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The Longitudinal Frequency Response to Elevator of an Aircraft over the Short Period Frequency Range

by

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THE LONGITUDINAL FREQUENCY RESPONSE TO ELEVATOR OF AN AIRCRAFT OVER THE SHORT PERIOD FREQUENCY RANGE

by

D. M. Ridland

SUMMARY

Formulae for the evaluation of aircraft longitudinal frequency response characteristics are presented. The simple short period frequency response is illustrated for three different types of aircraft and the effects of neglecting elevator lift and speed variations associated with the phugoid motion are separately determined and discussed.

LIST OF CONTENTS

LIST OF APPENDICES

LIST OF ILLUSTRATIONS

1 INTRODUCTION

With modern stability analysis techniques the actual aircraft is frequently treated as part of a system which may also include such items as servocontrols and a human pilot. To apply these techniques the response characteristics of the aircraft may be expressed in general terms, by the aircraft transfer functions, or specifically, by frequency response curves. The object of this report is to summarise the formulae most frequently used in assessments of aircraft longitudinal response to elevator at frequencies near the natural frequency of the aircraft short period pitching oscillation. The high frequency band which will be affected by structural modes is outside the scope of the paper.

The emphasis is generally on simplicity and the principal formulae are derived for the basic two degrees of freedom approximation. The effects of neglecting elevator lift and the phugoid freedom on this simplified short period frequency response are expressed as correction factors and discussed in detail. Examples of the correction factors are given for typical aircraft types.

The responses to elevator are derived for aircraft incidence, normal acceleration at the centre of gravity and rate of pitch, to cover the principal variables normally considered.

2 METHOD OF ANALYSIS

The method by which the formulae in the report have been obtained is that commonly used in this work[1,2,3]; it is described briefly below* .

The equations of motion are written in the usual D operator notation. This form is adopted as the equations of motion for most cases will already have been stated elsewhere in this way. They are then solved for the ratio of the relevant output variable to the input variable. The resulting expression is a transfer function[4]. The Laplace transform of the expression, obtained simply by substituting p for D, gives the Laplace transfer function[5] or simply the 'transfer function' of the system, while the Fourier transform, obtained by substituting $i\omega$ for D, gives on resolution into modulus (amplitude) and argument (phase angle), the frequency response formulae. The output variables normally considered in aircraft stability work are incidence $\left(\frac{w}{V}\right)$, normal acceleration n and rate of pitch q, while the input is negative elevator angle $(-\eta)$. Negative elevator angle has been used because it produces positive pitching moments. Pitching moment could equally well have been chosen as the input variable, but elevator angle is more easily measured in practice.

The transfer functions in this report have been expressed in terms of the usual aerodynamic parameters in the concise Neumark notation[6] whereas the frequency response functions are given in terms of damping and frequency parameters.

3 SHORT PERIOD FREQUENCY RESPONSE

Transfer functions for the general longitudinal motion are given in Appendix 1. They can be simplified for the short period frequency range

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*An excellent discussion of the underlying theory and the practical aspects of analysing and programming the results of response experiments is given in Convair Report FZA-36-195 (P72631) entitled "Dynamic response programme on the B-36 airplane: Part III. Presentation and theoretical considerations of the transient analysis method employed for obtaining frequency response functions from flight data", by T.P. Breaux and E.L. Zeiller (1952).

by neglecting both elevator lift and the phugoid freedom. When the motion of the aircraft is represented in this manner the drag equation is redundant. Further, the small valued terms $\frac{z_q}{\mu}$ and k' are omitted; $k' = -\frac{1}{2} C_L \tan \gamma_e$, γ_e being the angle of climb. $\frac{z_q}{\mu}$ is commonly neglected for both conventional and tailless aircraft because of the large values of μ appropriate to modern aircraft and k' is zero for level flight. The restriction to level flight is, however, not stringent; even at large angles of climb the values of the short period stability coefficients B and C are hardly changed by the term k'. As will be shown later, responses computed on this simple basis are reasonably correct in the case of tailed aircraft for frequencies near the short period frequency, but for tailless aircraft it would be advisable to include elevator lift.

The simplified equations of motion from which the short period transfer functions are derived are stated below, together with the assumptions governing them; the relevant response formulae follow.

3.1 Assumptions and equations of motion

It is assumed that

1 speed is constant (no phugoid mode),

2 the aircraft is trimmed to level flight,

3 there is no coupling with lateral modes,

4 the aircraft is rigid,

5 elevator lift is neglected and

6 the system is linear.

With the above assumptions the equations of motion with respect to the stability axes are

$$\left.\begin{aligned} \left(D - z_w\right)\left(\frac{w}{V}\right) \qquad\qquad - q\,\hat{t} \qquad\qquad &= 0 \\ -\left(\frac{m_{\dot{w}}}{i_B} D + \frac{\mu\, m_w}{i_B}\right)\left(\frac{w}{V}\right) + \left(D - \frac{m_q}{i_B}\right) q\,\hat{t} - \frac{\mu\, m_\eta}{i_B}\,\eta &= 0 \end{aligned}\right\} \qquad (1)$$

3.2 Transfer functions

The resulting transfer functions for incidence, normal acceleration and rate of pitch are respectively

$$\frac{\left(\frac{\bar{w}}{V}\right)}{(-\bar{\eta})} = \frac{\delta}{p^2 + Bp + C} , \tag{2}$$

$$\frac{\bar{n}}{(-\bar{\eta})} = \frac{a \rho S V^2}{2W} \frac{\delta}{p^2 + Bp + C} , \tag{3}$$

$$\frac{\bar{q}}{(-\bar{\eta})} = \frac{g \rho S V}{W} \frac{\delta\left(p + \frac{a}{2}\right)}{p^2 + Bp + C} , \tag{4}$$

where

$$\delta = -\frac{\mu m_\eta}{i_B} = -\frac{W \bar{\bar{c}}}{2 g \rho S k_B^2} \frac{\partial C_m}{\partial \eta} ,$$

$$B = \frac{a}{2} + \nu + \chi = \frac{a}{2} - \frac{m_q}{i_B} - \frac{m_{\dot{w}}}{i_B} ,$$

$$C = \omega + \frac{a}{2} \nu = -\frac{\mu m_w}{i_B} - \frac{a}{2} \frac{m_q}{i_B} .$$

δ is the elevator moment coefficient, while B and C are the short period stability parameters.

3.3 Frequency response formulae

Incidence, $\left(\frac{w}{V}\right)$

The aircraft incidence frequency response obtained from Equation (2) is given by:

Modulus

$$\left| \frac{\left(\frac{w}{V}\right)}{(-\eta)} \right| = \left| \frac{\delta}{C} \frac{1}{\sqrt{(1-x^2)^2 + (2 \zeta x)^2}} \right| . \tag{5}$$

Phase lag (of incidence behind negative elevator)

$$\phi_{\left(\frac{w}{V}\right)} = \tan^{-1} \frac{2 \zeta x}{1 - x^2} \tag{6}$$

In these expressions, which have been non-dimensionalised, the non-dimensional frequency is

$$x = \frac{2 \pi f \hat{t}}{\sqrt{C}}$$

where

$$\hat{t} = \frac{W}{g \rho S V}.$$

x is the ratio of the frequency, f, to the natural frequency, $\frac{\sqrt{C}}{2 \pi \hat{t}}$, of the short period oscillation. The damping ratio

$$\zeta = \frac{B}{2\sqrt{C}},$$

is the ratio of actual to critical damping.

It is apparent from equations 5 and 6 that the incidence frequency response is of exactly the same form as that of a simple, one degree of freedom, mass spring and damper system. The frequency response curves (one for modulus and one for phase lag) plotted against non-dimensional frequency, x, are uniquely defined by the damping ratio, ζ, and the main features of these curves are shown in Fig. 1. It can be seen that the phase lag is always 90° at the natural undamped frequency, $x = 1$. Hence, from measured frequency responses the value of C can be estimated and the value of B can be determined from the damping ratio, ζ. With B and C known, δ, the elevator moment coefficient, can then be evaluated.

Normal acceleration, n

The aircraft frequency response in terms of normal acceleration at the centre of gravity, corresponding to Equation (3) is:

Modulus

$$\left| \frac{n}{(-\eta)} \right| = \left| \frac{a \rho S V^2}{2 W} \frac{\delta}{C} \frac{1}{\sqrt{(1-x^2)^2 + (2 \zeta x)^2}} \right|. \tag{7}$$

Phase lag

$$\phi_n = \tan^{-1} \frac{2 \zeta x}{1 - x^2}. \tag{8}$$

Apart from the constant, $\frac{a \rho S V^2}{2 W}$, in the modulus expression these equations are identical to those for incidence response.

Rate of pitch, q

The rate of pitch frequency response derived from Equation (4) is given by:

Modulus

$$\left|\frac{q}{(-\eta)}\right| = \left|\frac{a\,\rho\,S\,V}{2\,W}\,g\,\frac{\delta}{C}\sqrt{\frac{1+\left(\frac{x}{2\,\zeta_a}\right)^2}{(1-x^2)^2+(2\,\zeta\,x)^2}}\right| . \tag{9}$$

Phase lag

$$\phi_q = \tan^{-1}\frac{(4\,\zeta\,\zeta_a + x^2 - 1)x}{2\,\zeta_a(1-x^2) + 2\,\zeta\,x^2} . \tag{10}$$

In this case the frequency response curves are again defined by the non-dimensional frequency x, the damping ratio ζ, and an additional parameter ζ_a. The term ζ_a, defined by

$$\zeta_a = \frac{\frac{a}{2}}{2\sqrt{C}} = \zeta\frac{\frac{a}{2}}{B} ,$$

is the damping ratio due to the heaving motion alone; the ratio ζ_a/ζ is thus the ratio of damping due to heaving to the total damping.

The main characteristics of the rate of pitch frequency response curves are given in Fig.2. It will be seen that the value of ζ_a can be obtained from measured rate of pitch responses and, since B is known from the acceleration or incidence response, the value of the lift slope, a, can be determined. From frequency response measurements, therefore, it is feasible to evaluate B, C, a and δ.

It can be seen from the foregoing that the incidence and acceleration responses depend only on the damping ratio ζ. The rate of pitch response depends not only on ζ, but also on the distribution of damping between that derived from heaving (a) and that derived from rotation ($m_{\dot\theta}$) as defined by ζ_a/ζ.

3.4 Examples of short period frequency response curves

Examples of short period frequency response curves for normal acceleration and rate of pitch are given in Figs.3 and 4 for three different types of aircraft. The data used in the calculations are given in Table 1 for each of the aircraft.

Aircraft A, a tailed aircraft with a ratio of restoring margin[6]* to elevator lift arm of 1/20 ($\bar\gamma$ = 0.05), has high total damping (ζ = 0.6) with a reasonably normal distribution of the contributions to damping from heaving, rate of pitch and rate of change of incidence (ζ_a = 0.3, ζ_ν = 0.2 and ζ_χ = 0.1 respectively). This could be a typical tailed aircraft in flight at medium altitude.

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*Restoring margin is equal to static margin when Mach number effects are negligible.

Aircraft B, a tailless aircraft with a ratio of restoring margin to elevator lift arm of 3/20 ($\bar{\gamma}$ = 0.15) has low total damping (ζ = 0.2) and this consists of equal contributions from heaving and pitching (ζ_a = 0.1 and ζ_ν = 0.1 respectively); there is no damping contribution from rate of change of incidence (ζ_χ = 0). This could be regarded as a tailless aircraft in supersonic flight at high altitude.

Aircraft C is again tailless with the same ratio between restoring margin and elevator lift arm and the same total damping as the second aircraft. The difference between the second and third aeroplanes is in the rotary damping contributions. It is assumed that the effective damping is entirely supplied by the heaving motion ($\zeta_a = \zeta$ = 0.2), the damping contributions due to rate of pitch and rate of change of incidence being numerically equal but of opposite sign ($\zeta_\nu = -\zeta_\chi$ = 0.05). This could represent aircraft B at transonic speeds.

Figs.3 and 4 give a general impression of short period frequency response curves and show the effects of different values of ζ and ζ_a.

4 EFFECT OF ELEVATOR LIFT

The effect of elevator lift, $z_\eta \eta$, has been determined by including elevator lift in the equations of motion (Equation (1)), otherwise the approximations made here are identical to those made in Section 3. The effect of neglecting elevator lift has been represented by corrections, for both modulus and phase angle, to the simplified short period frequency responses derived in the previous section.

As in the previous case the simplified equations of motion and governing assumptions are stated below and the various formulae follow.

4.1 Assumptions and equations of motion

It is assumed that

1 speed is constant (no phugoid mode),

2 the aircraft is trimmed to level flight,

3 there is no coupling with lateral modes,

4 the aircraft is rigid and

5 the system is linear.

With the above assumptions the equations of motion with respect to the stability axes are

$$\left.\begin{aligned} \left(D - z_w\right)\left(\frac{w}{V}\right) \qquad\qquad - q\,\hat{t} - z_\eta\,\eta &= 0 \\ -\left(\frac{m_{\dot{w}}}{i_B} D + \frac{\mu\, m_w}{i_B}\right)\left(\frac{w}{V}\right) + \left(D - \frac{m_q}{i_B}\right) q\,\hat{t} - \frac{\mu\, m_\eta}{i_B}\,\eta &= 0 \end{aligned}\right\} \qquad (11)$$

4.2 Transfer functions

The transfer functions for incidence, normal acceleration and rate of pitch obtained from the above equations are respectively

$$\frac{\left(\frac{\bar{w}}{V}\right)}{(-\bar{\eta})_E} = -\frac{z_\eta\, p - (\delta - z_\eta\, \nu)}{p^2 + B\,p + C}\,, \tag{12}$$

$$\frac{\bar{n}}{(-\bar{\eta})_E} = -\frac{a\,\rho\,S\,V^2}{2\,W}\;\frac{\left[\frac{z_\eta}{z_w}\right]p^2 + \left[(\chi + \nu)\,\frac{z_\eta}{z_w}\right]p + \left[\omega\,\frac{z_\eta}{z_w} - \delta\right]}{p^2 + B\,p + C}\,, \qquad \ldots\ldots (13)$$

$$\frac{\bar{q}}{(-\bar{\eta})_E} = \frac{g\,\rho\,S\,V}{W}\;\frac{(\delta + z_\eta\,\chi)\,p + \frac{a}{2}\,\delta + \omega\,z_\eta}{p^2 + B\,p + C}\,, \tag{14}$$

where

$\chi = -\dfrac{m_{\dot{w}}}{i_B}$, the rate of change of incidence damping coefficient,

$\nu = -\dfrac{m_q}{i_B}$, the rotary damping coefficient and

$\omega = -\dfrac{\mu\,m_w}{i_B}$, the static stability coefficient.

It can be seen that retaining the elevator lift term, $-z_\eta\,\eta$, in the equations of motion results in complicated transfer functions (compare with equations 2 to 4). The expressions for incidence and acceleration are now quite different and no longer strictly resemble a simple mass, spring and damper system as in Section 3.3.

4.3 Frequency response formulae

Incidence, $\left(\frac{w}{V}\right)$

The aircraft incidence frequency response corresponding to Equation (12) is given by:

Modulus

$$\left|\frac{\left(\frac{w}{V}\right)}{(-\eta)}\right|_E = \left|\frac{\delta}{C}\sqrt{\frac{(1 + 4\,\zeta_a\,\zeta_\nu\,\gamma)^2 + (2\,\zeta_a\,\gamma\,x)^2}{(1 - x^2)^2 + (2\,\zeta\,x)^2}}\right| \tag{15}$$

Phase lag

$$\phi_{\left(\frac{w}{V}\right)_E} = \tan^{-1} \frac{2\,\zeta(1 + 4\,\zeta_a\,\zeta_\nu\,\gamma) - 2\,\zeta_a\,\gamma(1-x^2)}{(1 + 4\,\zeta_a\,\zeta_\nu\,\gamma)\,(1-x^2) + 4\,\zeta\,\zeta_a\,\gamma\,x^2}\,x\,. \tag{16}$$

As in Section 3.3 x is non-dimensional frequency, ζ, ζ_a and ζ_ν are damping parameters, while γ is a non-dimensional coefficient defining the ratio of elevator lift to elevator moment derivatives.

$$\zeta_\nu = \frac{\nu}{2\sqrt{C}} = \zeta\,\frac{\nu}{B} \text{ and}$$

$$\gamma = \frac{z_\eta}{z_w}\,\frac{C}{\delta}\,.$$

<u>Normal acceleration, n</u>

The normal acceleration frequency response at the aircraft centre of gravity derived from Equation (13) is:

Modulus

$$\left|\frac{n}{(-\eta)}\right|_E = \left|\frac{a\,\rho\,S\,V^2}{2\,W}\;\frac{\delta}{C}\sqrt{\frac{(1 + \gamma\,x^2 - \bar{\gamma})^2 + 4(\zeta_\nu + \zeta_\chi)^2\,\gamma^2\,x^2}{(1-x^2)^2 + (2\,\zeta\,x)^2}}\right|. \tag{17}$$

Phase lag

$$\phi_{n_E} = \tan^{-1} \frac{2(\zeta_\nu + \zeta_\chi)\,\gamma(1-x^2) + 2\,\zeta(1 + \gamma\,x^2 - \bar{\gamma})}{(1 + \gamma\,x^2 - \bar{\gamma})\,(1-x^2) - 4\,\zeta(\zeta_\nu + \zeta_\chi)\,\gamma\,x^2}\,x, \tag{18}$$

where

$$\zeta_\chi = \frac{\chi}{2\sqrt{C}} = \zeta\,\frac{\chi}{B} \text{ and}$$

$$\bar{\gamma} = \frac{z_\eta}{z_w}\;\frac{m_w}{m_\eta}\,.$$

$\bar{\gamma}$ is the ratio of restoring margin to elevator lift arm and ζ_χ is the component of the damping ratio due to rate of change of incidence.

<u>Rate of pitch, q</u>

The rate of pitch frequency response obtained from Equation (14) is:

Modulus

$$\left|\frac{q}{(-\eta)}\right|_E = \left|\frac{a\,\rho\,S\,V}{2W}\,g\,\frac{\delta}{c}\sqrt{\frac{(1-\bar{\gamma})^2 + \left(\frac{x}{2\,\zeta_a}\right)^2 (1 - 4\,\zeta_a\,\zeta_\chi\,\gamma)^2}{(1-x^2)^2 + (2\,\zeta\,x)^2}}\right| . \qquad (19)$$

Phase lag

$$\phi_{q_E} = \tan^{-1} \frac{4\,\zeta\,\zeta_a(1-\bar{\gamma}) - (1 - 4\,\zeta_a\,\zeta_\chi\,\gamma)\,(1-x^2)}{2\,\zeta_a(1-\bar{\gamma})(1-x^2) + 2\,\zeta\,x^2(1 - 4\,\zeta_a\,\zeta_\chi\,\gamma)}\,x . \qquad (20)$$

4.4 Corrections for elevator lift

The effect of elevator lift on the simplified short period frequency responses of Section 3 can be represented by correction factors which are defined as

$$K_{\left(\frac{w}{V}\right)_E} = \frac{\left|\frac{\left(\frac{w}{V}\right)}{(-\eta)}\right|_E}{\left|\frac{\left(\frac{w}{V}\right)}{(-\eta)}\right|} ,$$

$$\Delta\,\phi_{\left(\frac{w}{V}\right)_E} = \phi_{\left(\frac{w}{V}\right)_E} - \phi_{\left(\frac{w}{V}\right)} ,$$

$$K_{n_E} = \frac{\left|\frac{n}{(-\eta)}\right|_E}{\left|\frac{n}{(-\eta)}\right|} , \text{ etc.}$$

$\left|\frac{\left(\frac{w}{V}\right)}{(-\eta)}\right|$, $\phi_{\left(\frac{w}{V}\right)}$, ... are defined by Equations (5) to (10) and

$\left|\frac{\frac{w}{V}}{(-\eta)}\right|_E$, $\phi_{\left(\frac{w}{V}\right)_E}$, ... are defined by Equations (15) to (20).

These correction factors are discussed below.

Incidence, $\left(\frac{w}{V}\right)$

Modulus correction factor

$$K_{\left(\frac{w}{V}\right)_E} = \sqrt{(1 + 4\,\zeta_a\,\zeta_\nu\,\gamma)^2 + (2\,\zeta_a\,\gamma\,x)^2}\ . \tag{21}$$

Phase angle correction

$$\Delta\,\phi_{\left(\frac{w}{V}\right)_E} = -\tan^{-1}\frac{2\,\zeta_a\,\gamma\,x}{1 + 4\,\zeta_a\,\zeta_\nu\,\gamma}\ . \tag{22}$$

For a tailed aircraft all the parameters in these expressions are normally positive. In this case, therefore, the effect of elevator lift is to increase the amplitude ratio $\left|\left(\frac{w}{V}\right)\Big/(-\eta)\right|$ over the whole frequency range and to decrease the lag of incidence behind elevator deflection. At zero frequency ($x = 0$) the modulus correction factor is $(1 + 4\,\zeta_a\,\zeta_\nu\,\gamma)$ and the phase angle correction is zero; when frequency approaches infinity ($x \to \infty$) the phase lag approaches 90° and not 180° as when elevator lift is neglected. No such general observations can be made for unorthodox aircraft; each must be considered on its merits.

Normal acceleration, n

Modulus correction factor

$$K_{n_E} = \sqrt{(1 + \gamma\,x^2 - \bar{\gamma})^2 + 4(\zeta_\nu + \zeta_\chi)^2\,\gamma^2\,x^2}\ . \tag{23}$$

Phase angle correction

$$\Delta\,\phi_{n_E} = \tan^{-1}\frac{2\,\gamma\,x(\zeta_\nu + \zeta_\chi)}{1 + \gamma\,x^2 - \bar{\gamma}}\ . \tag{24}$$

Here when frequency is zero the modulus and phase angle corrections are $(1 - \bar{\gamma})$ and zero respectively. At very high frequencies the modulus correction factor becomes very large and the phase angle correction becomes negligible.

Rate of pitch, q

Modulus correction factor

$$K_{q_E} = \sqrt{\frac{(1 - 4\,\zeta_a\,\zeta_\chi\,\gamma)^2\,x^2 + 4\,\zeta_a(1-\bar{\gamma})^2}{x^2 + 4\,\zeta_a^2}} \quad . \tag{25}$$

Phase angle correction

$$\Delta\,\phi_{q_E} = \tan^{-1}\frac{2\,\zeta_a\,x(4\,\zeta_a\,\zeta_\chi\,\gamma - \bar{\gamma})}{(1 - 4\,\zeta_a\,\zeta_\chi\,\gamma)\,x^2 + 4\,\zeta_a^2(1 - \bar{\gamma})} \quad . \tag{26}$$

(It may be noted that

$$\frac{\bar{\gamma}}{\gamma} = \frac{\omega}{C} = \frac{K_m}{H_m} = \frac{\text{restoring margin}}{\text{manoeuvre margin}} \; .)$$

The effect of elevator lift in this case is to reduce slightly the amplitude ratio, $\left|\frac{q}{(-\eta)}\right|$, and to decrease the phase lag. At zero frequency ($x = 0$) the amplitude reduction factor is $(1-\bar{\gamma})$ and the phase angle correction is zero. These corrections are identical to those for the acceleration at zero frequency and the static relationship $- n\,g = \dot{w} - q\,V$ is retained. The factor $(1-\bar{\gamma})$ is roughly 0.95 for tailed aircraft and 0.85 for tailless aircraft.

4.5 Examples of corrections for elevator lift

The effects of elevator lift are shown in Figs.5, 6 and 7, for incidence, normal acceleration and rate of pitch respectively. The three aircraft, A, B and C, which were described in Section 3.4 have again been used to indicate the variation in elevator lift effect with aircraft type.

Incidence

In Fig.5, which illustrates the corrections to incidence response, elevator lift is seen to increase the modulus and to reduce the phase lag throughout the frequency range. The increase in modulus is, however, less than two per cent in all the cases considered at the aircraft short period natural frequency and, as it increases only slowly with frequency, it could reasonably be neglected. The phase lag reduction, on the other hand, increases rapidly with frequency asymptotically approaching 90°. At the short period frequency, however, it is still only 4° in the worst example of Fig.5 and in comparison with experimental errors this is not a large amount.

These effects have a simple explanation. When elevator is applied to increase incidence it produces a negative lift. This lift accelerates the aircraft bodily downwards, resulting in an additional increase in incidence. The aircraft then pitches nose down, because of its inherent stability, partially alleviating this increase in incidence; the net result is an increase in incidence.

The correction factor expresses basically the ratio between the aircraft response to the combined effects of elevator lift plus moment and the response to elevator moment only. Changes in the motion of the aircraft are resisted by aircraft inertia and, as with increasing frequency the response to elevator moment approaches zero more rapidly than the response to elevator lift, the correction factor, being a ratio, tends to infinity as frequency is increased.

Acceleration

The effect of elevator lift on normal acceleration frequency response is shown in Fig. 6. The acceleration modulus is decreased at frequencies below approximately the short period frequency and increased above this frequency, while the phase lag is increased at all frequencies. The increase in phase lag in the worst case, however, is only about 5° which, as previously suggested, is not a very large amount from the experimental point of view. This value decreases with decreasing rotary damping, $(\zeta_\nu + \zeta_\chi)$, being smaller for the second, tailless aircraft, B, and zero for the third aircraft, C, for which the rotary damping is zero $(\zeta_\nu + \zeta_\chi = 0)$.

The physical interpretation of these effects is similar to that given above for incidence, but it can be seen that the acceleration correction (Equation (23)) is more complex, having a frequency component in each term. At low frequencies the primary effect of elevator lift is to produce a negative lift on the aircraft, thereby reducing the total acceleration. There will also be a secondary increase in acceleration, due to the incremental incidence induced by elevator lift, and this will be alleviated by the pitching motion of the aircraft as in the incidence case.

At very high frequencies the acceleration correction factor tends to infinity in much the same manner as the incidence correction factor, but, whereas the absolute value of the incidence response modulus including elevator lift tends to zero with increasing frequency, the corresponding acceleration modulus tends to a finite value, γ, as the elevator lift will always be apparent even though the aircraft can no longer respond in pitch. This effect of elevator lift on acceleration response at higher frequencies is shown clearly by plotting the moduli of the two acceleration responses, $\left|\frac{n}{(-\eta)}\right|$ and $\left|\frac{n}{(-\eta)}\right|_E$, on a logarithmic scale as in Fig. 8.

Rate of pitch

The effect of elevator lift on the rate of pitch frequency response is shown in Fig. 7. Both modulus and phase lag are, in general, decreased by elevator lift, but the corrections are appreciable only at frequencies below the short period frequency.

Physically, the effect of elevator lift on rate of pitch response follows directly from the explanations of the incidence and acceleration corrections. As the rate of pitch is unaffected by linear motion alone, only the pitching motion in response to the incidence induced by elevator lift contributes to the correction factor. This, as previously indicated, is only significant at low frequencies because of aircraft inertia, and as it is opposite in sense to the basic short period pitching motion, results in a decrease in rate of pitch response.

5 EFFECT OF THE PHUGOID

Unlike the assessment of the effect of elevator lift on the short period response characteristics, which was made simply by retaining the term $z_\eta \eta$ in the equations of motion, consideration of the effect of the phugoid oscillation involves an additional degree of freedom. The drag equation must be included and the system becomes one of fourth instead of second order. As, however, the aim is to examine the errors incurred at low frequencies by neglecting the phugoid, rather than to study the phugoid itself, several simplifications are made. Elevator lift is neglected ($z_\eta = 0$) and it is assumed that there is no Mach number effect ($m_u = 0$). Further, the small quantities $\frac{x_q}{\mu}$, $\frac{z_q}{\mu}$ and x_η are neglected and, in conformity with the two previous cases, z_w is defined as $-\frac{a}{2}$ instead of $-\frac{1}{2}(a + C_D)$ when deriving the frequency response formulae. This leads to some useful simplifications but for practical purposes leaves the value of z_w unchanged. k' too is omitted assuming as in Section 3 that the aircraft is trimmed for level flight.

As with elevator lift, the effects of the phugoid oscillation are represented by correction factors which can be applied to the simplified short period responses of Section 3.

The assumptions and equations of motion are stated below and followed by the transfer functions and response formulae.

5.1 Assumptions and equations of motion

It is assumed that:

1 the aircraft is trimmed to level flight,

2 there is no coupling with lateral modes,

3 the aircraft is rigid,

4 elevator lift is neglected and

5 the system is linear.

With these assumptions the equations of motion with respect to the stability axes are

$$\left.\begin{array}{lllll}
\left(D - x_u\right)\frac{u}{V} & - x_w\left(\frac{w}{V}\right) & + \frac{1}{2} C_L \theta & & = 0 \\
- z_u \frac{u}{V} & + \left(D - z_w\right)\left(\frac{w}{V}\right) & - q \hat{t} & & = 0 \\
 & - \left(\frac{m_{\dot{w}}}{i_B} D + \frac{\mu m_w}{i_B}\right)\left(\frac{w}{V}\right) & + \left(D - \frac{m_q}{i_B}\right) q \hat{t} & - \frac{\mu m_\eta}{i_B} \eta & = 0 \\
 & & - q \hat{t} + D \theta & & = 0
\end{array}\right\} \quad (30)$$

5.2 Transfer functions

The transfer functions obtained from the above equations are, for incidence, normal acceleration and rate of pitch respectively

$$\frac{\left(\frac{\bar{w}}{V}\right)}{(-\eta)_p} = \frac{\delta\left(p^2 + C_D\, p + \frac{C_L^{\,2}}{2}\right)}{\Delta}, \qquad (31)$$

$$\frac{\bar{n}}{(-\eta)_p} = \frac{a\,\rho\,S\,V^2}{2\,W}\;\frac{\delta\;p(p + C_D)}{\Delta}, \qquad (32)$$

$$\frac{\bar{q}}{(-\eta)_p} = \frac{g\,\rho\,S\,V}{W}\;\frac{\delta\;p\left(p^2 + \left\{\frac{a}{2} + C_D\right\}p + \frac{a}{2}\,C_D + \frac{C_L^{\,2}}{2}\right)}{\Delta}, \qquad (33)$$

where

$$\Delta = p^4 + (B + C_D)\,p^3 + \left(C + B\,C_D + \frac{C_L^{\,2}}{2}\right)p^2$$

$$+ \left(C\,C_D + \{\nu + \chi\}\,\frac{C_L^{\,2}}{2}\right)p + \omega\,\frac{C_L^{\,2}}{2}.$$

A better physical insight into the short period-phugoid relationship may be gained by re-writing Equations (31) to (33) in the following form

$$\frac{\left(\frac{\bar{w}}{V}\right)}{(-\bar{\eta})_p} = \frac{\delta\left(p^2 + C_D\,p + \frac{C_L^{\,2}}{2}\right)}{\Delta}, \qquad (34)$$

$$\frac{\bar{n}}{(-\bar{\eta})_p} = \frac{a\,\rho\,S\,V^2}{2W}\;\delta\;\frac{\left(p^2 + C_D\,p + \frac{C_L^{\,2}}{2}\right) - \frac{C_L^{\,2}}{2}}{\Delta} \qquad (35)$$

$$\frac{\bar{q}}{(-\bar{\eta})_p} = \frac{g\,\rho\,S\,V}{W}\;\delta\;\frac{\left(p^2 + C_D\,p + \frac{C_L^{\,2}}{2}\right)\left(p + \frac{a}{2}\right) - \frac{a}{2}\,\frac{C_L^{\,2}}{2}}{\Delta}, \qquad (36)$$

where

$$\Delta = \left(p^2 + C_D\,p + \frac{C_L^{\,2}}{2}\right)(p^2 + B\,p + C) - \frac{a}{2}\,\frac{C_L^{\,2}}{2}\,(p - \nu). \qquad (37)$$

It can be seen that when $C_L = 0$ these expressions reduce to the simple short period transfer functions of Section 3 (Equations (2) to (4)). At $C_L = 0$ the phugoid frequency is zero and the short period responses should not be affected. The last term in Equation (37), $-\frac{a}{2}\,\frac{C_L^{\,2}}{2}\,(p - \nu)$, can be regarded as a coupling term between the short period and phugoid oscillations; when $C_L = 0$ there is obviously no coupling.

5.3 Frequency response formulae

To simplify the derivation of the frequency response formulae, Equation (37) is considered in the form $(p^2 + B' p + C')(p^2 + bp + c)$ and the expressions for incidence, acceleration and rate of pitch are treated as products of two complex numbers. Approximate values of the coefficients, obtained by approximate factorisation[7] of the quartic (Equation (37)) are $B' \simeq B$ and $C' \simeq C$, where B and C are the short period stability parameters as defined in Section 3.1, and

$$b \simeq \frac{\left(C + B\,C_D + \frac{C_L^2}{2}\right)\left(C\,C_D + \{\nu + \chi\}\frac{C_L^2}{2}\right) - (B + C_D)\,\omega\,\frac{C_L^2}{2}}{\left(C + B\,C_D + \frac{C_L^2}{2}\right)^2} \quad \text{and}$$

$$c \simeq \frac{\omega\,\frac{C_L^2}{2}}{C + B\,C_D + \frac{C_L^2}{2}} \; .$$

To further simplify matters it is assumed that $B' = B$ and $C' = C$, the error in each case being extremely small (less than one per cent - see Appendix 2).

The transfer functions, Equations (31) to (33), describe two oscillatory modes, each with its own natural frequency. The previously considered transfer functions, involving only the short period oscillation, were, on conversion to frequency response functions, non-dimensionalised by dividing through by the short period frequency parameter, C. This is again done in the present case, because it is the short period response which is of primary concern.

Incidence, $\left(\frac{w}{V}\right)$

The incidence frequency response formulae obtained from Equation (31) are:

Modulus

$$\left|\frac{\left(\frac{w}{V}\right)}{(-\eta)}\right|_p = \left|\frac{\delta}{C}\sqrt{\frac{(R_2^2 - x^2)^2 + (2\,\zeta_D\,x)^2}{[(1 - x^2)^2 + (2\,\zeta\,x)^2][(R_1^2 - x^2)^2 + (2\,\zeta_b\,x)^2]}}\right| \qquad (38)$$

Phase angle

$$\phi_{\left(\frac{w}{V}\right)_p} = \tan^{-1} \frac{2x\left[4\zeta\zeta_b\zeta_D x^2+\zeta(R_1^2-x^2)(R_2^2-x^2)+\zeta_b(R_2^2-x^2)(1-x^2)-\zeta_D(1-x^2)(R_1^2-x^2)\right]}{(1-x^2)(R_1^2-x^2)(R_2^2-x^2)+4x^2\left[\zeta_b\zeta_D(1-x^2)+\zeta\zeta_D(R_1^2-x^2)-\zeta\zeta_b(R_2^2-x^2)\right]} \qquad \ldots\ldots (39)$$

The symbols, R_1, R_2, ζ_b and ζ_D, represent phugoid parameters and are defined below.

$$R_1 = \sqrt{\frac{c}{C}}.$$

R_1, as the ratio of phugoid frequency to short period frequency, is the most important parameter in the assessment of phugoid effects on short period response. The value of $1/R_1$, as obtained by the approximations given above, is plotted against C_L in Fig. 15 for aircraft A. The values given are for the present purpose, reasonably representative of conventional aircraft and show that R_1 may be taken as being proportional to C_L.

$$R_2 = \frac{C_L}{\sqrt{2C}} = R_1 \frac{C_L}{\sqrt{2c}}.$$

It can be shown that $R_2 \simeq R_1$, and R_2 is therefore an approximate ratio of phugoid frequency to short period frequency. Values of $1/R_2$ are compared with $1/R_1$ in Fig. 15 for aircraft A. At low C_L the absolute values of R_1 and R_2 are small with respect to unity (in Fig. 15, $R_1 = 0.0367$ at $C_L = 0.2$); as in the response equations R_1, R_2 occur only in the form $(R^2 - x^2)$, at frequencies near the short period frequency, when $x \simeq 1$, R_1 and R_2 may be neglected.

$$\zeta_b = \frac{b}{2\sqrt{C}} = R_1 \frac{b}{2\sqrt{c}}.$$

ζ_b is a phugoid damping parameter and is related here to the short period frequency, $\sqrt{C}$. Phugoid damping is very small and ζ_b is clearly a second order term. As $C_L \to 0$, $b \to C_D$ and at all C_L, $c \simeq \frac{\omega}{C}\frac{C_L^2}{2}$. At low C_L therefore, the order of ζ_b is given by $\left(\frac{C_D}{C_L} R_1\right)$, a small quantity which can usually be neglected.

$$\zeta_D = \frac{C_D}{2\sqrt{C}} = R_1 \frac{C_D}{2\sqrt{c}} .$$

ζ_D is an approximate form of ζ_b and for similar reasons can usually be neglected at low C_L.

Briefly then, for conventional aircraft

$$R_2 > R_1 \quad \text{and} \quad \zeta_D > \zeta_b .$$

At low lift coefficients and for frequencies well above the phugoid frequency R_1, R_2, ζ_b and ζ_D may be neglected.

It can now be seen that the combined phugoid and short period incidence response (Equations (38) and (39)) is, like the corresponding simple short period incidence response (Equations (5) and (6)), completely defined by frequencies and damping ratios. Assuming $x = 1$ and low values of C_L, and thence that $R_1 = R_2 = \zeta_b = \zeta_D = 0$, the phugoid terms disappear and the expressions become identical to the corresponding short period equations of Section 3. It can be expected therefore that, with low C_L, phugoid effects on incidence response will be negligible at short period frequencies.

Normal acceleration, n

The frequency response formulae for normal acceleration obtained from Equation (32) are:

Modulus

$$\left|\frac{n}{(-\eta)}\right|_p = \left|\frac{a\,\rho\, S\, V^2}{2\,W}\;\frac{\delta}{C}\sqrt{\frac{x^2(x^2 + 4\,\zeta_D^{\,2})}{[(1-x^2)^2 + (2\,\zeta\, x)^2][(R_1^{\,2} - x^2)^2 + (2\,\zeta_b\, x)^2]}}\right| . \qquad \ldots\ldots (40)$$

Phase angle

$$\phi_{n_p} = \tan^{-1} \frac{2[4\,\zeta\,\zeta_b\,\zeta_D\, x^2 - \zeta\, x^2(R_1^{\,2} - x^2) - \zeta_b\, x^2(1-x^2) - \zeta_D(1-x^2)(R_1^{\,2} - x^2)]}{x[4\,\zeta\,\zeta_b\, x^2 + 4\,\zeta_b\,\zeta_D(1-x^2) + 4\,\zeta\,\zeta_D(R_1^{\,2} - x^2) - (1-x^2)(R_1^{\,2} - x^2)]} . \qquad \ldots\ldots (41)$$

In the simplified short period case both incidence and acceleration responses were identical, apart from a constant, but in the present case they differ. However, making, as before, the approximation $R_1 = R_2 = \zeta_b = \zeta_D = 0$, the corresponding short period acceleration equations are obtained (Equations (7) and (8)). It can again be expected therefore that the phugoid will have negligible effect on the short period acceleration response at short period frequencies.

Rate of pitch, q

The rate of pitch frequency response derived from Equation (33) is given by:

Modulus

$$\left|\frac{q}{(-\eta)}\right|_p = \left|\frac{a \rho S V}{2W} g \frac{\delta}{C} \frac{x}{2 \zeta_a} \sqrt{\frac{4 x^2(\zeta_a + \zeta_D)^2 + (4 \zeta_a \zeta_D + R_2^2 - x^2)^2}{[(1 - x^2)^2 + (2 \zeta x)^2][(R_1^2 - x^2)^2 + (2 \zeta_b x)^2]}}\right| . \qquad \ldots\ldots (42)$$

Phase angle

$$\phi_{q_p} = - \tan^{-1} \frac{4 x^2(\zeta_a+\zeta_D)[\zeta(R_1^2 -x^2) + \zeta_b(1-x^2)] + (4 \zeta_a \zeta_D+R_2^2 -x^2)[(1-x^2)(R_1^2-x^2) - 4 \zeta \zeta_b x^2]}{2 x[4 \zeta \zeta_b(\zeta_a+\zeta_D)x^2 + (4 \zeta_a\zeta_D+R_2^2-x^2)\{\zeta(R_1^2-x^2) + \zeta_b(1-x^2)\} - (\zeta_a+\zeta_D)(1-x^2)(R_1^2-x^2)]} \qquad \ldots\ldots (43)$$

On making the approximation $R_1 = R_2 = \zeta_b = \zeta_D = 0$, Equations (42) and (43) reduce to the short period rate of pitch expressions, Equations (9) and (10). Only negligible interference by the phugoid need therefore be anticipated in the rate of pitch response at short period frequencies.

5.4 Corrections for the phugoid

As in the elevator lift case, the above phugoid frequency response expressions can be used to derive correction factors for the effect of the phugoid on frequency response in the short period range. The phugoid response expressions are cumbersome, however, and as the object here is to determine the frequency above which the phugoid can be neglected without significant error rather than to enable corrections to be made, only the modulus correction factors are examined. The examples of phugoid frequency responses given in Figs. 9 to 11 (which are discussed in the next section) show that it is reasonable to assume that where modulus errors are small, phase angle errors are also small. An individual assessment can of course be made in any particular case.

Incidence, $\left(\frac{w}{V}\right)$

Modulus correction factor

$$K_{\left(\frac{w}{V}\right)_p} = \sqrt{\frac{(R_2^2 - x^2)^2 + (2 \zeta_D x)^2}{(R_1^2 - x^2)^2 + (2 \zeta_b x)^2}} \qquad (44)$$

At zero frequency this expression reduces to

$$\frac{R_2}{R_1} = \frac{C + B\,C_D + \frac{C_L^{\,2}}{2}}{\omega}$$

$$\simeq \frac{C}{\omega}$$

$$= \frac{H_m}{K_m} = \frac{\text{manoeuvre margin}}{\text{restoring margin}} \,. \qquad (45)$$

At the short period frequency, $x = 1$, the effect of the phugoid as expressed by the correction factor is very small and at higher frequencies it will tend to diminish altogether.

Differentiation of Equation (44) shows that maximum and minimum values of the correction factor occur at frequencies given by the smaller and larger positive values of x respectively obtained from

$$x^2 = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a} \qquad (46)$$

where

$$a = R_1^{\,2} - R_2^{\,2} + 2(\zeta_D^{\,2} - \zeta_b^{\,2}) \,,$$

$$b = R_2^{\,4} - R_1^{\,4} ,$$

$$c = R_1^{\,2}\,R_2^{\,2}(R_1^{\,2} - R_2^{\,2}) + 2(\zeta_b^{\,2}\,R_2^{\,4} - \zeta_D^{\,2}\,R_1^{\,4}).$$

When $R_1 = 0.06$, $R_2 = 0.07$, $\zeta_b = 0.0036$ and $\zeta_D = 0.0044$, Equation (46) gives $x = 0.0601$ and 0.0701 (in fact, R_1 and R_2 approximately) for maximum and minimum values respectively of incidence modulus. These figures are in agreement with the turning points in Fig. 12.

If y is defined as the proportional error in the modulus incurred by neglecting the phugoid freedom in the incidence response, it follows that

$$K_{\left(\frac{w}{V}\right)_p} = 1 + y \,. \qquad (47)$$

Choosing as the maximum permissible error a value $y = |y|$, Equation (47) can be solved to give the limiting frequencies outside which neglect of the phugoid freedom is no longer permissible. In the case of the incidence response y is negative (i.e. $K_{\left(\frac{w}{V}\right)} < 1.0$) near the short period natural frequency and in consequence Equation (47) must be solved for negative values of y.

Equation (47) has been used to obtain the incidence curves of Figs. 13 and 14 for aircraft A. As aircraft A is considered representative of conventional aircraft, the curves will indicate the magnitude of the errors to be expected with conventional aircraft. Fig. 13 shows, for ranges of lift coefficient and phugoid/short period frequency ratios, the frequency at which neglect of the phugoid causes a 5% error in modulus. Fig. 14 similarly shows the error, incurred by neglecting to take account of the phugoid, at short period frequency.

Normal acceleration, n

Modulus correction factor

$$K_{n_p} = \sqrt{\frac{x^2(x^2 + 4\zeta_D^2)}{(R_1^2 - x^2)^2 + (2\zeta_b x)^2}} \quad . \tag{48}$$

This factor becomes zero at zero frequency, almost unity at the short period frequency and tends nearer to one as frequency is further increased.

The maximum value of the acceleration modulus correction factor is obtained at a frequency x given by the positive root of

$$x = \sqrt{\frac{R_1^4 + \sqrt{R_1^8 + 4(R_1^2 - 2\zeta_b^2 + 2\zeta_D^2)(2\zeta_D^2 R_1^4)}}{2(R_1^2 - 2\zeta_b^2 + 2\zeta_D^2)}} \tag{49}$$

$$\simeq R_1 \ .$$

R_1 is the phugoid natural frequency, expressed in terms of the short period frequency, and one would physically expect the correction factor to have its maximum value at this frequency. This can be seen in Fig. 12 for which $R_1 = 0.06$.

By putting

$$K_{n_p} = 1 + y \ , \tag{50}$$

where, as before, y is the maximum acceptable percentage error in the acceleration response modulus and solving for x, the highest value obtained will be the frequency ratio at which neglect of the phugoid causes this maximum error; as frequency is increased beyond this value the error becomes smaller. Curves of acceleration errors assessed in this way are given in Figs. 13 and 14 along with the incidence error curves.

Rate of pitch, q

Modulus correction factor

response tests are made at $R_1 < 0.07$ and simple short period theory is applied, the maximum possible error at the short period frequency due to neglect of the phugoid is 0.6%, and this occurs in the acceleration response.

Generalising, as the coupling term of Equation (37), $-\frac{a}{2}\frac{C_L^2}{2}(p - \nu)$, is mainly a function of C_L and thus of frequency, the interaction between the short period and phugoid oscillations is governed by the frequency ratio, R_1. ($1/R_1$ is more convenient in practice.) If the frequency ratio $1/R_1$ is more than 10 (i.e. with a short period frequency of say $\frac{1}{2}$ c.p.s., for a phugoid period of more than 20 seconds) the phugoid can be neglected with less than 5% error down to $x = 0.5$, and less than $1\frac{1}{4}$% error at the short period frequency, $x = 1$.

$1/R_1$ should be greater than 30 if the analysis is to have less than 5% error down to $x = 0.2$.

6 CONCLUDING REMARKS

The formulae derived in this report can be used either to estimate aircraft response characteristics or to obtain information from measured aircraft responses. The first application is straightforward and the simplified short period formulae should be adequate for the relevant range of frequencies; the second needs a little care. It has been shown that in interpreting experimental results the phugoid effects can be neglected providing that tests are made at a high value of the ratio of short period frequency to phugoid frequency. This separates the two frequencies and will generally be accomplished in flight tests if these are made at low lift coefficients, say $C_L < 0.2$. The effects of elevator lift can also be neglected if errors of 5% in amplitude and 5° in phase angle can be tolerated. These errors can, however, be much reduced if the analysis is restricted to a small frequency range around the short period frequency of the aircraft.

Complete transfer functions for the three degrees of freedom response to elevator are given in Appendix 1.

LIST OF SYMBOLS

a per rad	lift slope of the aircraft
a_2 per rad	lift slope of the elevator (referred to tail area for tailed aircraft and to wing area for tailless aircraft)
a_T per rad	lift slope of the tailplane ($a_T = a_2$ for all moving tail)
B slug ft^2	pitching moment of inertia
$B = \frac{a}{2} + \nu + \chi$ $= \frac{a}{2} - \frac{m_q}{i_B} - \frac{m_{\dot{w}}}{i_B}$	short period damping stability coefficient
b	phugoid damping stability coefficient

LIST OF SYMBOLS (Contd.)

$C = \omega + \frac{a}{2}\nu$	short period stiffness stability coefficient
$= -\frac{\mu\, m_w}{i_B} - \frac{a}{2}\frac{m_q}{i_B}$	
$\sqrt{C} = 2\pi f_o \hat{t}$	natural undamped short period frequency of the aircraft in aerodynamic time
C_D	drag coefficient
C_L	lift coefficient
c	phugoid stiffness stability coefficient
$\bar{\bar{c}}$ ft	mean aerodynamic chord
$D = \frac{d}{d\tau}$	operator
f c.p.s.	frequency of longitudinal oscillations
$H_m = K_m - \frac{\ell}{\bar{\bar{c}}}\frac{m_q}{\mu}$	manoeuvre margin
$i_B = \left(\frac{k_B}{\ell}\right)^2$	coefficient of pitching moment of inertia
$K_m = -\frac{\partial C_m}{\partial C_L}$	restoring margin (Restoring margin is equal to static margin when there are no Mach number effects)
$K_{\left(\frac{w}{V}\right)_E}$, K_{n_E}, K_{q_E}	correction factors to allow for the effect of elevator lift on short period incidence, acceleration and rate of pitch response moduli respectively
$K_{\left(\frac{w}{V}\right)_p}$, K_{n_p}, K_{q_p}	correction factors to allow for the effect of the phugoid on short period incidence, acceleration and rate of pitch response moduli respectively
$k_B = \sqrt{\frac{g\,B}{W}}$ ft	radius of gyration in pitch
$k' = -\frac{1}{2} C_L \tan\gamma_e$	
ℓ ft	reference length, usually tail arm length
$m_u = \frac{\bar{\bar{c}}}{2\,\ell}\frac{\partial C_m}{\partial\left(\frac{u}{V}\right)}$	pitching moment derivative due to forward velocity, u

$m_w = \frac{\bar{\bar{c}}}{2\,\ell} \frac{\partial C_m}{\partial \left(\frac{w}{V}\right)}$	pitching moment derivative due to heaving velocity, w
$m_{\dot{w}} = \frac{\bar{\bar{c}}}{2\,\ell} \frac{\partial C_m}{\partial \left(\frac{\left(\frac{\dot{w}}{V}\right)\ell}{V}\right)}$	pitching moment derivative due to heaving acceleration, $\dot{w}$
$m_q = \frac{\bar{\bar{c}}}{2\,\ell} \frac{\partial C_m}{\partial \left(\frac{q\,\ell}{V}\right)}$	pitching moment derivative due to rate of pitch, q
$m_\eta = \frac{\bar{\bar{c}}}{2\,\ell} \frac{\partial C_m}{\partial \eta}$	pitching moment derivative due to elevator deflection, η
$m_{\dot{\theta}} = m_q + m_{\dot{w}}$	full rotary damping derivative
n 'g'	normal acceleration increment above 1g
q rad/sec	rate of pitch
$R_1 = \sqrt{\frac{c}{C}}$	ratio of aerodynamic stiffnesses
$R_2 = \sqrt{\frac{C_L^2}{2\,C}}$	approximate ratio of aerodynamic stiffnesses
S ft^2	wing area
S_T ft^2	tail area
$t = \hat{t}\,\tau$ sec	time
$\hat{t} = \frac{W}{g\,\rho\,S\,V}$	unit of aerodynamic time
u ft/sec	increment of velocity along x axis in disturbed flight
V ft/sec	forward velocity of aircraft in undisturbed flight
W lb	weight of aircraft
w ft/sec	increment of velocity along z axis in disturbed flight
$\left(\frac{w}{V}\right)$ rad	incremental aircraft incidence
$x = \frac{2\,\pi\,f\,\hat{t}}{\sqrt{C}}$	non-dimensional frequency of short period oscillation

LIST OF SYMBOLS (Contd.)

$x_u = \frac{1}{2} \frac{\partial C_x}{\partial \left(\frac{u}{V}\right)}$	longitudinal force derivative due to forward velocity, u
$x_w = \frac{1}{2} \frac{\partial C_x}{\partial \left(\frac{w}{V}\right)}$	longitudinal force derivative due to incidence, $\frac{w}{V}$
$x_q = \frac{1}{2} \frac{\partial C_x}{\partial \left(\frac{q\ell}{V}\right)}$	longitudinal force derivative due to rate of pitch, q
$x_\eta = \frac{1}{2} \frac{\partial C_x}{\partial \eta}$	longitudinal force derivative due to elevator deflection, η
$z_u = \frac{1}{2} \frac{\partial C_z}{\partial \left(\frac{u}{V}\right)}$	normal force derivative due to forward velocity, u
$z_w = \frac{1}{2} \frac{\partial C_z}{\partial \left(\frac{w}{V}\right)}$	normal force derivative due to heaving velocity, w
$z_q = \frac{1}{2} \frac{\partial C_z}{\partial \left(\frac{q\ell}{V}\right)}$	normal force derivative due to rate of pitch, q
$z_\eta = \frac{1}{2} \frac{\partial C_z}{\partial \eta}$	normal force derivative due to elevator deflection, η
γ_e rad	angle of climb
$\delta = -\frac{\mu m_\eta}{i_B}$	elevator moment coefficient
ε rad	downwash angle
$\zeta = \frac{B}{2\sqrt{C}}$	short period damping ratio, ratio of actual to critical damping
$= \zeta_a + \zeta_\nu + \zeta_\chi$	
$\zeta_a = \frac{\frac{a}{2}}{2\sqrt{C}}$	lift contribution to damping ratio
$\zeta_\nu = \frac{\nu}{2\sqrt{C}}$	rate of pitch contribution to damping ratio
$\zeta_\chi = \frac{\chi}{2\sqrt{C}}$	rate of change of incidence contribution to damping ratio

LIST OF SYMBOLS (Contd.)

$\zeta_b = \frac{b}{2\sqrt{C}}$	phugoid damping term
$\zeta_D = \frac{C_D}{2\sqrt{C}}$	phugoid damping term
η rad	elevator deflection
θ rad	angular displacement in pitch from equilibrium position
$\mu = \frac{W}{g \rho S \ell}$	aircraft density parameter
$\nu = -\frac{m_q}{i_B}$	rotary damping coefficient
ρ slugs/ft^3	air density
$\tau = \frac{t}{\hat{t}}$	aerodynamic time
ϕ degs	phase angle
$\Delta\phi_{\left(\frac{w}{V}\right)_E}$, $\Delta\phi_{n_E}$, $\Delta\phi_{q_E}$	corrections to short period phase angle for incidence, acceleration and rate of pitch respectively, to allow for the effect of elevator lift
$\chi = -\frac{m_{\dot{w}}}{i_B}$	rate of change of incidence damping coefficient
$\omega = -\frac{\mu\, m_w}{i_B}$	static stability coefficient
$\gamma = \frac{z_\eta}{z_w} \frac{C}{\delta}$	non-dimensional coefficient defining the ratio of the elevator lift derivative to elevator moment derivative
$\bar{\gamma} = \frac{z_\eta}{z_w} \frac{m_w}{m_\eta}$	ratio of restoring margin to elevator lift arm

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APPENDIX 1

TRANSFER FUNCTIONS FOR THE GENERAL LONGITUDINAL CASE

Within the assumptions stated complete transfer functions are given below for each of the dependent variables in the commonly accepted equations of motion in the longitudinal plane (apart from that for $\bar{\theta}/\bar{\eta}$ which is simply the integral of the expression for $\bar{q}/\bar{\eta}$). Additionally, the transfer functions for incidence and rate of pitch have been combined to give the corresponding functions for normal acceleration.

Assumptions

It is assumed that

1 there is no coupling with lateral modes,

2 the aircraft is rigid and

3 the system is linear.

Equations of motion

With these assumptions the equations of motion written in stability axes are

$$\left.\begin{aligned}
\left(D - x_u\right)\frac{u}{V} - x_w\frac{w}{V} - \frac{x_q}{\mu}\, q\,\hat{t} + \tfrac{1}{2} C_L\,\theta - x_\eta\,\eta &= 0\\
- z_u\frac{u}{V} + (D - z_w)\frac{w}{V} - \left(1 + \frac{z_q}{\mu}\right) q\,\hat{t} - k'\,\theta - z_\eta\,\eta &= 0\\
- \frac{\mu\, m_u}{i_B}\frac{u}{V} - \left(\frac{m_{\dot{w}}}{i_B} D + \frac{\mu\, m_w}{i_B}\right)\frac{w}{V} + \left(D - \frac{m_q}{i_B}\right) q\,\hat{t} - \frac{\mu\, m_\eta}{i_B}\,\eta &= 0\\
- q\,\hat{t} + D\theta &= 0
\end{aligned}\right\} \qquad (53)$$

Transfer functions

The following transfer functions for forward speed, incidence and rate of pitch are obtained from the above equations by the methods of Section 2.

$$\frac{\frac{\bar{u}}{V}}{(-\bar{\eta})} = - \frac{B_2 p^3 + C_2 p^2 + D_2 p + E_2}{\Delta} \tag{54}$$

$$\frac{\frac{\bar{w}}{V}}{(-\bar{\eta})} = - \frac{B_3 p^3 + C_3 p^2 + D_3 p + E_3}{\Delta} \tag{55}$$

$$\frac{\bar{q}}{(-\bar{\eta})} = - \frac{g \rho S V}{W} \frac{p(C_4 p^2 + D_4 p + E_4)}{\Delta} . \tag{56}$$

The corresponding expression for normal acceleration, obtained from equations (55), (56) and

$$-n g = \dot{w} - q V , \tag{57}$$

is

$$\frac{\bar{n}}{(-\bar{\eta})} = - \frac{\rho S V^2}{W} \frac{p(B_5 p^3 + C_5 p^2 + D_5 p + E_5)}{\Delta} . \tag{58}$$

In these equations Δ is the characteristic expression

$$A_1 p^4 + B_1 p^3 + C_1 p^2 + D_1 p + E_1$$

and the coefficients A_1, B_1, B_2 etc. are as follows:

$$A_1 = 1$$

$$B_1 = - (x_u + z_w) - \frac{m_q}{i_B} - \left(1 + \frac{z_q}{\mu}\right) \frac{m_{\dot{w}}}{i_B}$$

$$C_1 = (x_u z_w - x_w z_u) + \frac{m_q}{i_B} (x_u + z_w) + \frac{m_{\dot{w}}}{i_B} \left\{ x_u \left(1 + \frac{z_q}{\mu}\right) - \frac{x_q}{\mu} z_u - k' \right\}$$

$$- \frac{m_w}{i_B} (\mu + z_q) - \frac{m_u}{i_B} x_q$$

$$D_1 = -\frac{m_q}{i_B}(x_u\ z_w - x_w\ z_u) + \frac{m_{\dot{w}}}{i_B}\left(\tfrac{1}{2}\ C_L\ z_u + k'\ x_u\right)$$

$$+ \frac{m_w}{i_B}\left\{x_u(\mu + z_q) - x_q\ z_u - \mu\ k'\right\}$$

$$+ \frac{m_u}{i_B}\left\{\mu\ \tfrac{1}{2}\ C_L - x_w(\mu + z_q) + x_q\ z_w\right\}$$

$$E_1 = \frac{\mu\ m_w}{i_B}\left(\tfrac{1}{2}\ C_L\ z_u + k'\ x_u\right) - \frac{\mu\ m_u}{i_B}\left(\tfrac{1}{2}\ C_L\ z_w + k'\ x_w\right)$$

$$B_2 = x_\eta$$

$$C_2 = -(x_\eta\ z_w + x_w\ z_\eta) - \frac{m_q}{i_B}x_\eta - \frac{m_{\dot{w}}}{i_B}\left\{x_\eta\left(1 + \frac{z_q}{\mu}\right) - \frac{x_q}{\mu}\ z_\eta\right\}$$

$$+ \frac{\mu\ m_\eta}{i_B}\frac{x_q}{\mu}$$

$$D_2 = -\frac{m_q}{i_B}(x_w\ z_\eta - x_\eta\ z_w) - \frac{m_{\dot{w}}}{i_B}\left(\tfrac{1}{2}\ C_L\ z_\eta + k'\ x_\eta\right)$$

$$- \frac{\mu\ m_w}{i_B}\left\{x_\eta\left(1 + \frac{z_q}{\mu}\right) - \frac{x_q}{\mu}\ z_\eta\right\} + \frac{\mu\ m_\eta}{i_B}\left\{x_w\left(1 + \frac{z_q}{\mu}\right) - \frac{x_q}{\mu}\ z_w - \tfrac{1}{2}\ C_L\right\}$$

$$E_2 = -\frac{\mu\ m_w}{i_B}\left(\tfrac{1}{2}\ C_L\ z_\eta + k'\ x_\eta\right) + \frac{\mu\ m_\eta}{i_B}\left(\tfrac{1}{2}\ C_L\ z_w + k'\ x_w\right)$$

$$B_3 = z_\eta$$

$$C_3 = (x_\eta\ z_u - x_u\ z_\eta) - \frac{m_q}{i_B}\ z_\eta + \frac{\mu\ m_\eta}{i_B}\left(1 + \frac{z_q}{\mu}\right)$$

$$D_3 = \frac{m_q}{i_B}(x_u\ z_\eta - x_\eta\ z_u) + \frac{\mu\ m_u}{i_B}\left\{x_\eta\left(1 + \frac{z_q}{\mu}\right) - \frac{x_q}{\mu}\ z_\eta\right\}$$

$$- \frac{\mu\ m_\eta}{i_B}\left\{x_u\left(1 + \frac{z_q}{\mu}\right) - \frac{x_q}{\mu}\ z_u - k'\right\}$$

$$E_3 = \frac{\mu\ m_u}{i_B}(\tfrac{1}{2}\ C_L\ z_\eta + k'\ x_\eta) - \frac{\mu\ m_\eta}{i_B}(\tfrac{1}{2}\ C_L\ z_u + k'\ x_u)$$

$$C_4 = \frac{\mu\ m_\eta}{i_B} + \frac{m_{\dot{w}}}{i_B}\ z_\eta$$

$$D_4 = \frac{\mu\ m_u}{i_B}\ x_\eta + \frac{\mu\ m_w}{i_B}\ z_\eta + \frac{m_{\dot{w}}}{i_B}(x_\eta\ z_u - x_u\ z_\eta) - \frac{\mu\ m_\eta}{i_B}(x_u + z_w)$$

$$E_4 = \frac{\mu\ m_u}{i_B}(x_w\ z_\eta - x_\eta\ z_w) + \frac{\mu\ m_w}{i_B}(x_\eta\ z_u - x_u\ z_\eta)$$

$$+ \frac{\mu\ m_\eta}{i_B}(x_u\ z_w - x_w\ z_u)$$

$$B_5 = -\ z_\eta$$

$$C_5 = -\ (x_\eta\ z_u - x_u\ z_\eta) + \left(\frac{m_q}{i_B} + \frac{m_{\dot{w}}}{i_B}\right) z_\eta - \frac{\mu\ m_\eta}{i_B}\ \frac{z_q}{\mu}$$

$$D_5 = -\ \frac{m_q}{i_B}(x_u\ z_\eta - x_\eta\ z_u) - \frac{m_u}{i_B}(x_\eta\ z_q - x_q\ z_\eta) + \frac{\mu\ \ m_w}{i_B}\ z_\eta$$

$$+ \frac{m_{\dot{w}}}{i_B}(x_\eta\ z_u - x_u\ z_\eta) + \frac{\mu\ m_\eta}{i_B}\left(\frac{x_u\ z_q}{\mu} - \frac{x_q\ z_u}{\mu} - k' - z_w\right)$$

$$E_5 = -\frac{\mu\, m_u}{i_B}\left\{x_\eta(k' + z_w) + z_\eta(\tfrac{1}{2}\, C_L - x_w)\right\} + \frac{\mu\, m_w}{i_B}\,(x_\eta\, z_u - x_u\, z_\eta)$$

$$+ \frac{\mu\, m_\eta}{i_B}\left\{x_u(k' + z_w) + z_u(\tfrac{1}{2}\, C_L - x_w)\right\}$$

APPENDIX 2

RANGE OF VALIDITY OF THE APPROXIMATE VALUES OF THE SHORT PERIOD STABILITY PARAMETERS B AND C

In Section 5.3 the quartic equation (Equation (37))

$$\Delta = \left(p^2 + C_D\, p + \frac{C_L^{\,2}}{2}\right)(p^2 + B\,p + C) - \frac{a}{2}\frac{C_L^{\,2}}{2}(p - \nu) \tag{59}$$

was expressed as

$$\Delta = (p^2 + b\,p + c)(p^2 + B'p + C') \tag{60}$$

and in the subsequent analysis it was assumed that $B' = B$ and $C' = C$, where B and C were the two degrees of freedom short period damping and stiffness parameters respectively. The divergence of the true values, B', C', from the approximate values, B, C, of these parameters with increasing C_L is shown in Fig. 16.

To obtain these true values, Equation (59), with the selected coefficients, $B = 5$, $C = 16$, $a = 5$, $\nu = 2$ and $C_D = 0.02 + 0.08\,C_L^{\,2}$, was expressed in the form of Equation (60) by iteration.

It can be seen from Fig. 16 that the errors incurred by making the above assumption are negligible at $C_L = 0.33$ (i.e. in the examples given in Section 5), and increases with C_L. At high lift coefficients, therefore, correct values of the coefficients must be used.

TABLE 1

Values of aircraft parameters

Aircraft	A	B	C
Type	Tailed, medium altitude, subsonic	Tailless, high altitude, supersonic	Tailless, high altitude, transonic
ζ	0.6	0.2	0.2
ζ_a	0.3	0.1	0.2
ζ_ν	0.2	0.1	0.05
ζ_χ	0.1	0	-0.05
Υ	0.066	0.156	0.156
$\bar{\Upsilon}$	0.05	0.15	0.15

For aircraft A only:

a	B	C
4	4	11.1

$$C_D = 0.022 + 0.0692\ C_L^{\ 2}$$

when $C_L = 0.33$ (as in Figs.9 to 12)

$C_D = 0.029$, $R_1 = 0.06$, $R_2 = 0.07$,

$\zeta_b = 0.0036$ and $\zeta_D = 0.0044$

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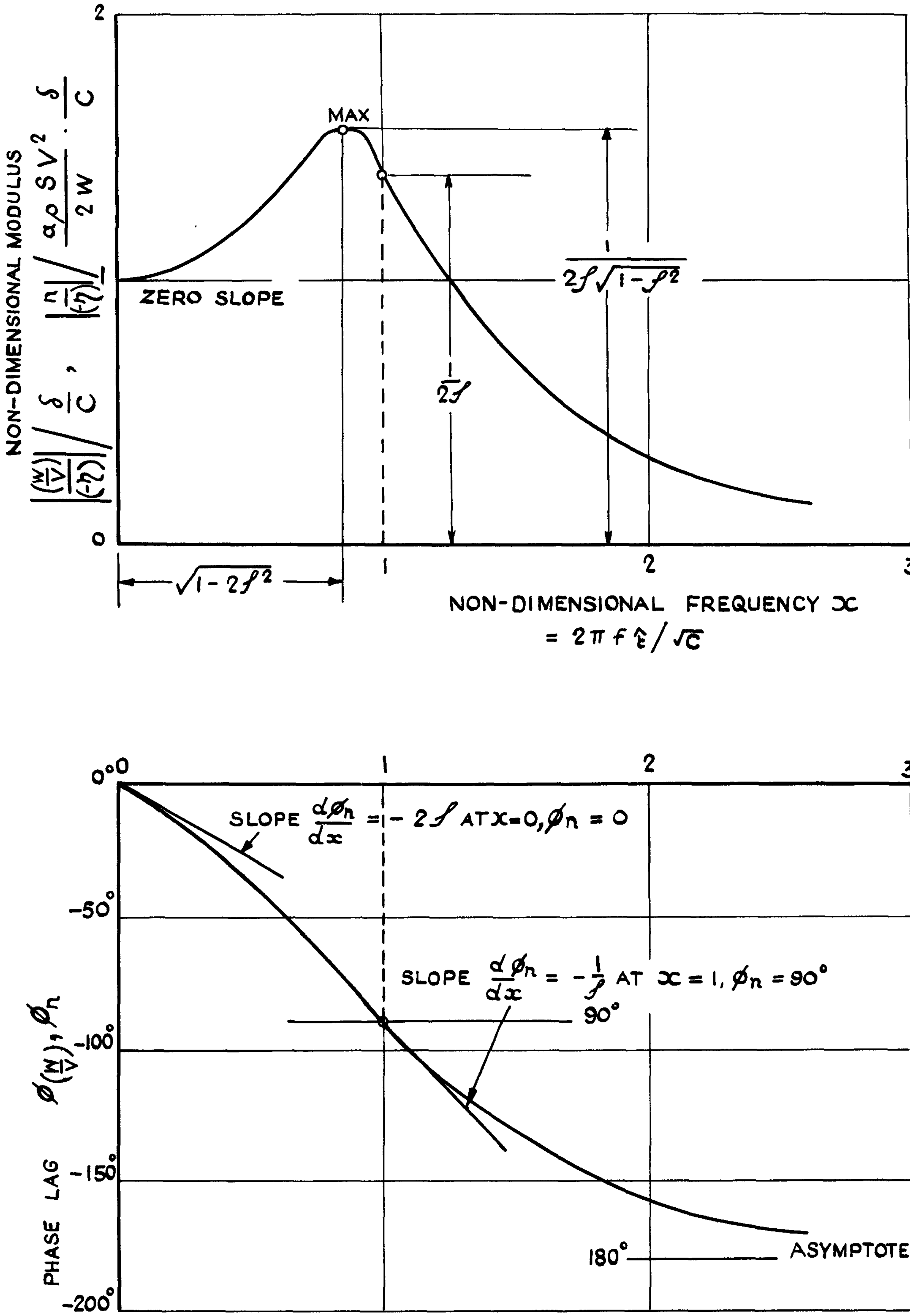


FIG. 1. THE CHARACTERISTICS OF THE NON-DIMENSIONAL SHORT PERIOD FREQUENCY RESPONSE OF INCIDENCE AND NORMAL ACCELERATION

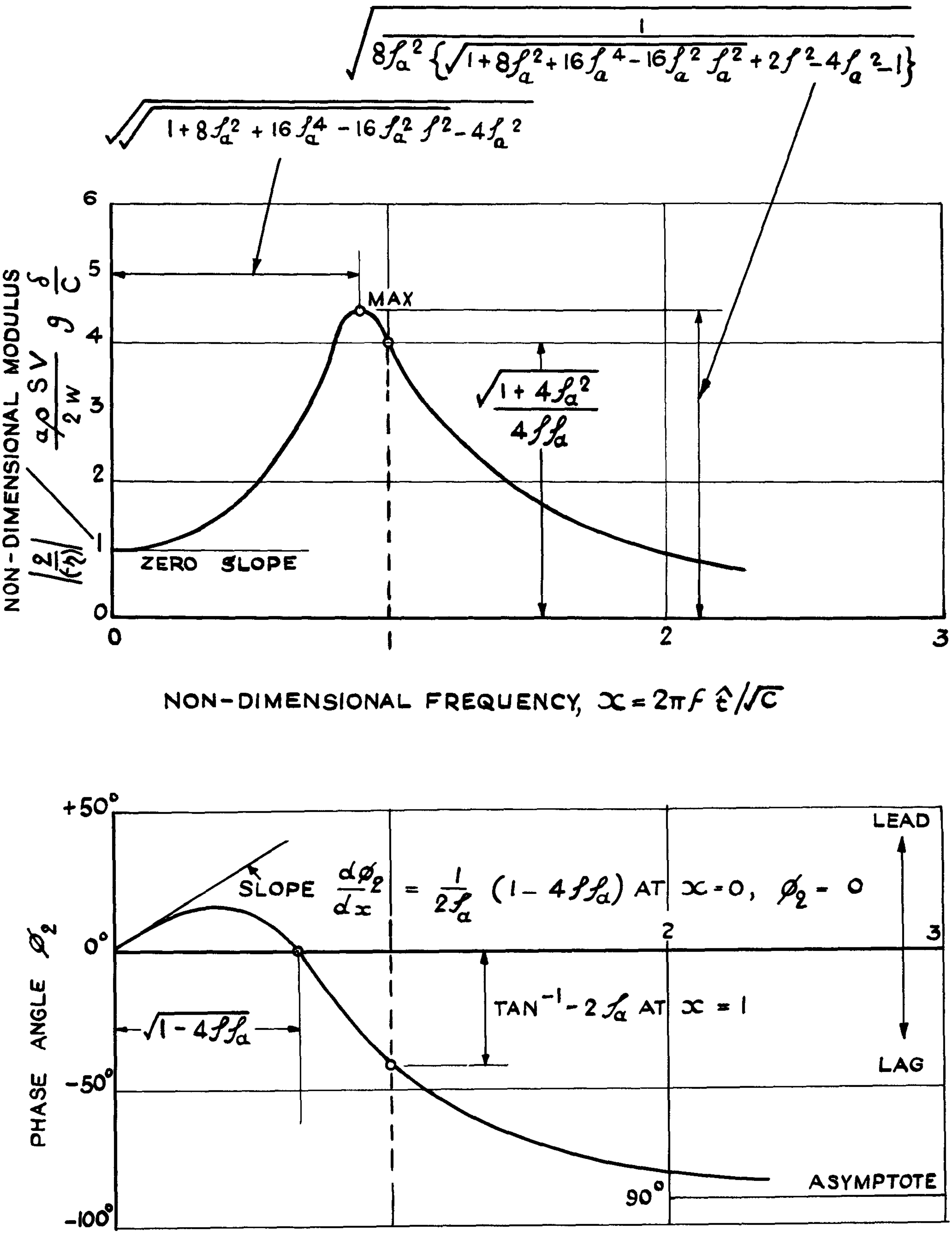


FIG. 2. THE CHARACTERISTICS OF THE NON-DIMENSIONAL SHORT PERIOD FREQUENCY RESPONSE OF RATE OF PITCH

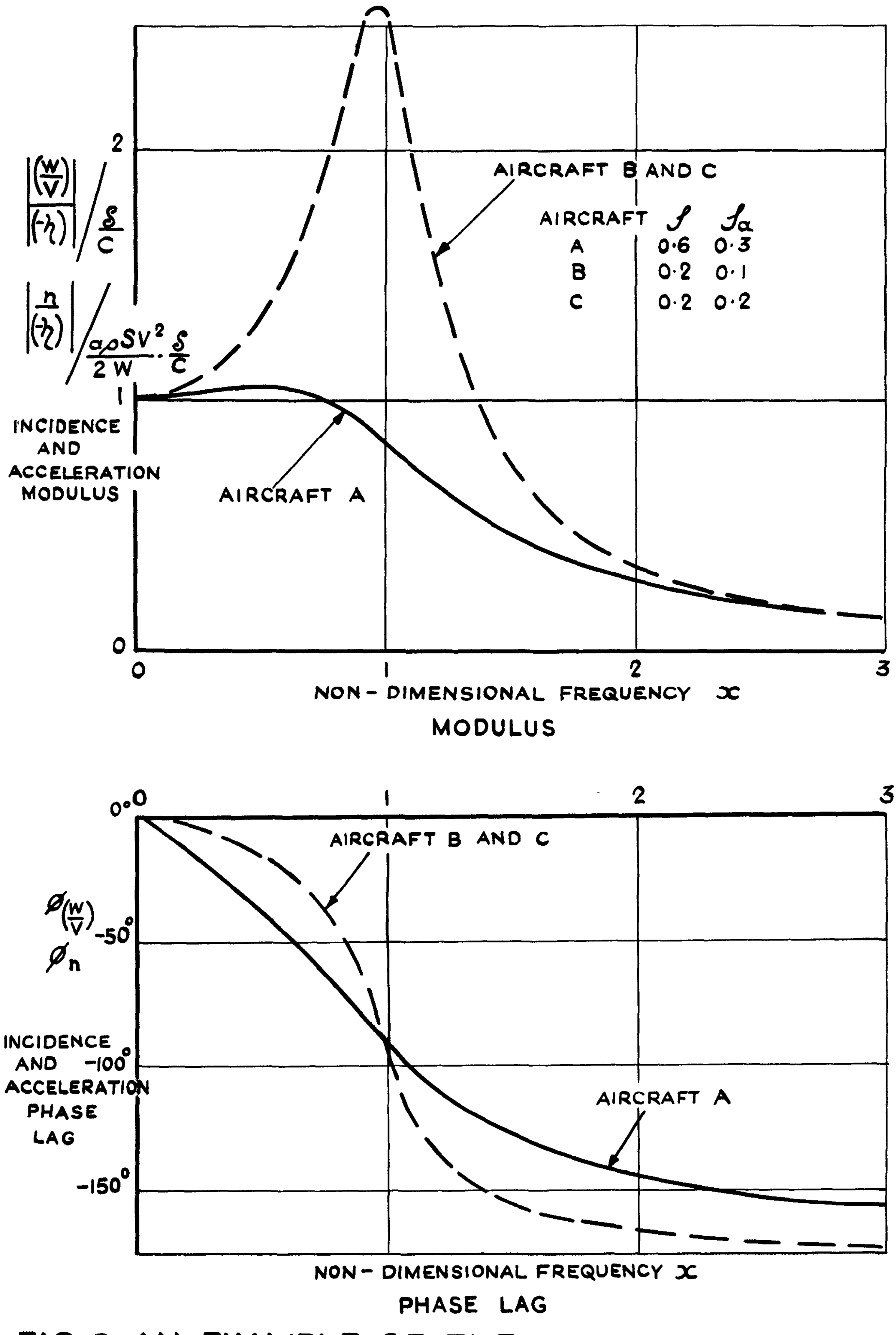


FIG. 3. AN EXAMPLE OF THE NON-DIMENSIONAL SHORT PERIOD FREQUENCY RESPONSE OF INCIDENCE AND NORMAL ACCELERATION

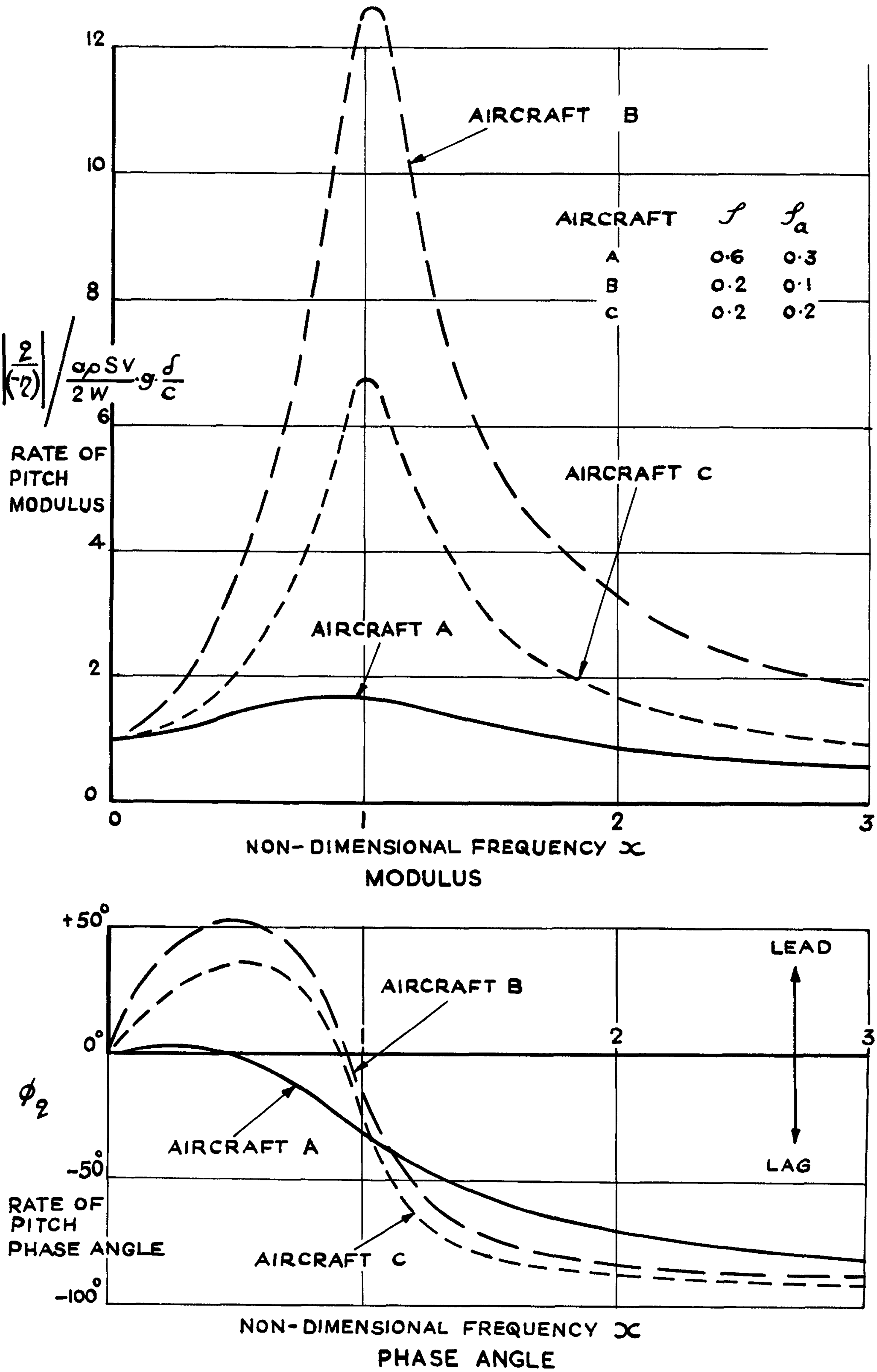
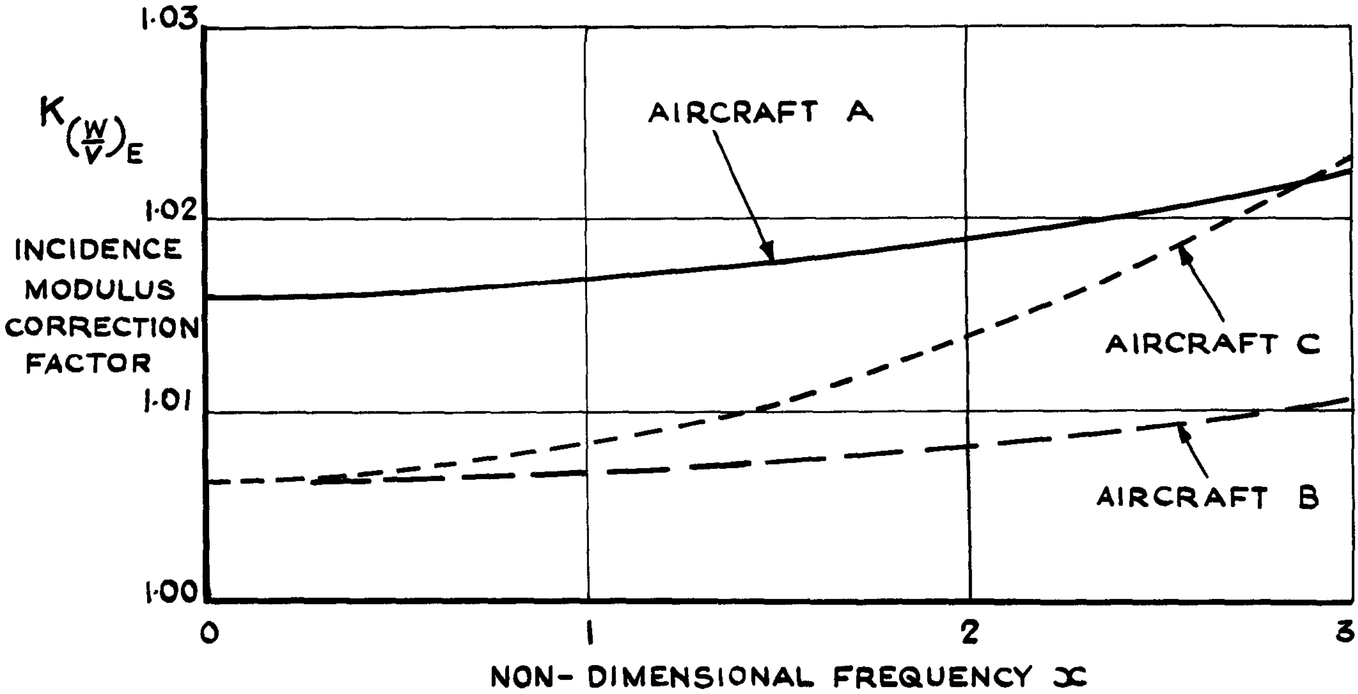
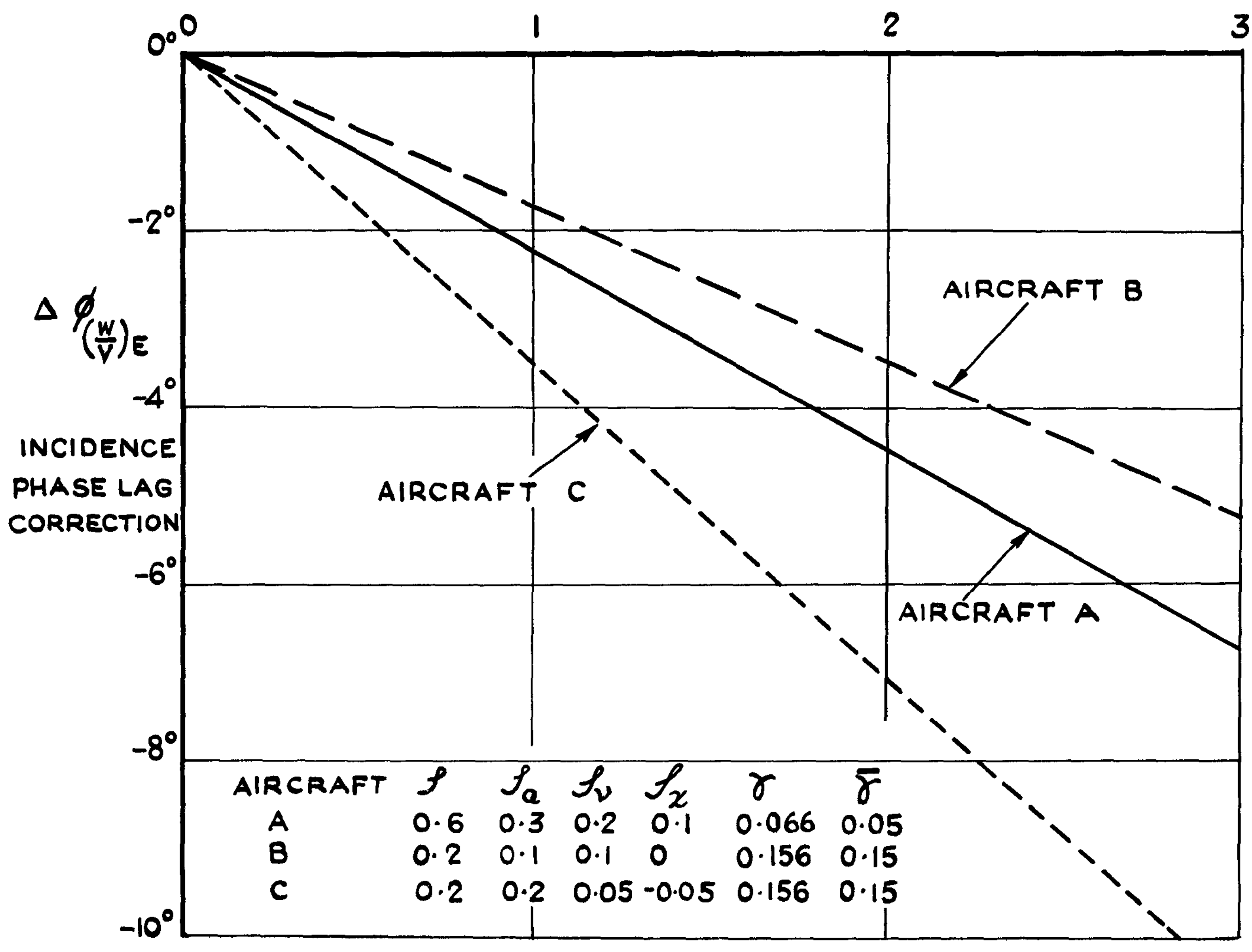


FIG. 4. AN EXAMPLE OF THE NON-DIMENSIONAL SHORT PERIOD FREQUENCY RESPONSE OF RATE OF PITCH

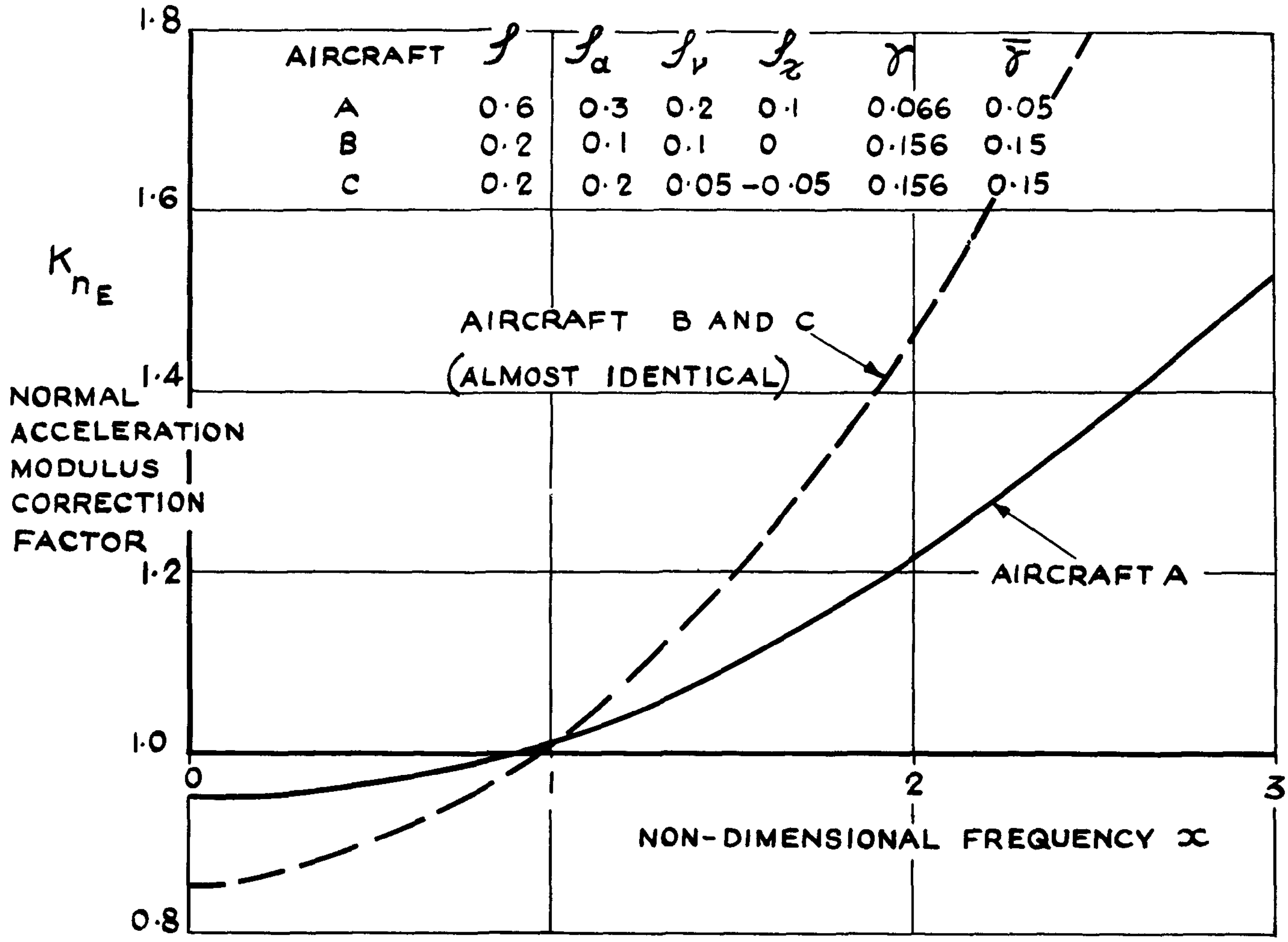


MODULUS CORRECTION FACTOR

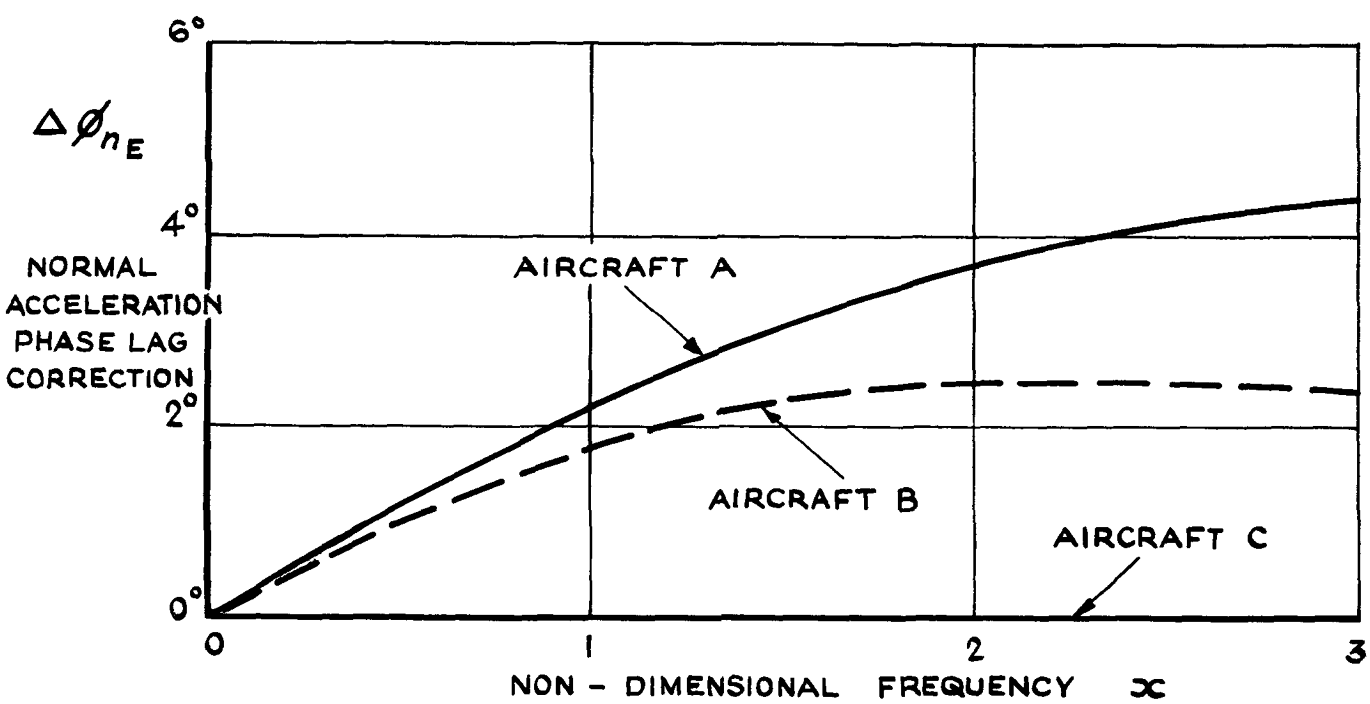


PHASE LAG CORRECTION

FIG. 5 AN EXAMPLE OF THE EFFECT OF ELEVATOR LIFT ON THE FREQUENCY RESPONSE OF INCIDENCE TO ELEVATOR

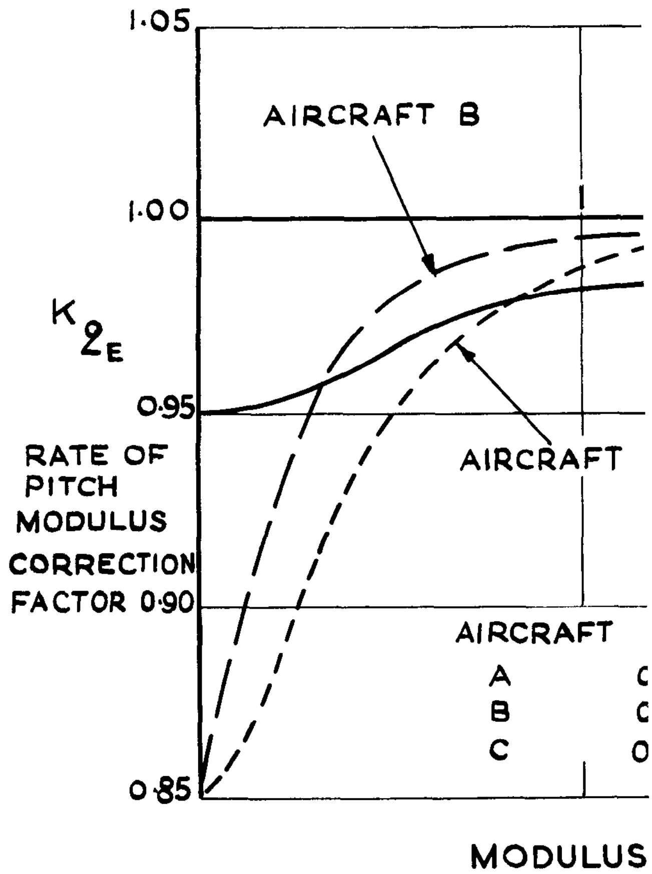


MODULUS CORRECTION FACTOR



PHASE LAG CORRECTION

FIG. 6. AN EXAMPLE OF THE EFFECT OF ELEVATOR LIFT ON THE FREQUENCY RESPONSE OF NORMAL ACCELERATION TO ELEVATOR



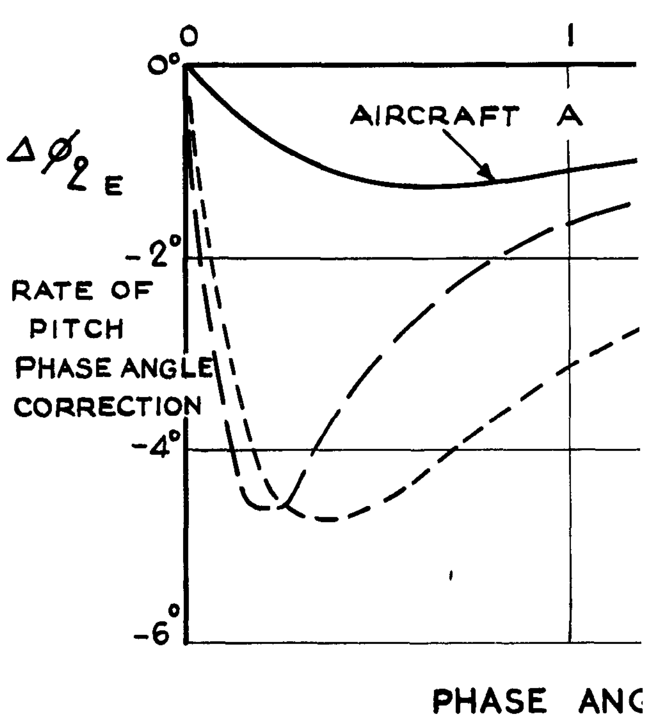
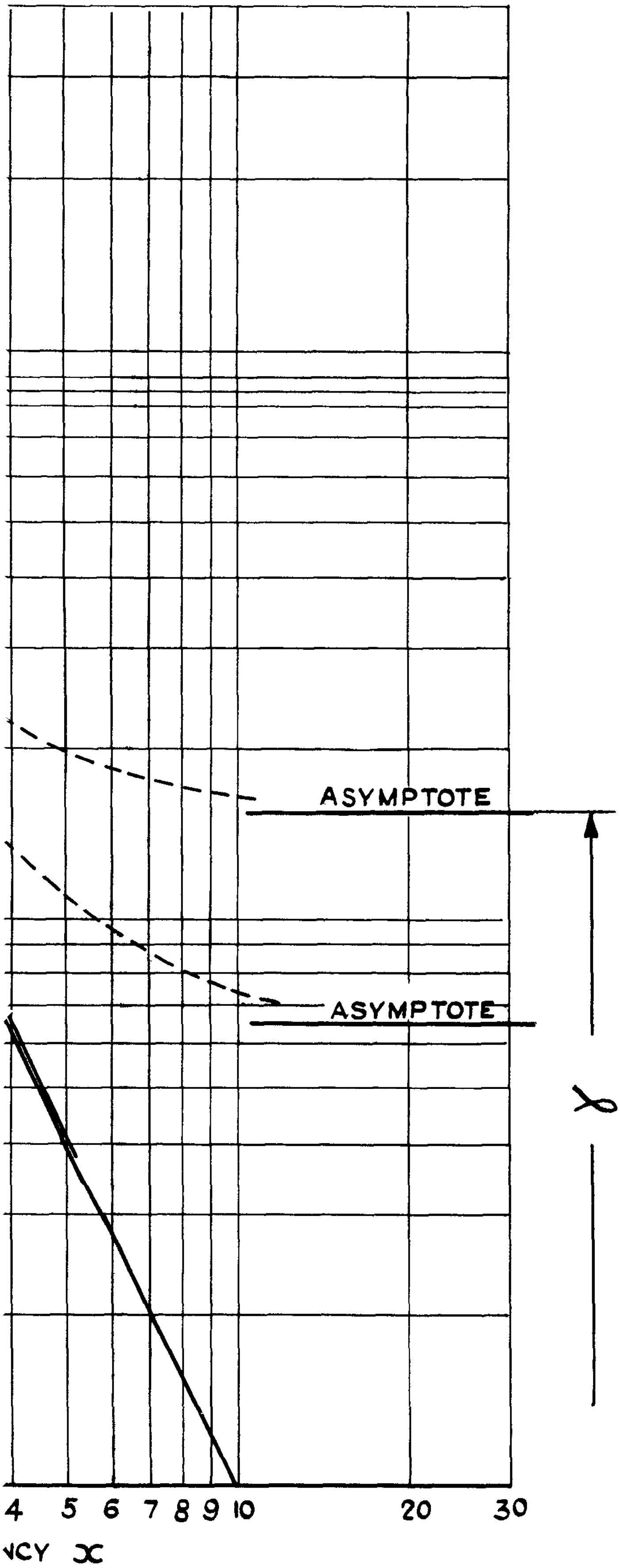


FIG. 7 AN EXAMPI
ELEVATOR LIFT ON TH
OF RATE OF PI

AIRCRAFT	$\mathcal{S}$	$\mathcal{S}_a$	$\mathcal{S}_v$	$\mathcal{S}_z$	γ	$\bar{\gamma}$
A	0·6	0·3	0·2	0·1	0·066	0·05
B	0·2	0·1	0·1	0	0·156	0·15



'ATOR LIFT ON THE MODULUS OF THE
ACCELERATION TO ELEVATOR

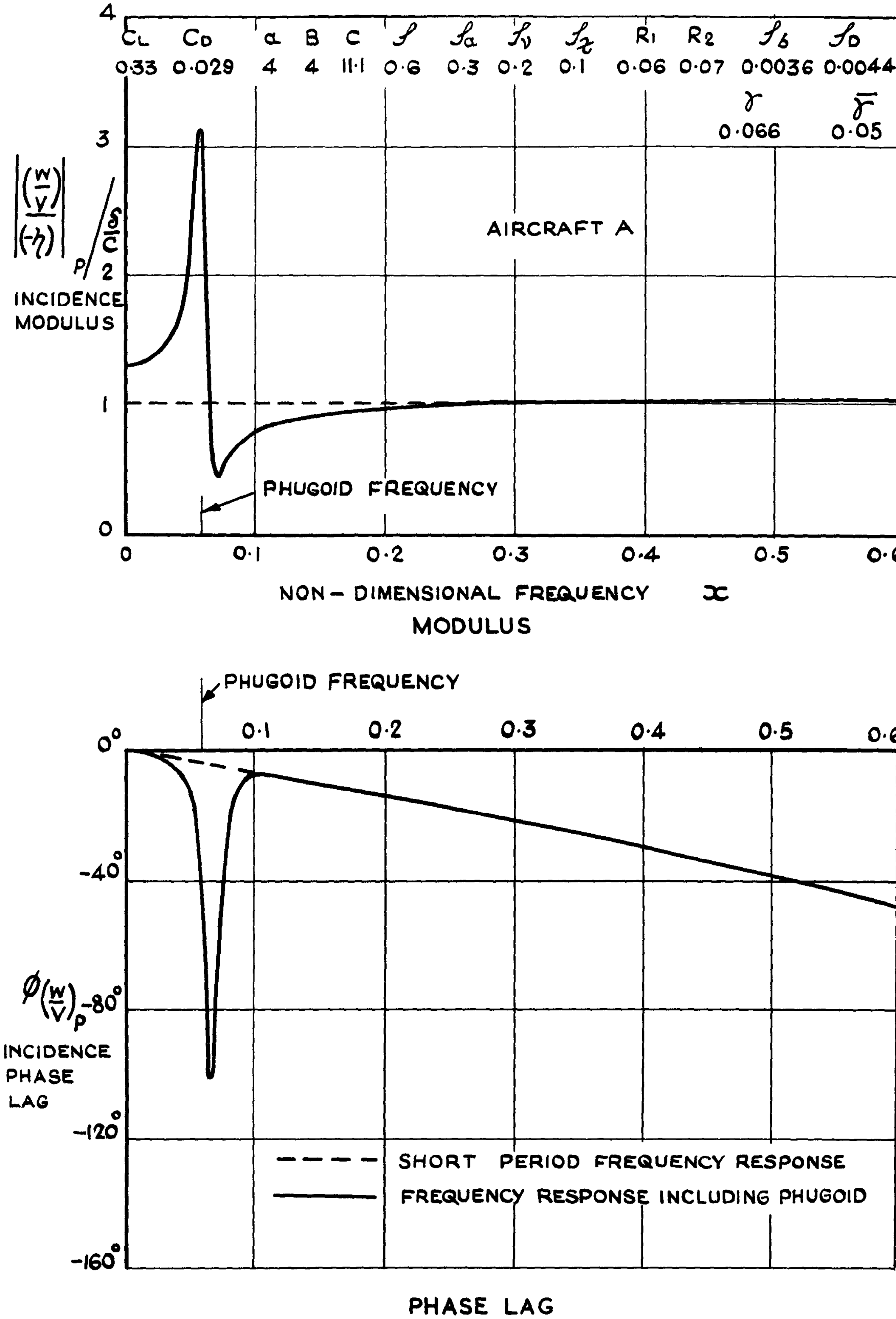
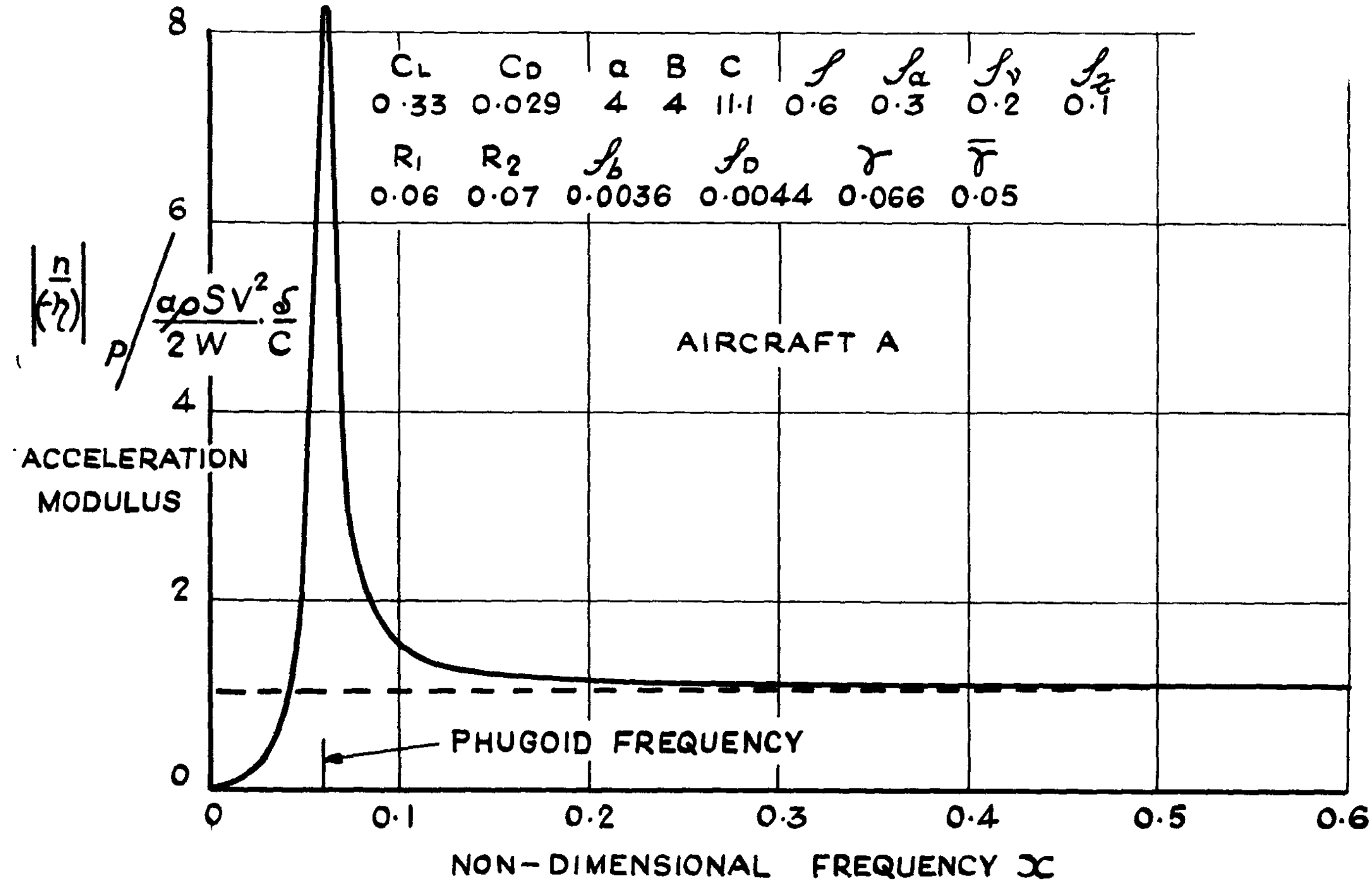
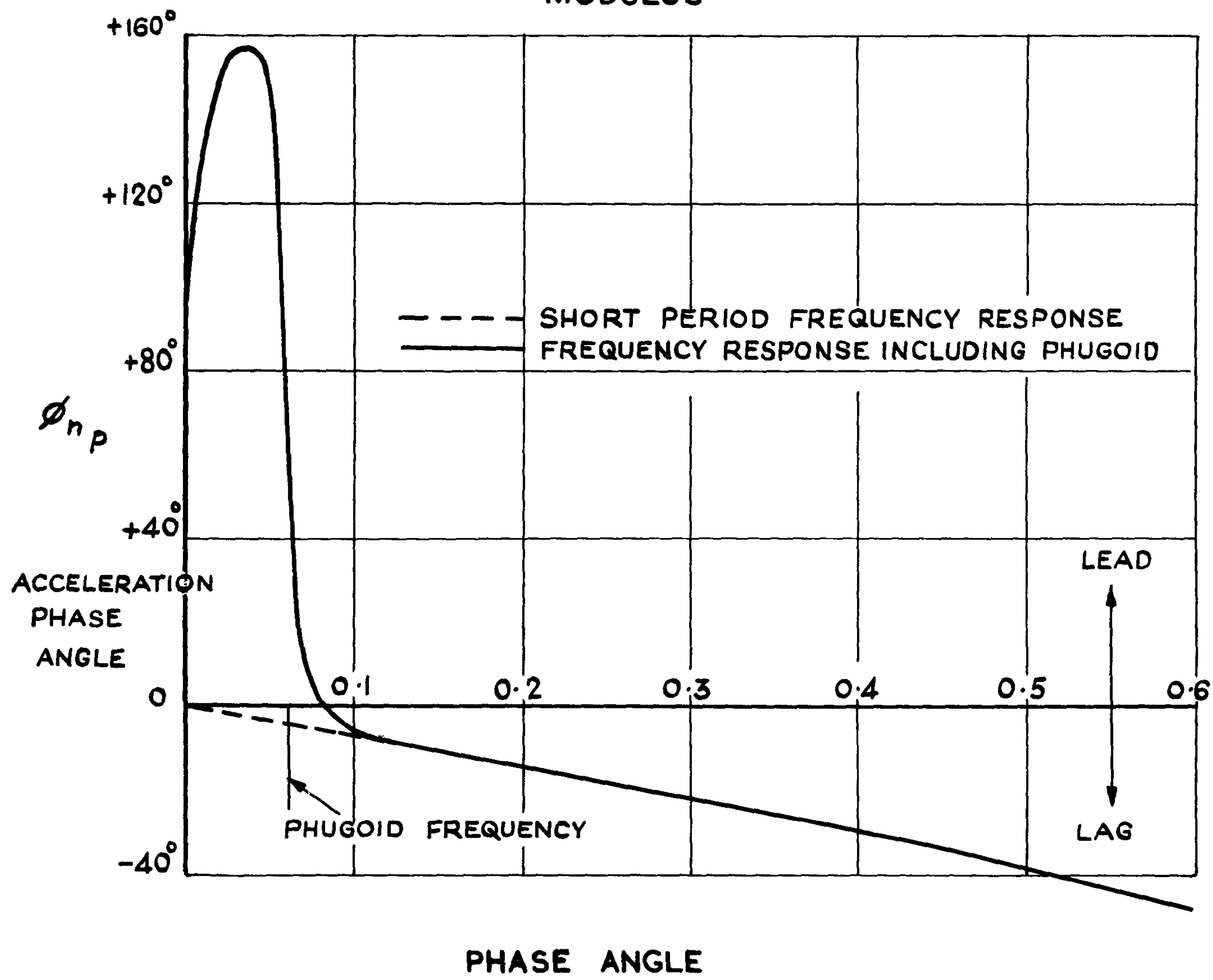


FIG.9. AN EXAMPLE OF PHUGOID FREQUENCY RESPONSE CURVES FOR INCIDENCE



MODULUS



PHASE ANGLE

FIG. 10. AN EXAMPLE PHUGOID FREQUENCY RESPONSE CURVES FOR NORMAL ACCELERATION

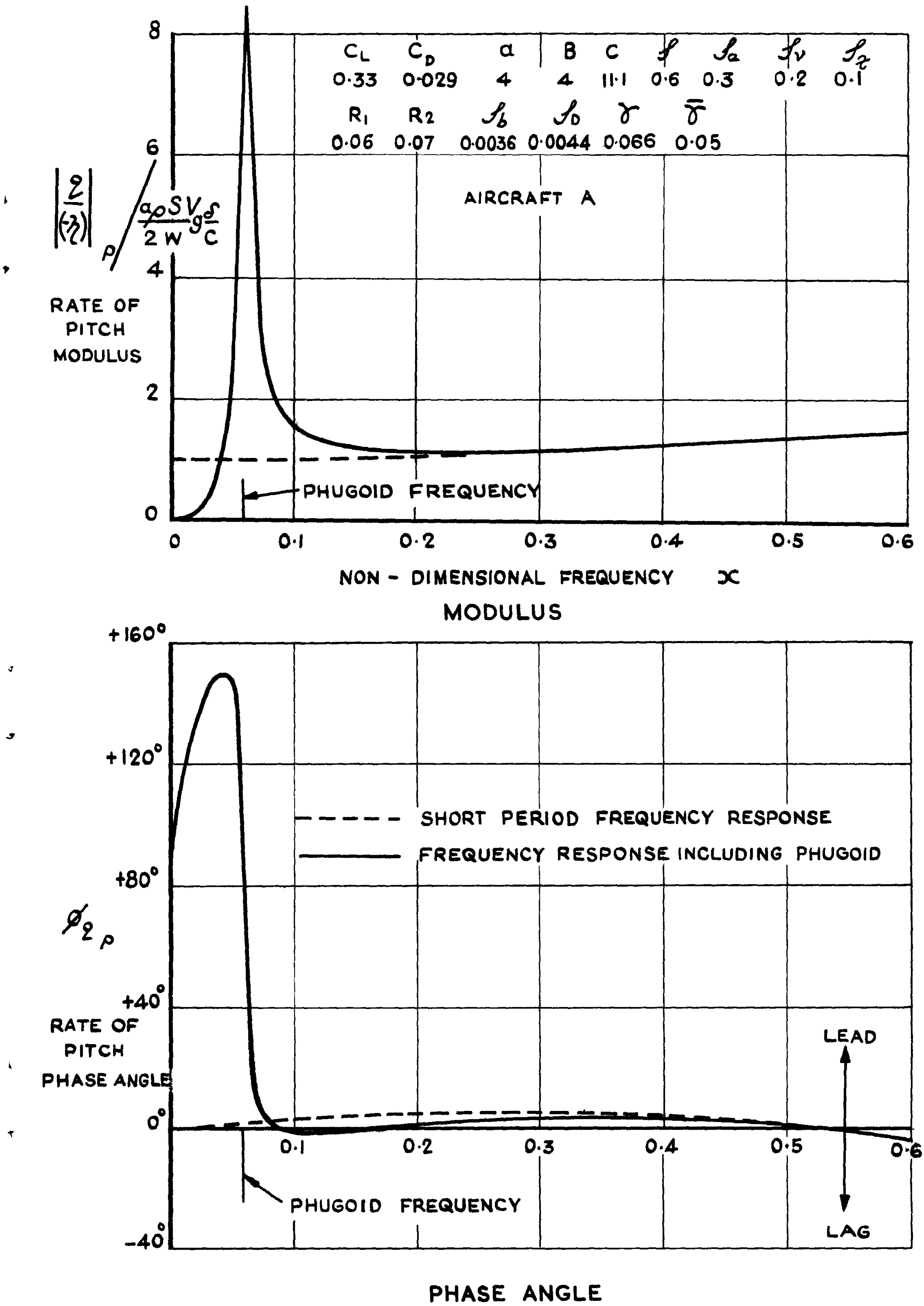


FIG. 11. AN EXAMPLE OF PHUGOID FREQUENCY RESPONSE CURVES FOR RATE OF PITCH

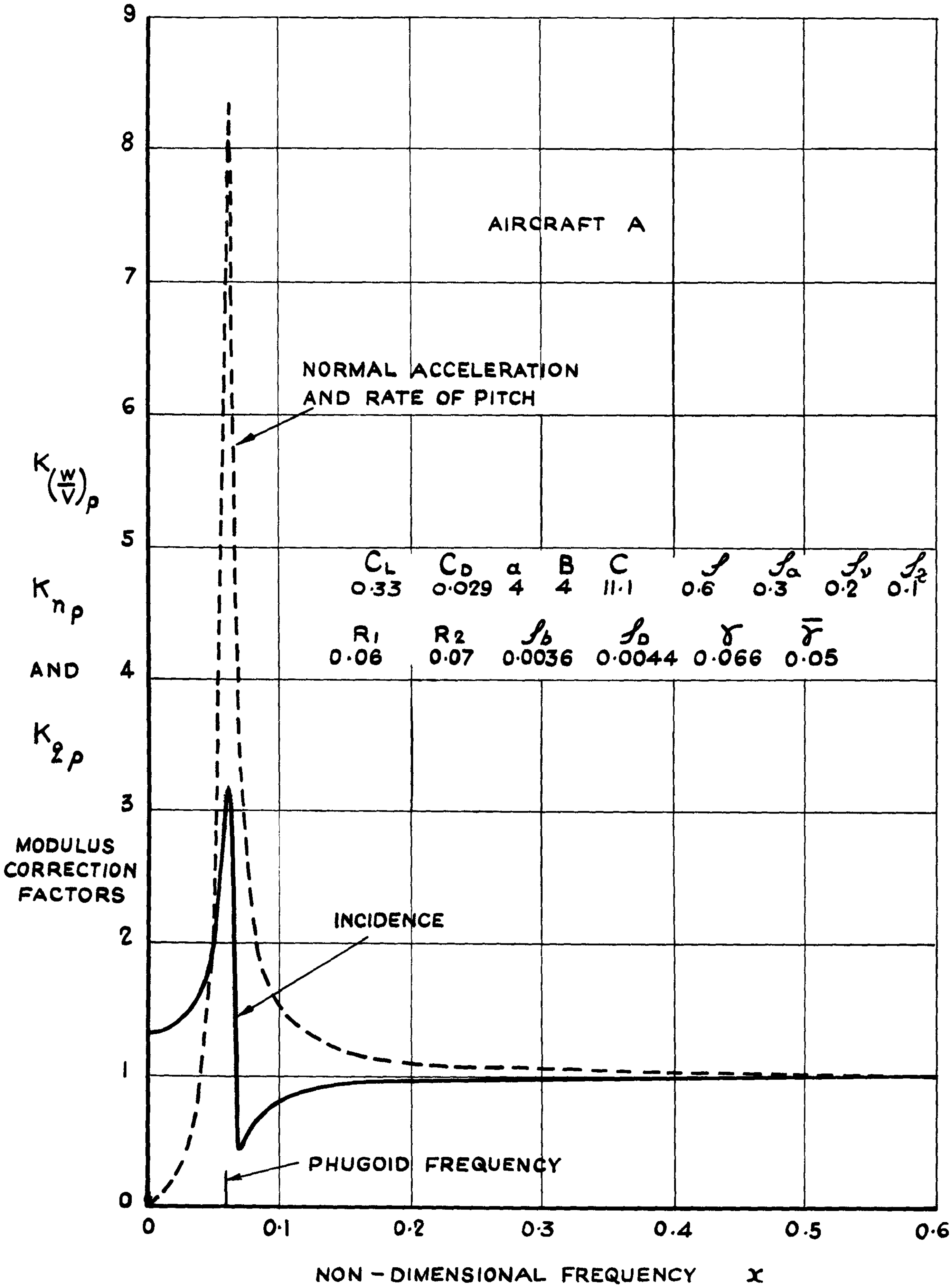


FIG.12. AN EXAMPLE OF THE EFFECT OF THE PHUGOID OSCILLATION ON SHORT PERIOD FREQUENCY RESPONSE MODULI

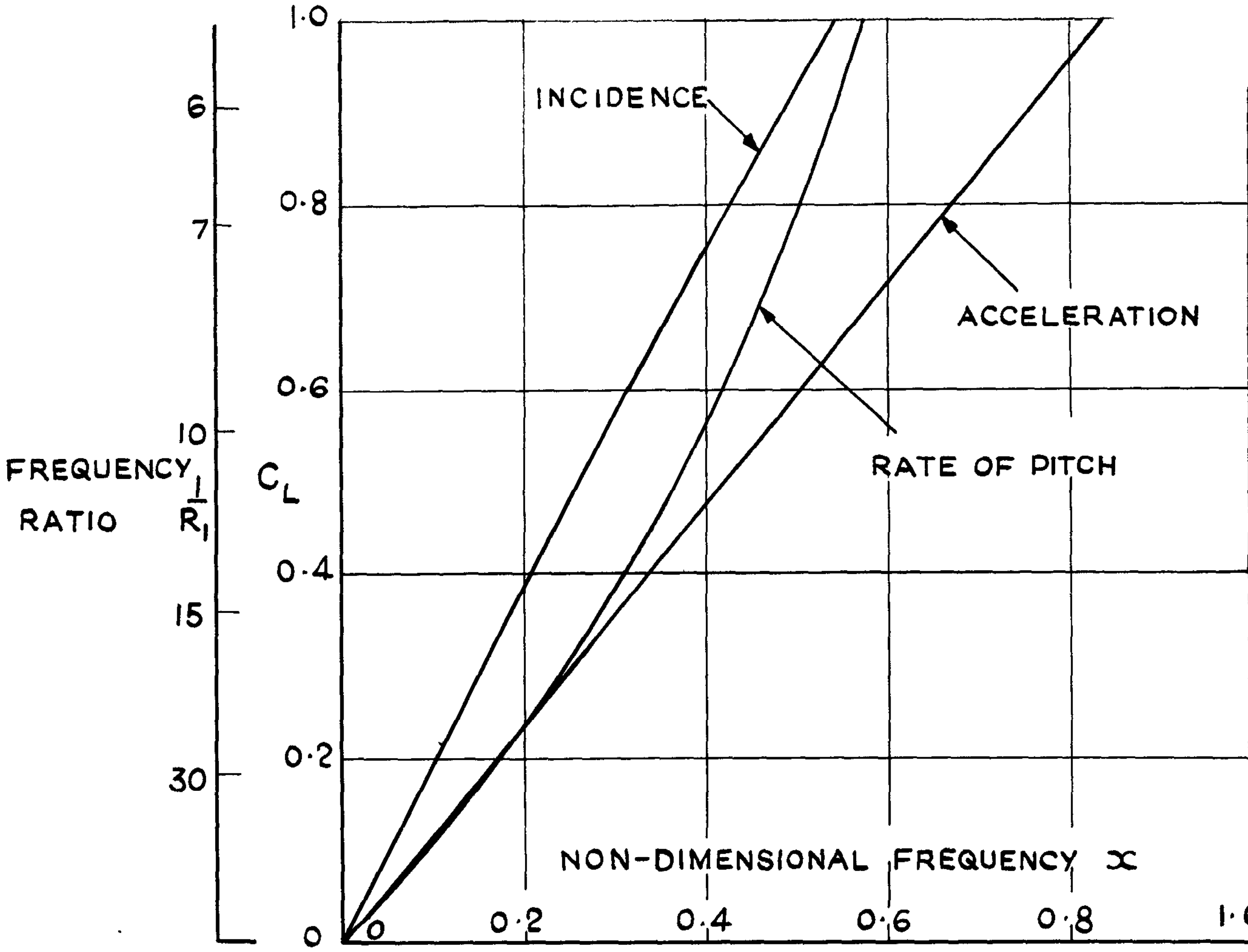


FIG. 13. FREQUENCY AT WHICH NEGLECT OF PHUGOID CAUSES 5% ERROR IN MODULUS

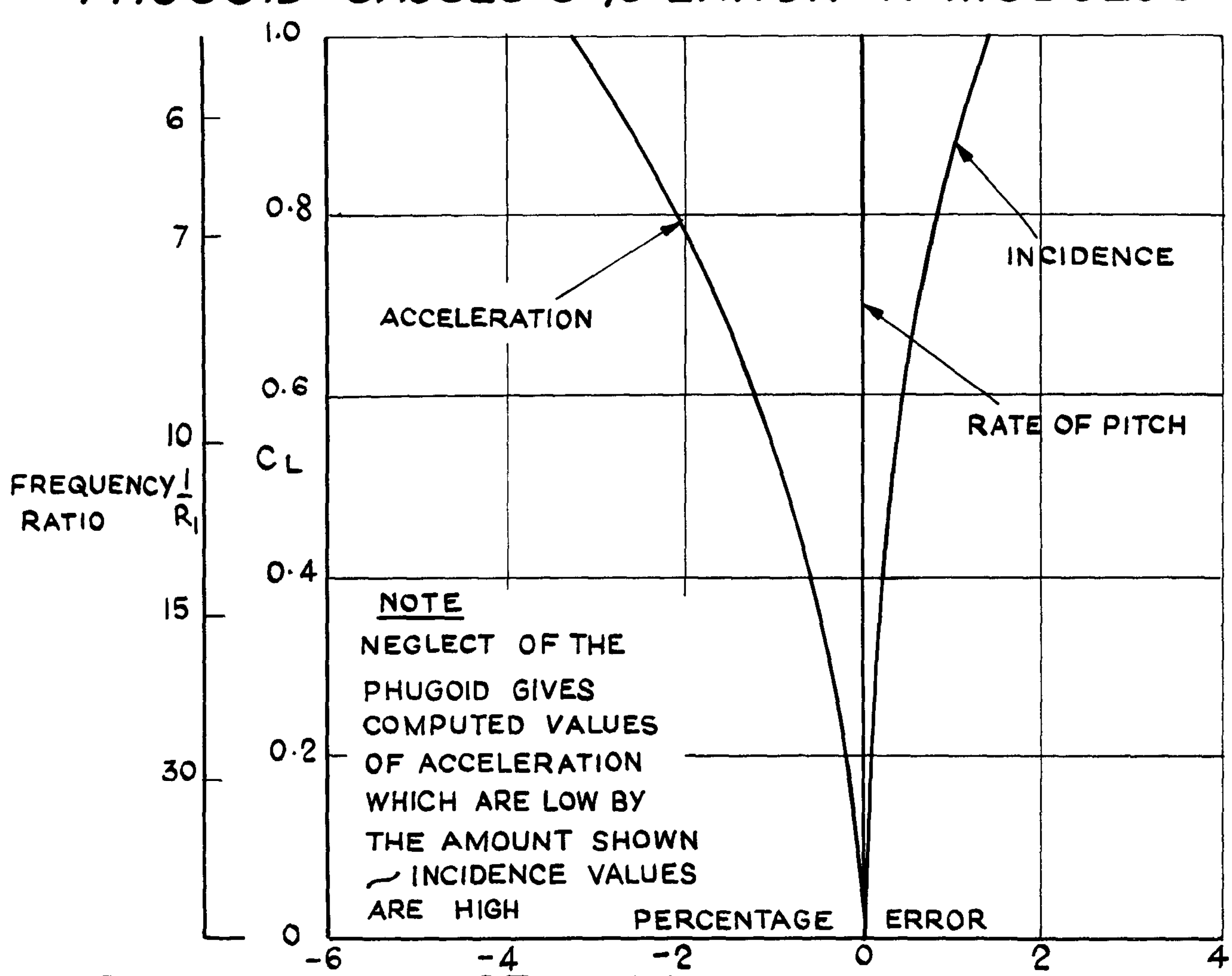


FIG. 14. PERCENTAGE ERROR IN MODULUS INCURRED AT SHORT PERIOD FREQUENCY ($x = 1$) BY NEGLECT OF PHUGOID

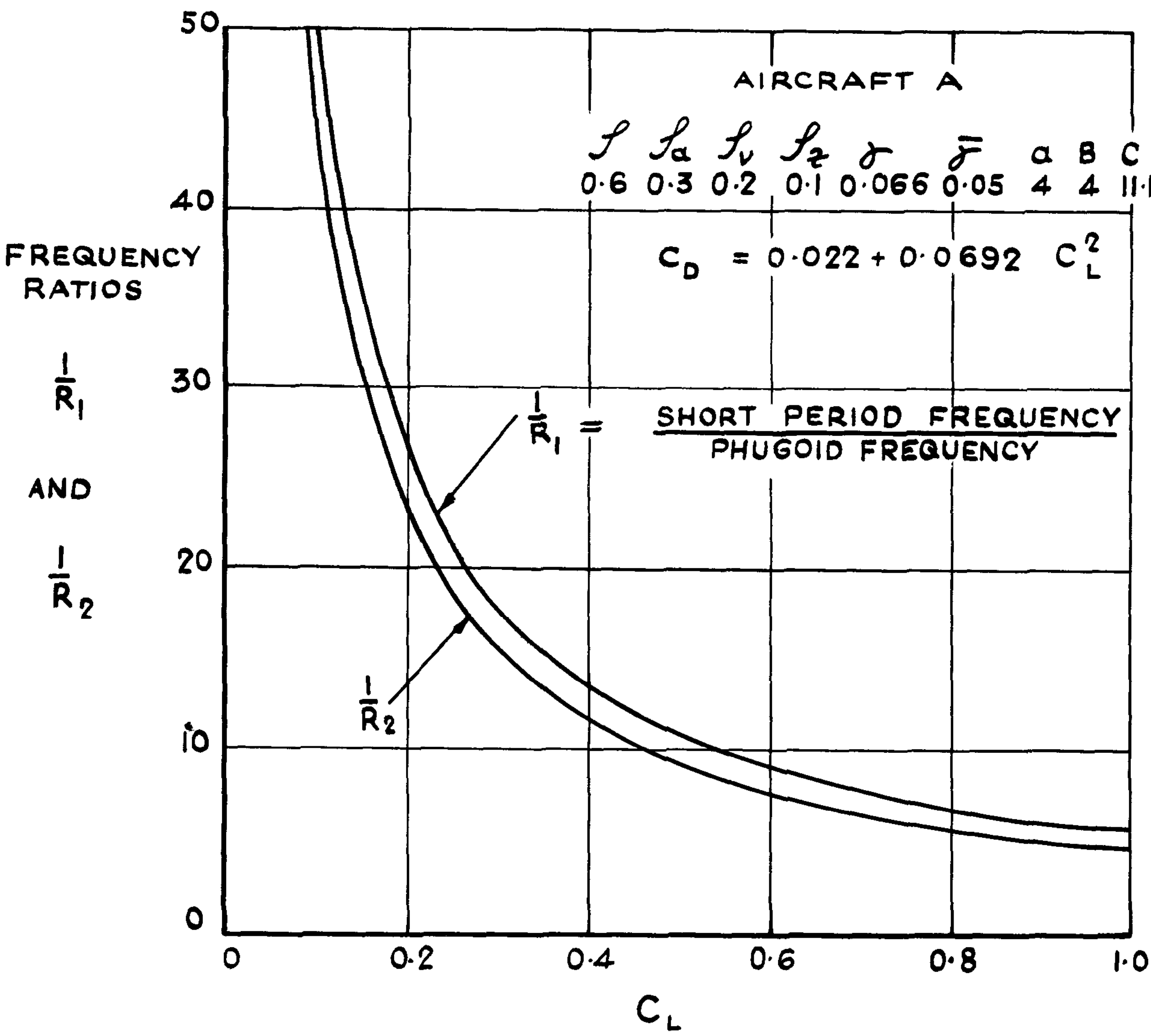
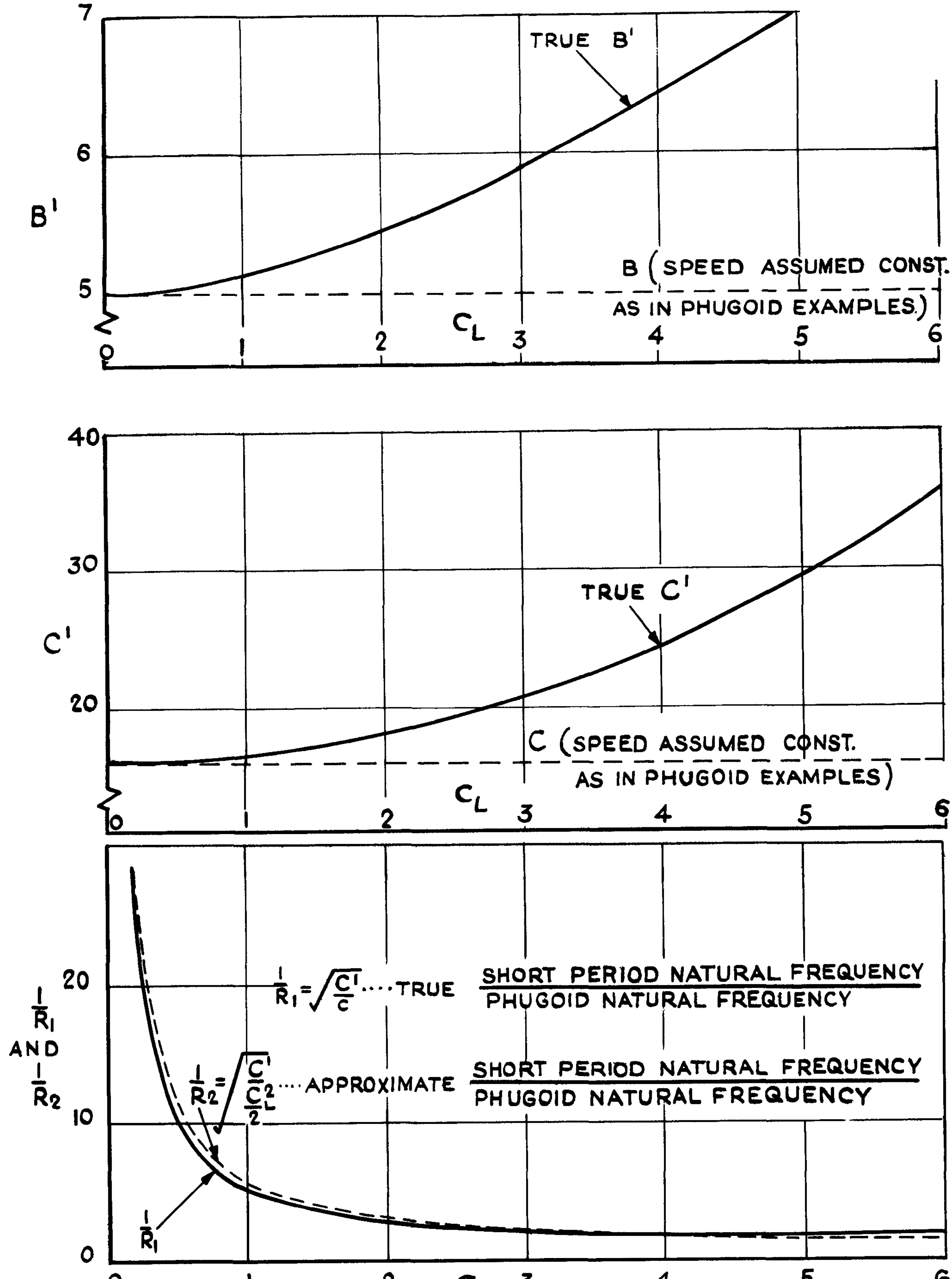


FIG. 15 RATIO OF SHORT PERIOD FREQUENCY TO PHUGOID FREQUENCY, $\frac{1}{R_1}$ AND THE RATIO $\frac{1}{R_2}$ FOR DIFFERENT LIFT COEFFICIENTS (AIRCRAFT A)



THE ASSUMED VALUES $B = 5$, $C = 16$, $\frac{a}{2} = 2.5$, $\nu = 2$,

$C_D = 0.02 + 0.08\, C_L^2$

WERE USED IN EQ. 37 $\left(p^2 + C_{D}p + \frac{C_L^2}{2}\right)\left(p^2 + Bp + C\right) - \frac{a}{2}\frac{C_L^2}{2}(p - \nu) = \Delta$

TO OBTAIN $\left(p^2 + bp + c\right)\left(p^2 + B'p + C'\right)$ BY ITERATION

FIG. 16. COMPARISON OF TRUE VALUES B', C' AND APPROXIMATE VALUES B, C OF THE SHORT PERIOD STABILITY PARAMETERS

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