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Summary.

A study is made of the linearized differential equation for supersonic flow of a gas relaxing in one mode, assuming a linear rate equation, in a two-dimensional non-uniform channel. An exact solution to this equation is found which includes the corner flow problem as a special case. This solution clearly demonstrates the exponential decay of disturbances along the frozen characteristics associated with the relaxation process. The results obtained for the corner flow problem agree with the earlier results of J. F. Clarke and J. J. Der. Approximate solutions are also obtained which are shown to be adequate for most practical values of the ratio of the equilibrium to the frozen speed of sound.

Similar exact and approximate solutions are also found for the linearized case of a two dimensional jet expanding into a uniform pressure field.

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1. *Introduction.*

Non-equilibrium effects in gas flows may arise from chemical reactions between the various species comprising the gas or from a redistribution of energy among the internal energy modes of the molecules after the gas has been perturbed from an equilibrium condition. Kirkwood and Wood[1] have shown the basic similarity between the two types of relaxation. Both processes introduce a source of dissipation into the flow (i.e. the flow is no longer isentropic), and if the processes take a time of the same order of magnitude as the time for a typical molecule to pass through the flow field considered, the relaxation effects become important.

In the present work, a simplified model of the gas, in which only one type of relaxation is present, is used in order to formulate the problem. In many cases, this approximation to the real gas behaviour is not unreasonable as one type of relaxation is found to dominate all the others. For instance, in a dissociation relaxation region the change in energy associated with the internal energy modes is very small compared with the change in energy associated with the dissociation; and similarly in the case of a gas in chemical equilibrium, in certain temperature ranges one internal mode is found to have a much longer relaxation time than the others which are treated as active modes, i.e. reach equilibrium instantaneously.

Gunn[2] investigated the effect of heat capacity lag in one-dimensional nozzle flows by linearising the equations to find the loss in available energy due to the temperature lag and hence the loss in total energy and found it to be a small effect. Chu[3] indicates how the problem of a relaxing gas may be solved by a step-by-step numerical calculation using the method of characteristics. Bray[6] and others showed that in the case where the amount of energy in the lagging mode is small, a criterion could be established for the 'freezing' position in the nozzle. This sudden 'freezing-out' of the flow where the lagging mode, having followed the equilibrium distribution closely at first, suddenly breaks away and rapidly approaches an apparently steady non-equilibrium value (frozen flow), is also borne out by the numerical calculations of Stollery and Smith[5] for vibrational temperature lag, Freeman[7] and Hall and Russo[8] for atomic recombination together with the recent analytic formulation of the problem by Blythe[4].

The governing linearised differential equation satisfied by the two-dimensional perturbation velocity potential $\phi\,(x, y)$ has been derived by Vincenti[9] and Clarke[10] in the form

$$K\,(B_f^{\,2}\,\phi_{xx}-\phi_{yy})_x+B_e^{\,2}\,\phi_{xx}-\phi_{yy} = 0$$

where K is the relation length, $B_f^{\,2} = M_f^{\,2}-1$, $B_e^{\,2} = M_e^{\,2}-1$, where M_f and M_e are the free-stream Mach numbers based on the frozen and equilibrium speeds of sound respectively. The importance of the two speeds of sound was shown by Chu[3] and Clarke[10] while Vincenti solved the equation for flow past wavy walls. The method of Laplace transforms has also been used by Clarke[11] to solve the equation for flow past a corner, and by Der[12] for flow past an arbitrary boundary. Moore and Gibson[13] approximate to the third-order equation by the second-order linear telegraph equation for flow past a wedge and a wavy wall. Clarke and Cleaver[14] find solutions to the third-order equation for $\phi(x,y)$ by use of a Green's Function technique for the flow past thin aerofoils.

Clarke has also used the axisymmetric form of the equation to investigate relaxation effects on slender bodies[15].

In the sections that follow, the method of Laplace transforms is used to solve the equation for relaxing flow through a two-dimensional channel with sharp corners at $x = 0$. The solution includes, as a special case, the flow round an isolated corner, and thus Clarke's solution for the latter problem which will apply up to the first reflected characteristic from the opposite corner is obtained directly. The solution for a general point in the flow field in the case of the isolated corner is compared with that obtained from the analysis of Morrison[16]. Approximate solutions are also obtained for most practical values of the ratio of the frozen and equilibrium speeds of sound.

At the suggestion of Professor N. H. Johannesen of Manchester University, the method is also applied to the case of a two-dimensional jet expanding into a uniform pressure field, and exact and approximate solutions are obtained.

2. *The Differential Equation, Boundary Conditions, and Solution by the Laplace Transform Method.*

Differential Equation.

It has been shown by Vincenti that for the two-dimensional flow of an inviscid, non-heat-conducting, non-radiating gas, relaxing in one mode, when perturbations from an undisturbed uniform supersonic flow and deviations from equilibrium are both small, a perturbation velocity potential $\phi(x,y)$ can be defined by

$$u' = \frac{\partial \phi}{\partial x}, v' = \frac{\partial \phi}{\partial y}$$

where

$$v = (U_\infty, 0) + (u', v')$$

which satisfies the linearized differential equation

$$K(B_f^2 \phi_{xx} - \phi_{yy})_x + B_e^2 \phi_{xx} - \phi_{yy} = 0 \tag{2.1}$$

K is a parameter proportional to the 'relaxation length', τU_∞, and B_e, B_f are the equilibrium and frozen Prandtl-Glauert factors respectively. τ, the relaxation time, is assumed to be constant and the rate equation is

$$\tau \frac{D}{Dt} e_i(T_i) = e_i(T_a) - e_i(T_i)$$

where $e_i(T_i)$ is the internal energy of the inert (relaxing) mode specified by the temperature T_i, and $e_i(T_a)$ is the energy of the active modes specified by the translational temperature T_a.

In 'equilibrium flow', when the relaxation processes are infinitely fast and all modes reach equilibrium instantaneously, $\tau \to 0$, $K \to 0$, and equation (2.1) reduces to the Prandtl-Glauert equation

$$B_e^2 \phi_{xx} - \phi_{yy} = 0$$

At the other extreme, when the relaxation process takes a very long time $\tau \rightarrow \infty$, $K \rightarrow \infty$, and equation (1.1) becomes

$$(B_f^2 \phi_{xx} - \phi_{yy})_x = 0$$

i.e.

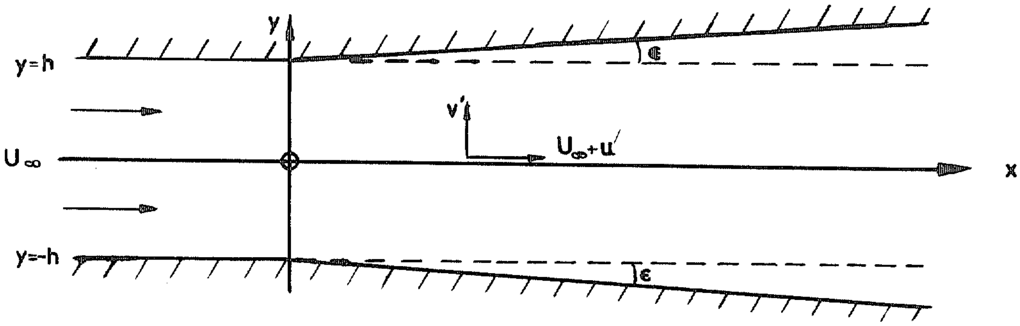
$$B_f^2 \phi_{xx} - \phi_{yy} = f(y).$$

But the equation must hold for all x, including the region of undisturbed flow, hence $f(y) = o$, and the equation reduces to the Prandtl-Glauert equation

$$B_f^2 \phi_{xx} - \phi_{yy} = 0.$$

This other limit of isentropic flow is known as 'frozen flow'

Boundary Conditions.



It is assumed that the flow is uniform and in equilibrium upstream of the station $x = 0$, and the boundary conditions are therefore

$$\phi, \phi_x, \phi_{xx} = 0, \; x \leqslant 0$$

$$v' = \frac{\partial \phi}{\partial y} = 0, \; y = 0, \text{ by symmetry, and}$$

$$\frac{v'}{U_\infty + u'} = \frac{dy}{dx} \text{ on } y = \pm(h + \varepsilon x), \text{ or with sufficient accuracy}$$

$$\frac{1}{U_\infty} \frac{\partial \phi}{\partial y} \doteq \frac{dy}{dx} \text{ on } y \doteq \pm h.$$

where the walls of the channel are given by $y = \pm(h + \varepsilon x)$ and ε is a small parameter.

Solution by the Laplace Transform Method.

The Laplace transform $\bar{\phi}(y,p)$ of $\phi(x,y)$ is defined by

$$\bar{\phi}(y,p) = \int_0^\infty e^{-px} \phi(x,y) \, dx \, .$$

Equation (1.1) transforms to

$$\frac{d^2\bar{\phi}}{dy^2} - p^2 B_f^2 \left(\frac{Kp+a}{Kp+1}\right)\bar{\phi} = 0$$

where $a = B_e{}^2/B_f{}^2$ and is greater than unity.

This equation has the solution for $\bar{\phi}\,(y,p)$,

$$\bar{\phi}\,(y,p) = A(p)\, e^{p\sqrt{\frac{Kp+a}{Kp+1}}B_f y} + B(p)\, e^{-p\sqrt{\frac{Kp+a}{Kp+1}}B_f y}.$$

The boundary condition

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = 0$$

transforms to

$$\frac{d\bar{\phi}}{dy} = 0 \quad \text{on } y = 0$$

and this implies

$$A(p) = B(p)\ ;$$

while the condition

$$\frac{\partial \phi}{\partial y} = \pm U_\infty \varepsilon \quad \text{on } y = \pm h$$

transforms to

$$\frac{d\bar{\phi}}{dy} = U_\infty \frac{\varepsilon}{p} \quad \text{on } y = h$$

Thus

$$\phi(x,y) = \frac{U_\infty \varepsilon}{B_f} L^{-} \left\{ \frac{e^{px} \cosh p\sqrt{\dfrac{Kp+a}{Kp+1}}\,B_f y}{p^2 \sqrt{\dfrac{Kp+a}{Kp+1}} \sinh p\sqrt{\dfrac{Kp+a}{Kp+1}}\,B_f h} \right\},$$

where L^{-1} denotes the Laplace transform inversion operator, viz:

$$L^{-1}\left\{\bar{\phi}(y,p)\, e^{px}\right\} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{px}\, \bar{\phi}\,(y,p)\, dp\,.$$

In the linearized theory

$$c_p = -\frac{2u'}{U_\infty}, \text{ where } u' = \frac{\partial \phi}{\partial x}.$$

Hence the pressure coefficient is given by

$$-\frac{c_p B_f}{2\varepsilon} = L^{-1}\left\{\frac{e^{px}\cosh p\sqrt{\frac{Kp+a}{Kp+1}}B_f y}{p\sqrt{\frac{Kp+a}{Kp+1}}\sinh p\sqrt{\frac{Kp+a}{Kp+1}}B_f h}\right\}. \tag{2.2}$$

The above equation (2.2) reduces to the pressure coefficient transform for 'equilibrium flow' on putting $K = 0$, and for 'frozen flow' on putting $K = \infty$. The inversion for these cases is performed in Appendix I by a straight-forward contour integration giving

$$-\frac{c_p B}{2\varepsilon} = \frac{x}{Bh} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{Bh}\cos\frac{n\pi y}{h}$$

where $B = B_e$ or B_f.

The frozen and equilibrium pressure coefficients on the wall and on the axis are presented in Figures 1 and 2 for comparison with the relaxing case.

However, the same method cannot readily be applied to the inversion of (1.2). The integrand has a non-isolated essential singularity at $p = -1/K$, and although the correct answer was obtained by integrating round a 'dumb-bell' contour around $p = -1/K$ and $p = -a/K$, it could not be proved rigorously that the integrand remained well-behaved at all points of this contour; nor could an alternative suitable contour be found.

In the case of one wall, the pressure coefficient for the flow round a sharp corner is

$$-\frac{c_p B_f}{2\varepsilon} = L^{-1}\left\{\frac{e^{px}e^{-p\sqrt{\frac{Kp+a}{Kp+1}}B_f y}}{p\sqrt{\frac{Kp+a}{Kp+1}}}\right\} \text{ with } y \text{ now measured from the wall,} \tag{2.3}$$

and this has been solved by Clarke for Cp_w, the pressure coefficient on the wall ($y = 0$). The evaluation for a general point in the flow field can be obtained from an analysis of Morrison (as mentioned above) which is outlined in Appendix II.

In Section 3 below, an analysis by series expansion is given for the channel flow, which includes the corner flow as a special case; and in Section 3, an approximate method is presented.

3. *Exact Evaluation of the Pressure Coefficient.*

The pressure coefficient for the relaxing gas in the two dimensional channel is given by equation (2.2),

$$-\frac{c_p B_f}{2\varepsilon} = L^{-1}\left\{\frac{e^{px}\cosh p\sqrt{\frac{Kp+a}{Kp+1}}B_f y}{p\sqrt{\frac{Kp+a}{Kp+1}}\sinh p\sqrt{\frac{Kp+a}{Kp+1}}B_f h}\right\},$$

and this can be simplified on transforming to the normalized co-ordinates,

$$x' = x/K\,,\ y' = \frac{B_f y}{K}$$

to
$$-\frac{c_p B_f}{2\varepsilon} = L^{-1}\left\{\frac{e^{p'x'}\cosh p'\sqrt{\dfrac{p'+a}{p'+1}}\,y'}{p'\sqrt{\dfrac{p+a}{p+1}}\sinh p'\sqrt{\dfrac{p'+a}{p'+1}}\,h'}\right\},$$

where $p' = Kp.$

Let $\zeta = 1+p'$, then,

$$-\frac{c_p B_f}{2\varepsilon} = L^{-1}\left\{\frac{e^{-x'}\,e^{\zeta x'}\cosh(\zeta-1)\sqrt{\dfrac{\zeta+a-1}{\zeta}}\,y'}{(\zeta-1)\sqrt{\dfrac{\zeta+a-1}{\zeta}}\sinh(\zeta-1)\sqrt{\dfrac{\zeta+a-1}{\zeta}}\,h'}\right\}. \tag{3.1}$$

Write $r = a-1$, and noting that

$$\frac{1}{\sinh x} = \frac{2}{e^x}\sum_{n=o}^{\infty} e^{-2nx},$$

(3.1) becomes

$$-\frac{c_p B_f}{2\varepsilon} = L^{-1}\left\{\frac{e^{-x'}\,e^{\zeta x'}\left[e^{(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}y'}+e^{-(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}y'}\right]}{(\zeta-1)\sqrt{\dfrac{\zeta+r}{\zeta}}\;e^{(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}h'}}\sum_{n=o}^{\infty} e^{-2n(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}h'}\right\}$$

$$= \bar{c}_{p_1}+\bar{c}_{p_2},$$

where

$$\bar{c}_{p_1} = e^{-x'}L^{-1}\left\{\frac{e^{\zeta x'}}{\zeta-1}\sqrt{\frac{\zeta}{\zeta+r}}\sum_{n=o}^{\infty} e^{-(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}(h'-y'+2nh')}\right\},$$

$$\bar{c}_{p} = e^{-x'}L^{-1}\left\{\frac{e^{\zeta x'}}{\zeta-1}\sqrt{\frac{\zeta}{\zeta+r}}\sum_{n=o}^{\infty} e^{-(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}(h'+y'+2nh')}\right\}.$$

For simplicity of notation, let

$$y_n = h'-y'+2nh',$$

$$z_n = h'+y'+2nh'.$$

Then

$$\bar{c}_{p_1} = e^{-x'} L^{-1}\left\{ \frac{e^{\zeta x'}}{\zeta-1}\sqrt{\frac{\zeta}{\zeta+r}}\sum_{n=o}^{\infty} e^{-(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}y_n}\right\},$$

$$\bar{c}_{p_2} = e^{-x'} L^{-1}\left\{ \frac{e^{\zeta x'}}{\zeta-1}\sqrt{\frac{\zeta}{\zeta+r}}\sum_{n=o}^{\infty} e^{-(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}z_n}\right\}$$

and $$-\frac{c_p B_f}{2\varepsilon} = \bar{c}_{p_1}+\bar{c}_{p_2}$$

$$= \sum_{n=o}^{\infty} b_n + \sum_{n=o}^{\infty} c_n \quad \text{say,} \tag{3.2}$$

where $$b_n = e^{-x'} L^{-1}\left\{ \frac{e^{\zeta x'}}{\zeta-1}\sqrt{\frac{\zeta}{\zeta+r}} e^{-\sqrt{\zeta(\zeta+r)}\,y_n+\sqrt{\frac{\zeta+r}{\zeta}}y_n}\right\} \tag{3.3}$$

but

$$\sqrt{\zeta(\zeta+r)} = \sqrt{(\zeta+r/2)^2-r^2/4},$$

so that if $\lambda = \zeta+r/2$,

and $s = \sqrt{\lambda^2-r^2/4}$,

(3.3) becomes

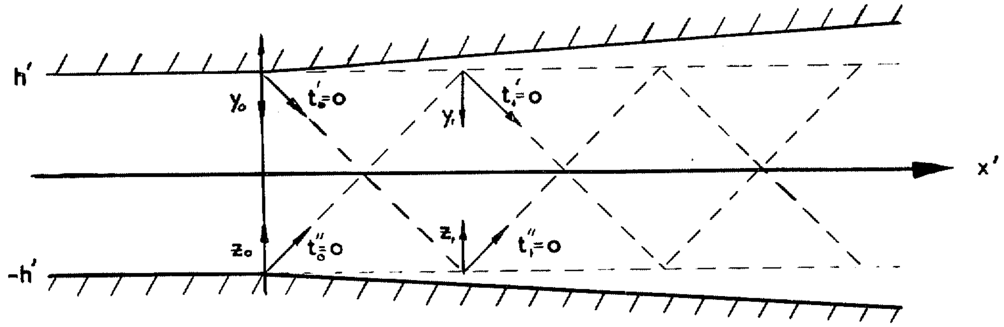
$$b_n = e^{-x'} L^{-1}\left\{ \frac{e^{\lambda x'}e^{-\frac{r}{2}x'}}{\lambda-1-r/2}\sqrt{\frac{\lambda-r/2}{\lambda+r/2}}\, e^{-sy_n}\, e^{\sqrt{\frac{\lambda+r/2}{\lambda-r/2}}y_n}\right\}$$

$$= e^{-\alpha x'} L^{-1}\left\{ \frac{e^{\lambda x'}}{\lambda-\alpha}\sqrt{\frac{\lambda-\beta}{\lambda+\beta}}\left[e^{(\lambda-s)y_n}-1\right] e^{-\lambda y_n}\, e^{\sqrt{\frac{\lambda+\beta}{\lambda-\beta}}y_n}\right\}$$

$$+e^{-\alpha x'} L^{-1}\left\{ \frac{e^{\lambda x'}}{\lambda-\alpha}\sqrt{\frac{\lambda-\beta}{\lambda+\beta}}\, e^{-\lambda y_n}\, e^{\sqrt{\frac{\lambda+\beta}{\lambda-\beta}}y_n}\right\} \tag{3.4}$$

where $\alpha = 1+r/2,\ \beta = r/2.$

Define $t'_n = x'-y_n,\ t''_n = x'-z_n$, and let the right hand side of $\bar{c}_{p_1}$ transform to semi-characteristic co-ordinates from the 'top' corner and for each n, along the $(n+1)^{th}$ characteristic parallel to the first; and similarly for $\bar{c}_{p_2}$:



Let $$L_1(t'_n, y_n) = L^{-1}\left\{ \frac{e^{t'_n \lambda}}{\lambda - \alpha} \sqrt{\frac{\lambda-\beta}{\lambda+\beta}}\, e^{\sqrt{\frac{\lambda+\beta}{\lambda-\beta}} y_n} \right\}$$

$$L_2(t'_n, y_n) = L^{-1}\left\{ e^{t'_n \lambda} \left[e^{(\lambda - s) y_{n-1}} \right] \right\}$$

So that if L_1 and L_2 are known, b_n is given by the convolution formula from equation (3.4), or

$$b_n = e^{-\alpha x'} L_1(t'_n, y_n) + e^{-\alpha x'} \int_0^{t'_n} L_2(\tau, y_n)\, L_1(t'_n - \tau, y_n)\, d\tau \tag{3.5}$$

and similarly for c_n.

The inversion for $L_2(t'_n, y_n)$ is given in Reference 17, or

$$L_2(t'_n, y_n) = \frac{r y_n}{2} \frac{\mathrm{I}_1\left(\beta\sqrt{(t'_n + y_n)^2 - y_n^{\,2}}\right)}{\sqrt{(t'_n + y_n)^2 - y_n^{\,2}}}$$

where I_1 is a modified Bessel Function of the first kind.

Therefore

$$b_n = e^{-\alpha x'} \left[L_1(t'_n, y_n) + \frac{r y_n}{2} \int_0^{t'_n} L(t'_n - \tau, y_n) \frac{\mathrm{I}_1\left(\beta\sqrt{(\tau + y_n)^2 - y_n^{\,2}}\right)}{(\tau + y_n)^2 - y_n^{\,2}}\, d\tau \right]$$

$$= e^{-\alpha x'} \left[L_1(t'_n, y_n) + \frac{r y_n}{2} \int_{y_n}^{x'} L_1(x' - \tau, y_n) \frac{\mathrm{I}_1\left(\beta\sqrt{\tau^2 - y_n^{\,2}}\right)}{\sqrt{\tau^2 - y_n^{\,2}}}\, d\tau \right] . \tag{3.6}$$

Now when $n = 0$, corresponding to the flow field between the corner and the first characteristic from the opposite corner, it is found from (3.6) that the solution in this case is similar to that found by Morrison (*see* Appendix II). However, the solution on the wall $y = h$, corresponding to $y_0 = 0$, found from (3.6) is

$$e^{-\alpha x'} L_1(t'_0, 0) = e^{-\alpha x'} \int_{L'} \frac{e^{t'_0 \lambda}}{\lambda - \alpha} \sqrt{\frac{\lambda - \beta}{\lambda + \beta}}\, d\lambda,$$

since the second term vanishes, and from Reference 17,

$$L_1(t'_0, 0) = e^{\alpha x'} \left[e^{-\alpha x'} \mathrm{I}_0(\beta x') + \int_0^{x'} e^{-\alpha\mu} \mathrm{I}_0(\beta\mu)\, d\mu \right] H(t'_0),$$

where I_0 is the modified Bessel function of the first kind. Hence Clarke's result,

$$-\frac{c_{pw} B_f}{2\varepsilon} = e^{-\alpha x'} \mathrm{I}_0(\beta x') + \int_0^{x'} e^{-\alpha\mu} \mathrm{I}_0(\beta\mu)\, d\mu, \tag{3.7}$$

is recovered directly.

For duct flow it is seen from (3.6) that terms corresponding to the double system of reflected characteristics for the frozen flow are added to the solution when $n = 0$. In other words the duct flow-solution is constructed from a series of isolated corner-flow solutions. It remains to evaluate L_1 (t'_n, y_n). This requires some manipulation to reduce it to a standard form. The inversions are obtained in Reference 17 and the evaluation is performed in Appendix III giving

$$L_1(t'_n, y_n) = e^{\alpha t'_n} \sum_{m=o}^{\infty} \frac{y_n^m}{m!} P_m(t'_n) \tag{3.8}$$

where $P_m(t)$ is a function defined in Appendix III which satisfies the recurrence relation

$$P_m(t) = P_{m-2}(t) + re^{-t} \int_0^t e^{\tau} P_{m-2}(\tau)\, d\tau \tag{3.9}$$

and the infinite series $\sum_{m=o}^{\infty} \frac{y_n^m}{m!} P_m(t'_n)$ is absolutely and uniformly convergent for all y_n and t_n since

$|P_m(t'_n)| \leqslant a^{\frac{m-1}{2}}$ for all m and t'_n and $\sum_{m=o}^{\infty} \frac{1}{\sqrt{a}} \frac{(y\sqrt{a})^m}{m!}$ is absolutely convergent for all y_n.

The exact expression for the pressure coefficient is therefore from (3.2), (3.6) and (3.8),

$$-\frac{c_p B_f}{2} = \sum_{n=o}^{\infty} \left[e^{-\alpha x'} e^{\alpha t'_n} \sum_{m=o}^{\infty} \frac{y_n^m}{m!} P_m(t'_n) + \right.$$

$$+ e^{-\alpha x'} \int_{y^n}^{x'} \beta y_n e^{\alpha(x'-\tau)} \sum_{m=o}^{\infty} \frac{y_n^m}{m!} P_m(x'-\tau) \frac{\mathrm{I}_1(\beta\sqrt{\tau^2 - y_n^2})}{\sqrt{(\tau^2 - y_n^2)}}\, d\tau +$$

$$+ e^{-\alpha x'} e^{\alpha t''_n} \sum_{m=o}^{\infty} \frac{z_n^m}{m!} P_m(t''_n) +$$

$$\left. + e^{-\alpha x'} \int_{z_n}^{x'} \beta z_n e^{\alpha(x'-\tau)} \sum_{m=o}^{\infty} \frac{z_n^m}{m!} P_m(x'-\tau) \frac{\mathrm{I}_1(\beta\sqrt{\tau^2 - z_n})}{\sqrt{(\tau^2 - z_n^2)}}\, d\tau \right].$$

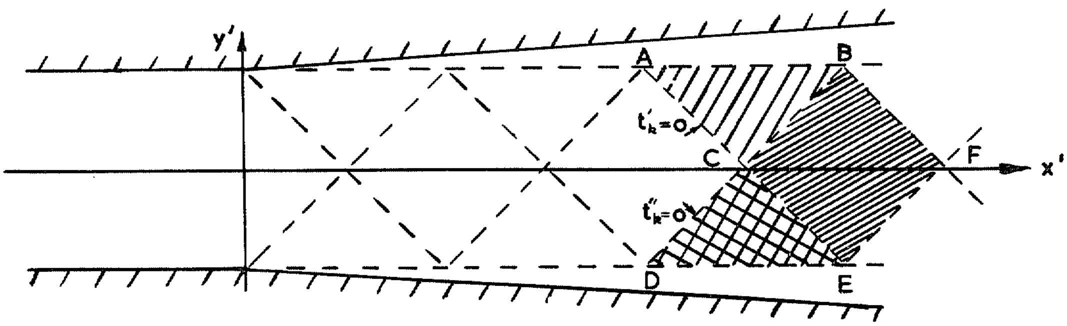
Because $P_m(t) = 0$ for $t < 0$ for all m, $b_n = 0$ for all $n \geqslant k+1$ whenever $t'_{k+1} < 0$ and $c_n = 0$ for all $n \geqslant k'+1$ whenever $t''_{k'+1} < 0$. Thus the first two terms are summed from $n = o$ to $n = k$ only, and the last two from $n = o$ to $n = k'$ only, in the region where $t'_k \geqslant 0$, $t''_{k'} \geqslant 0$ so that

$$-\frac{c_p B_f}{2\varepsilon} = \sum_{n=o}^{k} \left[e^{-\alpha y_n} \sum_{m=o}^{\infty} \frac{y_n^m}{m!} P_m(t'_n) + \right.$$

$$\left. + \beta y_n \int_{y_n}^{x'} e^{-\alpha\tau} \sum_{m=o}^{\infty} \frac{y_n^m}{m!} P_m(x'-\tau) \frac{I_1(\beta\sqrt{\tau^2 - y_n^2})}{\sqrt{\tau^2 - y_n^2}} d\tau \right] +$$

$$+ \sum_{n=o}^{k'} \left[e^{-\alpha z_n} \sum_{m=o}^{\infty} \frac{z_n^m}{m!} P_m(t''_n) + \beta z_n \int_{z_n}^{x'} e^{-\alpha\tau} \sum_{m=o}^{\infty} \frac{z_n^m}{m!} P_m(x'-\tau) \frac{I_1(\beta\sqrt{\tau^2 - z_n^2})}{\sqrt{\tau^2 - z_n^2}} d\tau \right] .$$

(3.10)



It should be noted that $k' = k$, $k+1$, or $k-1$, so that in the region ABC in the diagram above the first sum is from o to k, the second from o to $k-1$. In the region CDE the first sum is from $n = o$ to $k-1$, and the second from $n = o$ to k. In the region BCEF, both sums are from $n = o$ to $n = k$.

The pressure coefficient on the wall and on the axis is shown in Figures 1 and 2 respectively, together with the frozen and equilibrium values. In computing these pressure coefficients, the half-width of the channel was taken to be $h' = 0{\cdot}5$, $a = 1{\cdot}5$, and it was found that ten terms of the series $\sum_{m=o}^{\infty} \frac{y_n^m}{m!} P_m(t)$ were sufficient to obtain a value accurate to two decimal places for values of y_n (or z_n) up to 3.

The pressure coefficient at a general point in the flow round a corner is, from (3.10)

$$-\frac{c_p B_f}{2\varepsilon} = e^{-\alpha y_0} \sum_{m=o}^{\infty} \frac{y_0^m}{m!} P_m(t'o) + \beta y_0 \int_{y_0}^{x'} e^{-\alpha\tau} \sum_{m=o}^{\infty} \frac{y_0^m}{m!} P_m(x'-\tau) \frac{I_1(\beta\sqrt{\tau^2 - y_0^2})}{\sqrt{\tau^2 - y_0^2}} d\tau \,, \qquad (3.11)$$

which is equivalent to

$$-\frac{c_p B_f}{2\varepsilon} = H(t_0')\left\{e^{-\beta x'} + \beta e^{-\alpha x'}\int_{y_0}^{x'} \tau e^{-\tau}\frac{I_1(\beta\sqrt{x'^2-\tau^2})}{\sqrt{x'^2-\tau^2}}\,d\tau\right.$$

$$\left.+\beta\sqrt{r}\,e^{-\alpha x'}\int_{y_0}^{x'} e^{-\tau}\int_0^\infty e^{-\beta\mu}\frac{(\mu+\tau)\tau^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}\,I_1(2\sqrt{r\tau\mu}\,\frac{I_1(\beta\sqrt{(x'+\mu)^2-(\mu+\tau)^2})}{\sqrt{(x'+\mu)^2-(\mu+\tau)^2}}\,d\mu d\tau\right\}$$

as obtained by Morrison (*see* Appendix II).

Both expressions show that $c_p = 0$ for $y_0 < x'$ since the disturbance from the corner is confined to the region downstream of the Mach wave associated with the frozen Mach number M_f, a result which is well known.

For $y = 0$, Morrison's method gives,

$$-\frac{c_p B_f}{2\varepsilon} = e^{-\beta x'} + \beta e^{-\alpha x'}\int_0^{x'} \tau e^{-\tau}\frac{I_1(\beta\sqrt{x^{12}-\tau^2})}{\sqrt{x^{12}-\tau^2}}\,d\tau +$$

$$+\beta e^{-\alpha x'}\int_0^{x'} e^{-\tau}\int_0^\infty e^{-\beta\mu}\frac{(\mu+\tau)\sqrt{r\tau}}{\sqrt{\mu}}\,I_1(2\sqrt{r\tau\mu})\frac{I_1(\beta\sqrt{(x'+\mu)^2-(\tau+\mu)^2})}{\sqrt{(x'+\mu)^2-(\tau+\mu)^2}}\,d\mu\,d\tau,$$

and although this is equivalent to (3.7) the equality can only be established with some difficulty. (3.11) above has the merit that the simple form for c_{p_w} found by Clarke, equation (3.7), is obtained directly in place of the rather intractable relation above.

4. *Approximate Evaluation of Pressure Coefficient.*

The solution for small differences between the frozen and equilibrium sound speeds.

It is possible to obtain a very much simpler expression than (3.10) for the pressure coefficient if it is assumed that $r = a - 1 = (B_e^2/\ \ _f\ - 1 \ll 1$.

Equation (3.2) for the pressure coefficient is

$$-\frac{c_p B_f}{2\varepsilon} = L^{-1}\left\{\frac{e^{\zeta x'}}{\zeta - \Gamma}\sqrt{\frac{\zeta}{\zeta+1}}\sum_{n=o}^{\infty}\left[e^{-(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}\cdot y_n} + e^{-(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}\,z_n}\right]\right\} e^{-x'}.$$

Now suppose $r \ll 1$, so that

$$\sqrt{\frac{\zeta+r}{\zeta}} = (1+r/\zeta)^{\frac{1}{2}} = 1 + \frac{r}{2\zeta} + 0(r^2).$$

The equation for c_p when terms $0(r^2)$ are neglected then becomes, remembering that $\beta = \frac{r}{2}$,

$$-\frac{c_p B_f}{2\varepsilon} = L^{-1}\left\{\sum_{n=o}^{\infty}\left[e^{\zeta t_n' - t_n' - \beta y_n + \beta \frac{y_n}{\zeta}} + e^{\zeta t_n'' - t_n'' - \beta z_n + \beta \frac{z_n}{\zeta}}\right]\left(\frac{\zeta}{\zeta+\beta}\right)\left(\frac{1}{\zeta-1}\right)\right\} \tag{4.1}$$

$$= \sum_{n=o}^{\infty} (b_n + c_n) \text{ as before, where now}$$

$$b_n = e^{-t_n' - \beta y_n}\left[L^{-1}\left\{\frac{e^{\zeta t_n'}(e^{\beta y_n/\zeta} - 1)}{(\zeta-1)}\right\} + L^{-1}\left\{\frac{e^{\zeta t_n'}}{\zeta-1}\right\} - \beta L^{-1}\left\{\frac{e^{\zeta t_n'}(e^{\beta y_n/\zeta}\zeta^{-1})}{(\zeta-1)}\right\}\right].$$

Equation (4.1) retains all the important features of equation (3.1). It possesses an isolated essential singularity at $\zeta = 0$ and simple poles at $\zeta = \beta$, $\zeta = 1$.

The following results are obtained from Reference 17, with the aid of the convolution formula

$$\int_{L'} \frac{e^{px}(e^{y/p} - 1)}{p-1}\,dp = \int_0^x y^{\frac{1}{2}} t^{\frac{1}{2}} \mathrm{I}_1(2y^{\frac{1}{2}} t^{\frac{1}{2}})\, e^{x-t}\,dt\,,$$

$$\int_{L'} \frac{e^{px}.\, e^{y/p}\, p^{-1}}{p-1}\,dp = \int_0^x \mathrm{I}_0(2y^{\frac{1}{2}} t^{\frac{1}{2}})\, e^{x-t}\,dt$$

and

$$\int_{L'} \frac{e^{px}}{p-1}\,dp = e^x\,.$$

Therefore, using the above results,

$$b_n = e^{-t_n'} e^{-\beta y_n}\left\{\int_0^{t_n'} (\beta y_n)^{\frac{1}{2}} \mu^{\frac{1}{2}} \mathrm{I}_1(2\sqrt{\beta\, y_n \mu})\, e^{t_n' - \mu}\,d\mu + e^{t_n'} - \right.$$

$$\left. -\beta \int_0^{t_n'} \mathrm{I}_0(2\sqrt{\beta\, y_n \mu})\, e^{t_n' - \mu}\,d\mu\right\}$$

$$= e^{-\beta y_n}\left\{\int_0^{t_n'} \frac{d}{d\mu}\mathrm{I}_0(\sqrt{2ry_n\mu})e^{-\mu}\,d\mu + 1 - \beta\int_0^{t_n'} \mathrm{I}_0(\sqrt{ry_n\mu})\, e^{-\mu}\,d\mu\right\}$$

$$= e^{-\beta y_n}\mathrm{I}_0(\sqrt{2ry_n t_n'})\, e^{-t_n'} + (1-\beta)\, e^{-\beta y_n}\int_0^{t_n'} \mathrm{I}_0(\sqrt{2ry_n\mu})\, e^{-\mu}\,d\mu$$

after integration by parts.

Therefore, substituting in equation (4.1),

$$-\frac{c_p B_f}{2\varepsilon} = \sum_{n=o}^{k} \left\{ e^{-\beta y_n} \mathrm{I}_0(\sqrt{2ry_n t'_n})\, e^{-t'_n} + (1-\beta)\, e^{-\beta y_n} \int_0^{-t'_n} \mathrm{I}_0(\sqrt{2ry_n \mu})\, e^{-\mu}\, d\mu \right\}$$

$$+\sum_{n=o}^{k'} \left\{ e^{-\beta z_n} \mathrm{I}_0(\sqrt{2rz_n t''_n})\, e^{-t''_n} + (1-\beta)\, e^{-\beta z_n} \int_0^{t''_n} \mathrm{I}_0(\sqrt{2rz_n \mu})\, e^{-\mu}\, d\mu \right\} \tag{4.2}$$

where the summation is over a finite number of terms for the same reasons as in Section 3.

This approximation to the pressure coefficient gives excellent agreement with the exact result, and remains a good approximation even for values of r as high as 0·5 as shown on Figures 1 and 2. It is particularly good up to the first reflected characteristic and therefore for the corner flow problem (4.2) reduces to

$$-\frac{c_p B_f}{2\varepsilon} = e^{-\beta y_0}\, e^{-t'_0}\, \mathrm{I}_0(\sqrt{2ry_0 t'_0}) + (1-\beta) e^{-\beta y_0} \int_0^{t'_0} \mathrm{I}_0(\sqrt{2ry_0 \mu})\, e^{-\mu}\, d\mu \tag{4.2a}$$

and when $y_0 = 0$, (on the wall $y = h$)

$$-\frac{c_{p_w} B_f}{2\varepsilon} = (1-\beta) + \beta e^{-x'} \tag{4.2b}$$

whereas from (3.7), with $r \ll 1$,

$$-\frac{c_{p_w} B_f}{2\varepsilon} = \frac{1}{\alpha}(1 + \beta e^{-\alpha x'})$$

but since $\alpha = 1+\beta$, it is easily verified that these are the same to $0(r^2)$, provided $x' \ll 0(1/\beta)$.

The approximation (4.2b) for the pressure coefficient on the wall is compared with the exact expression (3.7) for various values of r in Figure 3.

Figure 4 shows the variation of the pressure coefficient with t for different values of y (at different distances from the corner) for $a = 1{\cdot}1$, calculated from equation (4.2a). Since for large values of t,

$$P_m(\infty) = (1+r)^{\frac{m-1}{2}}$$

and

$$\beta y_0 \int_{y_0}^{\infty} \frac{\mathrm{I}_1(\beta\sqrt{\tau^2 - y_0^2})}{\sqrt{\tau^2 - y_0^2}} e^{-\alpha\tau}\, d\tau = e^{-y_0\sqrt{1+r}} - e^{-\alpha y_0},$$

it follows that the exact expression (3.11) gives,

$$-\frac{c_p B_f}{2\varepsilon} \to \frac{e^{-\alpha y_0}}{\sqrt{1+r}} e^{\sqrt{1+r}\,y_0} + \frac{1}{\sqrt{1+r}}(1-e^{-(1+r/2-\sqrt{1+r})\,y_0})$$

$$= \frac{1}{\sqrt{1+r}}(e^{-(1+r/2-\sqrt{1+r})\,y_0} + 1 - e^{-(1+r/2-\sqrt{1+r})\,y_0}) \quad \text{as } t \to \infty,$$

showing that, for all y,

$$-\frac{c_p B_f}{2\varepsilon} \to \frac{B_f}{B_e}.$$

Thus the flow reaches equilibrium far downstream of the initial frozen Mach wave, corresponding to large values of t for a fixed y, and this is demonstrated in Figure 4 where the pressure coefficient approaches $(1-\beta)$, for all y. $(1-\beta)$ is the first order approximation to $\dfrac{1}{\sqrt{1+r}} = \dfrac{B_f}{B_e}$.

It is shown below that for small values of t the disturbance decays exponentially as $e^{-\beta y_0}$ away from the corner. The approximation to c_p given by (4.2) is equivalent to replacing 2.1) by

$$\left(\alpha+\frac{\partial}{\partial x'}\right)^2 \frac{\partial^2 \phi}{\partial x'^2} - \left(1+\frac{\partial}{\partial x'}\right)\frac{\partial^2 \phi}{\partial y'^2} = 0 \tag{4.3}$$

in the normalised co-ordinates. But if (1.1) (in the normalised co-ordinates) is differentiated with respect to x', it becomes

$$\frac{\partial^4 \phi}{\partial x'^4} + (1+r)\frac{\partial^3 \phi}{\partial x'^3} - \frac{\partial^3 \phi}{\partial x'^2 \partial y'^2} - \frac{\partial^3 \phi}{\partial x' \partial y'^2} = 0$$

and adding a term $\dfrac{r^4}{4}\dfrac{\partial^2 \phi}{\partial x'^2}$ to both sides and rearranging gives

$$\left(\alpha+\frac{\partial}{\partial x'}\right)^2 \frac{\partial^2 \phi}{\partial x'^2} - \left(1+\frac{\partial}{\partial x'}\right)\frac{\partial^2 \phi}{\partial y'^2} = \frac{r^2}{4}\frac{\partial^2 \phi}{\partial x'^2} \tag{4.4}$$

so that (3.3) is a form of the exact linearised equation when $r \ll 1$.

It cannot be assumed that inclusion of the additional term in (4.3) modifies the rate equation. However the term involving $r^2/4$ in (4.4) is a diffusion term and it is therefore this weak diffusion effect which is neglected in the approximate equation (4.3). Because this diffusion term is of small order, solutions to this simplified equation give reasonable agreement with these obtained from the original equation.

Solution for small values of t.

Certain useful results can be obtained from an investigation of the flow in the vicinity of the leading characteristics, i.e. for small values of t' or t''.

From Section 3, since $P_m(0) = 1$ for all m,

$$L_1(t'_n, y_n)\, e^{-\alpha t'_n} \to e^{y_n} \text{ as } t'_n \to 0,$$

so, for small values of t'_n,

$$e^{-\alpha x'} L_1(t'_n, y_n) = e^{-\alpha y_n} \left[e^{-\alpha t'_n} \mathrm{I}_0(\beta t'_n) + \int_0^{t'_n} e^{-\alpha\mu} \mathrm{I}_0(\beta\mu)\, d\mu \right] + e^{-\beta y_n} - e^{-\alpha y_n} \tag{4.4}$$

as $t'_n \to 0$

Hence from (3.6) together with (4.5),

$$b_n = e^{-\alpha x'} \mathrm{I}_0(\beta t'_n) + e^{-\alpha y_n} \int_0^{t'_n} e^{-\alpha\mu} \mathrm{I}_0(\beta\mu)\, d\mu + e^{-\beta y_n} - e^{-\alpha y_n} +$$

$$+ \beta y_n e^{-\alpha x'} \int_{y_n}^{x'} \frac{\mathrm{I}_1(\beta\sqrt{\tau^2 - y_n^2})}{\sqrt{\tau^2 - y_n^2}} \left\{ \mathrm{I}_0(\beta(x' - \tau)) + e^{\alpha(x'-\tau)} \int_0^{x'-\tau} \mathrm{I}_0(\beta\mu)\, d\mu + \right.$$

$$\left. + e^{y_n} e^{\alpha(x'-\tau)} - e^{\alpha(x'-\tau)} \right\} d\tau.$$

But

$$\frac{\mathrm{I}_1(\beta\sqrt{\tau^2 - y_n^2})}{\sqrt{\tau^2 - y_n^2}} = \frac{1}{\beta\tau} \frac{d}{d\tau} \mathrm{I}_0(\beta\sqrt{\tau^2 - y_n^2}).$$

Therefore

$$b_n = e^{-\beta y_n} + e^{-\alpha y_n} \left\{ e^{-t'_n} \mathrm{I}_0(\beta t'_n) + \int_0^{t'_n} e^{-\alpha\mu} \mathrm{I}_0(\beta\mu)\, d\mu - 1 \right\} +$$

$$+ y_n e^{-\alpha y_n} \int_0^{t'_n} \frac{e^{-\alpha\tau}}{\tau + y_n} \frac{d}{d\tau} \left[\mathrm{I}_0(\beta\sqrt{(\tau + y_n)^2 - y_n^2}) \right] \left\{ e^{-\alpha(t'_n - \tau)} \mathrm{I}_0(\beta(t'_n - \tau)) + \right.$$

$$\left. + \int_0^{t'_n - \tau} e^{-\alpha\mu} \mathrm{I}_0(\beta\mu)\, d\mu + e^{y_n} - 1 \right\} d\tau.$$

Therefore as $t'_n \to 0$,

$$b_n = e^{-\beta y_n} + y_n e^{-\alpha y_n + y_n} \int_0^{t'_n} \frac{e^{-\alpha\tau}}{\tau + y_n} \frac{d}{d\tau} \left[\mathrm{I}_0(\beta\sqrt{\tau^2 + 2\tau y_n}) \right] d\tau$$

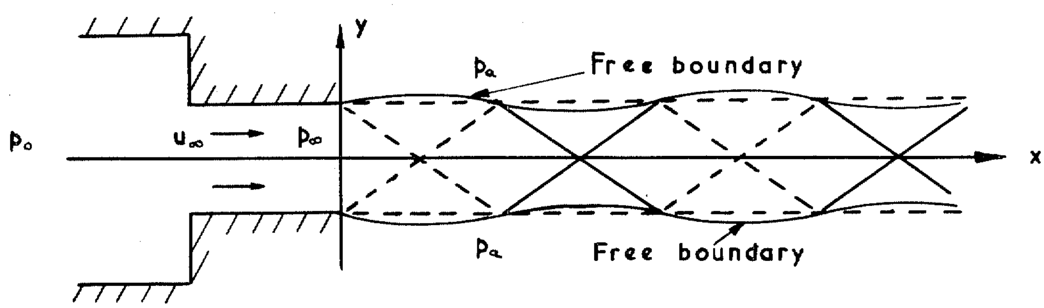
$$= e^{-\beta y_n} + e^{-\beta y_n} \int_0^{t'_n} e^{-\alpha\tau} \frac{d}{d\tau} \left[\mathrm{I}_0(\beta\sqrt{2\tau y_n}) \right] d\tau$$

$$= e^{-\beta y_n}(1 + 0(t)) . \qquad \text{Similarly } c_n \to e^{-\beta z_n},$$

showing that along leading characteristics, the disturbance decays exponentially with distance from the appropriate corner, as found previously by Clarke and others. In addition the reflected disturbances decay exponentially with distance from their points of reflection. As already noted above, it is seen that the nozzle flow solution is obtained from a series of isolated corner solutions. The results of numerical solution by Der[19] justify this approximation.

5. *Jet Expanding into a Uniform Pressure Field.*

Following a suggestion by Professor N. H. Johannesen of the University of Manchester, the case of a jet expanding into a uniform pressure field is also considered



The free boundary, for a linearized problem, is assumed to be at $y = \pm h$, and the boundary condition is

$$p = p_a$$

where p_a is the external pressure.

If p_0 is the stagnation pressure, p_∞ is the static pressure at the nozzle exit, and c_p is defined as

$$c_p = \frac{p - p_\infty}{p_0 - p_\infty},$$

$$c_{p_a} = \frac{p_a - p_\infty}{p_0 - p_\infty},$$

then the boundary condition on the free boundary is

$$c_p = c_{p_a} .$$

The case considered is when $p_\infty > p_a$. Thus the first Mach wave is one of expansion, followed by one of compression and so on, but the analysis could equally apply to the case when $p_\infty < p_a$, when the first wave is a compression, provided that the pressure difference $p_\infty - p_a$ is small enough to prevent the formation of shocks beyond the mouth of the nozzle. This condition must apply anyway to keep the problem within the scope of a linearized analysis.

Applying the method of the Laplace transform as in Section 2, the equation for $\bar{\phi}(y,p)$ is

gives

$$\bar{\phi}(y,p) = A(p)e^{p\sqrt{\frac{Kp+a}{Kp+1}}B_f y} + B(p)\, e^{-p\sqrt{\frac{Kp+a}{Kp+1}}B_f y}$$

The boundary condition $v' = \dfrac{\partial \phi}{\partial y} = 0$ on $y = 0$ still applies and gives

$$A(p) = B(p)$$

or

$$\bar{\phi}(y,p) = 2A(p)\cosh p\sqrt{\frac{Kp+a}{Kp+1}}\,B_f y.$$

The boundary condition $c_p = c_{p_a}$ on $y = \pm h$ gives

$$\phi_x = -\frac{U_\infty}{2} c_{p_a} \text{ on } y = \pm h$$

which transforms to

$$p\,\bar{\phi}(y,p) = -\frac{U_\infty}{2}\frac{c_{p_a}}{p} \text{ on } y = \pm h$$

giving

$$\bar{\phi}(y,p) = -\frac{U_\infty}{2} c_{p_a} \frac{\cosh p\sqrt{\dfrac{Kp+a}{Kp+1}}\,B_f y}{p^2 \cosh p\sqrt{\dfrac{Kp+a}{Kp+1}}\,B_f h}.$$

Therefore

$$\phi(x,y) = \frac{U_\infty c_{p_a}}{2} L^{-1}\left\{\frac{e^{px}\cosh p\sqrt{\dfrac{Kp+a}{Kp+1}}\,B_f y}{p^2\cosh p\sqrt{\dfrac{Kp+a}{Kp+1}}\,B_f h}\right\}$$

in the notation used above, or

$$\frac{c_p}{c_{p_a}} = L^{-1}\left\{\frac{e^{px}\cosh p\sqrt{\dfrac{Kp+a}{Kp+1}}\,B_f y}{p\cosh p\sqrt{\dfrac{Kp+a}{Kp+1}}\,B_f h}\right\}. \tag{5.1}$$

The frozen and equilibrium solutions are obtained by putting $K = \infty$, and $K = 0$ respectively, giving

$$\frac{c_p}{c_{p_a}} = L^{-1}\left\{\frac{e^{px}\cosh(p\,By)}{p\cosh(p\,Bh)}\right\}, (B = B_f \text{ or } B_e)$$

which is evaluated in a similar way as in Appendix I, giving

$$\frac{c_p}{c_{p_a}} = 1-\frac{2}{\pi}\sum_{n=o}^{\infty}\frac{(-1)^n}{n+\frac{1}{2}}\cos\frac{(n+\frac{1}{2})\pi x}{Bh}\cos\frac{(n+\frac{1}{2})\pi y}{h}.$$

On the axis of the jet, making use of the well-known result (Reference 18),

$$\cos\theta-\frac{1}{3}\cos 3\theta+\frac{1}{5}\cos 5\theta \ldots . = \frac{\pi}{4}, -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$\frac{c_p}{c_{p_a}} = 0 \quad -Bh < x < Bh$$

$$= 2 \quad Bh < x < 3Bh$$

etc.

Equation 5.1 is evaluated in exactly the same way as 2.2. On transforming to normalized co-ordinates $x' = x/K, y' = \frac{B_f y}{K}$, and putting $\zeta = 1+p$,

$$\frac{c_p}{c_{p_a}} = L^{-1}\left\{\frac{e^{\zeta x}e^{-x'}\cosh(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}y'}{(\zeta-1)\cosh(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}h'}\right\}$$

Since

$$\frac{1}{\cosh x} = \frac{2}{e^x}\sum_{n=o}^{\infty}(-1)^n e^{-2nx},$$

$$\frac{c_p}{c_{p_a}} = \bar{c}_{p_1}+\bar{c}_{p_2}, \text{ where now}$$

$$\bar{c}_{p_1} = e^{-x'}L^{-1}\left\{\frac{e^{\zeta x'}}{\zeta-1}\sum_{n=o}^{\infty}(-1)^n e^{-(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}y_n}\right\},$$

$$\bar{c}_{p_2} = e^{-x'}L^{-1}\left\{\frac{e^{\zeta x'}}{\zeta-1}\sum_{n=o}^{\infty}(-1)^n e^{-(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}z_n}\right\}$$

$$= \sum_{n=o}^{\infty}(-1)^n b_n+\sum_{n=o}^{\infty}(-1)^n c_n \text{ say}$$

where

$$b_n = e^{-x'} L^{-1}\left\{\frac{e^{\zeta x'}}{\zeta-1} \; e^{\sqrt{\zeta(\zeta+r)}y_n} e^{\sqrt{\frac{\zeta+r}{\zeta}}y_n}\right\}.$$

Let

$$L_1(t'_n, y_n) = L^{-1}\left\{\frac{e^{t'_n\lambda}}{\lambda-\alpha} e^{\sqrt{\frac{\lambda-\beta}{\lambda+\beta}}y_n}\right\},$$

$$L_2(t'_n, y_n) = L^{-1}\left\{e^{t_n\lambda}\left[e^{(\lambda-s)y_n}-1\right]\right\},$$

where $\lambda = \zeta + r/2,\; s = \sqrt{\lambda^2 - {r^2}/4}.$

Then,

$$L_2(t'_n, y_n) = \beta y_n \frac{I_1(\beta\sqrt{(t'_n+y_n)^2-y_n{}^2})}{\sqrt{(t'_n+y_n)^2-y_n^2}} \text{ as before,}$$

but now

$$L_1(t'_n, y_n) = e^{\beta t'_n} L^{-1}\left\{\frac{e^{\zeta t'_n}}{\zeta-1} e^{\sqrt{\frac{\zeta+r}{\zeta}}y_n}\right\}$$

$$= e^{\beta t'_n}\sum_{m=o}^{\infty}\frac{y_n^m}{m!}L^{-1}\left\{\frac{e^{\zeta t'_n}}{\zeta-1}\left(\frac{\zeta+r}{\zeta}\right)^{m/2}\right\}$$

$$= e^{-\alpha t'_n}\sum_{m=o}^{\infty}\frac{y_n^m}{m!}L^{-1}\left\{\frac{e^{\mu t'_n}}{\mu}\left(\frac{\mu+a}{\mu+1}\right)^{m/2}\right\}$$

$$= e^{-\alpha t'_n}\sum_{m=o}^{\infty}\frac{y_n^m}{m!}P_{m+1}(t'_n),$$

where the $P_m(t)$ are the functions defined in Appendix III. Hence the exact expression for the pressure coefficient in the jet is

$$\frac{c_p}{c_{p_a}} = \sum_{n=o}^{k}(-1)^n\left[e^{-\alpha y_n}\sum_{m=o}^{\infty}\frac{y_n^m}{m!}P_{m+1}(t'_n) + \right.$$

$$\left. +\beta y_n\int_{y_n}^{x'} e^{-\alpha\tau}\sum_{m=o}^{\infty}\frac{y_n^m}{m!}P_{m+1}(x'-\tau)\frac{I_1(\beta\sqrt{\tau^2-y_n^2})}{\sqrt{\tau^2\, y_n^2}}\,d\tau\right] +$$

$$+\sum_{n=o}^{k'}(-1)^n\left[e^{-\alpha z_n}\sum_{m=o}^{\infty}\frac{z_n^m}{m!}P_{m+1}(t''_n) + \right.$$

$$\left. +\beta z_n\int_{z_n}^{x'} e^{-\alpha\tau}\sum_{m=o}^{\infty}\frac{z_n^m}{m!}P_{m+1}(x'-\tau)\frac{I_1(\beta\sqrt{\tau^2-z_n^2})}{\sqrt{\tau^2\, z_n^2}}\,d\tau\right]. \qquad (5.2)$$

As in the channel flow, a much simpler expression than (5.2) is obtained if it is assumed that the relaxation parameter $r \ll 1$, so that

$$\sqrt{\frac{\zeta+r}{\zeta}} = 1+\beta/\zeta+0(r^2)$$

and

$$\frac{c_p}{c_{p_a}} = \int_L \frac{e^{\zeta x'}}{\zeta-1} \sum_{n=o}^{\infty} (-1)^n \left\{ e^{-(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}y_n} + e^{-(\zeta-1)\sqrt{\frac{\zeta+r}{\zeta}}z_n} \right\} d\zeta$$

becomes

$$\frac{c_p}{c_{p_a}} = \int_L \sum_{n=o}^{\infty} (-1)^n \left\{ e^{\zeta t'_n - t'_n - \beta y_n + \beta^{y_n}/\zeta} + e^{\zeta t''_n - t''_n - \beta z_n + \beta^{z_n}/\zeta} \right\} \frac{d\zeta}{\zeta-1}$$

$$= \sum_{n=o}^{\infty} (-1)^n (b_n+c_n)$$

as before, but now

$$b_n = e^{-t'_n-\beta y_n} \left\{ \int_L \frac{e^{\zeta t'_n}(e^{\beta y_n/\zeta}-1)}{\zeta-1} d\zeta + \int_L \frac{e^{\zeta t'_n}}{\zeta-1} d\zeta \right\}$$

$$= e^{-\beta y_n} \left\{ \int_0^{t'_n} \frac{d}{d\mu} \left[\mathrm{I}_0(\sqrt{2r y_n \mu}) \right] e^{-\mu} d\mu + 1 \right\} \quad \text{(Reference 17)}$$

$$= e^{-\beta y_n} \mathrm{I}_0(\sqrt{2r y_n t'_n})\, e^{-t'_n} + e^{-\beta y_n} \int_0^{t_n} \mathrm{I}_0(\sqrt{2r y_n \mu})\, e^{-\mu} d\mu$$

after integration by parts.

Hence

$$\frac{c_p}{c_{p_a}} = \sum_{n=o}^{k} (-1)^n e^{-\beta y_n} \left[\mathrm{I}_0(\sqrt{2r y_n t'_n})\, e^{-t'_n} + \int_0^{-t'_n} \mathrm{I}_0(\sqrt{2r y_n \mu})\, e^{-\mu} d\mu \right] +$$

$$+ \sum_{n=o}^{k'} (-1)^n e^{-\beta z_n} \left[\mathrm{I}_0(\sqrt{2r z_n t''_n})\, e^{-t''_n} + \int_0^{t'_n} \mathrm{I}_0(\sqrt{2r z_n \mu})\, e^{-\mu} d\mu \right] , \quad (5.3)$$

for sufficiently small values of r.

The exact solution (5.2) and the approximate one above (5.3) are presented in Figure 5 together with the frozen and equilibrium values of the pressure coefficient along the axis of the jet for a = 1·5, $h' = 0{\cdot}5$.

The summation over n in both (5.2) and (5.3) is over a finite number of terms as in the channel flow, and it can be verified that

$$\frac{c_p}{c_{p_a}} = 1 \text{ on } y' = \pm h',$$

all terms but the first cancelling out.

6. *Discussion.*

An exact linearized solution has been found for the flow of a relaxing gas in a two dimensional channel, and in a jet, assuming a linearized rate equation where the relaxation time, τ, is assumed to be constant along the channel. Of course it might be expected that τ will vary with temperature and pressure and will increase with increasing distance downstream in a diverging channel. 'Freezing-out' is therefore not apparent in a solution with a constant τ. This is verified in the one-dimensional analysis which is performed in Appendix V, and the resulting pressure coefficient on the axis is shown in Figure 2. This shows no evidence of 'freezing' found by Blythe who used a rate equation of the form

$$\frac{de_i}{dt} = \omega(\rho, T)\,(\bar{e}_i(T) - e_i)$$

(where $\bar{e}_i$ is the equilibrium value and $\omega = \rho\,\Omega\,(T)$ where $\Omega\,(T)$ is assumed to be $\Omega \propto T^s$) for a one dimensional analysis (*see* Reference 4).

Figures 1 and 2 show that the pressure approaches the equilibrium value between each reflected disturbance, but does not remain between the frozen and equilibrium isentropic solutions as may at first be expected. This is because of the exponential decay of the disturbance along characteristics found in Section 4. In the isentropic case, disturbances are reflected with the same strength at each intersection with the channel wall, but by the relaxation process each reflection is weaker than the preceding one by an amount corresponding to the exponential decrease $e^{-\beta \cdot 2h'}$ where $2h'$ is the distance between reflections. This effect can also be observed for the flow in a jet in Figure 5.

The approximation for small values of the relaxation parameter of Section 4.1 follows the same trend, and the reason why it remains a good approximation even for values of r as large as 0·5 is because it is equivalent to adding a term of smaller order than the existing terms to the original differential equation, as explained in Section 4.

The method of solution of the isentropic channel flow by contour integration is immediately applicable to other wall shapes. It is just a matter of calculating residues at the poles of the integrand which are the origin, $\pm in\pi/Bh$, and any others introduced by the transform of the wall shape; *see* Appendix I. However, the method of solution of the relaxing flow given in Section 3, is not easily extended to other wall shapes, as additional terms in the integrand will change the form of $L_1\,(t,y)$ radically. However, when the wall boundary condition produces an additional term such as $\frac{1}{p}$ (for walls $\pm(h+\varepsilon\, x^2)$) or $\frac{1}{p+1}$ $\left\{ \text{for walls } \pm[h+\varepsilon\,(1-\cos x)] \right\}$, the solution is immediately obtained from the exact solution above for the walls $\pm(h+\varepsilon\, x)$, and the transform of the additional term with the aid of the convolution formula.

In addition an alternative method of solution of the equation in the form (4.4) is outlined in Appendix IV for the corner flow problem which gives results correct to $0(\beta^3)$ and it is shown that this method can also be extended to channels with different wall shapes with the aid of the convolution formula.

The linearized duct theory used in this report as well as being restricted to values of x not too far down this nozzle from the corners will only be valid for values of $\frac{\varepsilon M_e^2}{2B_e^3(\sqrt{a}-1)} \ll 1$, since for larger values of this parameter, non-linear inertial effects will introduce a streamwise displacement of the equilibrium characteristics of the order of the spread between the equilibrium and frozen characteristics in the linear problem. Thus the present analysis is restricted to not too large x', small values of ε, and values of M_e not close to unity.

7. *Conclusions.*

An exact linearized solution for supersonic flow in a two-dimensional diverging channel of a gas relaxing in one mode, assuming a linear rate equation, has been obtained which contains as a special case the solution for flow round a sharp corner.

The pressure coefficient for the relaxing gas, compared with the two limits of isentropic flow in the same channel, demonstrates the effects of damping introduced into the flow by the relaxation process.

An approximate solution, assuming small values of the relaxation parameter, which involves much simpler algebraic expressions and which remains in good agreement with the exact for values of r up to 0·5, is also obtained.

The solution for other wall shapes is indicated. Similar solutions are found for the case of a two-dimensional jet which also demonstrate the damping effect.

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LIST OF SYMBOLS

a	B_e^2/B_f^2
a_e, a_f	Equilibrium and frozen sound speeds
b_n, c_n	Defined in equation (3. 2)
c_p	Pressure coefficient
$e(T)$	Internal energy
h	Halfwidth of channel
p	Laplace operator; also pressure
u'	Perturbation velocity in x-direction
v'	Perturbation velocity in y-direction
r	$a-1$
t	$x'-y'$
t_n', t_n''	Defined in Section 3
x	Direction of axis of channel, x' normalized co-ordinate
y	Normal to x, y' normalized co-ordinate
y_n, z_n	Defined in Section 3
B	$\sqrt{M^2-1}$ for the perfect gas
B_f, B_e	$\sqrt{(M_f^2-1)}, \sqrt{(M_e^2-1)}$
I_0, I_1	Modified Bessel functions of the first kind
K	Relaxation length (proportional to τU_∞)
L_1, L_2	Functions defined in Section 3
M	Mach number; M_f, M_e, freestream Mach numbers based on frozen and equilibrium soundspeeds
$P_m(t)$	Function defined in Appendix III, equation (3)
T	Temperature
U	Velocity in x-direction
α	$1+{}^r/_2$
β	${}^r/_2$
ε	A small positive quantity: angle of the corner
ζ	$1+p$
τ	Relaxation time
ϕ	Perturbation velocity potential
$\bar{(\ )}$	Denotes a transformed quantity, except $\bar{c}_p$ which denotes $-\dfrac{c_p B_f}{2\varepsilon}$

Subscript

∞	Denotes freestream conditions

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APPENDIX I

Isentropic Flow in a Two-Dimensional Channel.

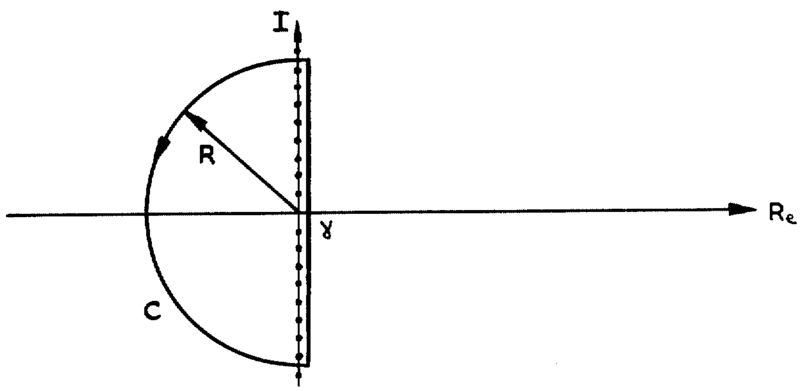
The pressure coefficient for the flow of a perfect gas in the two-dimensional channel is given by

$$-\frac{c_p B}{2\varepsilon} = L^{-1}\left\{ \frac{e^{px} \cosh Bpy}{p \sinh Bph} \right\},$$

where $B = B_e$ for equilibrium flow, $B = B_f$ for frozen flow.

The singularities of the integrand are a multipole at the origin, and simple poles at $p = \pm\frac{in\pi}{Bh}$, $n = 1, 2, \ldots\ldots$

The path of integration can be closed by a semicircle in the half plane in which $Re\, p$ is negative, the contribution from this part tending to zero as the radius tends to infinity, provided (x,y) lies downstream of the characteristics through $(0, \pm h)$ where $R = \frac{i(n+\frac{1}{2})}{Bh}$ and $R \to \infty$ through integral values of n, so that the contour does not pass through a pole.



Then, by Cauchy's theorem, the integral $= 2\pi i x$(sum of residues at singularities enclosed by C). The residue at the origin is obtained by expanding the integrand as a Laurent series about $p = o$ and is the coefficient of $1/_p$ in this expansion. It is $\frac{x}{Bh}$.

The residue at $p = \frac{in\pi}{Bh}$ is

$$\frac{e^{\frac{in\pi x}{Bh}} \cosh \frac{in\pi By}{Bh}}{\frac{in\pi}{Bh}\left[\frac{d}{dp}(\sinh Bph)\right]_{p = \frac{in\pi}{Bh}}}$$

$$= \frac{e^{\frac{in\pi x}{Bh}} \cos \frac{n\pi y}{h}}{in\pi \cosh in\pi} = \frac{(-1)^n}{in\pi} e^{\frac{in\pi x}{Bh}} \cos \frac{n\pi y}{h}.$$

Similarly the residue at $p = -\frac{in\pi}{Bh}$ is $-\frac{(-1)^n}{in\pi} e^{-\frac{in\pi x}{Bh}} \cos\frac{n\pi y}{h}$,

so that the sum of the residues at $p = \pm\frac{in\pi}{Bh}$ for all n is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{Bh}\cos\frac{n\pi y}{h}.$$

Therefore

$$-\frac{c_pB}{2\varepsilon} = \frac{x}{Bh}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{Bh}\cos\frac{n\pi y}{h}.$$

Hence, on the axis $y = o$, using the well-known result,

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{Bh} = -\frac{\pi x}{2Bh}, \quad -\pi < \frac{\pi x}{Bh} < \pi, \text{ (Reference 18)}$$

$$-\frac{c_pB}{2\varepsilon} = \frac{x}{Bh} - \left(\frac{x}{Bh}\right)_{-Bh < x < Bh.}$$

The resultant of the two terms is a step function. The perturbation pressure coefficient is zero up to the first Mach lines from the corners, and changes discontinuously by an amount $-\frac{4\varepsilon}{B}$, where the Mach lines

$$x = \pm B(y+(2n+1)h)$$

(the characteristics of the differential equation) cut the axis.

Similarly the pressure coefficient on the wall $y = h$ is given by

$$-\frac{c_pB}{2\varepsilon} = \frac{x}{Bh}+\left(\frac{Bh-x}{Bh}\right)_{0 < x < 2Bh}$$

again using a known result

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{Bh} = \frac{\pi(Bh-x)}{2Bh},$$

$o < \frac{\pi x}{Bh} < 2\pi$, from Reference 18.

On the wall, the first pressure drop at the corner is half the magnitude of the first pressure drop on the axis, but the subsequent steps are of magnitude $4\varepsilon/B$, i.e. there is a pressure drop of magnitude $\frac{2\varepsilon}{B}$ for each characteristic the flow passes through.

The discontinuity in the pressure coefficient is the result of linearization, which effectively approximates to the expansion fan by one of zero thickness parallel to the Mach line.

For a different wall shape, the contour remains the same but there will be additional poles introduced by the transform of the wall shape, e.g. for the walls $y = \pm(h+\varepsilon x^2)$

$$-\frac{c_p B}{2\varepsilon} = \int_L \frac{e^{px}\cosh Bpy}{p^2 \sinh Bph}\,dp$$

$$= \frac{x^2}{Bh} - \frac{1}{3}Bh + \frac{4Bh}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{Bh}\cos\frac{n\pi y}{h}.$$

On the axis, $y = o$, and the infinite series reduces to the well-known Fourier series for

$$\frac{\pi^2}{4B^2h^2}\left(-x^2+\frac{B^2h^2}{3}\right), \; -\pi < \frac{\pi x}{Bh} < \pi \text{ (Reference 18).}$$

The two terms cancel out for $x < Bh$, and in general, when $x = 2n\,Bh + x'$, $-Bh \leqslant x' \leqslant Bh$,

$$-\frac{c_p B}{2^{\varepsilon}} = 4n^2\,Bh + 4nx'.$$

There is a continuous linear pressure drop down the axis between $x = (2n-1)Bh$ and $x = (2n+1)Bh$ with discontinuities in slope at odd multiples of Bh.

APPENDIX II

Morrison's method applied to the corner flow problem.

Morrison (Reference 16) solved an equation analagous to (2.1) for the problem of wave propagation in rods of Voigt material and visco-elastic materials with three-parameter models. Morrison's method for solving equation (2.3) is as follows: if the Laplace transform of $f(x,y)$ is written $L\{f(x,y)\}$, then (2.3) is written

$$L\left\{ f(x,y) \right\} = \frac{1}{p}\sqrt{\frac{p+1}{p+a}}\, e^{-p\sqrt{\frac{p+a}{p+1}}y} \tag{II.1}$$

where (x,y) now denote the normalised co-ordinates of Section 3. The following general results for Laplace transforms are required;

$$\begin{aligned} \text{If} \quad G(\lambda) &= e^{-w\left(\lambda - \frac{2\beta}{\lambda}\right)} \\ &= L\left\{ g(x,w) \right\} \end{aligned} \tag{II.2}$$

and

$$e^{-yh(\lambda)} = L\left\{ \phi(x,y) \right\} \tag{II.3}$$

where

$$h(\lambda) = \sqrt{\lambda^2 - \beta^2} - \lambda + \beta,$$

then

$$\begin{aligned} G(y,h(\lambda)) &= e^{-y\left[h(\lambda) - \frac{2}{h(\lambda)}\right]} \\ &= L\left\{ \int_0^\infty \phi(x,y')\, g(y',y)\, dy' \right\} \end{aligned} \tag{II.4}$$

According to equation (II.1),

$$f(x,y) = L^{-1}\left\{ \frac{e^{px}}{p}\sqrt{\frac{p+1}{p+a}}\, e^{-p\sqrt{\frac{p+a}{p+1}}y} \right\} \tag{II.5}$$

and on applying the transformation $\lambda = p+\alpha$, equation (5) becomes

$$\begin{aligned} f(x,y) &= e^{-\alpha x} L^{-1}\left\{ \frac{e^{\lambda x}}{\lambda-\alpha}\sqrt{\frac{\lambda-\beta}{\lambda+\beta}}\, e^{-y(\lambda-\alpha)\sqrt{\frac{\lambda+\beta}{\lambda-\beta}}} \right\} \\ &= e^{-\alpha x}\int_y^\infty L^{-1}\left\{ e^{\lambda x}\, e^{-w(\lambda-\alpha)\sqrt{\frac{\lambda+\beta}{\lambda-\beta}}} \right\} dw \end{aligned} \tag{II.6}$$

But Morrison showed that

$$e^{-w(\lambda-\alpha)\sqrt{\frac{\lambda+\beta}{\lambda-\beta}}} = e^{-w\lambda - w(\alpha-2\beta) - w\left[h(\lambda)-\frac{2\beta}{h(\lambda)}\right]}$$

so that equation (II.6) can be written

$$f(x,y) = e^{-\alpha x}\int_y^\infty e^{-w(\alpha-2\beta)}\int_L e^{\lambda x - w\lambda}\,G(w,h(\lambda))\,d\lambda dw$$

$$= e^{-\alpha x}\int_y^\infty e^{-w(\alpha-2\beta)}\,H(x-w)\int_0^\infty \phi(x-w,y')\,g(y',w)\,dy'\,dw$$

where $H(x)$ is the unit Heaviside function.

Morrison gives the following results,

$$\phi(x,y) = L^{-1}\left\{e^{\lambda x}\,e^{-yh(\lambda)}\right\}$$

$$= e^{-\beta y}\left[\delta(x)+\beta y\frac{H(x)\mathrm{I}_1\left[\beta x^{\frac{1}{2}}(x+2y)^{\frac{1}{2}}\right]}{x^{\frac{1}{2}}(x+2y)^{\frac{1}{2}}}\right] \qquad \text{(II.7)}$$

where $\delta(x)$ is the Dirac delta function,
and

$$g(x,w) = L^{-1}\left\{e^{\lambda x}\,e^{-w\left(\lambda-\frac{2\beta}{\lambda}\right)}\right\}$$

$$= \delta(x-w)+w^{\frac{1}{2}}H(x-w)\sqrt{2\beta}\,\frac{\mathrm{I}_1\left[2\sqrt{2\beta w(x-w)}\right]}{\sqrt{x-w}} \qquad \text{(II.8)}$$

where $2\beta = r$.

Thus from the general results (II.2), (II.3) and (II.4), with (II.7) and (II.8),

$$\int_0^\infty \phi(x,y')\,g(y',w)\,dy' = L^{-1}\left\{e^{\lambda x}\,e^{-w\left[h(\lambda)-\frac{r}{h(\lambda)}\right]}\right\}$$

$$= \delta(x)\,e^{w(\alpha-2\beta)}+\beta\,w\,e^{-\beta w}\frac{\mathrm{I}_1\left[\beta\sqrt{x(x+2w)}\right]}{\sqrt{x(x+2w)}}+$$

$$+\beta\int_w^\infty e^{-\beta y'}\frac{y'\sqrt{rw}}{\sqrt{y'-w}}\;\frac{\mathrm{I}_1\left[\beta\sqrt{x(x+2y')}\right]}{\sqrt{x(x+2y')}}\,\mathrm{I}_1\left[2\sqrt{rw(y'-w)}\right]dy'$$

and

$$L^{-1}\left\{e^{\lambda x}\, e^{-w(\lambda-\alpha)\sqrt{\frac{\lambda+\beta}{\lambda-\beta}}}\right\} = \delta(x-w)\, e^{w} + \beta w H(x-w)\, e^{-w} \frac{\mathrm{I}_1[\beta\sqrt{x^2-w^2}]}{\sqrt{x^2-w^2}}$$

$$+\beta H(x-w)\, e^{-w} \int_0^{\infty} e^{-\beta y'} \frac{\sqrt{rw}\,(y'+w)}{\sqrt{y}}\, \mathrm{I}_1\,(2\sqrt{rwy'}\; \frac{\mathrm{I}_1\,\beta[\sqrt{(x+y')^2-(y'+w)^2}]}{\sqrt{(x+y')^2-(y'+w)^2}}\, dy'$$

Hence from (II.7), (II.1), and (2.3), Morrison's method gives

$$-\frac{c_p B_f}{2\varepsilon} = H(x-y)\left\{e^{-\beta x} + \beta e^{-\alpha x} \int_y^x w\, e^{-w} \frac{\mathrm{I}_1(\beta\sqrt{x^2-w^2})}{\sqrt{x^2-w^2}}\, dw + \right.$$

$$\left. +\beta\sqrt{r}\, e^{-\alpha x} \int_y^x e^{-w} \int_0^{\infty} e^{-\beta y'} \frac{\sqrt{y'+w}\,\sqrt{w}}{\sqrt{y}}\, \mathrm{I}_1(2\sqrt{rwy'})\, \frac{\mathrm{I}_1[\beta\sqrt{(x+y')^2-(y'+w)^2}]}{\sqrt{(x+y')^2-(y'+w)^2}}\, dy'\, dw\, .\right.$$

APPENDIX 3

Evaluation of $L\,(t'_n, y_n)$.

$$L_1(t'_n, y_n) = L^{-1}\left\{ \frac{e^{t'_n\lambda}}{\lambda-\alpha}\sqrt{\frac{\lambda-\beta}{\lambda+\beta}}\,e^{\sqrt{\frac{\lambda-\beta}{\lambda+\beta}}y_n} \right\}.$$

Replace $\lambda = \zeta + r/2$, and for simplicity write $t'_n = t$, $y_n = y$.

Then $$L_1(t,y) = e^{\beta t}\,L^{-1}\left\{ \frac{e^{\zeta t}}{\zeta-1}\sqrt{\frac{\zeta}{\zeta+r}}\,e^{\sqrt{\frac{\zeta+r}{\zeta}}y} \right\}. \qquad \text{(III.1)}$$

Expand $e^{y\sqrt{\frac{\zeta+r}{\zeta}}} = \sum_{m=o}^{\infty} \frac{y^m}{m!}\left(\frac{\zeta+r}{\zeta}\right)^{\frac{m}{2}}$ for all values of y and ζ, so that (III.1) becomes

$$L_1(t,y) = e^{\beta t}\sum_{m=o}^{\infty}\frac{y^m}{m!}L^{-1}\left\{ \frac{e^{\zeta t}}{\zeta-1}\left(\frac{\zeta+r}{\zeta}\right)^{\frac{m-1}{2}} \right\}.$$

(The order of summation and integration can be inverted, since the integral is shown to converge below). (This expansion procedure can only be used to evaluate (II.1); it cannot conveniently be applied for the evaluation of 3.3) say, because of the extra $(\zeta-1)$ term within the exponential). If $\mu = \zeta - 1$, then

$$L_1(t,y) = e^{\alpha t}\sum_{m=o}^{\infty}\frac{y^m}{m!}L^{-1}\left\{ \frac{e^{\mu t}}{\mu}\left(\frac{\mu+a}{\mu+1}\right)^{\frac{m-1}{2}} \right\}. \qquad \text{(III.2)}$$

Let $$P_m(t) = L^{-1}\left\{ e^{\mu t}\left(\frac{\mu+a}{\mu+1}\right)^{\frac{m-1}{2}}\frac{1}{\mu} \right\} \qquad \text{(III.3)}$$

$$= L^{-1}\left\{ e^{\mu t}\left(\frac{\mu+a}{\mu+1}\right)^{\frac{m-3}{2}}\frac{1}{\mu} \right\} + r\,L^{-1}\left\{ e^{\mu t}\left(\frac{\mu+a}{\mu+1}\right)^{\frac{m-3}{2}}\frac{1}{\mu(\mu+1)} \right\}$$

so $P_m(t)$ satisfies the recurrence relation,

$$P_m(t) = P_{m-2}(t) + r\,e^{-t}\int_0^t e^{\tau}\,P_{m-2}(\tau)\,d\tau. \qquad \text{(III.4)}$$

$P_0(t)$ and $P_1(t)$ are given in Reference 17,

$$P_0(t) = H(t)\left[e^{-\alpha t}\,\mathrm{I}_0(\beta t) + \int_0^t e^{-\alpha\mu}\,\mathrm{I}_0(\beta\mu)\,d\mu \right]$$

$$P_1(t) = H(t)$$

where $H(t)$ is the Heaviside unit step function, and from the recurrence relation (III.4), all the $P_m(t)$ can be evaluated.

Thus

$$L_1(t,y) = e^{\alpha t} \sum_{m=o}^{\infty} \frac{y^m}{m!} P_m(t) \tag{III.5}$$

$P_2(t)$ to $P_9(t)$ are given below,

$$P_2(t) = H(t) \left[e^{-\alpha t} \mathrm{I}_0(\beta t) + (1+r) \int_0^t e^{-\alpha\mu} \mathrm{I}_0(\beta\mu)\, d\mu \right],$$

$$P_3(t) = H(t) \left[a - re^{-t} \right],$$

$$P_4(t) = H(t) \left[e^{-\alpha t} \mathrm{I}_0(\beta t) + a^2 \int_0^t e^{-\alpha\mu} \mathrm{I}_0(\beta\mu)\, d\mu - r^2 e^{-t} \int_0^t e^{-\beta\mu} \mathrm{I}_0(\beta\mu)\, d\mu \right],$$

$$P_5(t) = H(t) \left[e^{-t} + a^2(1-e^{-t}) - r^2\, t\, e^{-t} \right],$$

$$P_6(t) = H(t) \left[e^{-\alpha t} \mathrm{I}_0(\beta t) + a^3 \int_0^t e^{-\alpha\mu} \mathrm{I}_0(\beta\mu)\, d\mu - r^2(3+r)e^{-t} \int_0^t e^{-\beta\mu} \mathrm{I}_0(\beta\mu)\, d\mu - \right.$$

$$\left. - r^3 e^{-t} \int_0^t \int_0^\tau e^{-\beta\mu} \mathrm{I}_0(\beta\mu)\, d\mu\, d\tau \right],$$

$$P_7(t) = H(t) \left[e^{-t} + a^2(1-e^{-t}) - r^2\, t\, e^{-t} + r(1-a^2)\, te^{-t} - \tfrac{1}{2} r^3\, t^2\, e^{-t} + a^2\, r(1-e^{-t}) \right],$$

$$P_8(t) = H(t) \left[e^{-\alpha t} \mathrm{I}_0(\beta t) + a^4 \int_0^t e^{-\alpha\mu} \mathrm{I}_0(\beta\mu) d\mu - e^{-t} r^2(6+4r+r^2) \int_0^t e^{-\beta\mu} \mathrm{I}_0(\beta\mu)\, d\mu - \right.$$

$$\left. - r^3(4+r)e^{-t} \int_0^t \int_0^\tau e^{-\beta\mu} \mathrm{I}_0(\beta\mu)\, d\mu\, d\tau - r^4 e^{-t} \int_0^t \int_0^\tau \int_0^{\tau'} e^{-\beta\mu} \mathrm{I}_0(\beta\mu)\, d\mu\, d\tau\, d\tau' \right],$$

$$P_9(t) = H(t) \left[P_7(t) - r(a^3-1)\, te^{-t} + \tfrac{1}{2} r^2\, (1-a^2-r)\, t^2\, e^{-t} - \frac{1}{6} r^4\, t^3\, e^{-t} + a^3\, r(1-e^{-t}) \right].$$

APPENDIX IV

Solution of the Exact Differential Equation in the Form (4.4).

As shown in Section 3, the exact differential equation can be rewritten in the form,

$$\left(\alpha+\frac{\partial}{\partial x}\right)\frac{\partial^2\phi}{\partial x}-\left(1+\frac{\partial}{\partial x}\right)^2\frac{\partial^2\phi}{\partial y^2}=\frac{r^2}{4}\frac{\partial^2\phi}{\partial x^2} \tag{IV.1}$$

where (x,y) denote the normalized co-ordinates of Section 3. This form of the equation can be solved as follows; writing the Laplace transform $\bar{\phi}(y,p)$ of $\phi(x,y)$ as

$$\bar{\phi}(y,p)=\int_0^\infty e^{-px}\,\phi(x,y)\,dx,$$

equation (IV.1) transforms to

$$\left[\frac{(\alpha+p)p}{1+p}\right]^2\bar{\phi}-\frac{\partial^2\bar{\phi}}{\partial y^2}=-A(y,p) \tag{IV.2}$$

where

$$A(y,p)=-\frac{r^2}{4(1+p)^2}\int_0^\infty e^{-px}\left(\frac{\partial^2\phi}{\partial x^2}\right)dx. \tag{IV.3}$$

The boundary condition on the wall for the corner flow problem has the transform

$$\frac{B_f}{U_\infty}\frac{\partial\phi}{\partial y}=\frac{\varepsilon}{p}\text{ on }y=0 \tag{IV.4}$$

and since the disturbance must be bounded at $y=\infty$,

$$\bar{\phi}(\infty,p)=0 \tag{IV.5}$$

The solution of (IV.2) is an integral equation for $\bar{\phi}$ and has the form

$$\bar{\phi}(y,p)=\tfrac{1}{2}\bar{\phi}(o,p)\left[e^{qy}+e^{-qy}\right]+\tfrac{1}{2}\frac{\varepsilon/p}{qB_f/U_\infty}(e^{qy}-e^{-qy})+$$

$$+\frac{e^{qy}}{2q}\int_0^y e^{-qy'}A(y',p)\,dy'-\frac{e^{-qy}}{2q}\int_0^y e^{qy'}A(y',p)\,dy'. \tag{IV.6}$$

In order to satisfy the boundary condition (IV.5),

$$\bar{\phi}(o,p)=-\frac{\varepsilon/p}{qB_f/U_\infty}-\frac{1}{q}\int_0^\infty e^{-qy'}A(y',p)\,dy' \tag{IV.7}$$

and hence

$$\bar{\phi}(y,p) = -\frac{\varepsilon/p}{qB_f/U_\infty} e^{-qy} + \frac{r^2 q^2}{4(1+p)^2} \frac{\cosh qy}{q} \int_y^\infty e^{-qy'} \bar{\phi}(y',p)\, dy' +$$

$$+\frac{r^2 p^2}{4(1+p)^2} \frac{e^{-qy}}{q} \int_0^y \cosh q y' \, \bar{\phi}(y',p)\, dy' \tag{IV.8}$$

where

$$q(p) = \frac{(\alpha+p)p}{1+p}. \tag{IV.9}$$

Therefore

$$\phi(x,y) = -\frac{\varepsilon}{B_f/U_\infty} L^{-1}\left\{\frac{e^{px} e^{-qy}}{qp}\right\} + \frac{r^2}{4} L^{-1}\left\{\int_y^\infty \frac{e^{px} p^2 \cosh qy}{(1+p)\, q} e^{-qy'} \bar{\phi}(y',p)\, dy'\right\} +$$

$$+\frac{r^2}{4} L^{-1}\left\{\int_0^y \frac{e^{px} p^2 e^{-qy}}{(1+p)^2 q} \cosh q y' \, \bar{\phi}(y',p)\, dy'\right\},$$

or since $c_p = -\frac{2u'}{U_\infty} = -\frac{2}{U_\infty}\frac{\partial \phi}{\partial x}$,

$$-\frac{c_p B_f}{2\varepsilon} = L^{-1}\left\{\frac{e^{px} e^{-qy}}{q}\right\} + \frac{r^2}{4} L^{-1}\left\{\int_y^\infty \frac{e^{px} p^2 \cosh qy}{(1+p)^2 q} e^{-qy'} \left(-\frac{\bar{c}_p B_f}{2\varepsilon}\right) dy'\right\} +$$

$$+\frac{r^2}{4} L^{-1}\left\{\int_0^y \frac{e^{px} p^2 e^{-qy}}{(1+p)^2 q} \cosh qy' \left(-\frac{\bar{c}_p B_f}{2\varepsilon}\right) dy'\right\}. \tag{IV.10}$$

Now the last two terms in (10) are in the nature of correction terms and so a reasonable approximation to $-\frac{\bar{c}_p B_f}{2\varepsilon}$ in them will be obtained if the first approximation (correct to $0(\beta)$) is inserted in these terms,

$$-\frac{\bar{c}_p B_f}{2\varepsilon} = \frac{e^{-qy}}{q} + 0(\beta^2) \tag{IV.11}$$

so (IV.10) becomes

$$-\frac{c_p B_f}{2\varepsilon} = L^{-1}\left\{\frac{e^{px} e^{-qy}}{q}\right\} + \frac{\beta^2}{2} L^{-1}\left\{\frac{e^{px} p^2}{(1+p)^2 q^2}\left[\frac{e^{-qy}}{q} + ye^{-qy}\right]\right\}. \tag{IV.12}$$

Replace p by $\zeta-1$, and write $t = x-y$ to get

$$-\frac{c_p B_f}{2\varepsilon} = e^{-t} e^{-\beta y} L^{-1}\left\{\frac{e^{t\zeta} . e^{y\beta/\zeta}}{(\zeta-1)(\zeta+\beta)\zeta^{-1}}\right\} + \frac{\beta^2 e^{-t} e^{-\beta y}}{2} L^{-1}\left\{\frac{e^{t\zeta} e^{y\beta/\zeta}}{(\zeta-1)(\zeta+\beta)^3 \zeta^{-1}}\right\} +$$

$$+\frac{\beta^2 e^{-t} e^{-\beta y}}{2} L^{-1}\left\{\frac{e^{t\zeta} e^{y\beta/\zeta}}{(\zeta+\beta)^2}\right\} . \qquad \text{(IV.13)}$$

But $\qquad L^{-1} \quad e^{t\zeta}(e^{y\beta/2\zeta}-1) \qquad = \frac{d}{dt} \mathrm{I}_0(\sqrt{2y\beta t})$

and $\qquad L^{-1}\left\{e^{t\zeta}\frac{\zeta}{(\zeta-1)(\zeta+\beta)}\right\} \qquad = \frac{\beta e^{-\beta t}}{1+\beta}+\frac{e^t}{1+\beta}$, from Reference 17.

Therefore

$$L^{-1}\left\{\frac{e^{t\zeta}}{(\zeta-1)(\zeta+\beta)^3\zeta^{-1}}\right\} = \frac{1-e^t}{1-\alpha^3} - \frac{1-e^{-\beta t}}{1-\alpha^3}\left(1+\alpha t - \frac{1-\alpha^2\beta}{1-2} t^2\right) \qquad \text{(IV.14)}$$

and

$$L^{-1}\left\{\frac{1-e^{t\zeta}}{1-(\zeta+\beta)^2}\right\} = te^{-\beta t}, \qquad \text{(IV.15)}$$

so that

$$-\frac{c_p B_f}{2\varepsilon} = \frac{e^{-t} e^{-\beta y}}{\alpha}\left\{e^t + \beta e^{-\beta t} + \int_0^t \left[e^{t-\tau}+\beta e^{-\beta(t-\tau)}\right]\frac{d}{d\tau}\mathrm{I}_0(\sqrt{4y\beta\tau})\,d\tau\right\} +$$

$$+\frac{\beta^2 e^{-t} e^{-\beta y}}{2\alpha^3}\left\{e^t - \left(1+\alpha t - \frac{\alpha^2\beta t^2}{2}\right)e^{-\beta t} + \right.$$

$$\left. + \int_0^t \left[e^{t-\tau} - \left(1+\alpha(t-\tau) - \frac{\alpha^2\beta}{2}(t-\tau)\right)e^{-\beta(t-\tau)}\right]\frac{d}{d\tau}\mathrm{I}_0(\sqrt{4y\beta\tau})\,d\tau\right\}$$

$$+\frac{\beta^2 e^{-t} e^{-\beta y}}{2\varepsilon} y \left\{te^{-\beta t} + \int_0^t (t-\tau)\,e^{-(t-\tau)}\frac{d}{d\tau}\mathrm{I}_0(\sqrt{4y\beta\tau})\,d\tau\right\} . \qquad \text{(IV.16)}$$

After integration by parts and rearrangement, (16) can be written

$$-\frac{c_p B_f}{2\varepsilon} = \frac{1}{\alpha}\left(1+\frac{\beta^2}{2\alpha^2}\right) L(t,y;\beta) + \left[\frac{\beta}{\alpha} - \frac{\beta^2}{2\alpha^2}\left(1+\alpha t - \frac{\alpha^2\beta t^2}{2}\right) + \frac{\beta^2 y t}{2}\right] J(t,y;\beta) +$$

$$+\left[\frac{\beta}{2\alpha}(1-\alpha\beta t) - \frac{\beta y}{2}\right] M(t,y;\beta) + \frac{\beta}{4\alpha} N(t,y;\beta), \qquad \text{(IV.17)}$$

where

$$L(t,y;\beta) = e^{-\beta y}\left\{e^{-t}\,\mathrm{I}_0(\sqrt{4y\beta t})+\int_0^t e^{-\tau}\,\mathrm{I}_0(\sqrt{4y\beta\tau})\,d\tau\right\},$$

$$J(t,y;\beta) = e^{-\beta y}\,e^{-t}\left\{\mathrm{I}_0(\sqrt{4y\beta t})-e^{-\beta t}\int_0^t e^{\beta\tau}\,\mathrm{I}_0(\sqrt{4y\beta\tau})\,d\tau\right\},$$

$$M(t,y;\beta) = \beta t\,J(t,y;\beta)-\beta e^{-\beta t}\,e^{-t}\int_0^t e^{\tau}\,e^{\beta\tau}\,J(\tau,y;\beta)\,d\tau\Big\},$$

$$N(t,y;\beta) = \beta t\,M(t,y;\beta)-\beta e^{-\beta t}\,e^{-t}\int_0^t e^{\tau}\,e^{\beta\tau}\,M(\tau,y;\beta)\,d\tau\Big\}.$$

For $\beta \ll 1$, (IV.17) reduces to (4.2a) and

$$-\frac{c_p B_f}{2\varepsilon} = \frac{1}{\alpha}\,L(t,y;\beta)+\frac{\beta}{\alpha}\,J(t,y;\beta) \qquad \text{(IV.17a)}$$

and when $y = 0$, this reduces to

$$-\frac{c_{p_w} B_f}{2\varepsilon} = \frac{1}{\alpha}+\frac{\beta^2}{2\alpha^3}+\left[\frac{\beta}{\alpha}-\frac{\beta^2}{2\alpha^3}\left(1+\alpha x-\frac{\alpha^2\beta}{2}x^2\right)\right]e^{-x}\,e^{-\beta x} \qquad \text{(IV.18)}$$

which could have been obtained directly from (IV.16),
since

$$L(x,o;\beta) = 1,$$

$$J(x,o;\beta) = e^{-x}\,e^{-\beta x},$$

$$M(x,o;\beta) = 0$$

$$N(x,o;\beta) = 0$$

equation (IV.18) gives

$$-\frac{c_{p_w}(o)B_f}{2\varepsilon} = 1$$

and

$$-\frac{c_{p_w}(\infty)\beta_f}{2\varepsilon} = \frac{1}{\alpha}+\frac{\beta^2}{2\alpha^3} = 1-\beta+\frac{3\beta^2}{2}-\frac{5\beta^2}{2}+4\beta^4+0(\beta^5)$$

where the exact results is, [given by $P_0(\infty)$]

$$-\frac{c_{p_w}(\infty)B_f}{2\varepsilon} = \frac{1}{\sqrt{1+2\beta}} = 1-\beta+\frac{3\beta^2}{2}-\frac{5}{2}\beta^3+\frac{105}{24}\beta^4+0(\beta^5),$$

so that the approximate theory is correct to $0(\beta^3)$. To apply this method to the channel flow, the co-ordinates (x,y) in the above analysis must be replaced by the normalized co-ordinates (x',y_n) and (x',z_n) of Section 3 and t replaced by t'_n, when the result (IV.16) is now b_n and c_n and

$$-\frac{c_p B_f}{2\varepsilon} = \sum_{n=o}^{k} b_n + \sum_{n=o}^{k'} c_n \text{ as before.}$$

For a different wall shape, e.g. $h = \pm(h+\varepsilon x^2)$, the boundary condition (IV.4) becomes

$$\frac{B_f}{U_\infty}\frac{\partial\bar{\phi}}{\partial y} = \frac{2\varepsilon}{p^2}$$

which will modify the first term on the right hand side of equation (IV.8) so that equation (IV.12) becomes

$$-\frac{c_p B_f}{2\varepsilon} = 2\,L^{-1}\left\{\frac{e^{px}\,e^{-qy}}{qp}\right\} + \beta^2\,L^{-1}\left\{\frac{e^{px}\,p}{(1+p)^2 q}\left[\frac{e^{-qy}}{q}+ye^{-qy}\right]\right\}$$

and (IV.13) becomes

$$-\frac{c_p B_f}{2\varepsilon} = 2e^{-t}\,e^{-\beta y}\,L^{-1}\left\{\frac{e^{t\zeta}\,e^{\beta y/\zeta}}{(\zeta-1)^2\,(\zeta+\beta)\,\zeta^{-1}}\right\} + \beta^2\,e^{-t}\,e^{-\beta y}\,L^{-1}\left\{\frac{e^{t\zeta}\,e^{\beta y/\zeta}}{(\zeta-1)^2\,(\zeta+\beta)^3\,\zeta^{-1}}\right\} +$$

$$+\beta^2\,e^{-t}\,e^{-\beta y}\,y\,L^{-1}\left\{\frac{e^{t\zeta}\,e^{\beta y/\zeta}}{(\zeta-1)\,(\zeta+\beta)^2}\right\}$$

and these inversions can still be performed with the aid of the convolution formula in a similar way as above.

APPENDIX V

One-Dimensional Analysis Using a Linear Rate Equation.

The pressure coefficient on the axis can be obtained from a one-dimensional analysis as follows: First, for a perfect gas, the basic equations are,

$$\text{Momentum:} \quad U\frac{dU}{dx}+\frac{1}{\rho}\frac{dp}{dx} = 0.$$

$$\text{Continuity:} \quad \frac{d}{dx}(\rho U A) = 0.$$

Normalize the co-ordinates to $x' = x/K$, $A' = \dfrac{B_f A}{K}$, and linearize,

$$\rho_\infty U_\infty B_f \frac{dA'}{dx'} + \rho_\infty A_\infty B_f \frac{du'}{dx'} + U_\infty B_f A_\infty \frac{d\rho'}{dx'} = 0$$

$$U_\infty \frac{du'}{dx'} + \frac{1}{\rho_\infty}\frac{dp'}{dx'} = 0$$

and eliminate $\dfrac{dp'}{dx'}$ by writing $\dfrac{d\rho'}{dx'} = \dfrac{1}{a_{f\infty}^2}\dfrac{dp'}{dx'}$, on the assumption that the flow is isentropic, to get

$$A'_\infty B_f(M_f^2-1)\frac{du'}{dx'} - U_\infty B_f \frac{dA'}{dx'} = 0,$$

where

$$M_f = \frac{U_\infty}{a_{f\infty}},$$

i.e.

$$\frac{du'}{dx'} = \frac{U_\infty}{A'_\infty B_f^2}\frac{dA'}{dx'}.$$

Take $A'_\infty = 1$, $A' = 1+2\varepsilon B_f x'$, and solving for $c_p = -\dfrac{2u'}{U_\infty}$,

the frozen solution is

$$-\frac{c_p B_f}{2\varepsilon} = 2x'.$$

The equilibrium solution is similarly obtained from the equation

$$A'_\infty \frac{du'}{dx'} = \frac{U_\infty}{B_e^2}\frac{dA'}{dx'}$$

so

$$-\frac{c_p B_e^2}{2\varepsilon} = 2B_f x'$$

or

$$-\frac{c_p B_f}{2\varepsilon} = \frac{2x' B_f^2}{B_e^2} = \frac{2}{a} x'$$

$$= 1{\cdot}333\, x' \text{ for } a = 1{\cdot}5.$$

These are shown in Figure 2.

For the relaxing gas, the basic equations are,

$$\text{Momentum:} \quad U_\infty \frac{du'}{dx} + \frac{1}{\rho_\infty} \frac{dp'}{dx} = 0. \tag{V.1}$$

$$\text{Continuity:} \quad \frac{d}{dx}(\rho U A) = 0$$

or

$$\rho_\infty U_\infty \frac{dA}{dx} + \rho_\infty A_\infty \frac{du'}{dx} + U_\infty A_\infty \frac{d\rho'}{dx} = 0. \tag{V.2}$$

$$\text{Energy:} \quad U_\infty \frac{du'}{dx} + \frac{dh'}{dx} = 0. \tag{V.3}$$

$$\text{Equation of State:} \quad h = h(p, \rho, q)$$

where q is the energy in the relaxing mode, so that

$$\frac{dh'}{dx} = h_{p\infty} \frac{dp'}{dx} + h_{\rho\infty} \frac{d\rho'}{dx} + h_{q\infty} \frac{dq'}{dx}. \tag{V.4}$$

$$\text{Rate Equation:} \quad U_\infty \frac{dq'}{dx} = \frac{\bar{q}' - q}{\tau_0} \tag{V.5}$$

where $\bar{q}' = \bar{q}'(p, \rho)$ = equilibrium energy in the relaxing mode, or

$$\frac{d\bar{q}'}{dx} = \bar{q}\rho_\infty \frac{d\rho'}{dx} + \bar{q}_{p\infty} \frac{dp'}{dx}. \tag{V.6}$$

Substitute (V.5) into (V.4) and then (V.4) into (V.3) and eliminate pressure and density gradients using (2) and (1) to get

$$\left[\left(\frac{1/\rho_x - h_{p\infty}}{h_{\rho_\infty}}\right) U_\infty^2 - 1\right] \frac{du'}{dx} - \frac{U_\infty}{A_\infty} \frac{dA}{dx} + \frac{h_{q\infty}}{h_{\rho\infty}\rho_\infty} \left(\frac{\bar{q}' - q'}{\tau_0}\right) = 0,$$

or

$$B_f^2 \frac{du'}{dx} - \frac{U_\infty}{A_\infty}\frac{dA}{dx} + \frac{h_{q\infty}}{h_{p_\infty}\rho_\infty}\left(\frac{\bar{q}'-q'}{\tau_0}\right) = 0 \qquad \text{(V.7)}$$

where

$$B_f^2 = \frac{U_\infty{}^2}{a_{f\infty}^2} - 1,\ a_{f\infty}^2 = \frac{h_{p_\infty}}{1/\rho_\infty - h_{p_\infty}} \text{ by definition.}$$

(V.7) is differentiated with respect to x, and (V.4) and (V.6) used to eliminate gradients of $\bar{q}'$, q' and then (V.1), (V.2) and (V.3) used to eliminate gradients of h', p', and ρ' in favour of velocity and area gradients.

Thus

$$B_f^2 \frac{d^2u}{dx^2} - \frac{U_\infty}{A_\infty}\frac{d^2A}{dx} + \left(\frac{h_{p_\infty} + h_{q_\infty}\bar{q}_{p_\infty}}{h_{p_\infty}\,.\,U_\infty\,\tau_0}\right)\left\{\left[\frac{U_\infty{}^2(1/\rho_\infty - \bar{q}_{p\infty}h_{q\infty} - h_{p\infty})}{h_{p_\infty} + h_{q_\infty}\bar{q}_{p_\infty}} - 1\right]\frac{du'}{dx} - \frac{U_\infty}{A_\infty}\frac{dA}{dx}\right\} = 0$$

or

$$K\left(B_f^2 \frac{d^2u}{dx^2} - \frac{U_\infty}{A_\infty}\frac{d^2A}{dx^2}\right) + B_e^2 \frac{du'}{dx} - \frac{U_\infty}{A_\infty}\frac{dA}{dx} = 0 \qquad \text{(V.8)}$$

where

$$K = \frac{h_{p_\infty} U_\infty \tau_0}{h_{p_\infty} + h_{q_\infty}\bar{q}_{p_\infty}}$$

and

$$B_e^2 = \frac{U_\infty{}^2}{a_{e_\infty}^2} - 1, \qquad a_{e_\infty}^2 = \frac{h_{p_\infty} + h_{q_\infty}\bar{q}_{p_\infty}}{1/\rho_\infty - \bar{q}_{p_\infty}h_{q_\infty} - h_{p_\infty}}$$

by definition.

Transform to normalized co-ordinates $x' = {}^x/K$, $A' = \dfrac{B_f A}{K}$ to get

$$\frac{d^2u'}{dx'^2} - \frac{U_\infty}{B_f^2 A'_\infty}\frac{d^2A'}{dx'^2} + a\frac{du'}{dx'} - \frac{U_\infty}{B_f^2 A'_\infty}\frac{dA'}{dx'} = 0$$

where $\qquad A' = A'_\infty + 2\,\varepsilon\, B_f\, x'\, H(x')$

and so $\qquad A'_\infty = 1.$

The equation for u' is therefore

$$\frac{d^2u'}{dx'^2} - 0 + a\frac{du'}{dx'} - \frac{U_\infty}{B_f^2} . 2\,\varepsilon B_f = 0 .$$

The solution for u' is

$$u' = \frac{2\,\varepsilon\,U_\infty}{a^2\,B_f}\left[ax' - 1 + e^{-ax'}\right] + \frac{2\,\varepsilon U_\infty}{B_f a}(1 - e^{-ax'})$$

so

$$-\frac{c_p B_f}{2\varepsilon} = \frac{2}{a^2}\left[ax' - 1 + e^{-ax'}\right] + \frac{2}{a}(1 - e^{-ax'}) .$$

This is also shown in Figure 2.

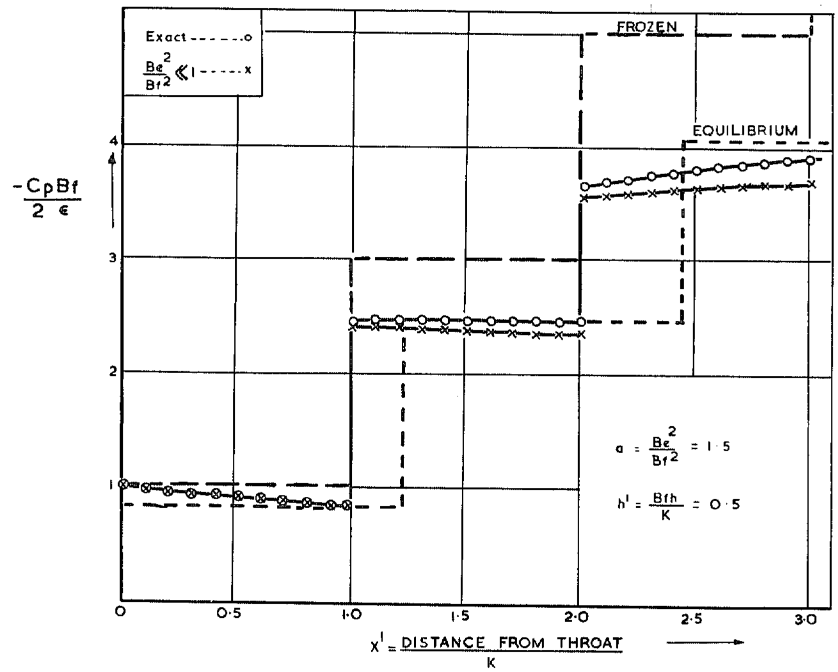


FIG. 1. Pressure coefficient on wall of divergent nozzle.

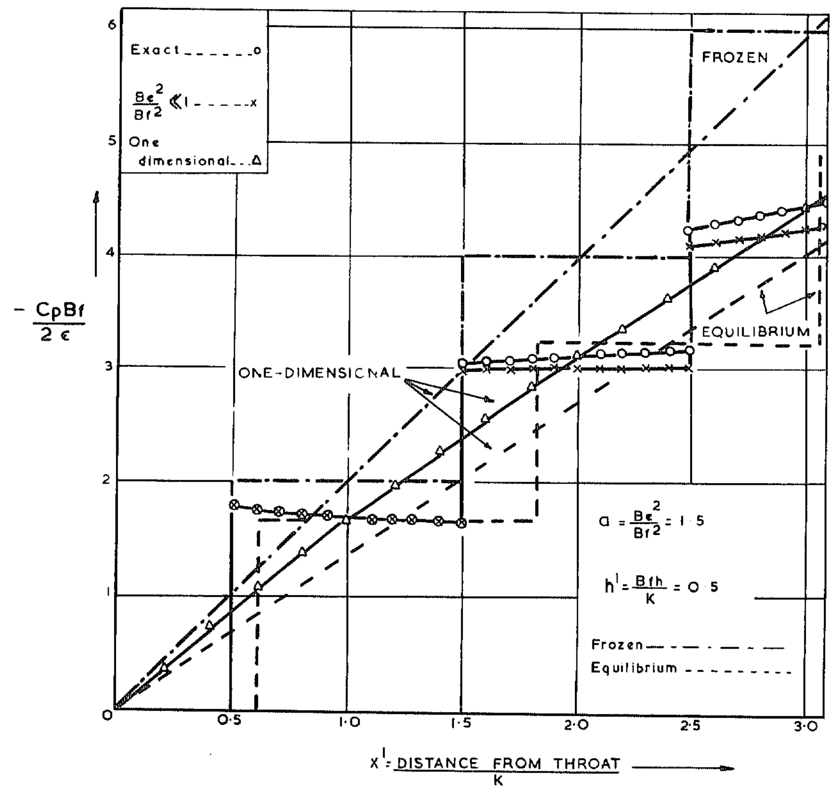


FIG. 2. Pressure coefficient on axis of divergent nozzle.

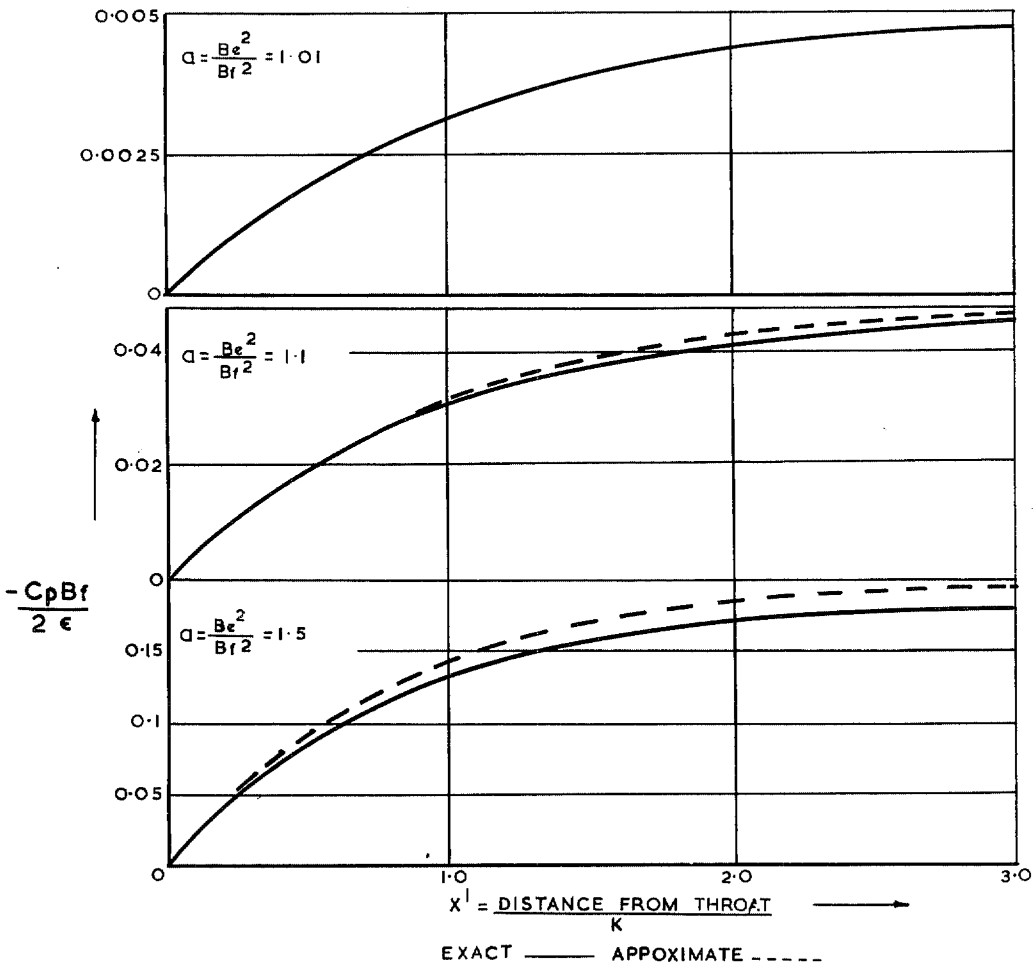


FIG. 3. Corner flow-pressure coefficient on wall.

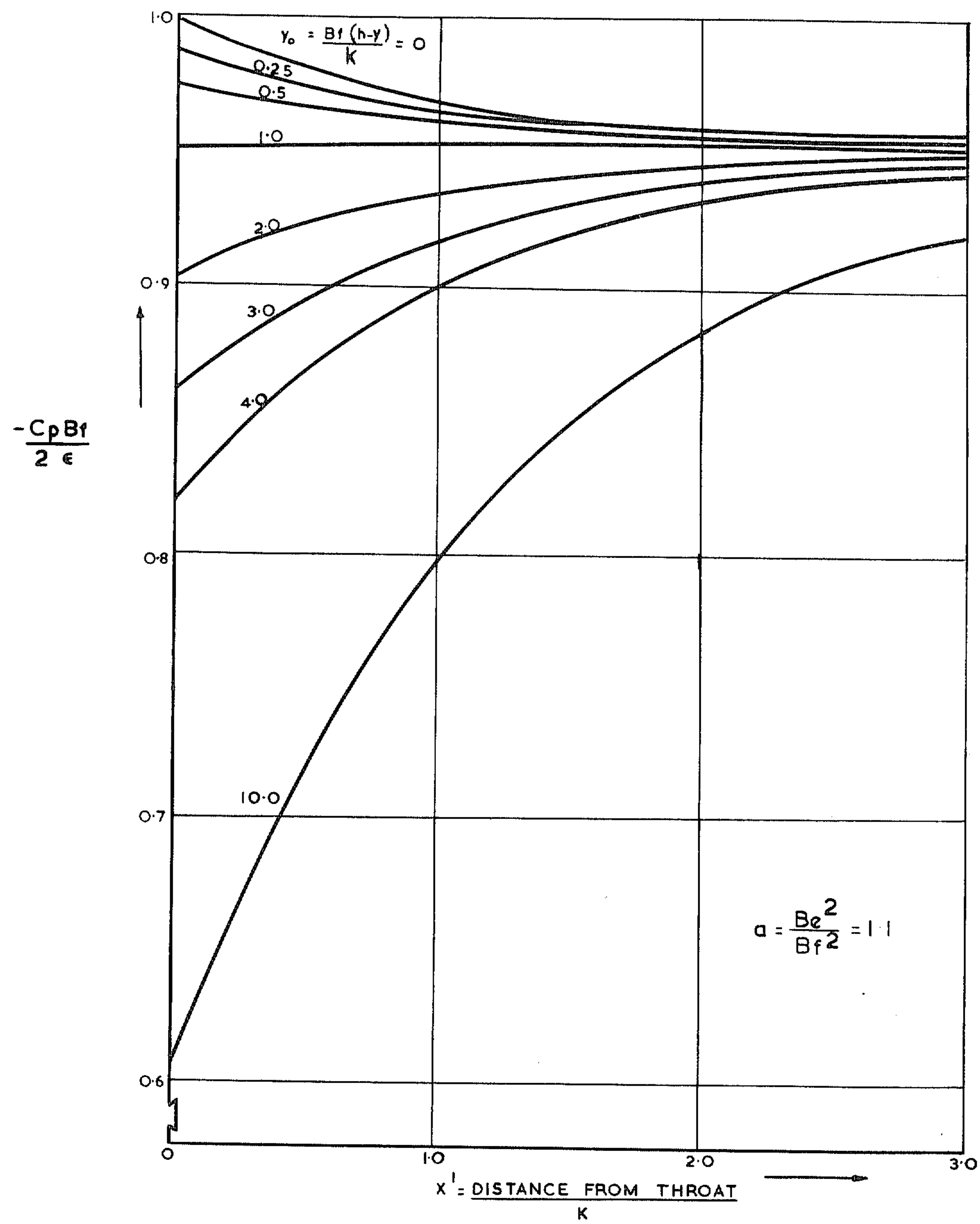


FIG. 4. Variation of pressure coefficient with distance from wall-equation (4.2a).

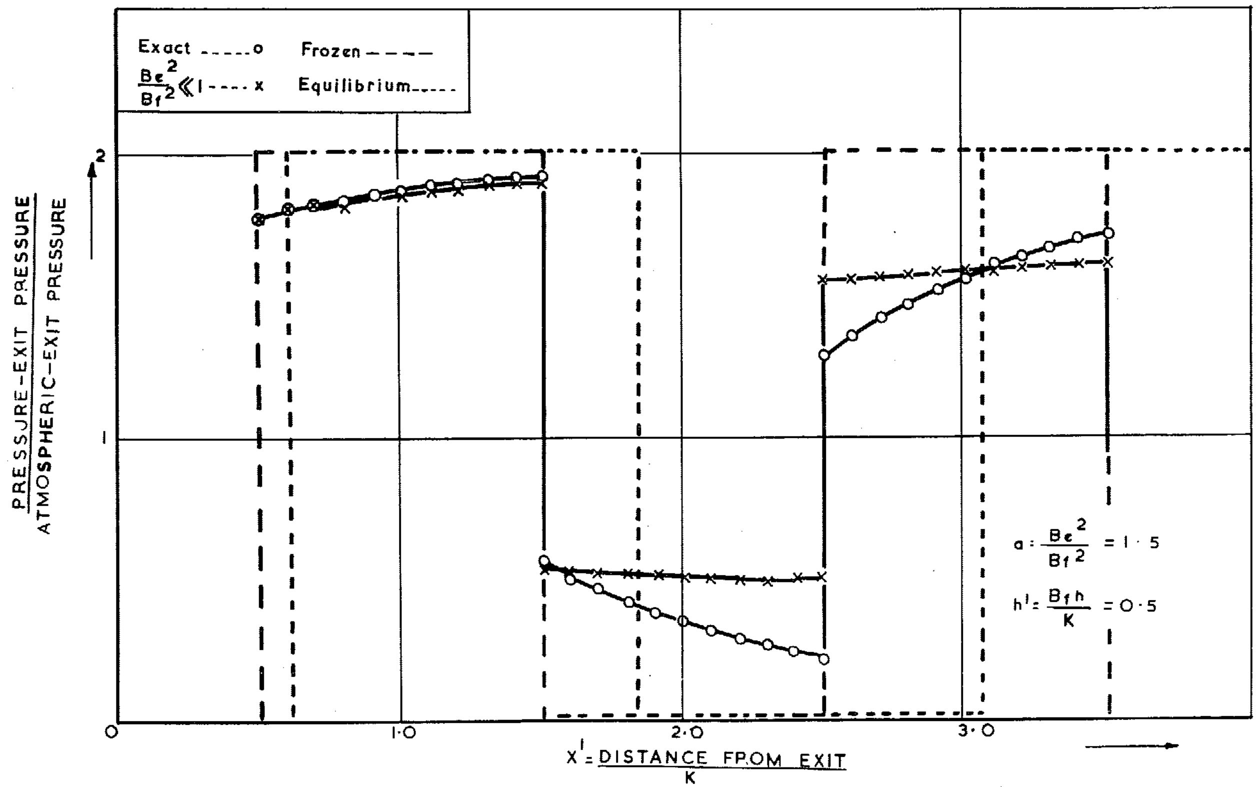


FIG. 5. Pressure coefficient on axis of jet.

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