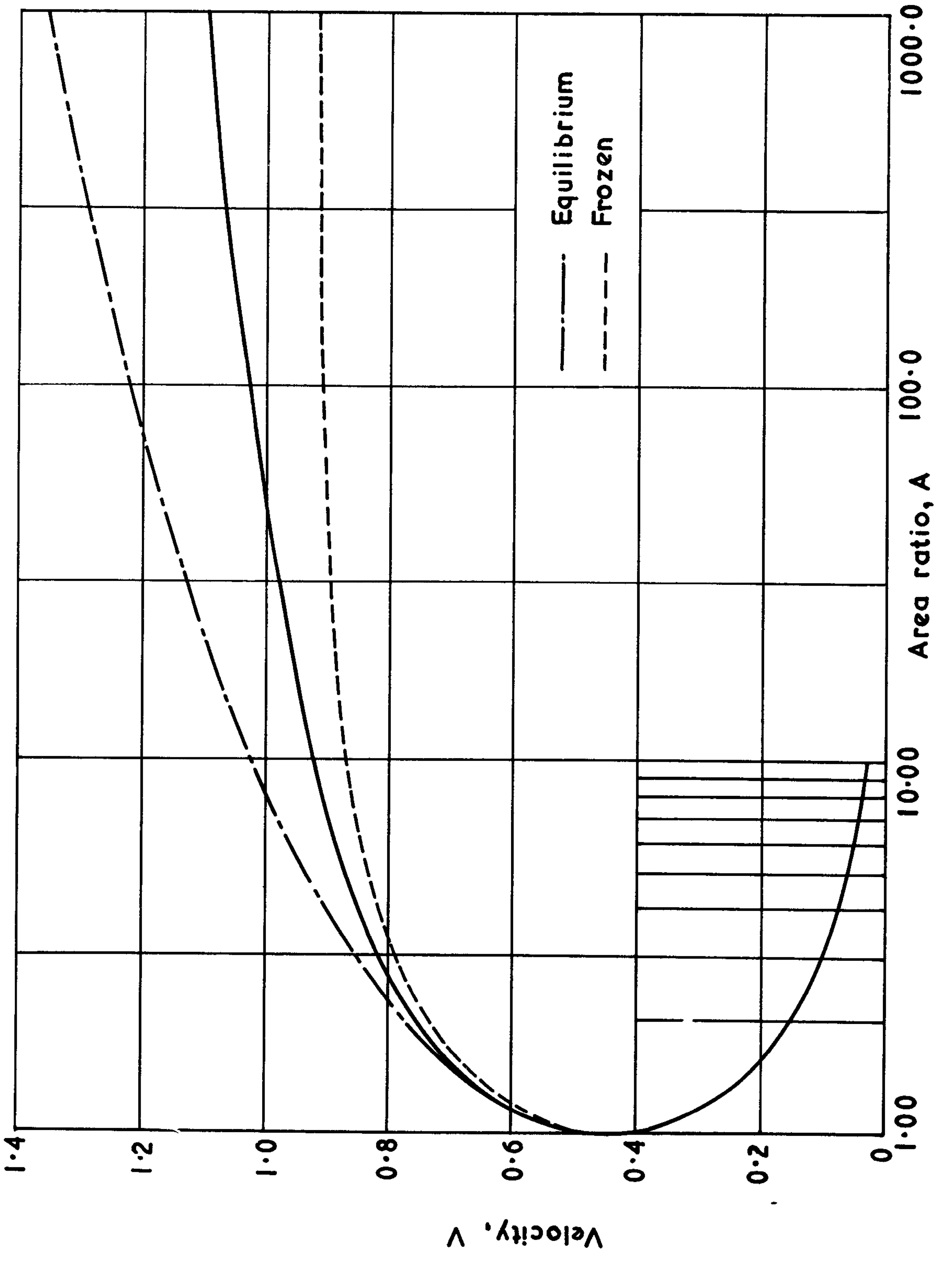
$\alpha$ varying between 0·1 and 0·9 in steps of 0·1

FIG. 5



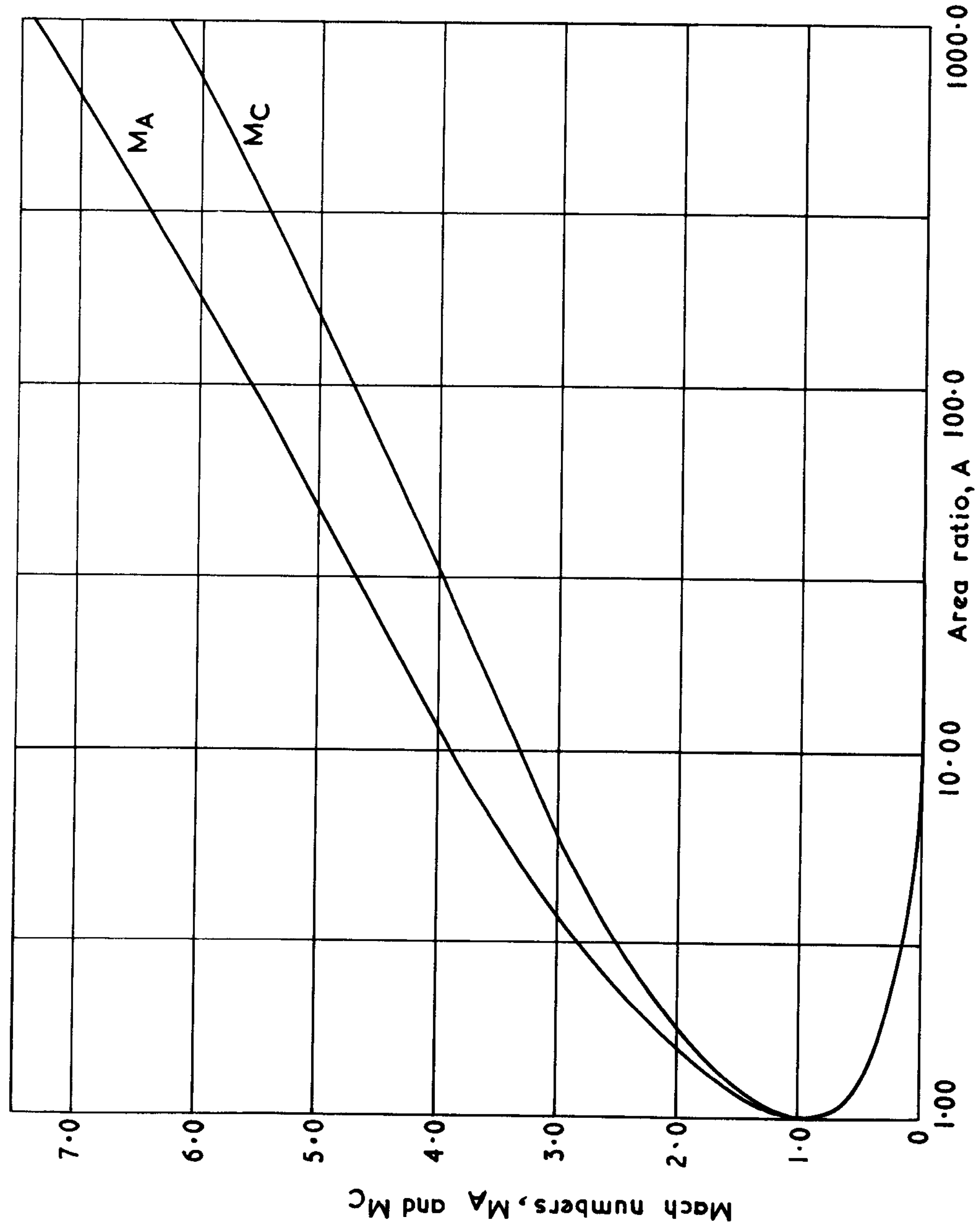

$P_o = 10^8$, $T_o = 0·09$, $\Phi_1 = 4·0 \times 10^{10}$, $\psi = 1·69 \times 10^{-8}$.

FIG. 6.



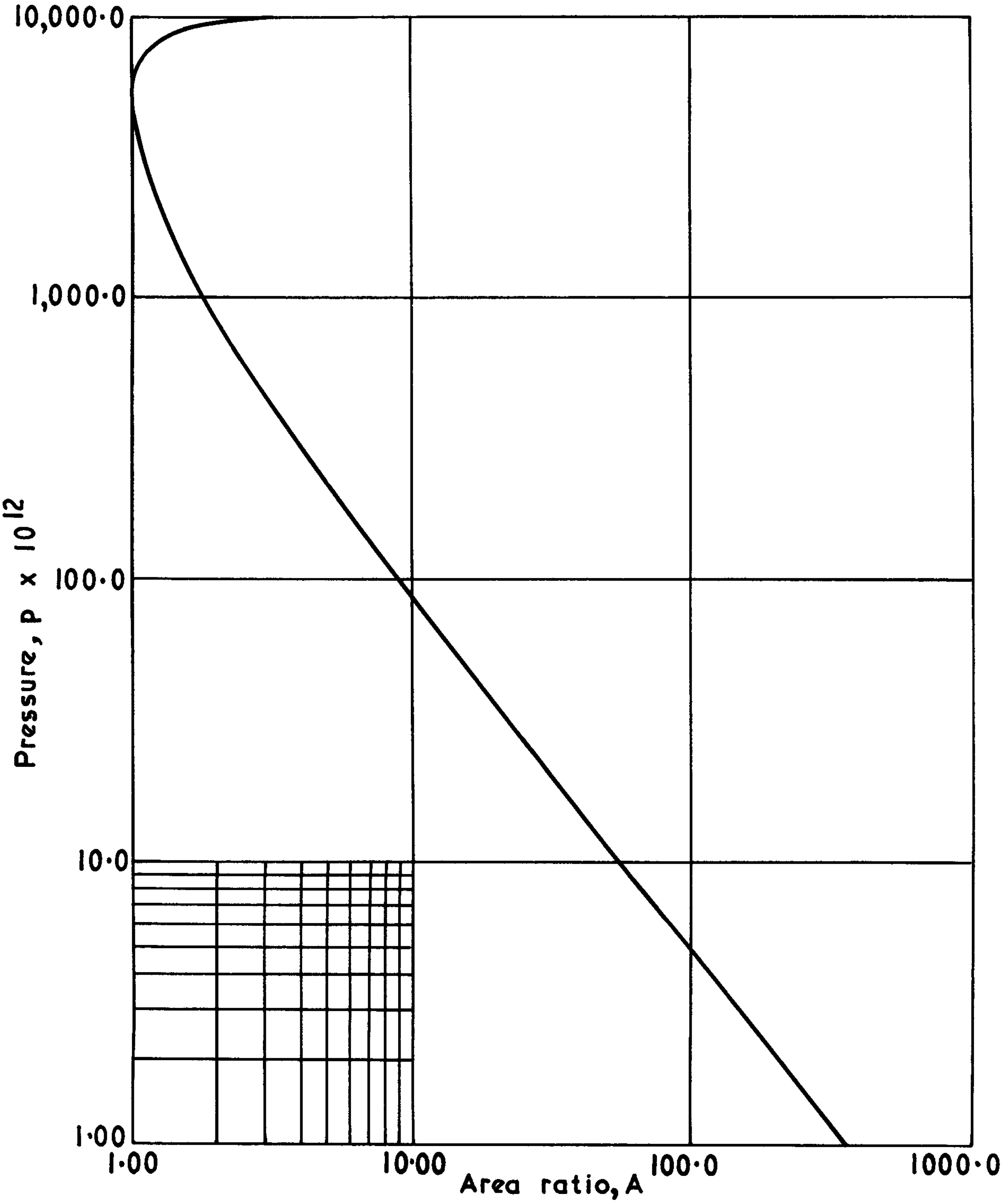

$P_o = 10^{-8}, \quad T_o = 0{\cdot}09, \quad \Phi_1 = 4{\cdot}0 \times 10^{-10}, \quad \psi = 1{\cdot}69 \times 10^{-8}$

FIG. 7



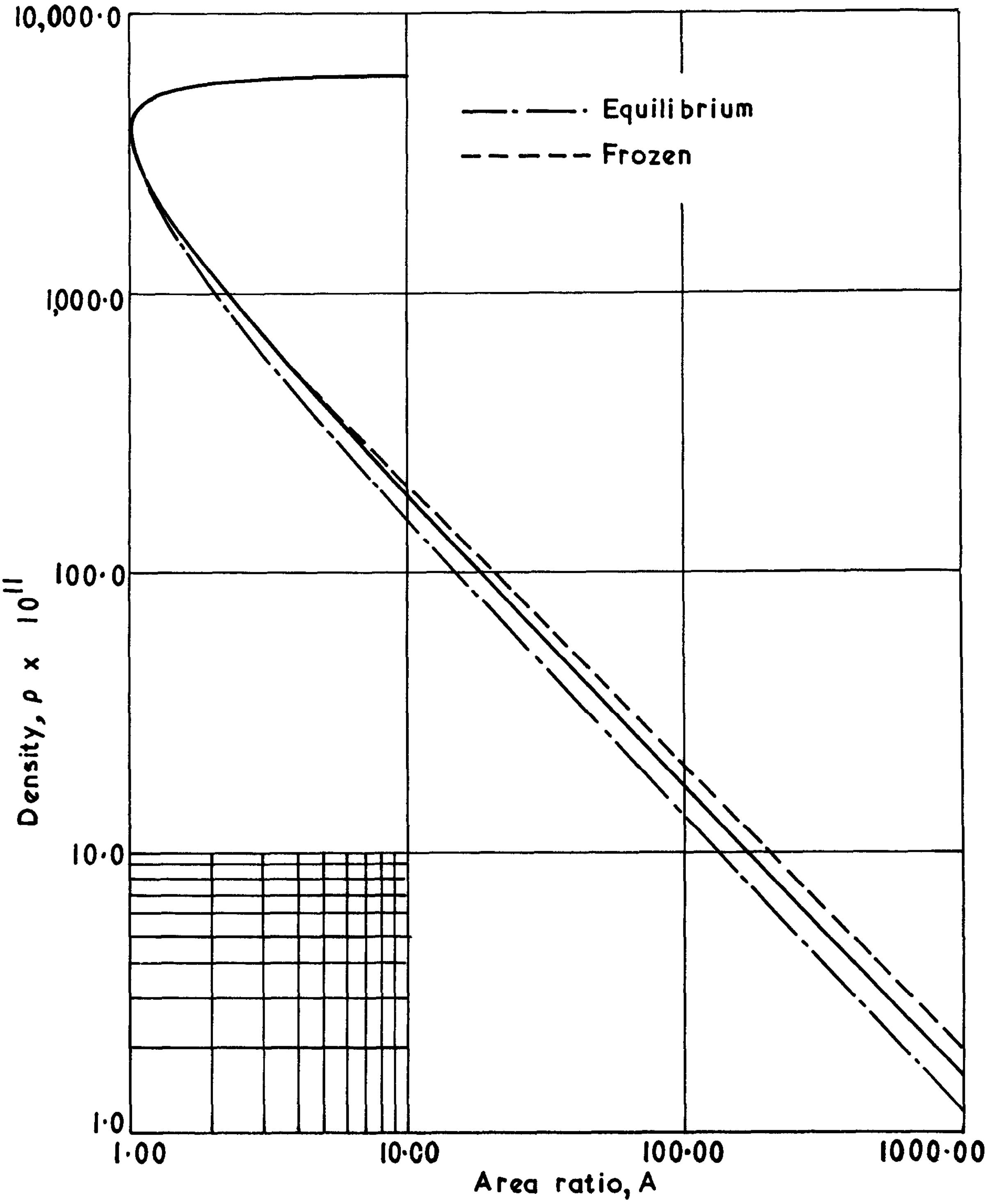

$P_o = 10^{-8}$, $T_o = 0{\cdot}09$, $\Phi_1 = 4{\cdot}0 \times 10^{10}$, $\psi = 1{\cdot}69 \times 10^{-8}$

FIG. 8.



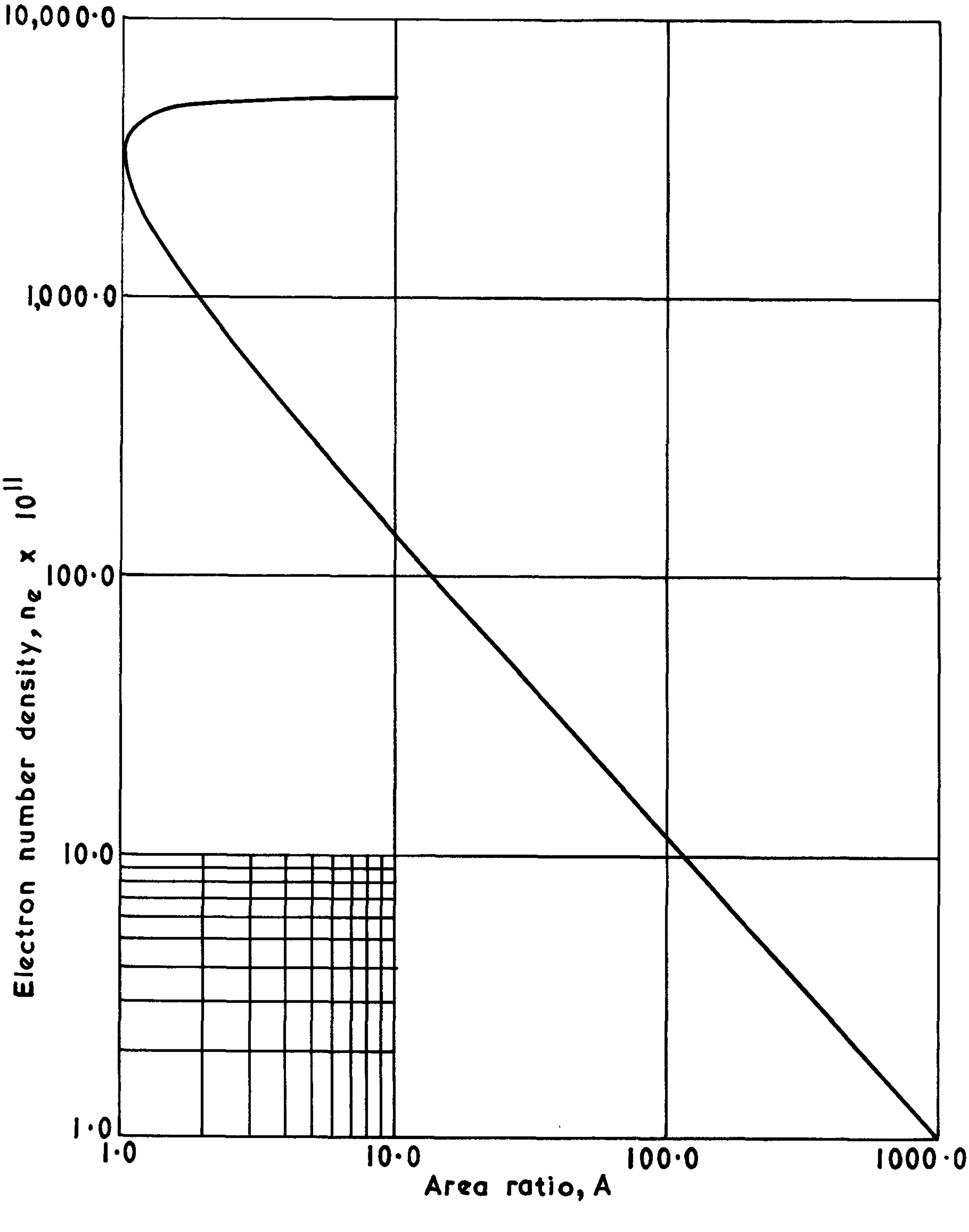

$p_o = 10^{-8}$, $T_o = 0{\cdot}09$, $\Phi_1 = 4{\cdot}0 \times 10^{10}$, $\psi = 1{\cdot}69 \times 10^{-8}$

FIG. 9.

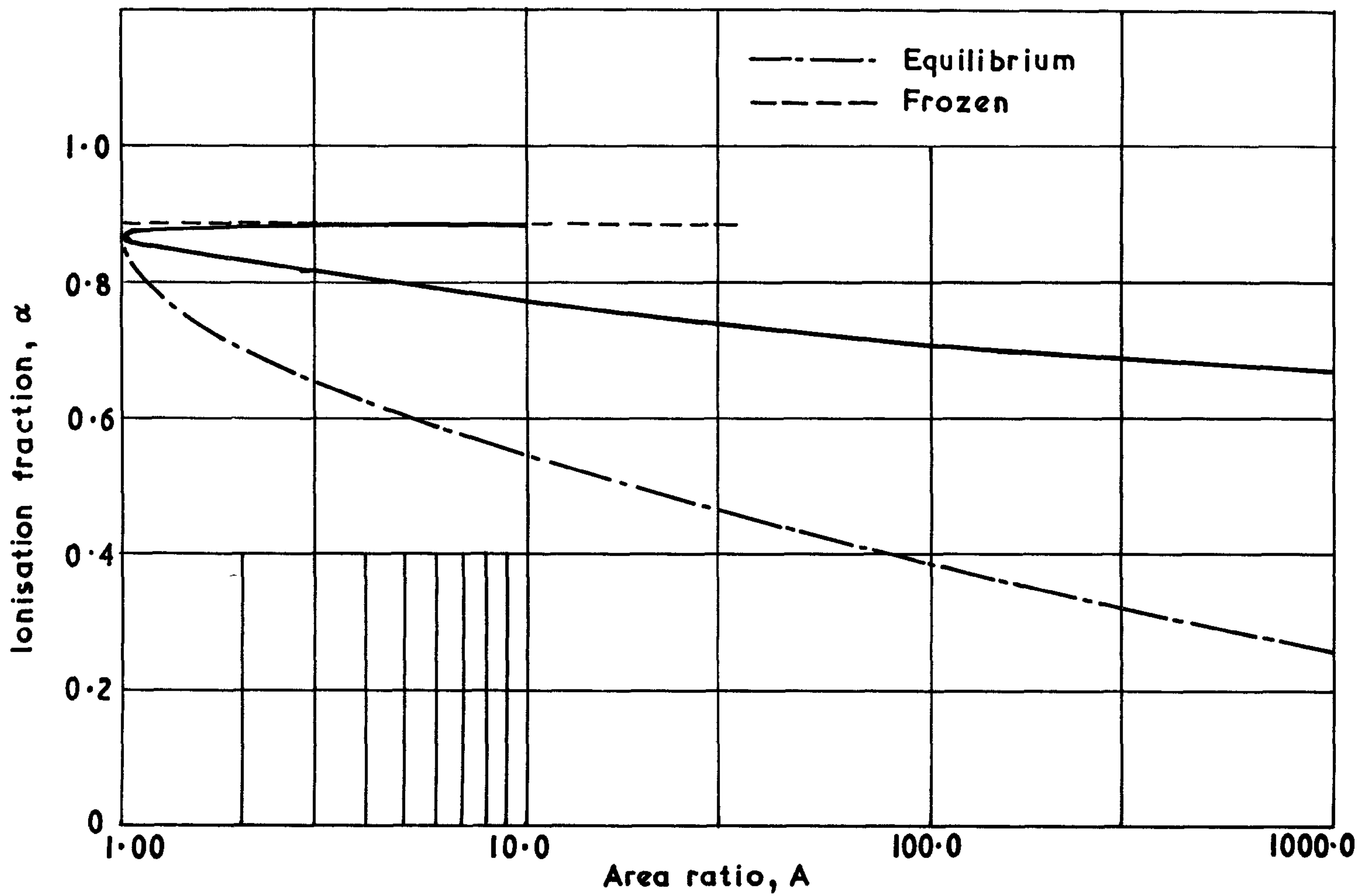



$p_o = 10^{-8}$, $T_o = 0{\cdot}09$, $\Phi_1 = 4{\cdot}0 \times 10^{10}$, $\psi = 1{\cdot}69 \times 10^{-8}$

FIG. 10



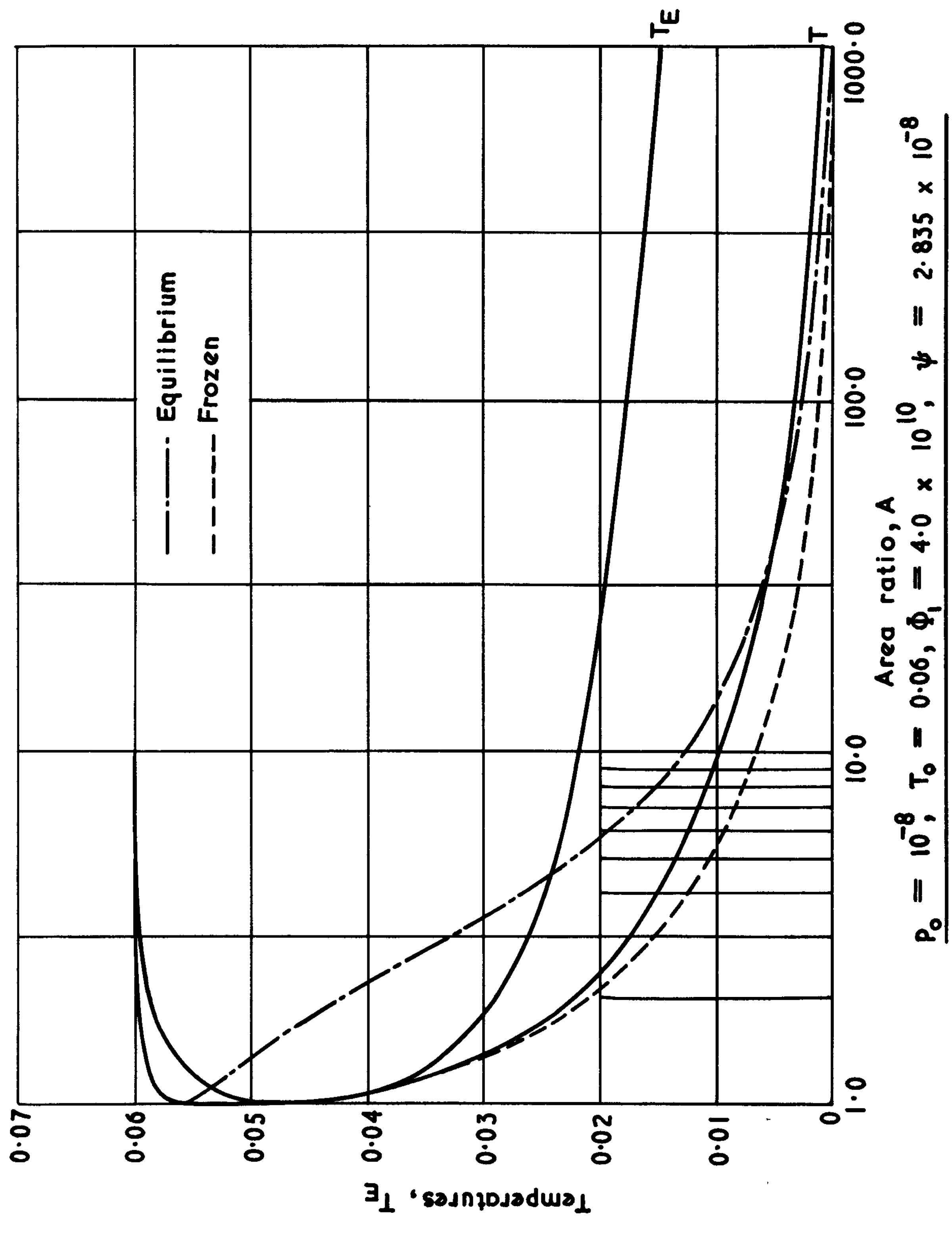

$p_o = 10^{-8}$, $T_o = 0{\cdot}09$, $\Phi_1 = 4{\cdot}0 \times 10^{10}$, $\psi = 1{\cdot}69 \times 10^{-8}$.




$p_o = 10^{-8}$, $T_o = 0{\cdot}09$, $\Phi_1 = 4{\cdot}0 \times 10^{10}$, $\psi = 1{\cdot}69 \times 10^{-8}$

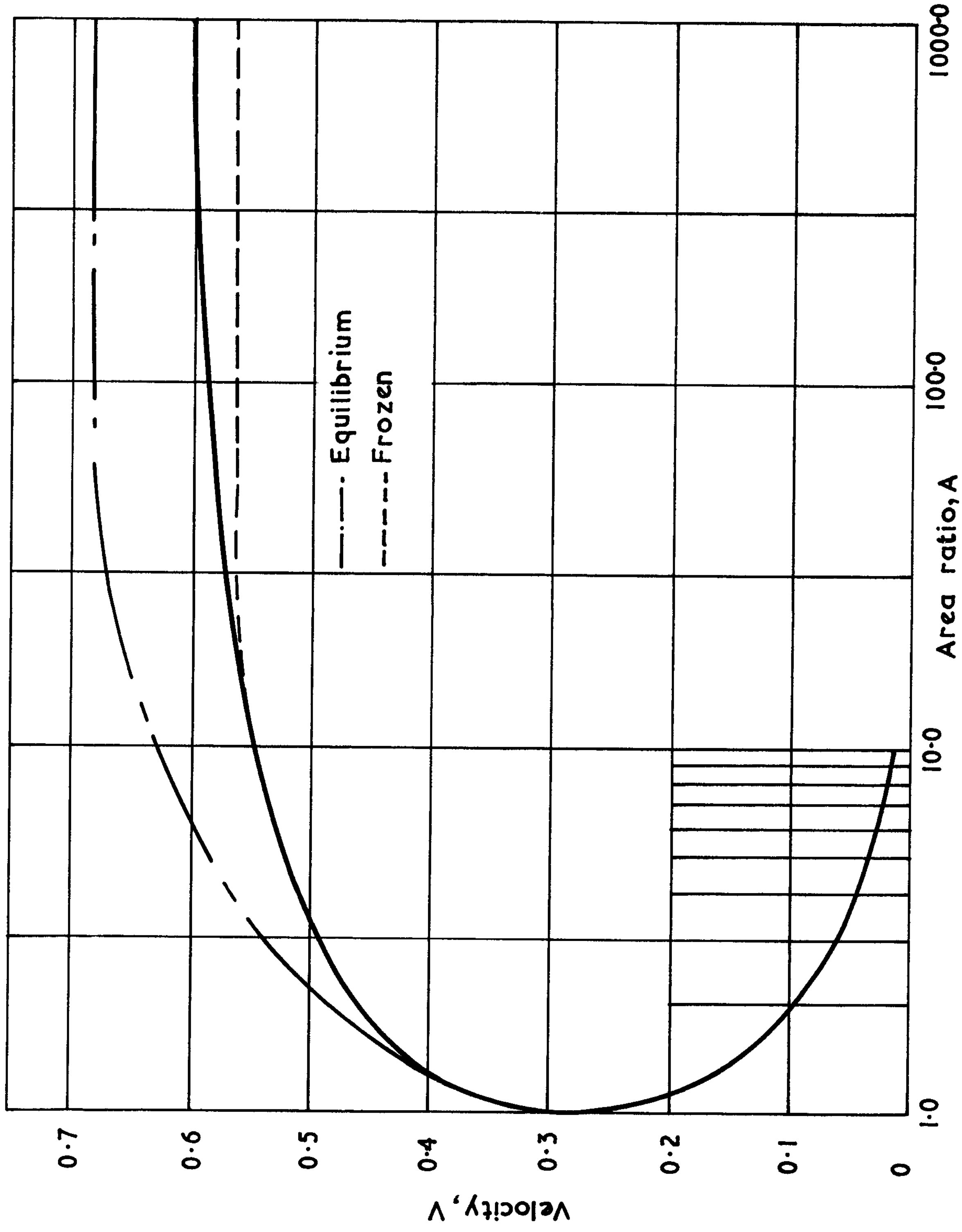
FIG. 11

FIG. 12



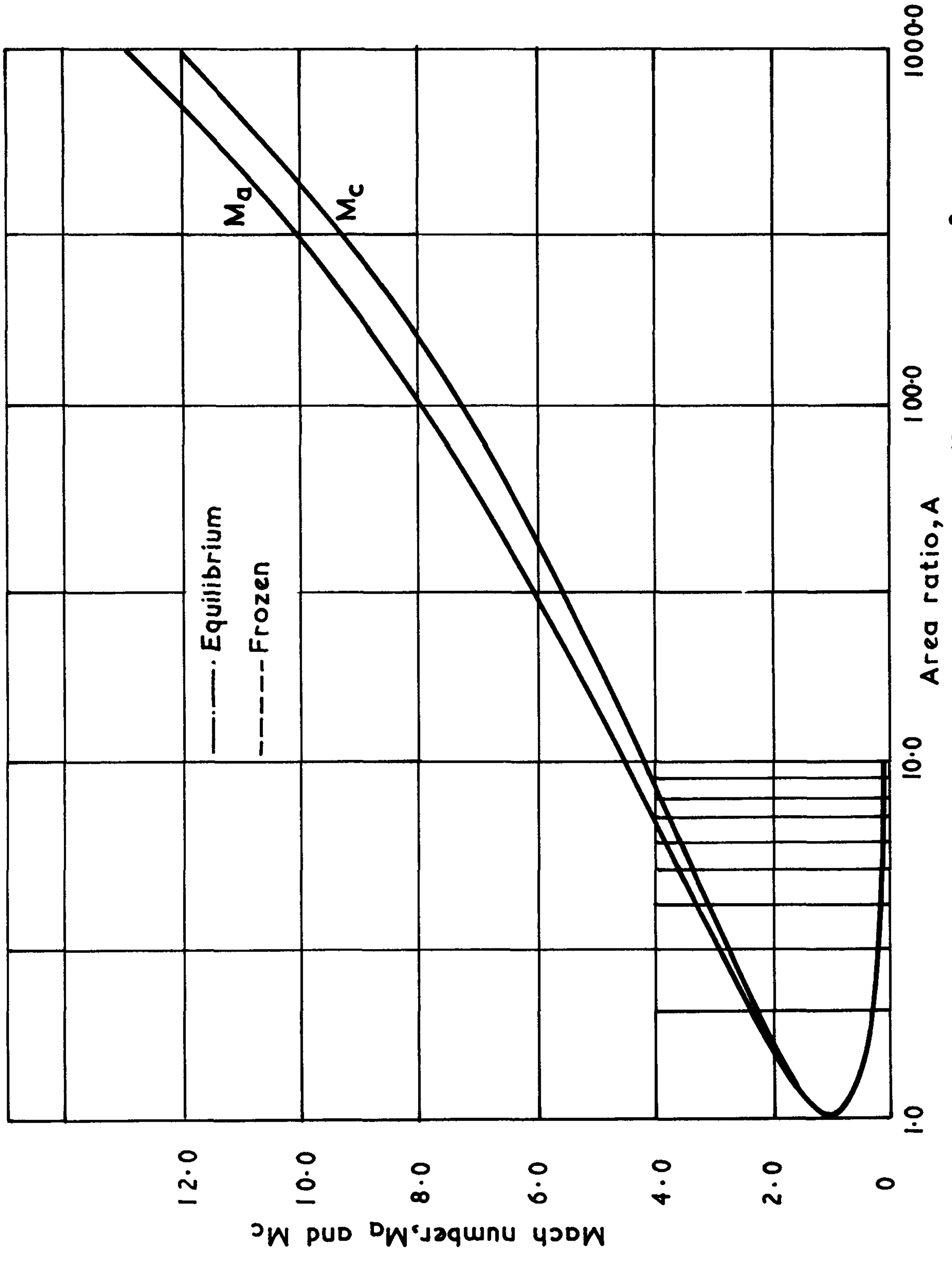
Equilibrium
Frozen
$T_E$
T
0·07
0·06
0·05
0·04
0·03
0·02
0·01
0
Temperatures, $T_E$
1·0
10·0
100·0
1000·0
Area ratio, A
$P_o = 10^{-8}$, $T_o = 0{\cdot}06$, $\Phi_1 = 4{\cdot}0 \times 10^{10}$, $\psi = 2{\cdot}835 \times 10^{-8}$

FIG. 13.



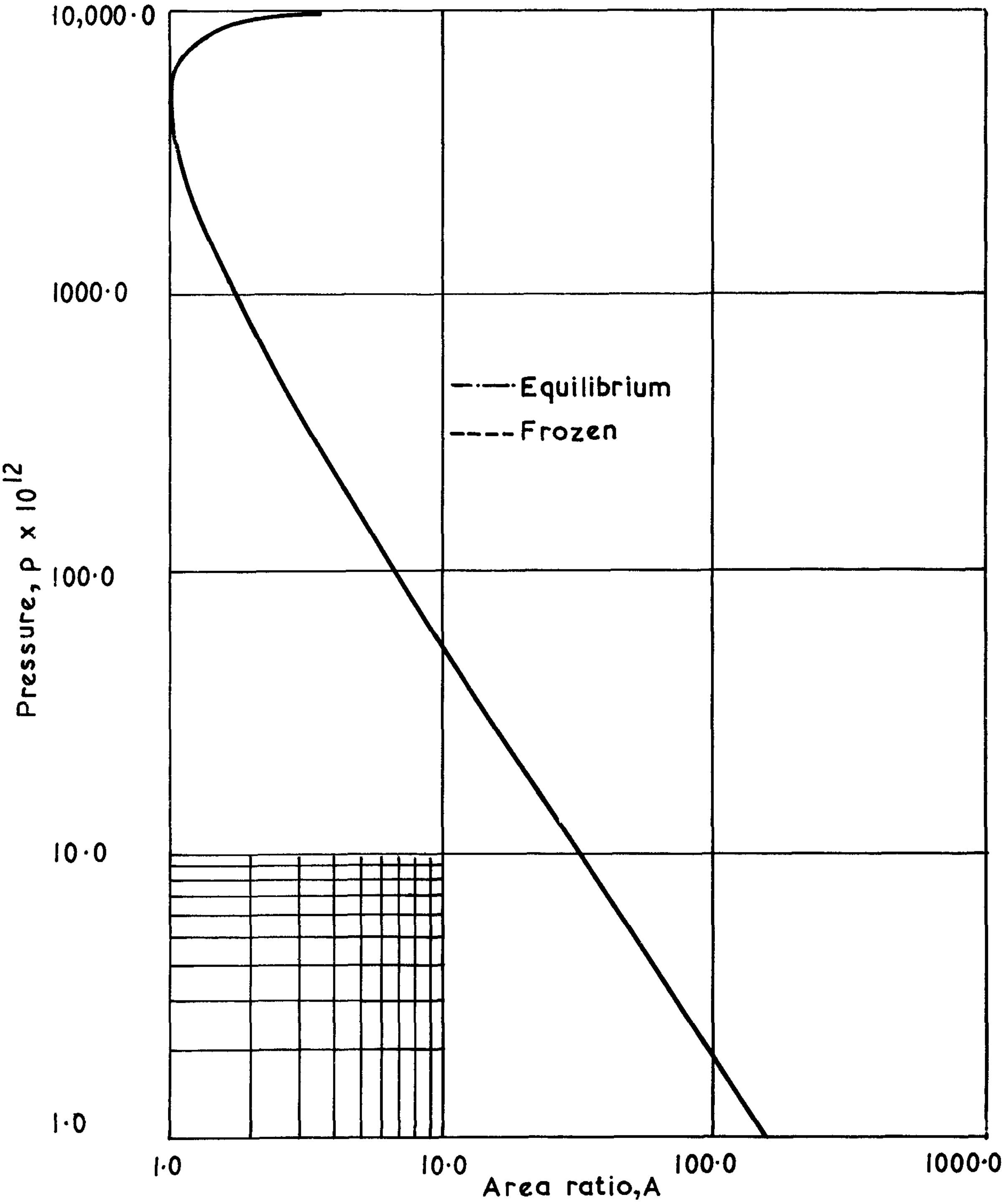

$P_o = 10^{-8}, T_o = 0{\cdot}06, \Phi_1 = 4{\cdot}0 \times 10^{10}, \psi = 2{\cdot}835 \times 10^{-8}$

FIG. 14.



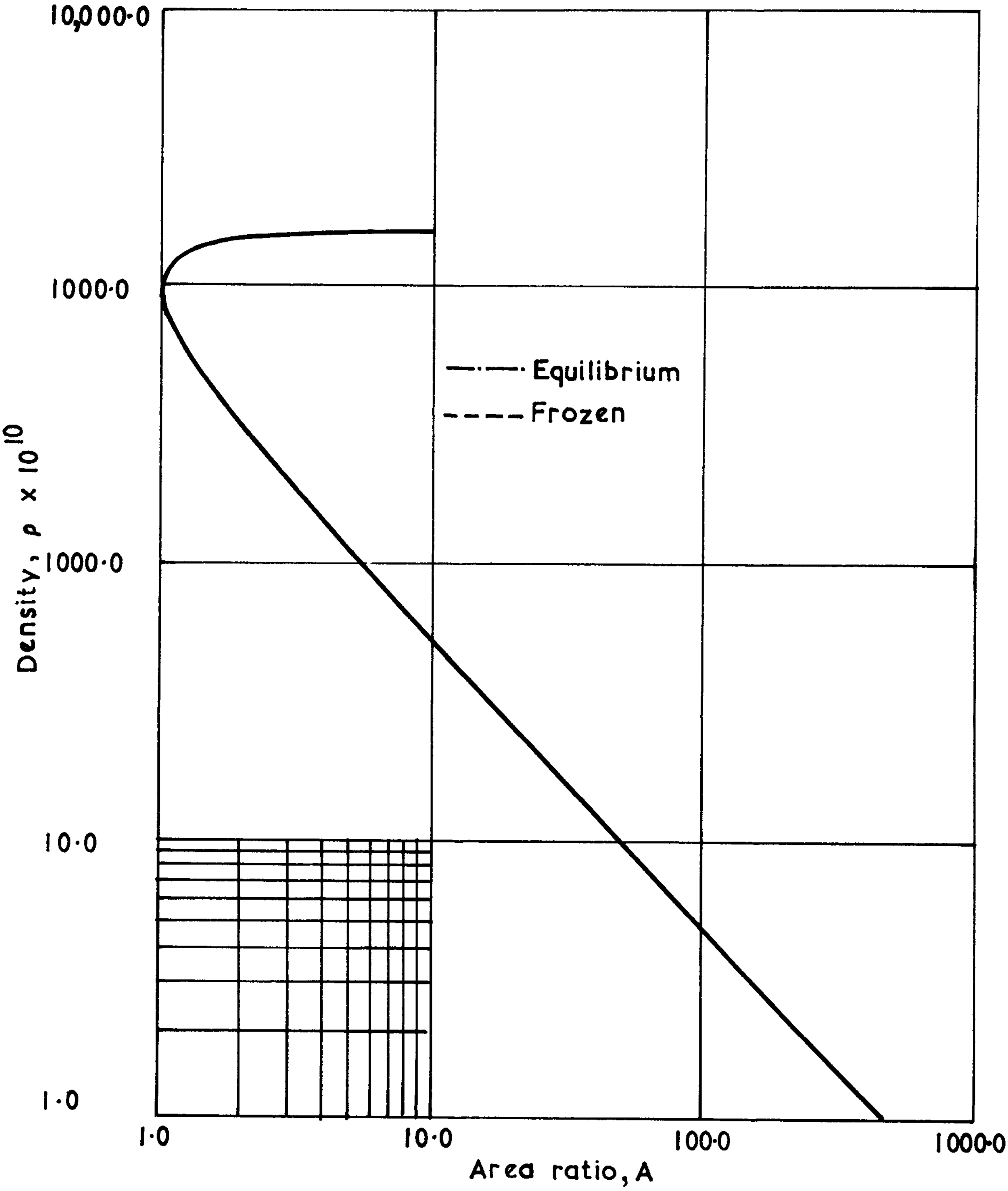

$P_0 = 10^{-8}$, $T_0 = 0{\cdot}06$, $\Phi_1 = 4{\cdot}0 \times 10^{10}$, $\psi = 2{\cdot}835 \times 10^{-8}$

FIG. 15.



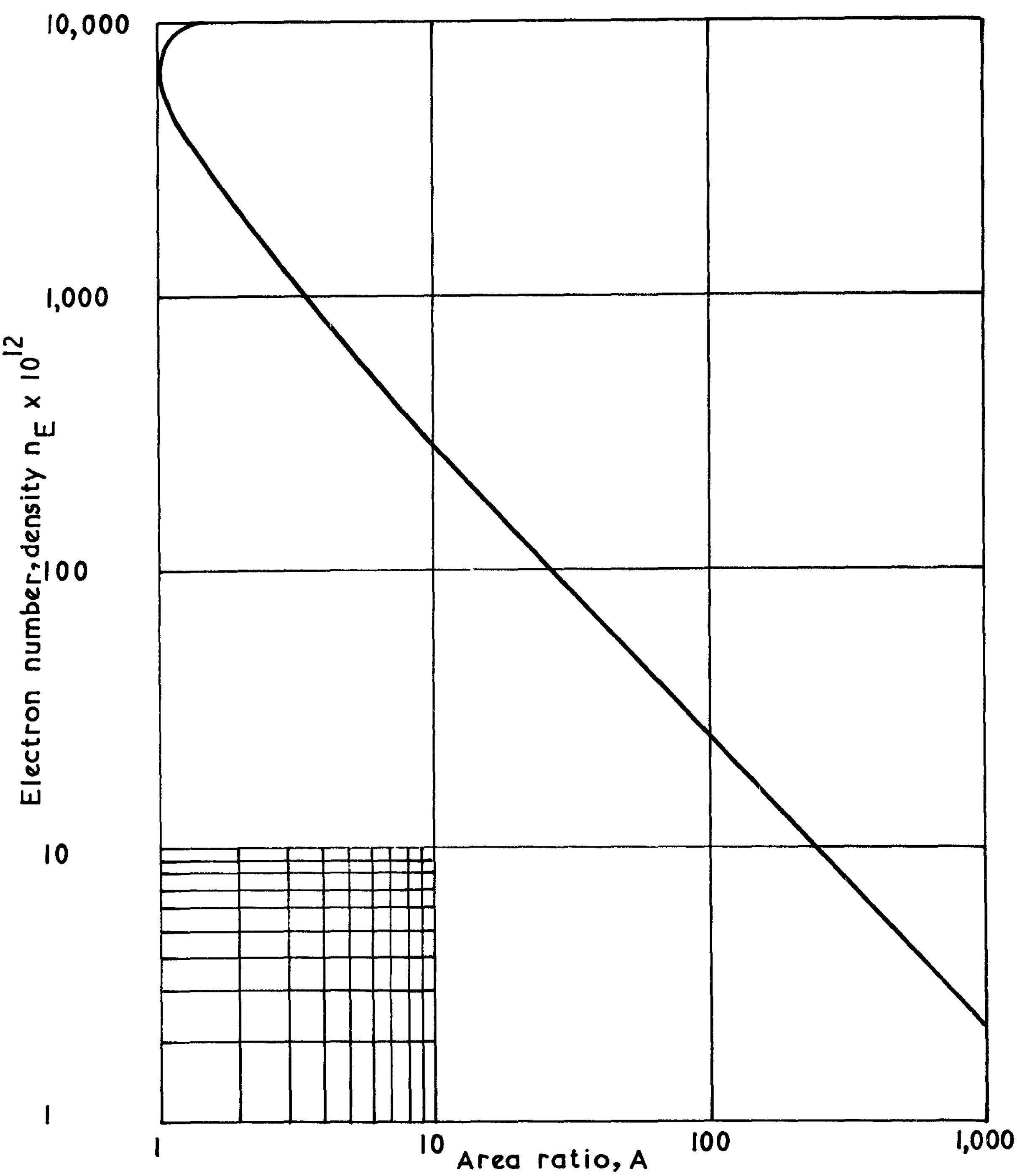

$p_0 = 10^{-8}$, $T_0 = 0{\cdot}06$, $\Phi_1 = 4{\cdot}0 \times 10^{10}$, $\psi = 2{\cdot}835 \times 10^{8}$

FIG. 16.

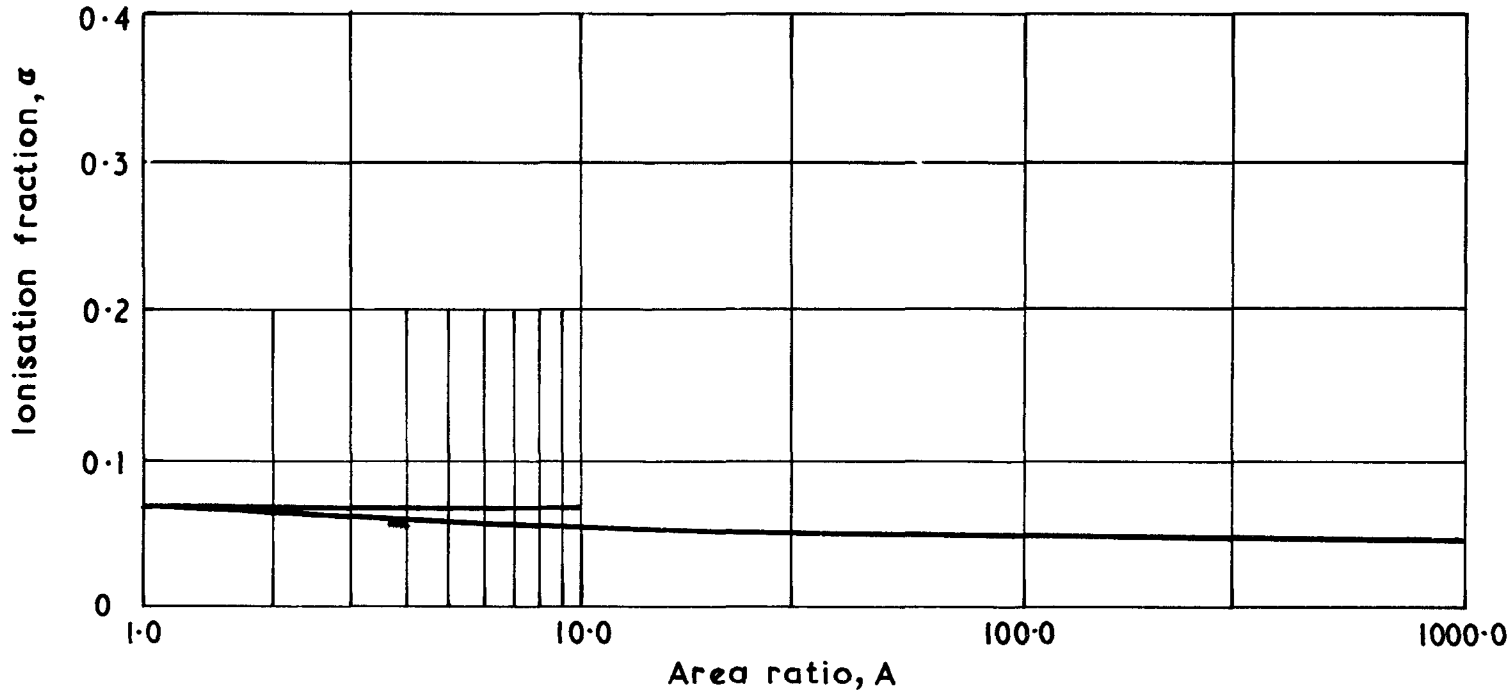



$P_0 = 10^{-8}$, $T_0 = 0{\cdot}06$, $\Phi_1 = 4{\cdot}0 \times 10^{10}$, $\psi = 2{\cdot}835 \times 10^{-8}$

FIG. 17.



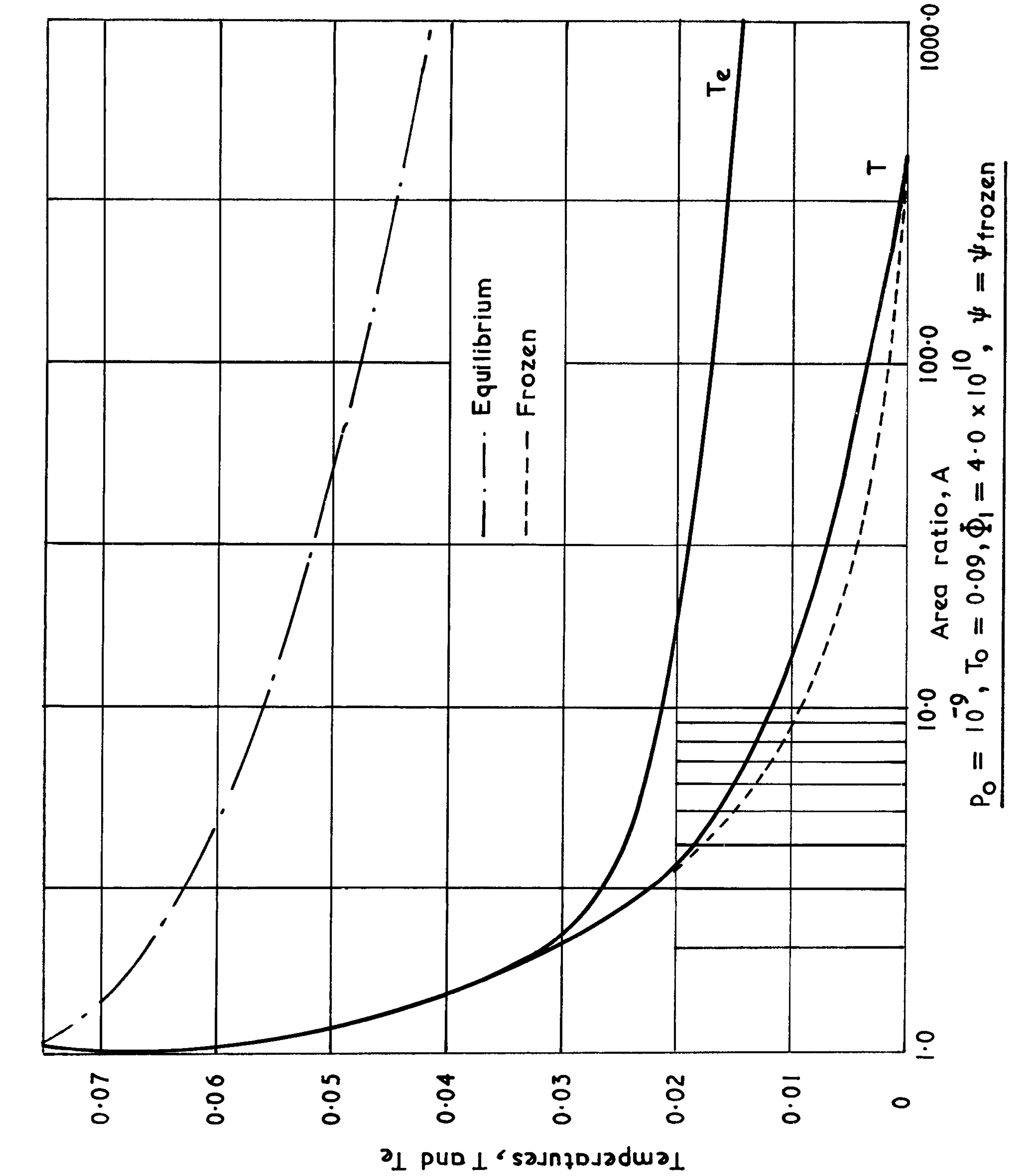

$P_o = 10^{-8}$, $T_o = 0{\cdot}06$, $\Phi = 4{\cdot}0 \times 10^{10}$, $\psi = 2{\cdot}835 \times 10^{-8}$




$p_0 = 10^{-8}$, $T_0 = 0{\cdot}06$, $\Phi = 4{\cdot}0 \times 10^{10}$, $\psi = 2{\cdot}835 \times 10^{-8}$

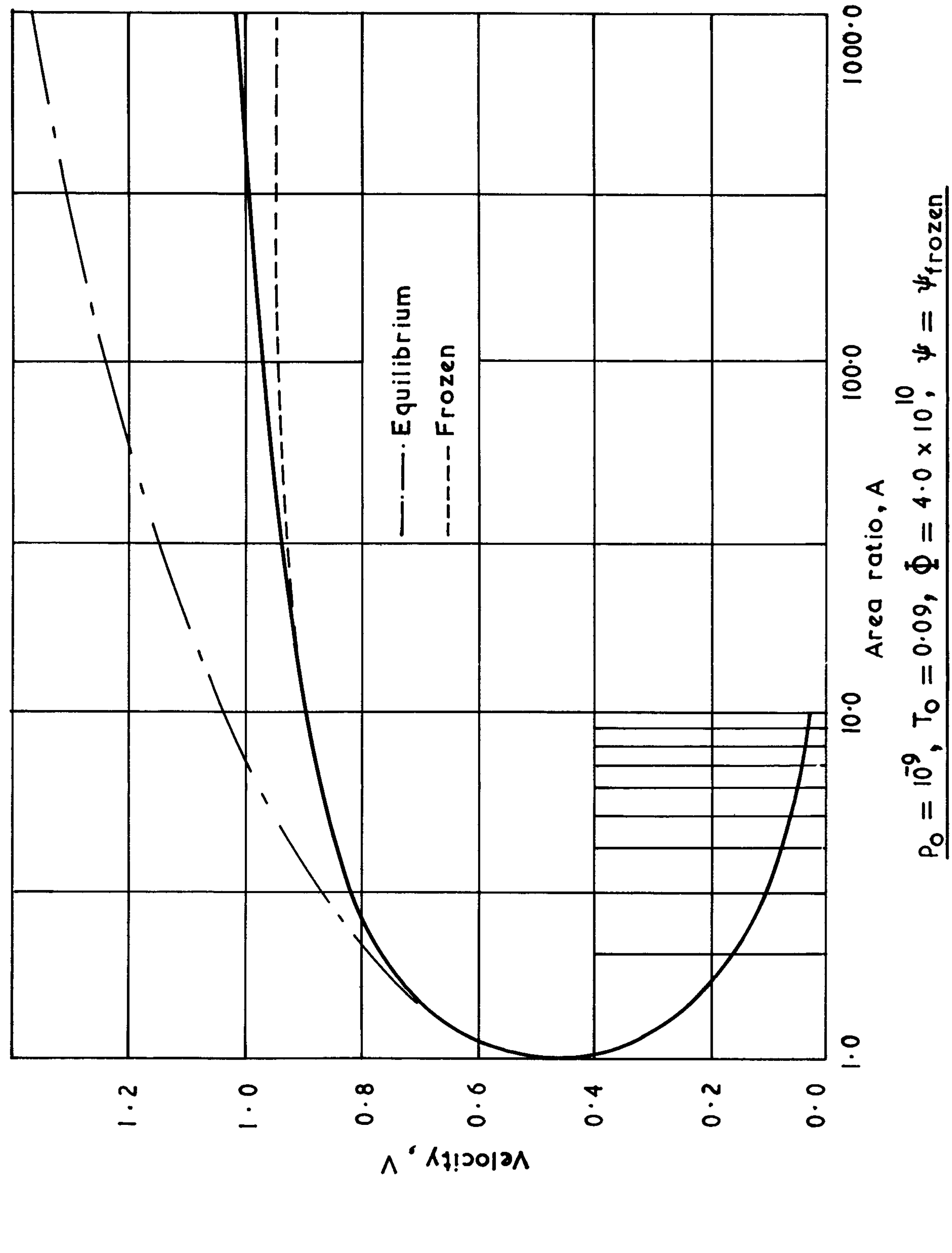
FIG 18.

FIG. 19.



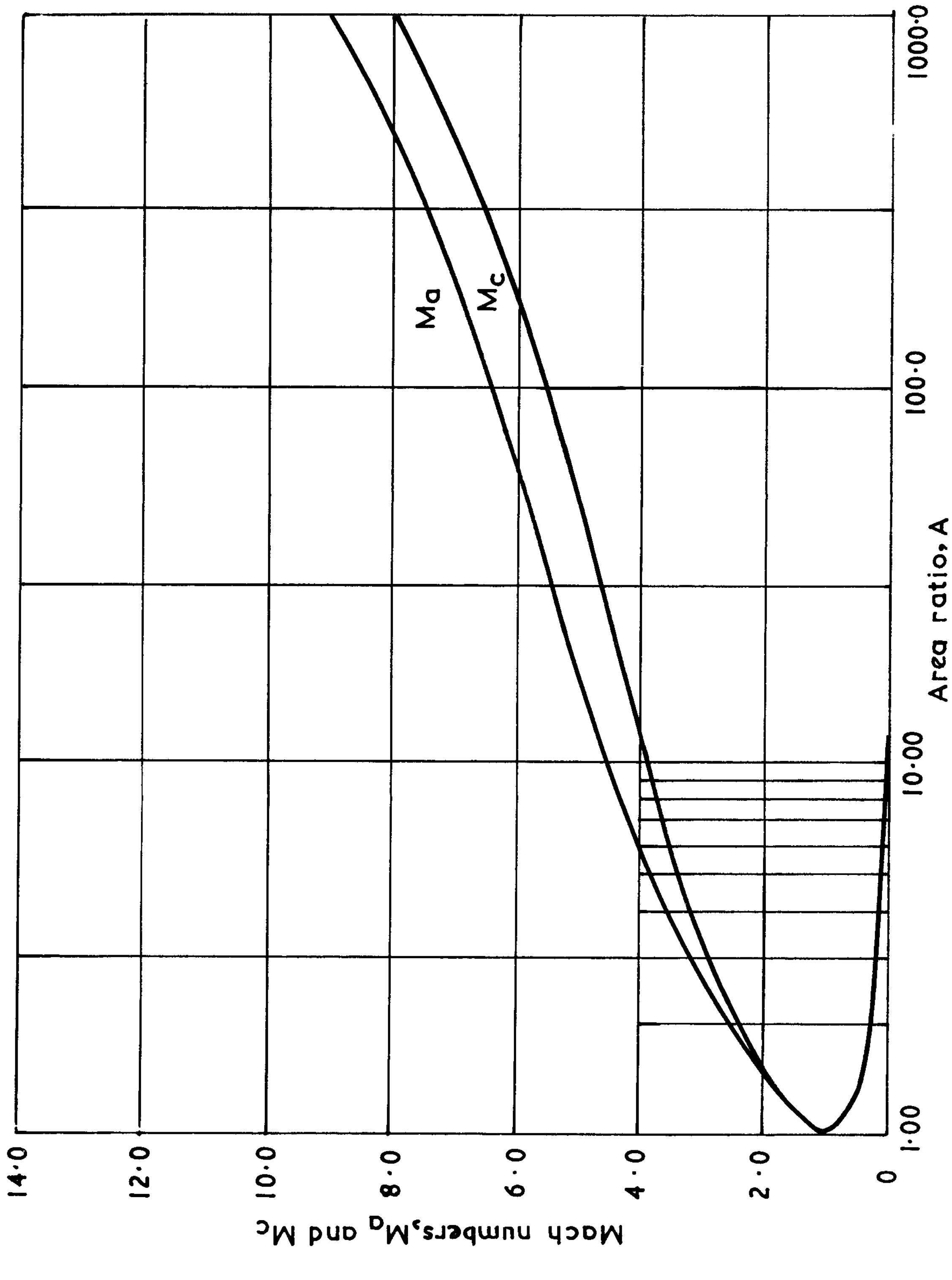
Temperatures, T and Te
0·07
0·06
0·05
0·04
0·03
0·02
0·01
0
1·0
10·0
100·0
1000·0
Area ratio, A
Equilibrium
Frozen
Te
T
$P_o = 10^{-9}$, $T_o = 0{\cdot}09$, $\Phi_1 = 4{\cdot}0 \times 10^{10}$, $\psi = \psi_{frozen}$

FIG. 20.



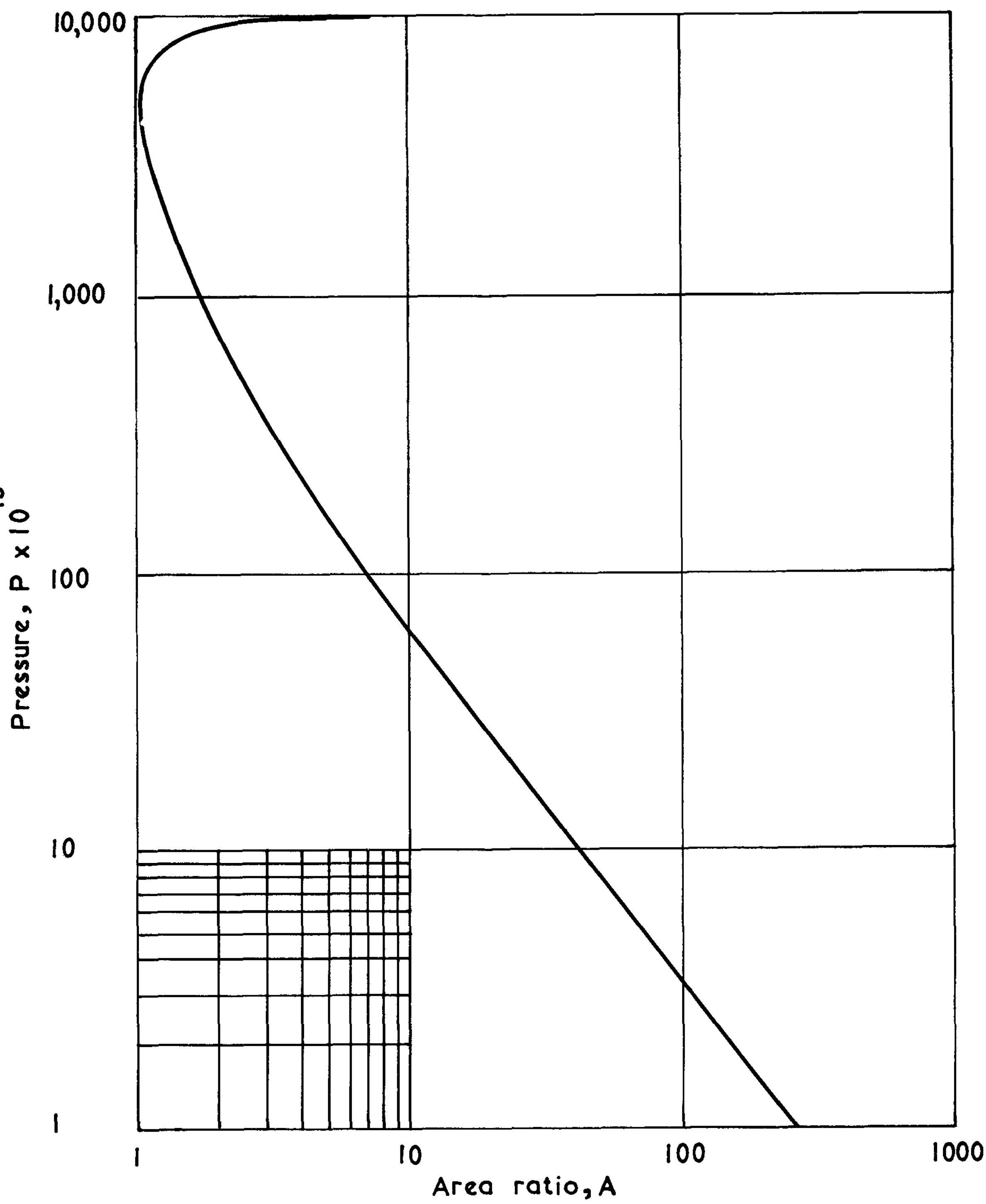
Equilibrium
Frozen
Area ratio, A
1·0
10·0
100·0
1000·0
Velocity, V
0·0
0·2
0·4
0·6
0·8
1·0
1·2
$P_O = 10^{-9}$, $T_O = 0{\cdot}09$, $\Phi = 4{\cdot}0 \times 10^{10}$, $\psi = \psi_{frozen}$

FIG. 21.



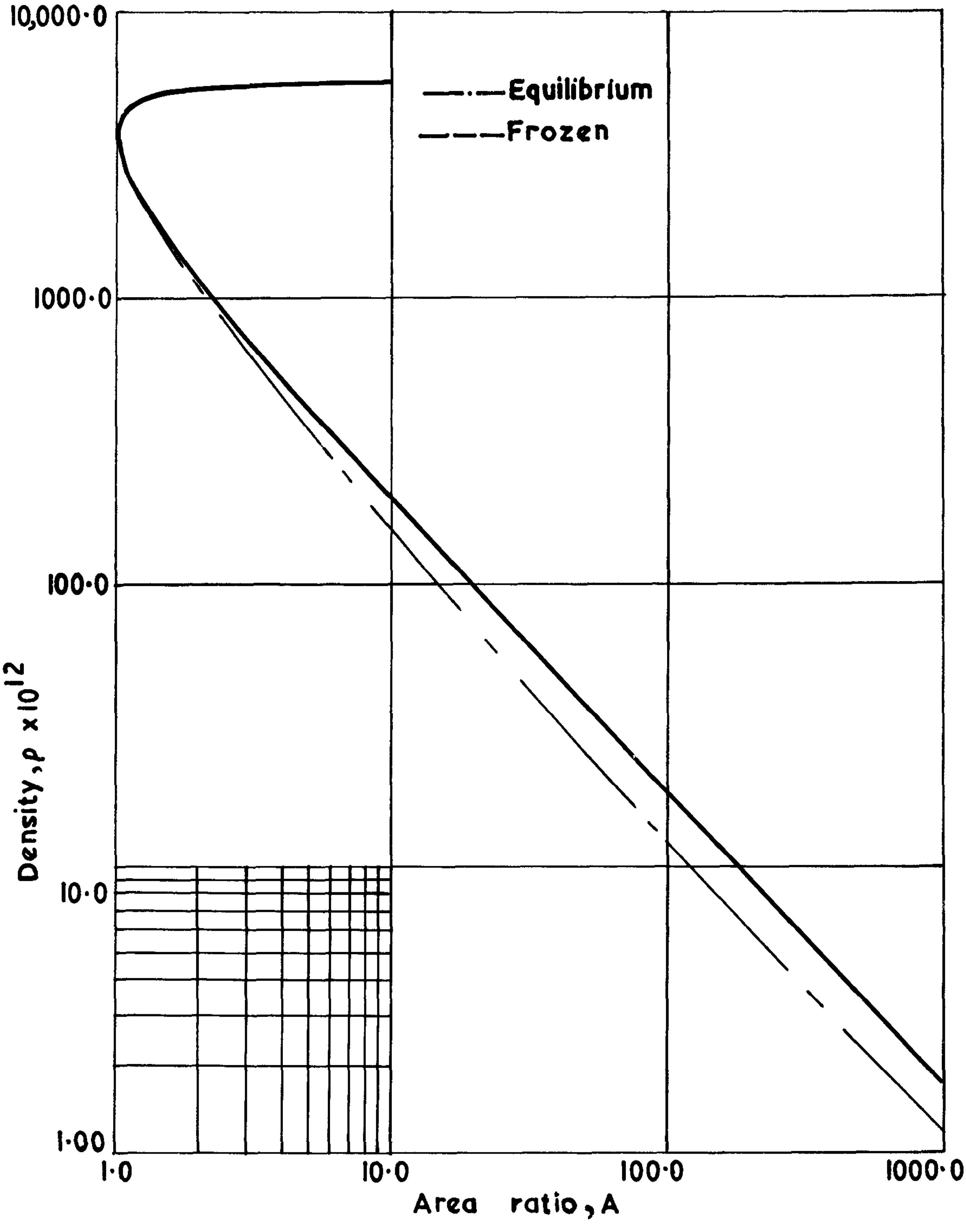

$P_o = 10^{-9}$, $T_o = 0{\cdot}09$, $\Phi_I = 4{\cdot}0 \times 10^{10}$, $\psi = \psi_{frozen}$

FIG.22.



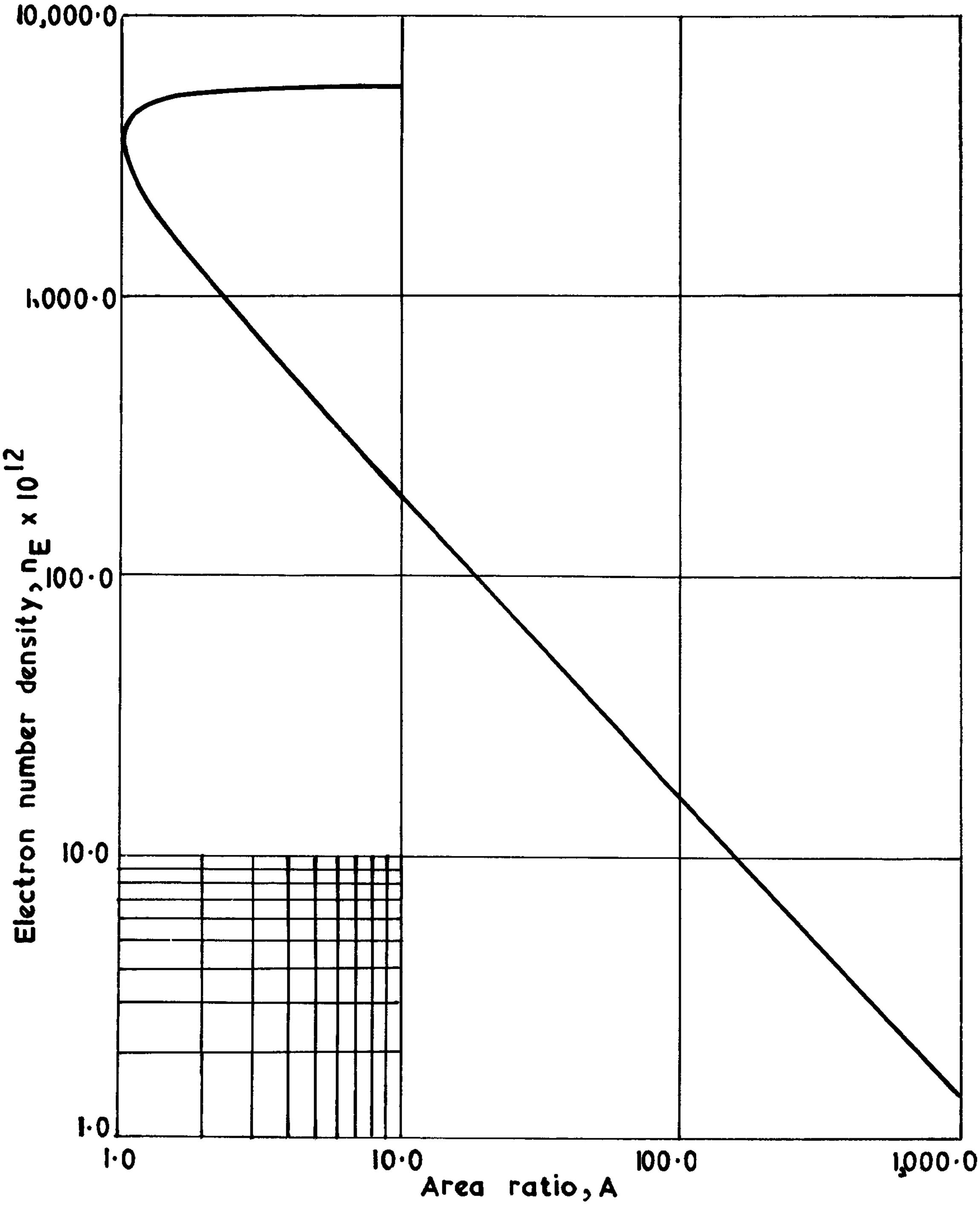

$P_0 = 10^{-9}$, $T_0 = 0{\cdot}09$, $\Phi = 4{\cdot}0 \times 10^{10}$, $\psi = \psi_{frozen}$

FIG. 23.



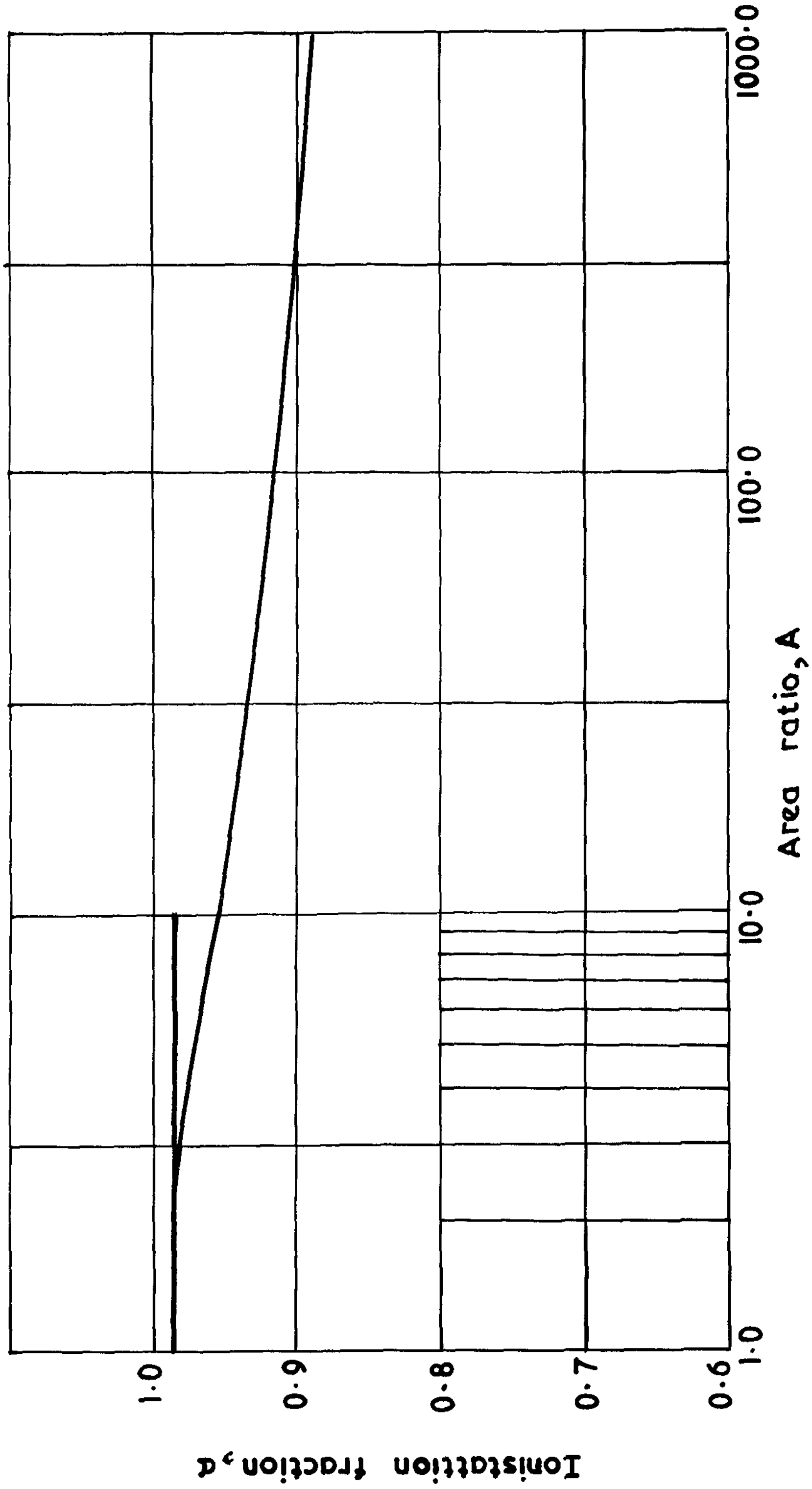

$p_o = 10^{-9}$, $T_o = 0{\cdot}09$, $\Phi = 4{\cdot}0 \times 10^{10}$, $\Psi = \Psi_{frozen}$.

FIG.24.



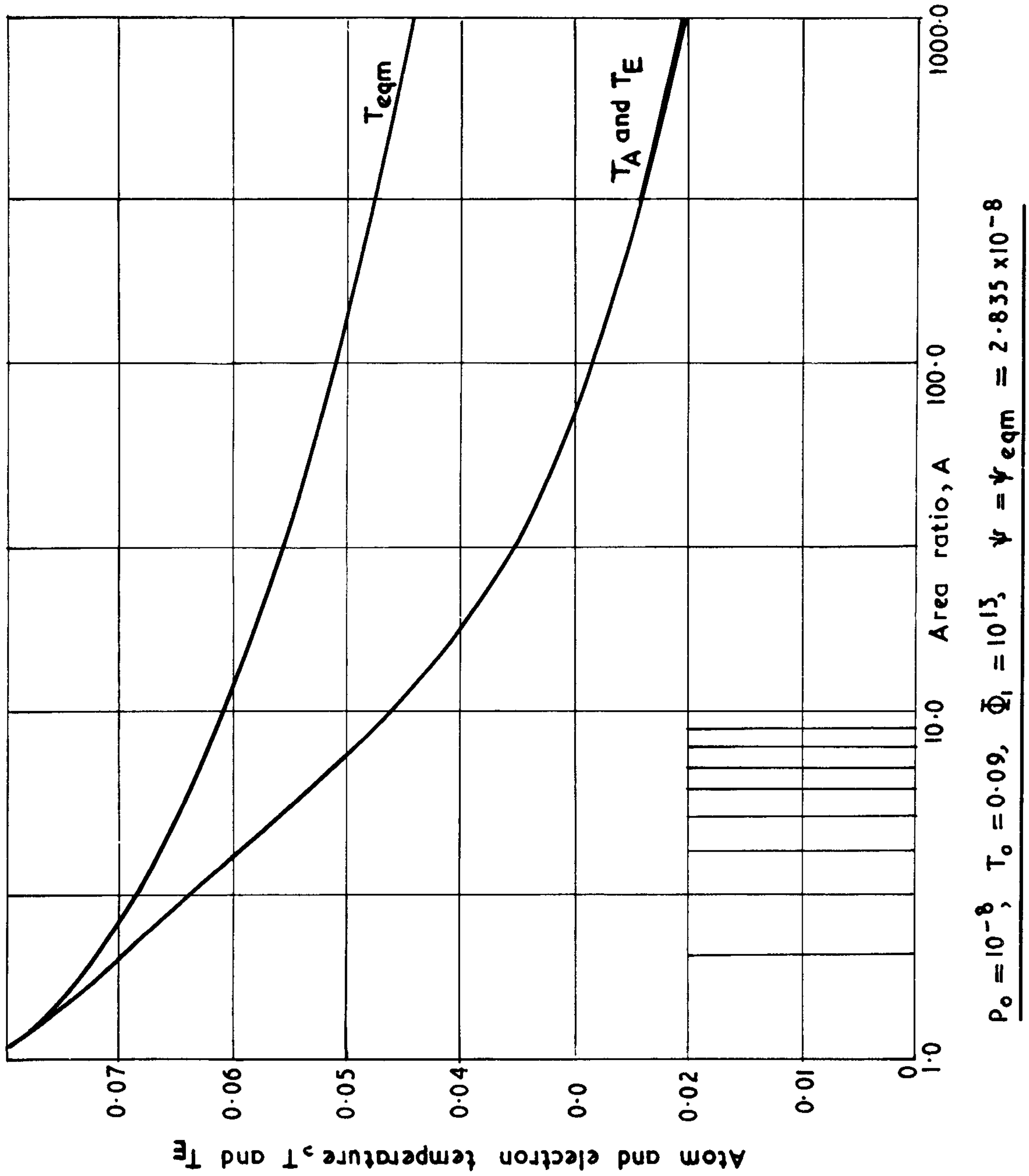

$p_o = 10^{-9}$, $T_o = 0.09$, $\Phi = 4.0 \times 10^{10}$, $\psi = \psi_{frozen}$

FIG.25.



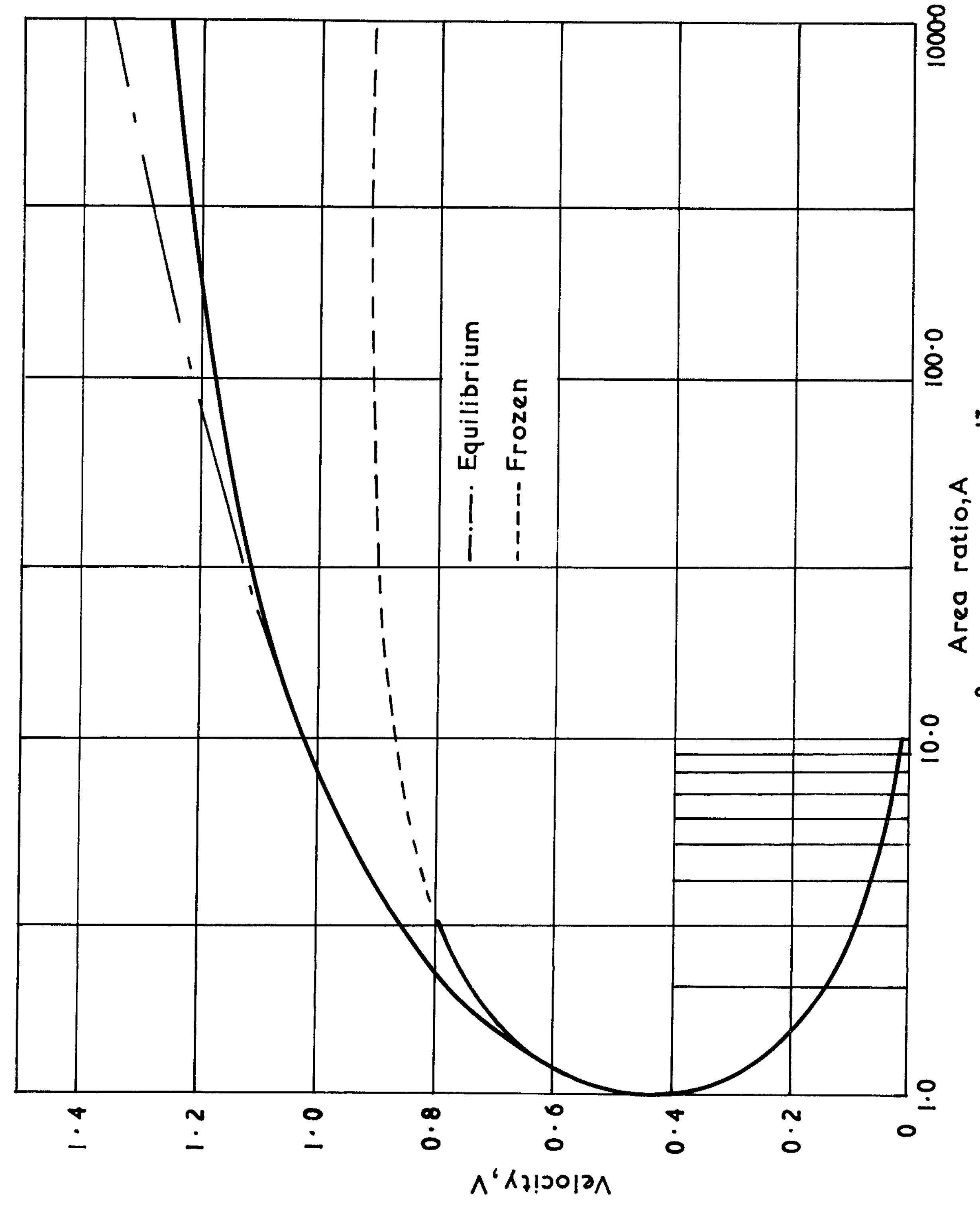

$P_o = 10^{-9}$, $T_o = 0{\cdot}09$, $\Phi = 4{\cdot}0 \times 10^{10}$, $\psi = \psi_{frozen}$

FIG. 26.



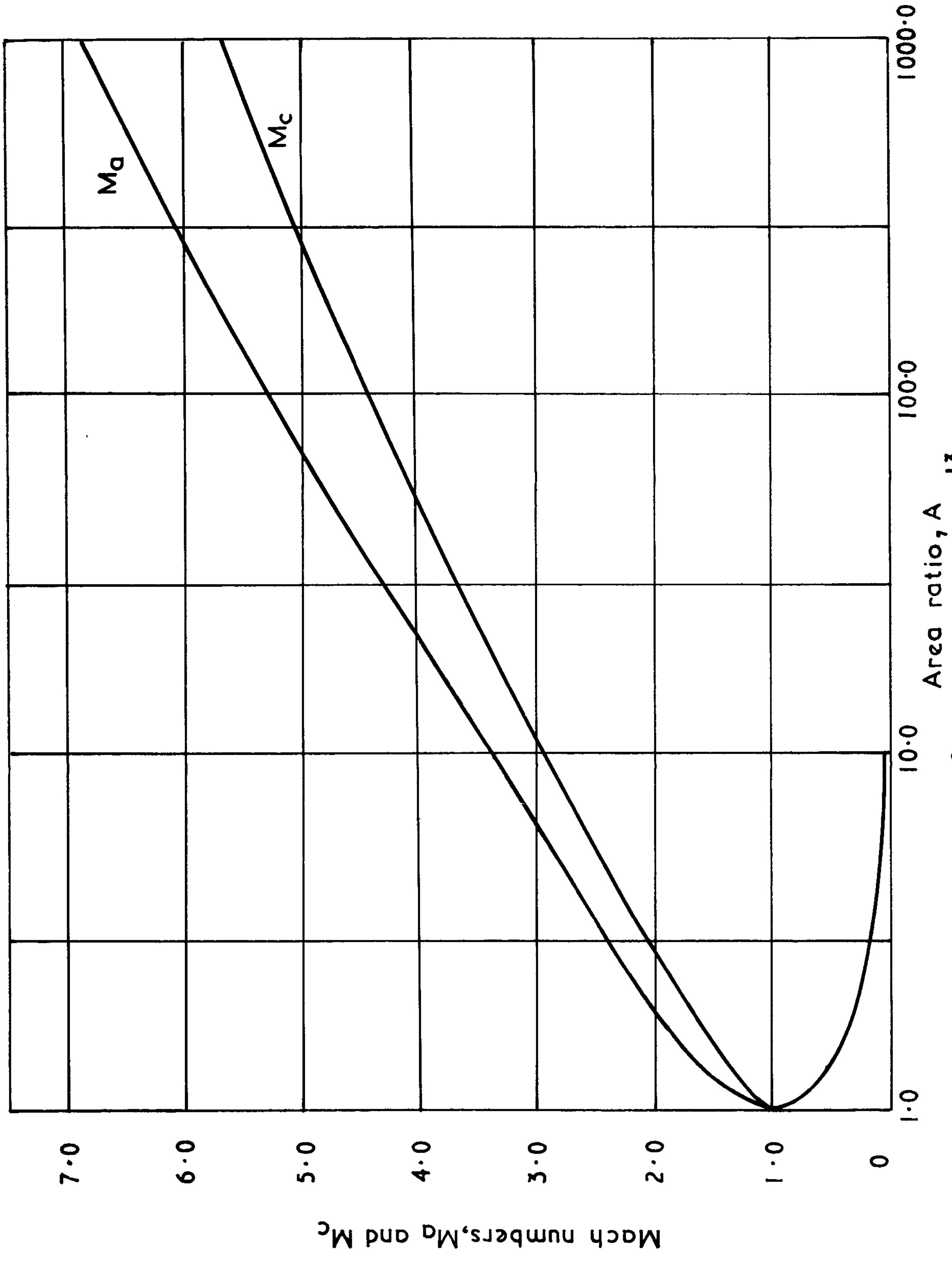
$T_{eqm}$
$T_A$ and $T_E$
0·07
0·06
0·05
0·04
0·0
0·02
0·01
0
1·0
10·0
100·0
1000·0
Area ratio, A
Atom and electron temperature, T and $T_E$
$P_o = 10^{-8}$, $T_o = 0·09$, $\Phi_1 = 10^{13}$, $\psi = \psi_{eqm} = 2·835 \times 10^{-8}$

FIG. 27.



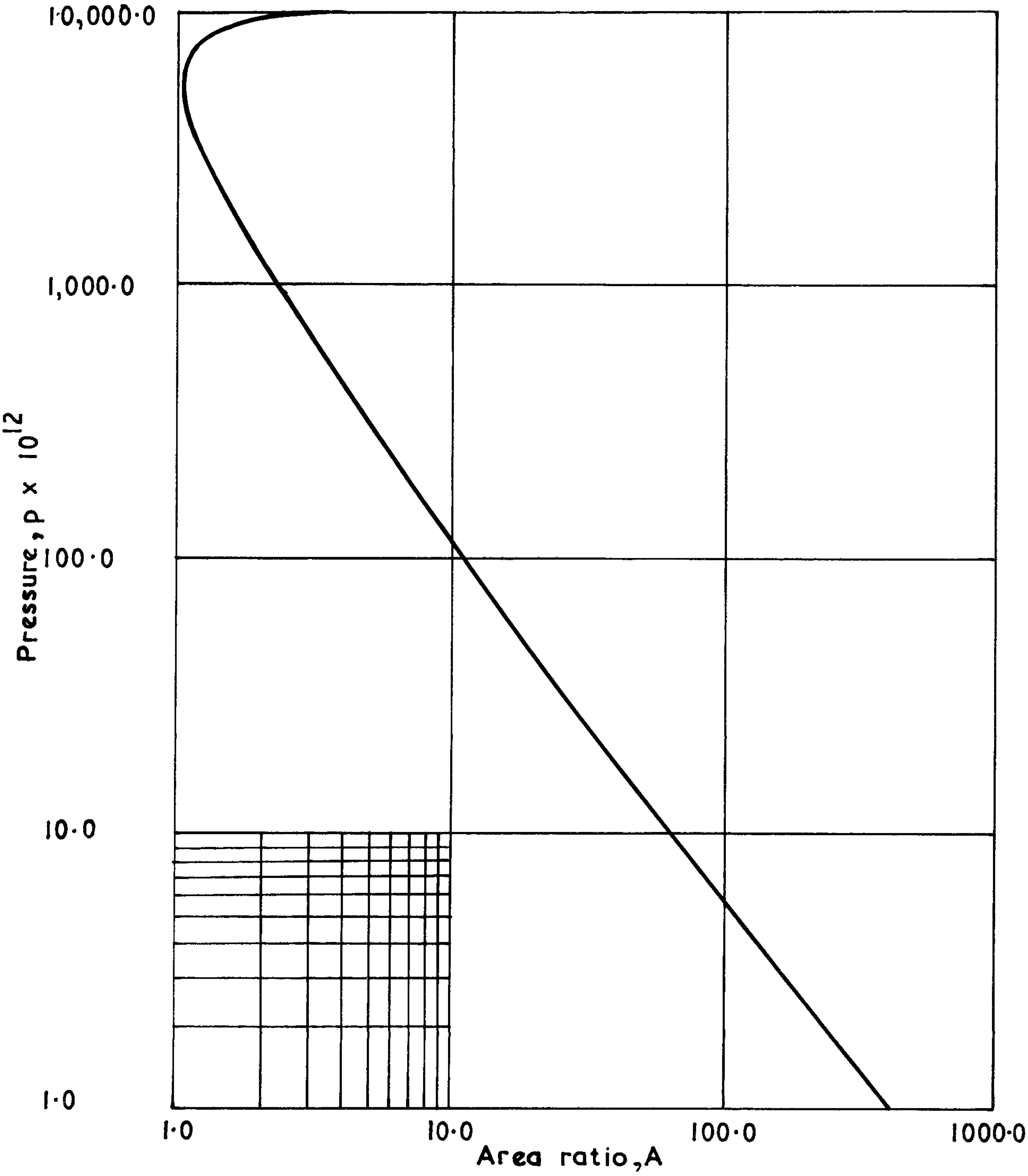

$P_o = 10^{-8}$, $T_o = 0{\cdot}09$, $\Phi = 10^{13}$, $\psi = \psi_{eqm.}$

FIG. 28.



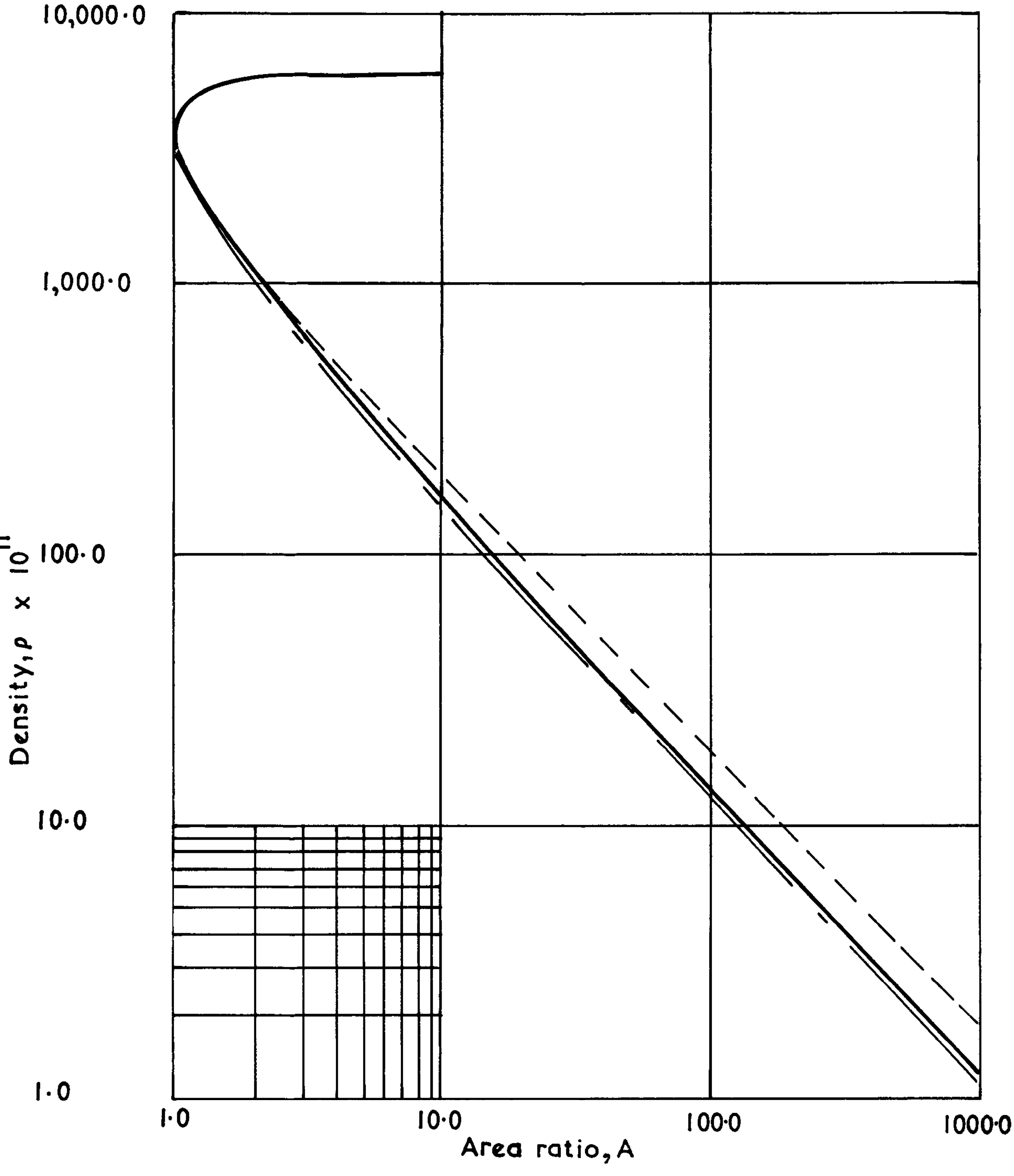

$P_o = 10^{-8}$, $T_o = 0{\cdot}09$, $\Phi = 10^{13}$, $\psi = \psi_{eqm.}$

FIG. 29.



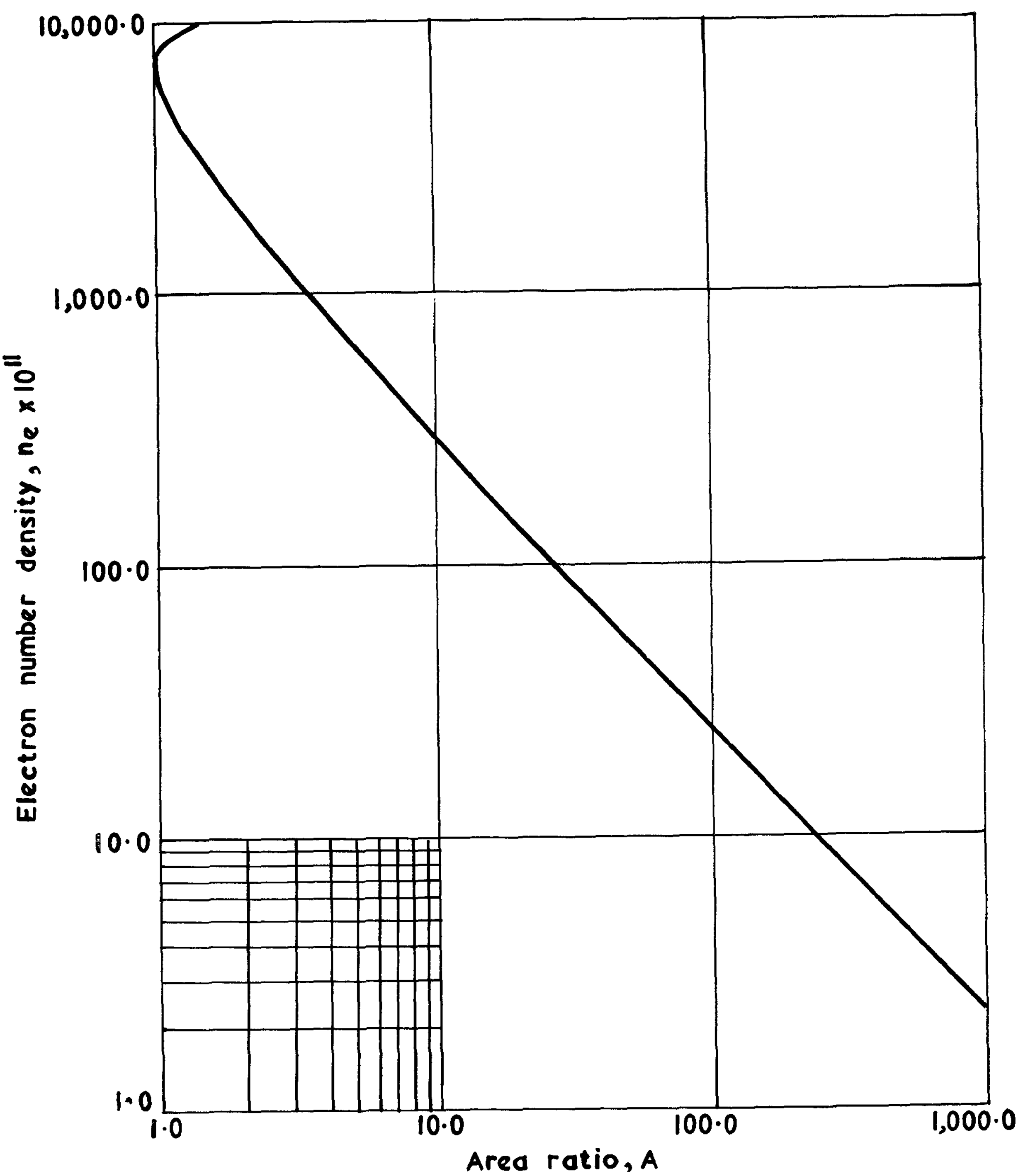

$P_0 = 10^{-8}$, $T_0 = 0{\cdot}09$, $\Phi = 10^{13}$, $\psi = \psi_{eqm.}$

FIG. 30.



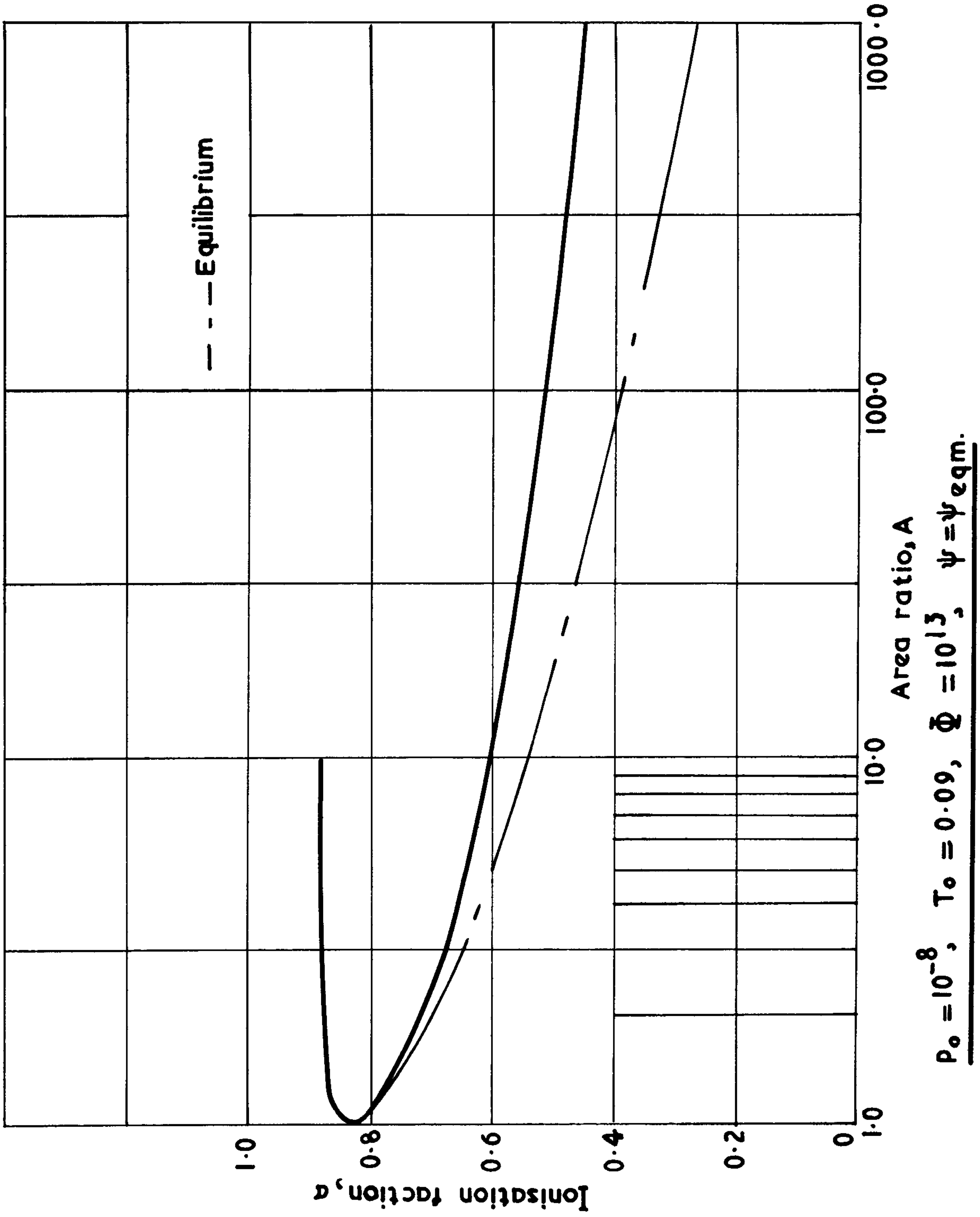

$\rho_0 = 10^{-8}$, $T_0 = 0{\cdot}09$, $\Phi = 10^{13}$, $\psi = \psi_{eqm.}$

FIG. 31.



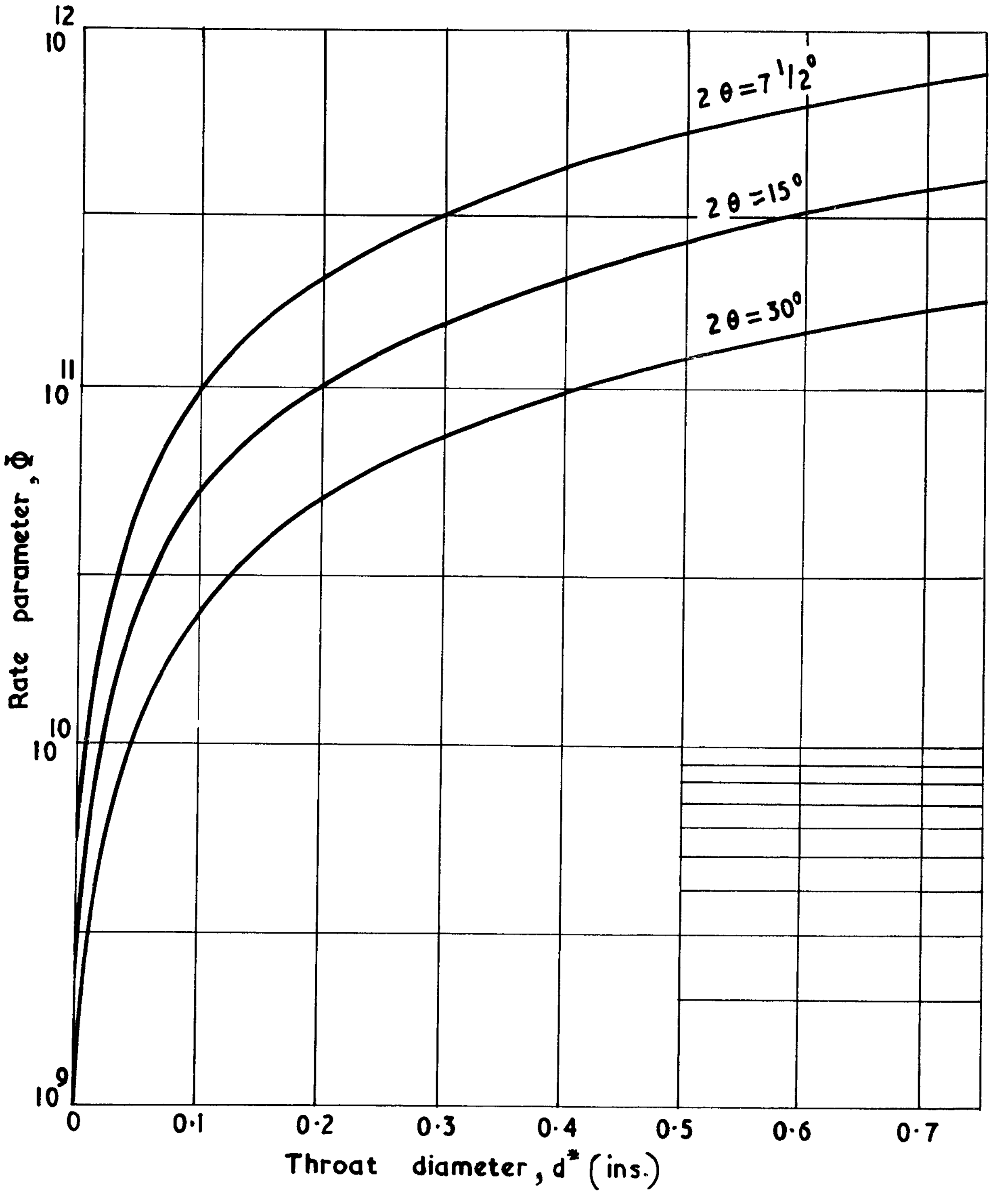

$P_o = 10^{-8}$
$T_o = 0{\cdot}09$
$\Phi = 10^{-13}$
$\psi = \psi$ eqn

FIG. 32.



Equilibrium
Ionisation faction, α
1·0
0·8
0·6
0·4
0·2
0
1·0
10·0
100·0
1000·0
Area ratio, A
$P_o = 10^{-8}$, $T_o = 0{\cdot}09$, $\Phi = 10^{13}$, $\psi = \psi_{eqm.}$

# FIG. 33.

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