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Calculation of the Response of a Flexible Aircraft to Harmonic and Discrete Gusts by a Transform Method

By C. G. B. Mitchell

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Calculation of the Response of a Flexible Aircraft to Harmonic and Discrete Gusts by a Transform Method

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Summary.

The equation of motion for a flexible aircraft responding to a step gust is derived, and the relation developed between this and the response to an array of harmonic gusts. A Fourier transform method is used to solve the equation of motion. This allows the rigorous inclusion of indicial effects in both the gust and response aerodynamic forces.

As an example the symmetric responses of a flexible slender-wing aircraft in subsonic flight to harmonic and step gusts are calculated. The aircraft is given two rigid and four flexible degrees-of-freedom. Aerodynamic forces are calculated by a lifting-surface theory and include unsteady effects.

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1. *Introduction.*

The formal requirement for the strength of an aircraft to withstand atmospheric turbulence is set by postulating that it encounters a single gust of defined shape and size. On the other hand, an increasing number of calculations are being made of the random response of aircraft flying in relatively continuous* random turbulence. The question as to which of these models is the best representation of atmospheric turbulence is beyond the scope of this Report and is discussed by Zbrozek[1,2] among others.

A semi-empirical method is usually used to calculate the peak incremental** normal acceleration Δn of an aircraft entering a discrete gust. This computes the acceleration that the aircraft would experience if the gust did not cause the aircraft to pitch or deform and if the airforces on the aircraft instantly adopted the values appropriate to the incidence induced by the gust. This acceleration is then multiplied by a 'gust alleviation factor' (which typically varies from 0·5 for a light aircraft to 0·9 for a dense military one) to allow for pitching, gust gradient length, and unsteady aerodynamic effects. Because aircraft flexibility is ignored the peak stress at some point on the structure during the gust entry is taken to be the steady level flight stress at that point multiplied by the peak acceleration, in g units, caused by the gust.

This method has worked in the past because of the general dynamic and aerodynamic similarity that existed between different aircraft designs. The changes from straight to swept to slender wings, the wider range of powerplant positions, and the increasing structural flexibility of modern aircraft have removed these similarities. This makes it necessary to include pitching, flexibility, and if possible unsteady aerodynamic effects in the rigorous calculation of the response of an aircraft to a discrete gust. Calculations of this type have been made[3,4] using normal modes to represent flexibility and exponential approximations for the unsteady (indicial) aerodynamic effects, solving the equation of motion by step-by-step methods.

This Report shows how a single calculation can be made to provide both the time history of the transient response of a flexible aircraft to a discrete gust and also the transfer function needed for the calculation of the random response of the aircraft to continuous turbulence. This calculation of the

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*That is, not a series of non-interacting discrete events.

**By convention, level flight is taken as a normal acceleration of $1g$.

response of the aircraft is performed in the frequency domain and the transient response in the time domain recovered by the use of the Fourier inverse transform.

The oscillatory airforces on a harmonically oscillating lifting surface vary with frequency as a result of the same physical processes that cause the variation with time of the indicial aerodynamic functions. Because the aircraft response calculation is made in the frequency domain unsteady aerodynamic effects appear as a variation with frequency of the oscillatory aerodynamic generalised forces. These can be calculated by existing lifting surface computer program for arbitrary planforms in compressible flow[5], supported by the extensive knowledge of oscillatory aerodynamics gained from the study of flutter phenomena.

As an example, this method is used to calculate the transient response and transfer function for a slender-wing transport aircraft in subsonic flight through turbulence.

2. *The Response of a Flexible Aircraft to a Discrete Gust.*

2.1. *The Flexible Aircraft.*

An aircraft in flight is a flexible structure on which are imposed force systems due, among other things, to gravity, acceleration, the propulsion system and the air. This report discusses disturbances of the aircraft about a steady flight path and it is assumed that a set of aerodynamic forces balance those due to gravity and the propulsion system in this steady condition. These are henceforth omitted and only the forces and motions of perturbations about a condition of steady flight considered.

The aircraft flexibility is represented approximately by a number of normal modes with generalised co-ordinates $q_i(t)$, and the generalised forces that disturb the aircraft from its steady flight path (at this stage these need not be due to gusts) are $\mathscr{F}_i(t)$. Because the aircraft is disturbed it will experience structural and aerodynamic forces which will, if it is not unstable, tend to restore it to its trimmed flight condition (this need not be the same as the initial steady flight path). The structural response forces are due to inertia, damping and stiffness and at a time t depend only on the response acceleration, velocity and displacement at that time. Because of indicial aerodynamic effects the response airforces at time t depend on the whole history of the response and must be represented by a superposition integral.

The system is assumed to be linear and the equation of motion derived from Lagrange's equation[6]

$$A\,\ddot{q}(t)+B\,\dot{q}(t)+C\,q(t) = \mathscr{F}(t)+\int_0^t F\{t-\tau,\,\dot{q}(\tau),\,q(\tau)\}d\tau \qquad (1)$$

where A, B, C are square matrices of generalised structural inertia, damping and stiffness, $q(t)$ is a column matrix of generalised co-ordinates, $\mathscr{F}(t)$ a column matrix of disturbing generalised forces and $F\{t-\tau,$ $\dot{q}\,(\tau),\,q(\tau)\}$ a column matrix of response aerodynamic generalised forces which is a linear function of q and $\dot{q}$.

The aircraft as a dynamic system is the same whatever form of excitation is applied to it. Thus the whole range of aeroelastic problems of aircraft are in principle represented by equation (1). For example, the stability of motion when $\mathscr{F}(t) = 0$ covers flutter, divergence and aircraft stability; $\mathscr{F}(t)$ being harmonic is the problem of forced response to excitation by out of balance rotating machinery; $\mathscr{F}(t)$ being the airforces due to a transient control movement or gust represent control runaway and gust cases; $\mathscr{F}(t)$ being the airforces due to random gusts is the turbulence response problem, and associated with this is the concept of $\mathscr{F}(t)$ being the airforces due to flight through an infinite array of harmonic gusts to give the matrix of transfer functions for the harmonic responses of the aircraft to excitation by gusts.

Although equation (1) describes all aspects of the dynamics of an elastic aircraft in practice it is solved by different methods for different types of phenomena. This Report develops the equation for the transient response of a flexible aircraft to a step gust and shows how this can be obtained from the corresponding transfer function. Throughout the analysis the aircraft's forward speed is assumed constant, the system to be linear, and the controls to be fixed. However the equations could easily be extended to include the effect of an autostabiliser.

2.2. *The Use of the Step Gust.*

One feature of the discrete gust response of an aircraft that can be studied by a full dynamic analysis is the variation in the response as the gust shape and length are altered. It is not efficient to solve the full equation of motion for each gust shape. If the equation is solved for the response to a step gust then the response to any other gust shape can be calculated by the use of Duhamel's superposition integral[6]. In this case it would take the form

$$z(t) = Z(t)\,w(0) + \int_0^t Z(t-\tau)\,\dot{w}(\tau)\,d\tau \tag{2}$$

where $z(t)$ is the displacement response of some point or generalised co-ordinate of the aircraft to a gust with velocity history $w(t)$ and $Z(t-\tau)$ is the response of that point or generalised co-ordinate at time t to a unit step gust at time τ. For multi-degree-of-freedom systems these terms are matrices.

2.3. *The Relation Between Harmonic and Transient Response.*

The responses of a linear system to harmonic excitation are described in terms of 'transfer functions'. For such a system, once the starting transients have decayed, any response parameter, such as displacement, acceleration, stress, etc., varies harmonically at the frequency of, although not necessarily in phase with, the excitation. That is, if the harmonic excitation is $\bar{h}(t) = \mathscr{R}[h\,e^{i\omega}t]$, and the response is $\bar{r}(t)$, then

$$\bar{r}(t) = \mathscr{R}[r(\omega)\,e^{i\omega t}]\,, \tag{3}$$

and the transfer function $T(\omega)$ is defined by

$$r(\omega) = T(\omega)\,h\,. \tag{4}$$

The transfer function relates to a condition of non-decaying harmonic oscillation. If the system concerned is an aircraft in flight the aerodynamic excitation and response forces can be calculated rigorously to the limit of accuracy of existing computer programmes. Alternatively, forces measured on an oscillating model in a wind tunnel may be used.

If the transfer functions of a system are known then its transient reponse to transient excitation can be calculated by the use of a relation that involves the use of the Fourier transform.

The Fourier transformation[6] is a method of representing a transient function of time as a function of frequency. If the function of time $f(t)$ satisfies the condition that

$$\int_{-\infty}^{\infty} |f(t)|dt \text{—is finite} \tag{5}$$

then the Fourier transform $F(\omega)$ of $f(t)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt\,. \tag{6}$$

The inverse transform is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{i\omega t}\,d\omega\,. \tag{7}$$

If $f(t)$ does not satisfy equation (5) its transform $F(\omega)$ may still exist, but is likely to contain improper functions.

The relation between any response parameter of a linear system that can be represented by equation (1) to transient excitation is[7]

$$R(\omega) = T(\omega)\, E(\omega) \qquad (8)$$

where $E(\omega)$ is the Fourier transform of the transient excitation, $R(\omega)$ that of the transient response and $T(\omega)$ is the transfer function already defined in terms of the harmonic response. In particular the Fourier transform of a unit step (taken for example as $f(t) = 0, t \leqslant 0; f(t) = \lim_{\beta \to 0} e^{-\beta t}, t > 0$) is $1/i\omega$, so the response to a unit step is given by

$$R(\omega) = \frac{T(\omega)}{i\omega}. \qquad (9)$$

Thus the relation between the responses of an aircraft to continuous harmonic gusts and to a single step gust is very simple and is illustrated in Figure 1.

This relation between the harmonic and transient response of linear systems has been used to determine the harmonic response of an aircraft from flight measurements of the transient response to a transient control movement[8].

2.4. *The Symmetric Response to a Step Gust.*

The aircraft in symmetric perturbed motion has two rigid-body degrees-of-freedom, normally represented by heave and pitch, and an infinite number of elastic degrees-of-freedom which can in practice be represented by a limited number of symmetric normal modes. These may be those calculated for the aircraft with structural damping neglected, in which case they will be orthogonal with respect to inertia. Alternatively, measured natural vibration modes may be used. These may be only approximately orthogonal.

For each mode the principal generalised structural inertia is

$$A_{ii} = \sum_{\substack{\text{over} \\ \text{aircraft}}} l^2 f_{ri}^2 m_r \qquad (10)$$

where A_{ii} is the generalised direct inertia in mode i, l is a reference length, f_{ri} is the non-dimensional displacement at point r associated with a unit displacement of the reference point in mode i, and m_r is the mass associated with point r.

The normal or natural mode will have associated with it a natural frequency ω_i. For a structure with small damping the direct structural stiffness is given approximately by

$$C_{ii} = A_{ii}\, \omega_i^2 \, . \qquad (11)$$

This expression is exactly true if the damping is zero.

If measured modes are used and cross inertia terms included these are

$$A_{ij} = \sum_{\substack{\text{over} \\ \text{aircraft}}} l^2 f_{ri} f_{rj}\, m_r. \qquad (12)$$

This introduces the problem of how to represent the cross-stiffness terms. It will be found that if the expression

$$C_{ij} = A_{ij}\,\omega_i\,\omega_j \tag{13}$$

is used the structural stiffness matrix is symmetric, as it must be. There is however no theoretical justification for the use of equation (13).

The aerodynamic generalised forces on the aircraft during a gust encounter are of two types, as noted in equation (1). The first, $\mathscr{F}(t)$ in equation (1), are those due to the gust forces acting on the aircraft which is restrained against responding. These can be described at time t by the expression

$$\mathscr{F}_i(t) = \rho V^2 Sl \frac{w}{V} K_{2i}(t)\, Q_i'(0) \tag{14}$$

where $\mathscr{F}_i(t)$ is the generalised aerodynamic force in mode i due to encountering a step gust of velocity w at time $t = 0$, ρ is the air density, V the flight velocity, $Q_i'(0)$ the zero frequency value of $Q_i'(v) + i\,Q_i''(v)$, the non-dimensional oscillatory aerodynamic generalised force in mode i due to harmonic gusts of reduced frequency $v = \frac{\omega l}{V}$, and K_{2i} is its associated indicial aerodynamic function. K_{2i} tends to unity as t tends to infinity, and is often referred to as a Küssner function to mark that Küssner evaluated the function $K_2(t)$ for the lift on a two-dimensional wing in incompressible flow encountering a step gust.

The second set of forces, $F[t-\tau, \dot{q}(\tau), q(\tau)]$, are due to the response of the aircraft. Because the response history is not a simple step function it is necessary to build up the response airforces by superposition from the histories of the airforces after step changes in the responses in the separate modes. If the displacement of the jth generalised co-ordinate is increased as a step from zero to q_j at time τ then the generalised force in mode i due to this can be written

$$\bar{F}_{ij}(t-\tau) = \rho V^2 Sl\, q_j\, K_{1(ij)}(t-\tau)\, Q_{ij}'(0) \tag{15}$$

where $\bar{F}_{ij}(t-\tau)$ is the aerodynamic generalised force in mode i at time t due to a step displacement in mode j at time τ, $Q_{ij}'(0)$ is the zero frequency value of $Q_{ij}'(v) + ivQ_{ij}''(v)$, the non-dimensional oscillatory aerodynamic generalised force in mode i due to motion in mode j at reduced frequency v, and $K_{1(ij)}(t-\tau)$ is the associated indicial aerodynamic function. $K_{1(ij)}(t-\tau)$ tends to unity as t tends to infinity, and is often referred to as a Wagner function. Wagner evaluated the function $K_1(t)$ for the lift on a two-dimensional wing in incompressible flow following a step change of heaving velocity.

The exception to equation (15) is for motion in translational modes such as heave, or an assumed bending mode with no associated torsional deformation. Here there is no force due to steady displacement but only due to steady motion. To maintain the relation that makes $K_{1(im)}(t-\tau)$ tend to unity as t tends to infinity (mode m being translation) requires the airforces in this case to be defined as

$$\bar{F}_{im}(t-\tau) = \rho V^2 Sl^2 \frac{\dot{q}_m}{V} K_{1(im)}(t-\tau)\, Q_{im}''(0). \tag{16}$$

For the case of general response the aerodynamic generalised forces at time t are built up by the use of Duhamel's integral

$$F_{ij}(t) = \rho V^2 Sl\, Q_{ij}'(0) \int_0^t K_{1(ij)}(t-\tau)\, \dot{q}_j(\tau)\, d\tau \tag{17}$$

and

$$F_{im}(t) = \rho V Sl^2\, Q_{im}''(0) \int_0^t K_{1(im)}(t-\tau)\, \ddot{q}_m(\tau)\, d\tau. \tag{18}$$

The equation of motion for the response of the aircraft to a step gust at time $t = 0$ can now be written. If the aircraft is represented by n degrees-of-freedom (of which p are translational modes) the single matrix equation of motion consists of n linear simultaneous differential equations, of which the ith is

$$\sum_{j=1}^{n} (A_{ij}\ddot{q}_j(t)+B_{ij}\dot{q}_j(t)+C_{ij}q_j(t))$$
$$= \rho VSlw\, K_{2i}(t)\, Q_i'(0)+\rho VSl^2 \sum_{m=1}^{p} Q_{im}''(0)\int_0^t K_{1(im)}(t-\tau)\ddot{q}_m(\tau)\, d\tau+$$
$$+\rho V^2Sl \sum_{j=p+1}^{n} Q_{ij}'(0)\int_0^t K_{1(ij)}(t-\tau)\, \dot{q}_j(\tau)\, d\tau. \qquad (19)$$

This equation can be conveniently reduced to non-dimensional form by making the substitutions

$$s = \frac{Vt}{l}, \qquad \sigma = \frac{V\tau}{l}, \qquad a_{ij} = \frac{A_{ij}}{\rho Sl^3}, \qquad b_{ij} = \frac{B_{ij}}{\rho Sl^2V}, \qquad c_{ij} = \frac{C_{ij}}{\rho SlV^2}$$

and noting that

$$\frac{d}{dt} = \frac{V}{l}\frac{d}{ds}, \qquad \frac{d^2}{dt^2} = \frac{V^2}{l^2}\frac{d^2}{ds^2}. \qquad (20)$$

The indicial aerodynamic functions are rewritten in terms of the new variable s, $K(s)$ meaning $K(t)$ at a value of t corresponding to s. Equation (19) can now be written in non-dimensional form as

$$\sum_{j=1}^{n} (a_{ij}\ddot{q}_j(s)+b_{ij}\,\dot{q}_j(s)+c_{ij}q_j(s))$$
$$= \frac{w}{V}K_{2i}(s)\, Q_i'(0)+\sum_{m=1}^{p} Q_{im}''(0)\int_0^s K_{1(im)}(s-\sigma)\ddot{q}_m(\sigma)\, d\sigma+$$
$$+ \sum_{j=p+1}^{n} Q_{ij}'(0)\int_0^s K_{1(ij)}(s-\sigma)\, \dot{q}_j(\sigma)\, d\sigma. \qquad (21)$$

Because of the superposition integrals it contains equation (21) can only be solved in the time domain by using step-by-step methods.

2.5. *Solution of the Equation of Motion by the use of the Fourier Transformation.*

It has been suggested previously[9, 10] that a transform relation can be used to solve the equation of motion, equation (21). One immediate difficulty is that the Fourier transforms of some of the rigid body responses of an aircraft to a step gust do not exist. The responses concerned are the heave velocity and displacement and the pitch displacement. This non-existence appears as a singularity in the transform at zero frequency. However, if calculations are always confined to the acceleration responses to a step gust, and if the aircraft is stable, transforms of all the response histories will exist. Once the transient acceleration response has been calculated the velocity and displacement histories can be found by integration.

Before solving equation (21) by the use of the Fourier transform certain basic relations, in addition to equations (6) and (7), are needed. These are firstly, addition

$$A(\omega)+B(\omega) = \int_{-\infty}^{\infty} [a(t)+b(t)]\, e^{-i\omega t}\, dt \tag{22}$$

and secondly, multiplication (also called convolution, because of the term $a(t-\tau)\, b(\tau)$)

$$A(\omega)\, B(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \int_{0}^{t} a(t-\tau)\, b(\tau)\, d\tau\, dt. \tag{23}$$

In this application the integral $a(t-\tau)\, b(\tau)$ is only non-zero in the interval $0<\tau<t$. The transforms of the indicial aerodynamic functions K will be denoted by $\bar{K}$; of q, $\bar{q}$; of $\ddot{q}$, $\bar{\ddot{q}}$ and we note that $\bar{\ddot{q}} = -v^2\, \bar{q}$. The non-dimensional parameters s and v will be used.

The Fourier transform of equation (21) may now be taken, applying equations (6), (7), (22) and (23) term by term and using $\ddot{q}$ as the response variable to ensure its existence. This leads to

$$\sum_{j=1}^{n} (-v^2\, a_{ij}+iv\, b_{ij}+c_{ij})\bar{\ddot{q}}_j(v)$$

$$= -v^2 \frac{w}{V} Q_i'(0)\, \bar{K}_{2i}(v) - v^2 \sum_{m=1}^{p} Q_{im}''(0)\, \bar{K}_{1(im)}(v)\, \bar{\ddot{q}}_m(v) +$$

$$+iv \sum_{j=p+1}^{n} Q_{ij}'(0)\, \bar{K}_{1(ij)}(v)\, \bar{\ddot{q}}_j(v). \tag{24}$$

Now a key relation in this work is that due to Garrick[11] connecting the oscillatory and indicial aerodynamic functions. In the notation of this Report the relation is

$$\left.\begin{aligned} \bar{K}_{1(im)} &= -\frac{1}{Q_{im}''(0)} \left\{ \frac{Q_{im}'(v)}{v^2} + \frac{i\, Q_{im}''(v)}{v} \right\} \\ \bar{K}_{1(ij)} &= \frac{1}{Q_{ij}'(0)} \left\{ Q_{ij}''(v) - \frac{i\, Q_{ij}'(v)}{v} \right\} \\ \bar{K}_{2i} &= \frac{1}{Q_i'(0)} \left\{ \frac{Q_i''(v)}{v} - \frac{i\, Q_i'(v)}{v} \right\} \end{aligned}\right\} \tag{25}$$

These relations are developed by superposing a number of steps to form a harmonic oscillation and calculating the resulting oscillatory aerodynamic generalised force. They can be used simply to calculate indicial aerodynamic functions from oscillatory aerodynamic forces. Alternatively, substituting them in equation (24) leads to

$$\sum_{j=1}^{n} \left(-v^2\, a_{ij}+iv\, b_{ij}+c_{ij} - Q_{ij}'(v) - iv\, Q_{ij}''(v) \right) \bar{q}_j(v) = -v \frac{w}{V} \left(Q_i''(v) - i\, Q_i'(v) \right). \tag{26}$$

Equation (26) is solved to give n sets of response transforms $\bar{\ddot{q}}_j(\nu)$. These are transformed back to the time domain to give the transient response $\ddot{q}_j(s)$ by the use of equation (7). Since $\bar{\ddot{q}}_j(\nu)$ is likely to be known as a function defined numerically rather than analytically the inverse transformation process is a simple numerical integration. A computer program to perform this integration is described in the Appendix.

Equation (26) can be obtained by a different approach. If the aircraft is flying through an infinite array of harmonic gusts the excitation forces $\mathscr{F}_i(t)$ of equation (1) are the gust forces of which the non-dimensional form is $Q_i'(\nu)+i\,Q_i''(\nu)$, while the response airforces F_{ij} are the normal oscillatory aerodynamic generalised forces $Q_{ij}'(\nu)+i\nu\,Q_{ij}''(\nu)$. Thus equation (1) gives

$$\sum_{j=1}^{n}\left(-\nu^2 a_{ij}+i\nu\, b_{ij}+c_{ij}-Q_{ij}'(\nu)-i\nu\, Q_{ij}''(\nu)\right)\ddot{q}_j(\nu) = -\nu^2\frac{w}{V}\left(Q_i'(\nu)+i\,Q_i''(\nu)\right). \qquad (27)$$

In this equation the matrix $\ddot{q}_j(\nu)\Big/\left(\frac{w}{V}\right)$ is the transfer function for the acceleration response in the generalised co-ordinates due to gust excitation and, following equation (9), if this is divided by $i\nu$ it will be found to give the transform of the transient response to a step gust as calculated by equation (26).

It will be remembered that equations (21), (26) and (27) apply to perturbations of the aircraft from a condition of steady flight; that the forward speed is assumed constant; that the aircraft is a linear dynamic system; and that the controls are fixed, both during harmonic motion (equation (27)) and during the response to a step gust at $s = 0$ (equations (21) and (26)).

3. *Application to a Slender-wing Aircraft.*

3.1. *The Aircraft.*

An example of the use of the method outlined in Section 2 is the calculation of the harmonic and transient response of the slender-wing aircraft shown in Fig. 2. This is a project design of which engineering aspects have been studied by the British Aircraft Corporation (Operating) Ltd., Filton Division. Principal data for the aircraft are given in Table 1.

The symmetric response only is considered and the aircraft is represented by the rigid-body degrees-of-freedom heave and pitch about the rear spar. Flexibility is represented by the first four calculated normal modes (Fig. 3). These were provided by the British Aircraft Corporation[12]. The flight condition considered is a Mach number of 0·8 at an altitude of 20 000 feet.

The wing oscillatory aerodynamic generalised forces were calculated by the computer program RAE 161A[5], using the 35 collocation points shown in Fig. 4. The same program was used to calculate the generalised forces due to flight through an infinite array of harmonic gusts. This cannot be done directly as the program calculates aerodynamic forces due to modes whose displacements are related to axes that move with the undisturbed motion of the aircraft

$$\frac{z_j(x,y,t)}{l} = f_{rj}(x,y)\, q_{j0}\, e^{i\omega t} \qquad (28)$$

where q_{j0} is the peak amplitude of the jth generalised co-ordinate. To represent the case of flight through gusts a mode shape is required that sweeps back over the wing at the flight speed. This should take the form[13]

$$\frac{w(x,y,t)}{V} = e^{i\omega(t-\frac{x}{V})}. \qquad (29)$$

It is found that if mode shapes of the form

$$\left.\begin{aligned} f_{1r}(x) &= \frac{x}{l}\sin\frac{\omega x}{V} \\ f_{2r}(x) &= \frac{x}{l}\cos\frac{\omega x}{V} \end{aligned}\right\} \tag{30}$$

are taken, and these are combined as

$$f_r(x) = f_{2r}(x) - i f_{1r}(x) \tag{31}$$

so that the displacements are

$$\frac{z}{l} = \left(\frac{x}{l}\cos\frac{\omega x}{V} - i\frac{x}{l}\sin\frac{\omega x}{V}\right)e^{i\omega t} \tag{32}$$

then the downwash due to this mode is

$$\frac{w}{V} = e^{i\omega(t-\frac{x}{V})} . \tag{33}$$

Fuselage and nacelle aerodynamic loading, and the effects of separated leading edge vortices, were ignored.

3.2. *Details of the Calculation.*

The calculation of the transient response of the aircraft to a step gust can be broken into several sections.

(i) Calculation of the oscillatory aerodynamic generalised forces by program RAE 161A for a number of reduced frequencies. This requires information about the aircraft geometry, mode shapes and Mach number. For the particular calculation described the typical length was 78 feet (the wing span) and the flight speed 830 ft/sec. Reduced frequencies of 0·01, 0·76, 1·6376, 2·8364, 4·9127, 6·8475 and 10·5856 were used, but the results for the highest frequency were considered to be very inaccurate and were not carried on to later sections of the calculation.

(ii) The aerodynamic generalised forces are plotted against reduced frequency to allow interpolation. Some typical examples are shown in Fig. 5.

(iii) Using a simple computer program, equation (27) is solved to give the transfer function. This requires the aircraft structural data and, for each frequency at which the transfer function is calculated, the aerodynamic generalised forces. At this stage of the calculation varying the kinetic pressure at constant Mach number is possible by straightforward scaling of the structural data. The calculation is in non-dimensional terms for a gust velocity w/V of unity. To obtain the acceleration in g units for a gust velocity w the non-dimensional acceleration must be factored by wV/gl.

(iv) The transfer functions are plotted as vector diagrams to allow interpolation. A typical example is shown in Fig. 6.

(v) At suitable frequencies the transform of a response is calculated

$$\bar{\ddot{q}}_R(\nu) + i\,\bar{\ddot{q}}_I(\nu) = \frac{\ddot{q}_I(\nu)}{\nu} - i\frac{\ddot{q}_R(\nu)}{\nu} \tag{40}$$

where the suffices R and I indicate real and imaginary parts.

(vi) At chosen values of s the physical response is recovered by the use of a computer program to evaluate separately

$$\ddot{q}(s) = \frac{2}{\pi}\int_0^\infty \bar{\ddot{q}}_R(\nu) \cos \nu s \, d\nu$$

$$\ddot{q}(s) = -\frac{2}{\pi}\int_0^\infty \bar{\ddot{q}}(\nu) \sin \nu s \, d\nu. \qquad (41)$$

Since this Report was prepared the process has been automated and it is only necessary to supply the response program with the aerodynamic generalised forces at the frequencies at which they were calculated. This automated program prints out the transfer function at preselected frequencies and the transient response at preselected times.

3.3. *Calculation Results.*

For the slender-wing aircraft of this example the non-dimensional transfer functions for each degree-of-freedom are shown in Fig. 7 and the dimensional transient responses to a step gust in Fig. 8. The responses plotted are the accelerations in pitch, in the four elastic modes at points on the structure for each mode at which $f_{rj} = 1{\cdot}0$, and at a point on the structure that coincides with the centre of gravity when the aircraft is in steady flight. This latter parameter is not used as a basic degree-of-freedom but is a linear combination of the six used. It is plotted as having more physical significance than the acceleration in the heave mode alone. These responses together with information on the mode shapes allow the transfer function and transient response at any point on the aircraft to be calculated. As examples Fig. 9 shows the transient responses of the structure at the centre of gravity, at the wing apex and at the wing tip. To show the effect of flexibility and pitch the transient acceleration response at a point on the structure near the centre of gravity for the aircraft with one (heave) and two rigid degrees-of-freedom are compared with that of the flexible aircraft in Fig. 10.

Calculation of the transient response to a step gust from the transfer function is a purely mathematical process, so that the accuracy of this step is set solely by the programs used. Three examples of what can be achieved are given. The first is to compare the response for pitching acceleration obtained from the real and imaginary parts of the transform separately. These should be the same, and the average error is only 5·7 per cent of the peak value of the response. This is shown in Fig. 11.

A second test is to use the same program to calculate the transform of this response. This should, after the insertion of a factor $2/\pi$, be the same as the original transform. These are compared in Fig. 12. This is a severe test of the transformation program as both the response and its transform are very erratic functions. The average error is 9·5 per cent of the peak value which is roughly double the error of the single transformation shown in Fig. 11. An example of a double transform of a smooth function is given in Fig. 13 and it will be seen that the accuracy for this is very much improved.

The third check is to compare the response of the aircraft to that obtained from a step-by-step solution. This latter was done by the British Aircraft Corporation[12] and differs from the transform solution in that the response aerodynamic forces are considered to depend only on the instantaneous response rather than the whole response history. On a low aspect ratio wing this is not an unreasonable assumption. It is of course possible to simulate this calculation exactly by the transform method by taking the response oscillatory aerodynamic generalised forces to be independent of frequency. In view of the actual small variation of these with frequency at the lower frequencies (Fig. 5) the author did not consider it worthwhile repeating the calculation simply to obtain exact equivalence in the comparison. The results of the two calculations are shown in Fig. 14.

4. *Aircraft Loading.*

Although the transient response of an aircraft to a gust is important and interesting, of even greater importance is the transient stress distribution during the gust encounter. This requires the calculation of the loading on the aircraft, and this can readily be obtained by the transform method.

A computer program RAE 263 A, exists[5] that calculates the pressure at chosen points on a lifting surface due to oscillatory motion in defined modes. These pressures are defined by

$$p_r = \rho V^2 e^{i\omega t} \sum_{j=1}^{n} (l'_{rj} + i\, l''_{rj})\, q_{j0} \tag{42}$$

where p_r is the oscillatory pressure at point r and $l'_{rj} + i\, l''_{rj}$ is the non-dimensional oscillatory pressure at point r due to mode j. The transform of the pressure at point r due to a step gust and the subsequent response of the aircraft is then

$$\bar{p}_r = \rho V w \left(\frac{l''_r}{\nu} + i \frac{l'_r}{\nu} \right) + \sum_{j=1}^{n} \rho V^2 (l'_{rj} + i\, l''_{rj})\, \bar{q}_j \tag{43}$$

where $l'_r + i\, l''_r$ is the non-dimensional oscillatory pressure at point r due to flight through harmonic gusts. To the loading due to pressure associated with point r can be added that due to inertia, and the whole transformed back into the time domain.

5. *Discussion.*

So far as the author is aware the calculations presented in this paper are the first for a flexible aircraft to allow rigorously for unsteady aerodynamic effects. Calculations by the British Aircraft Corporation (Operating) Ltd., Filton Division, for the same slender-wing aircraft have shown that these effects do modify the response considerably, if they are included in the forcing terms only. Their inclusion in the response aerodynamics has less effect. The modification is greatest in the elastic response terms and thus is most apparent at the extremities of the aircraft. Since a low aspect-ratio aircraft will be least affected by unsteady aerodynamic effects it can be concluded that these are, in general, worth including. Their effect on the loading as opposed to the response of the aircraft is not yet known.

The most dubious assumptions in the example calculation are both specific to the aircraft considered and are not associated with the method of analysis. These assumptions are that the aerodynamics are linear and that the flexibility of the structure can be represented by only four normal modes. The airflow over the wing does in fact separate to form leading edge vortices in the incidence range of this example, and this will produce aerodynamic non-linearities. However, because the response of the aircraft is a perturbation from a steady case it can be argued that the perturbation aerodynamics will be approximately linear. The whole system must, of course, be linear for the method described here to work.

The structural assumption that four modes adequately represent the flexibility of the aircraft is justified by the calculations of both the author and the British Aircraft Corporation. These have included up to five symmetric normal modes and it appears that provided the first to third modes are included the flexibility of the aircraft is indeed represented accurately, and that adding further modes does not modify the response significantly. To obtain the loading accurately may require the use of more modes, and this result only applies to the integrated shape of aeroplane used in this example. On a more conventional design Ref. 4 reports that it was necessary to use 18 modes to obtain adequate results for loading.

The calculated transient response of the slender-wing aircraft shows several marked differences to that of a more conventional design. The first is that because of the long root chord the aircraft initially tends to pitch nose up on entering an up-gust. The second is that, because there is relatively little lag in the build up of lift after the incidence is changed, the peak acceleration occurs early in the response history. At the time of the peak acceleration there is little heaving response and the details of the response aerodynamic forces due to heave have little effect on the peak acceleration. Thirdly, there is a large interaction between the elastic deformation of the wing and the overall pitching of the aircraft. This may be due to the elastic response modifying the wing camber, in which case it will only be a feature of the response of large, low aspect ratio, aircraft. The first and last of these results has been noted by Huntley[18].

The first two effects combine to give a much larger gust alleviation factor than is predicted by any of the empirical methods in general use. The factors predicted by Refs. 15, 16 and 17 are respectively 0·84, 0·72 and 0·73. The full dynamic calculation for the response to a gust with a ramp length of 100 feet gives an alleviation factor at the centre of gravity of the rigid aircraft of 0·99, with an elastic overshoot of the structure in this area to a factor of 1·12.

However, the effect of elasticity is such when the incremental acceleration at the centre of gravity due to a step gust is a maximum at $1{\cdot}57g$, that at the wing tip is $-8{\cdot}4g$ and at the wing apex $1{\cdot}8g$. These accelerations are reduced by increasing the gust ramp length. For a length of 100 feet the maximum acceleration at the wing tip comes down from $10{\cdot}5g$ to $4{\cdot}9g$ and occurs at a different point in the response history. On the structure near the centre of gravity, however, the peak acceleration is only reduced from $1{\cdot}57g$ to $1{\cdot}52g$. The variation of acceleration over the aircraft does show the degree to which the uniform acceleration concept of the conventional discrete gust approach is in error.

One point of interest is that the pitch angle of the aircraft does not return to zero, nor the heave velocity tend to the gust velocity, when the short period motion following gust entry has ceased. In general the aircraft weathercocks into the gust, but the final condition can vary from a nose down pitch attitude of $\frac{w}{V}$ with no heave velocity through a horizontal attitude with a heave velocity w to a nose up attitude with a heave velocity $> w$. These conditions occur as the manoeuvre margin is progressively decreased towards zero. This long term behaviour of the aircraft is determined very readily by the transformation method but would be difficult to calculate by any step-by-step method. In particular, the response an infinite time after gust entry can be found directly from the response transform without leaving the frequency domain. For pitch angle, as an example

$$q_2(\infty) = \int_0^\infty \dot{q}_2(t)\,dt = \bar{\dot{q}}_2(0).$$

The relation between the transient and harmonic responses emphasizes the fact that the aircraft is a single dynamic system, and that once the equation of motion of this system is written down any aspect of the dynamic characteristics can be calculated. As an example of this the transfer function has been used to calculate the flutter speed and sub-critical response of the slender-wing aircraft.

This is done by recalculating the transfer function at a number of values of the kinetic pressure, this corresponding to increasing the equivalent air speed at constant Mach number. The transfer function is divided by the ratio of kinetic pressures (to eliminate changes due solely to changing the ratio of the aerodynamic exciting forces to the structural inertia) and the result drawn as a vector plot. Fig. 15a shows the transfer function for the third elastic mode. This plots as a circle which at 360 kt is traced in a clockwise direction as frequency is increased. Increases of kinetic pressure change the circle radius until at the critical speed the radius becomes infinite. Beyond this speed the transfer function plots as a more complicated form with values tending to infinity at two frequencies. Between these frequencies it has finite values but is traversed in an anticlockwise direction with increasing frequency.

From the vector plots the damping in mode 3 is assessed at the various kinetic pressures and plotted against equivalent air speed in Fig. 15b. The critical flutter speed is 625 kt E.A.S. and the flutter frequency 4·25 cs. These compare well with a prediction by the British Aircraft Corporation of a flutter speed of 640 kt E.A.S. and a frequency of 4·4 cs at a Mach number of 0·85. This latter calculation included 10 normal modes and the agreement with the author's one using four modes only suggests that the higher frequency modes have very little effect on the lower frequency dynamic characteristics of the aircraft. It should also be noted that the predicted flutter speeds are well outside the flight envelope of a typical supersonic transport aircraft.

6. *Conclusions.*

(i) The transient response of an aircraft to discrete transient excitation can be obtained from the transfer function by means of a relationship involving the Fourier transform. By this means the bulk of the response calculation is carried out in the frequency domain and the aerodynamic data required can be obtained from existing lifting surface oscillatory aerodynamic programs. These aerodynamic terms will include unsteady effects, whether these appear as variation with frequency of oscillatory aerodynamic derivatives or as variation with time of indicial aerodynamic functions.

(ii) Calculations for both the transfer function and the response to a step gust, this latter by the Fourier transform method, have been made for a slender-wing aircraft the dynamics of which were represented by two rigid-body and four elastic degrees-of-freedom. These agree well with calculations for the gust response by a conventional step-by-step integration method when, in the latter, the forcing terms due to the gust are derived from an oscillatory lifting-surface theory and include unsteady aerodynamic effects.

(iii) Comparison with calculations of the gust response for the slender-wing aircraft based on the assumption that the local pressure on the wing is related only to the instantaneous local incidence shows that it is necessary to include unsteady aerodynamic effects in the exciting forces, even on a relatively slender aircraft, as these modify the elastic response of the aircraft significantly. It is not so necessary to include unsteady effects in the response aerodynamic terms on this aircraft.

(iv) Full dynamic calculations for the slender-wing aircraft show that the accelerations obtained from the empirical methods laid down by British and American airworthiness authorities are unconservative by factors of between 25 per cent and 35 per cent for this aircraft.

LIST OF SYMBOLS

A	Generalised inertia matrix
A_{ij}	Element of the generalised inertia matrix
a_{ij}	Element of the non-dimensional generalised inertia matrix, $a_{ij} = A_{ij}/\rho Sl^3$
B	Generalised structural damping matrix
B_{ij}	Element of the generalised structural damping matrix
b_{ij}	Non-dimensional element of the generalised structural damping matrix, $b_{ij} = B_{ij}/\rho Sl^2 V$
C	Generalised structural stiffness matrix
C_{ij}	Element of the generalised structural stiffness matrix
c_{ij}	Non-dimensional element of the generalised structural stiffness matrix, $c_{ij} = C_{ij}/\rho SlV^2$
$E(\omega)$	Fourier transform of excitation
F	Gust alleviation factor
$\mathscr{F}(t)$	Matrix of generalised excitation forces
$F\{t-\tau, \dot{q}(\tau), q(\tau)\}$	Matrix of generalised aerodynamic response forces
$F_{ij}(t)$	Element of $F\{t-\tau, \dot{q}(\tau), q(\tau)\}$
$F_{ij}(t-\tau)$	Generalised force in mode i due to a step in mode j at time τ
$F(\omega)$	Function of frequency
$f(t)$	Function of time
f_{rj}	Displacement at point r for a unit displacement of the jth generalised co-ordinate
g	Acceleration of gravity
h	Amplitude of harmonic exciting force
$I(\Omega)$	Value of an integral to a limit Ω
i	$\sqrt{-1}$
i, j	Mode reference parameters
$K_{2i}(t)$	Aerodynamic indicial function for the generalised force in mode i due to gust penetration

$K_{1(ij)}(t)$	Aerodynamic indicial function for the generalised force in mode i due to motion or displacement in mode j
l	Reference length
M	Mach number
m_r	Mass at point r
Δn	Incremental acceleration in g units
$Q'_{ij}+i\nu\, Q''_{ij}$	Non-dimensional oscillatory aerodynamic generalised force, $Q_{ij}(t) = \rho V^2 Sl\,(Q'_{ij}+i\nu\, Q''_{ij})\, q_{jo}\, e^{i\omega t}$
$Q'_i+i\, Q''_i$	Non-dimensional oscillatory aerodynamic generalised force due to harmonic gusts, $Q_i(t) = \rho VwSl\,(Q'_i+i\, Q''_i)\, e^{i\omega t}$
q_j	Generalised co-ordinate for the jth mode
q_{jo}	Maximum amplitude of q_j when q_j varies harmonically
$\bar{q}_j$	Fourier transform of q_j
$R(\omega)$	Fourier transform of the response
r	Location parameter
S	Reference area
s	Non-dimensional time or distance, $s = Vt/l$
$T(\omega)$	Transfer function
t	Time
V	Flight speed
w	Gust velocity
x	Chordwise co-ordinate
y	Spanwise co-ordinate
z	Normal co-ordinate

$Z(t-\tau)$	Response of a point at time t to an excitation step at time τ
γ	Damping ratio
ν	Reduced frequency, $\nu = \omega l/V$
ρ	Air density
σ	Non-dimensional time or distance, $\sigma = V\tau/l$
τ	Time
ω	Angular frequency
ω_j	Natural frequency of the jth mode

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16	.. ..	British civil airworthiness requirements Air Registration Board.
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TABLE 1

Principal Characteristics of the Slender-wing Aircraft

Span	78 ft
Length (omitting nose boom)	153·5 ft
Root chord	110·8 ft
Mean chord $\bar{c}$	49·2 ft
Wing area	3840 ft^2
Reference area S	3042 ft^2
Reference length l	78 ft
Centre of gravity (aft of wing apex)	70·8 ft
Static margin	3·04 per cent $\bar{c}$
Weight	219 600 lb
Air density at 20 000 ft	0·00127 slug/ft^3
Flight speed at $M = 0{\cdot}80$	830 ft/sec

APPENDIX

The Fourier Transformation Program

A Mercury Autocode program has been written to evaluate the transformation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega. \tag{A1}$$

In this work $f(t)$ is a real function which is zero for $t<0$ and $F(\omega) = F_R(\omega)+i\,F_I(\omega)$ where $F_R(\omega)$ is even in ω and $F_I(\omega)$ is odd. $f(t)$ is the sum of even and odd functions of t. These are equal in magnitude at a given t, add for $t>0$ and cancel when $t<0$. Thus for $t>0$ $f(t)$ can be evaluated separately from the real and imaginary parts of $F(\omega)$.

$$f(t) = \frac{2}{\pi}\int_{0}^{\infty} F_R(\omega)\cos\omega t\, d\omega \tag{A2}$$

$$= -\frac{2}{\pi}\int_{0}^{\infty} F_I(\omega)\sin\omega t\, d\omega. \tag{A3}$$

This transformation requires the existence of the function $F(\omega)$ in an integrable form. If $f(t)$ is one of several classes of function $F(\omega)$ will contain one or more singularities. For example, unless $f(t)\rightarrow 0$ as $t\rightarrow\infty$, $F(\omega)$ will contain improper functions, such as $\delta(\omega)$. Also, if $f(t) = \sin\omega_0 t$ then $F(\omega) = \pi\,(\delta(\omega-\omega_0)-\delta(\omega+\omega_0))$. For a marginally damped structure near its flutter condition $F(\omega)$ will have a tall, narrow, peak at the appropriate frequency.

These difficulties can be overcome in a number of ways. For the case of $f(\infty) \neq 0$ the transform evaluated can be that of the function $f(t)-f(\infty)$. Similarly, if $F(\omega)\rightarrow\infty$ as $\omega\rightarrow 0$ a higher derivative (obtained by multiplying $F(\omega)$ by ω to the appropriate power) can be transformed, and the solution for the required derivative obtained by integration in the time domain. In this case it is necessary to check that $F(\infty)$ still tends to zero; this is usually true for real systems.

When $F(\omega)$ has a narrow peak at ω_0 (corresponding to a lightly damped mode of the system with a circular frequency ω_0) it is necessary to use closely spaced points to define $F(\omega)$ near ω_0. As the peak gets narrower a stage is reached at which the integration step length in the existing program is too short to define the peak correctly. This will lead to distortion of the function $f(t)$ at small values of t, because the step length is proportional to $1/t$. With the present program and a peak at $\nu = \omega l/V = 2$, the first cycle of the oscillation $(0<t<3l/V)$ will be distorted if the damping is 2 per cent critical, and the first two cycles will be affected if the damping is only 1 per cent critical. This distortion can, of course, be overcome by the adoption of a shorter step length for the integration program. The present step length has been chosen to give moderate accuracy for normal systems while keeping the computing time small. It is in no way an optimum step length, and could be changed without affecting the principle of the program described here.

In detail, the task of the program is to store a function of frequency $F(\omega)$; to read a value of a time parameter t; to form the function $F(\omega)$ cos ωt or $F(\omega)$ sin ωt, and to integrate this function over the frequency range zero to infinity.

Since it is not possible actually to integrate numerically to infinity some finite limit must be introduced. If the function $F(\omega)$ shown in Fig. 16a is multiplied by sin ωt and integrated between the limits zero to Ω the variation of the value of the integral $I(\Omega)$ with the limit Ω is as shown in Fig. 16b. It will be noticed that $I(\Omega)$ oscillates about the value $I(\infty)$ with a period of π/t, and that the function crosses the final value near $\Omega t = (2n+1)\,\pi/2$, where n is an integer. If $F(\omega)$ decreases monotonically with increasing ω for large values of ω then for consecutive values of n $I(\Omega)$ is alternate sides of the final value.

Thus the procedure adopted is to integrate to a value of n of 3 and store the value of $I(7\pi/2)$; continue the integration to an n of 4 and note $I(9\pi/2)$; take the final value as the average of these two. In some cases it is necessary to integrate to beyond $\omega = 9\pi/2t$. The reason for this is that at large values of t the value of ω to give $\omega t = 9\pi/2$ is small and could be below values of ω at which the function $F(\omega)$ is still varying significantly.

The representation of the function $F(\omega)$ has been a problem. This was originally stored as a polynomial expression in ω that was calculated to give a least-squares fit to the curve of $F(\omega)$ read by the computer as a number of points $(\omega, F(\omega))$. This was extended to a two portion piecewise polynomial curve fitting which could be made to work for smooth functions but which was quite incapable of handling the rather erratic functions that occur in the case of the flexible aircraft.

The final program reads the values of $F(\omega)$ at a large number of values of ω and evaluates the function at intermediate points by linear interpolation. This, though crude, works remarkably well in practice.

A flow diagram for the program is given in Fig. 17. Points to note are that the integration limit and integration interval change at a preset value of t, this allowing for both the limit problems mentioned earlier and for the fact that at small values of t the interval is set by variation of $F(\omega)$, while for large values of t it is set by variation of the sin ωt term.

It is of interest to note that the change in the frequency limit of integration is very large, from $9\pi/2t$ to $57\pi/2t$, yet for points at which the transform has been evaluated using both limits the final results are very similar.

The integration limit is also subject to a cut-off at the highest frequency for which $F(\omega)$ is defined.

The program for the real part of the transform is similar except that the integral is near its final value at values of $\omega t = n\pi$.

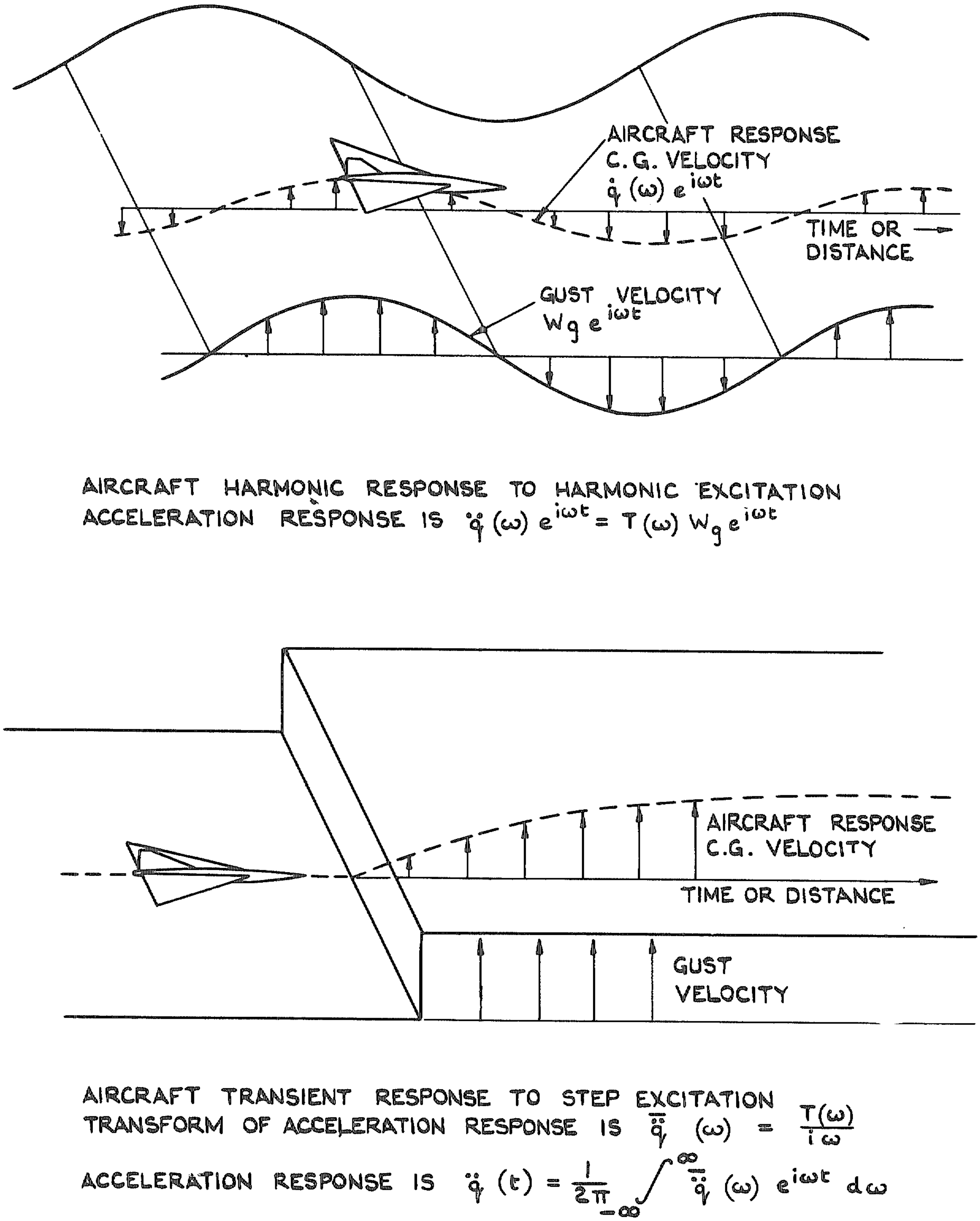


FIG. 1. Types of gust response.

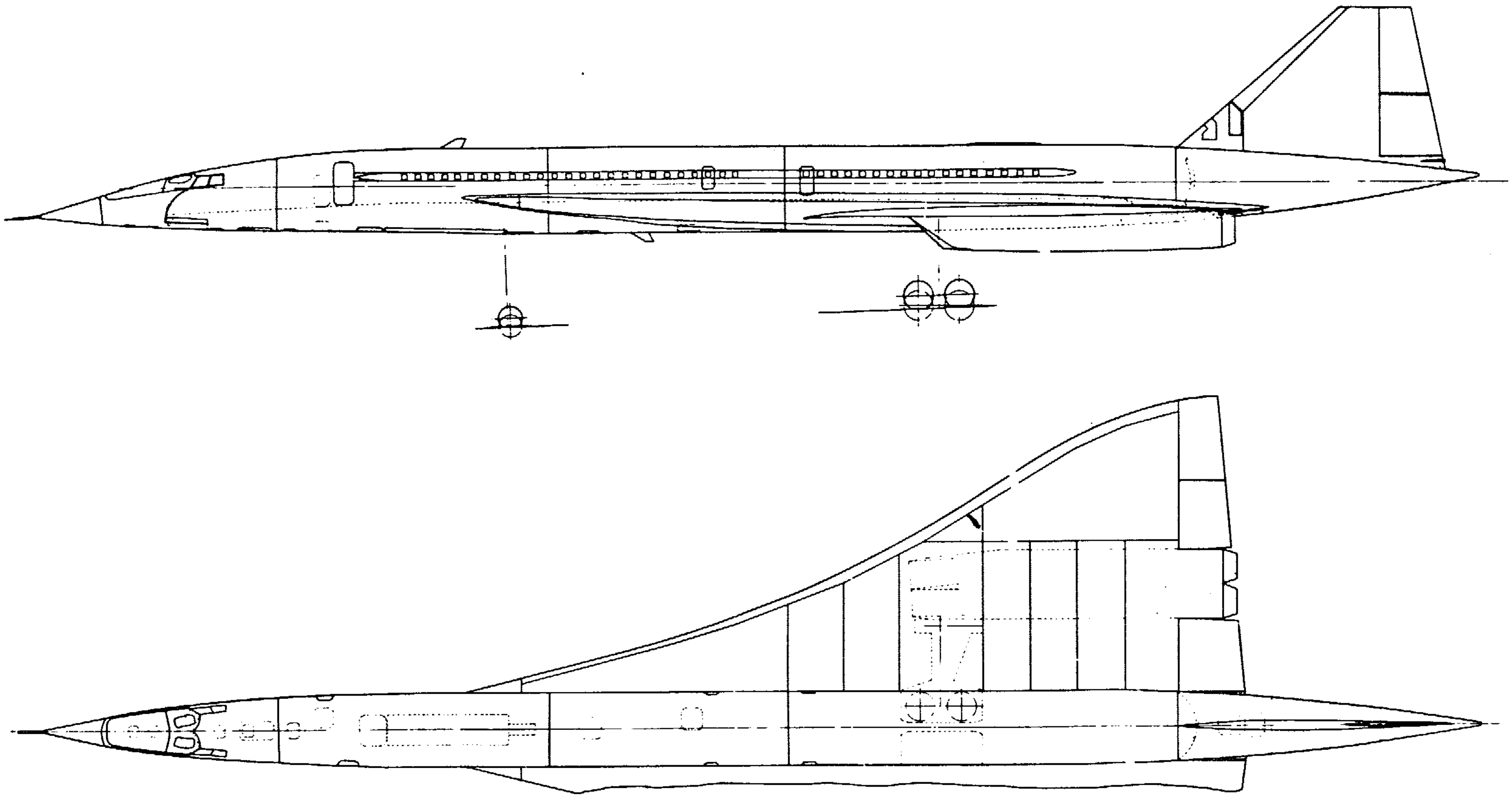


FIG. 2. The slender-wing aircraft.

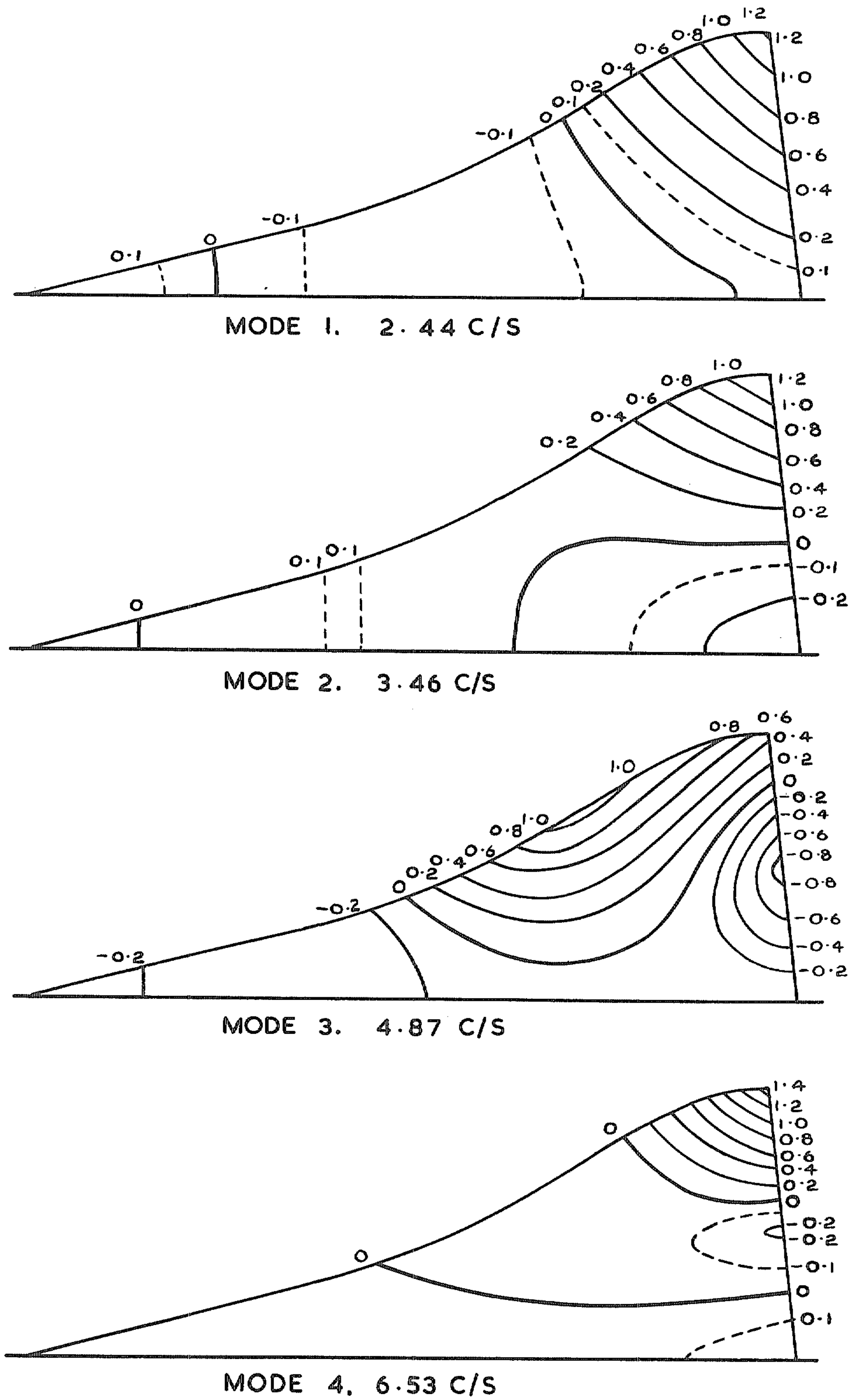


FIG. 3. Normal modes for the slender-wing aircraft.

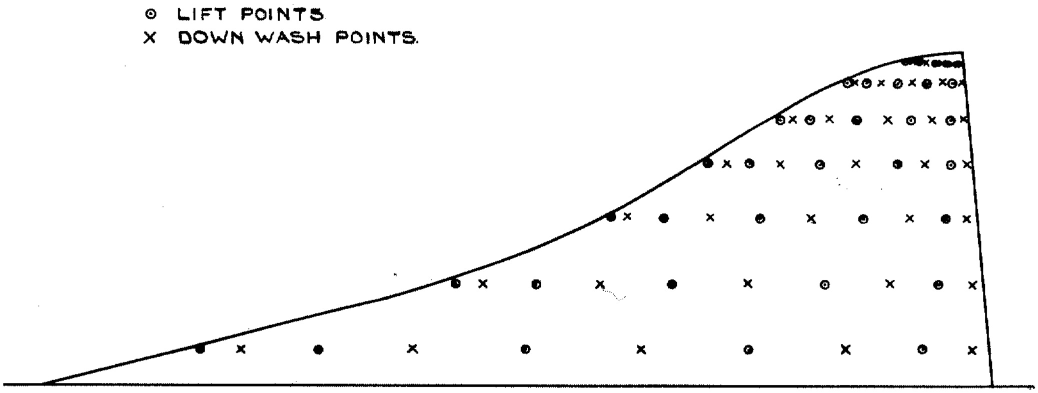


FIG. 4. Collocation points on the slender-wing aircraft.

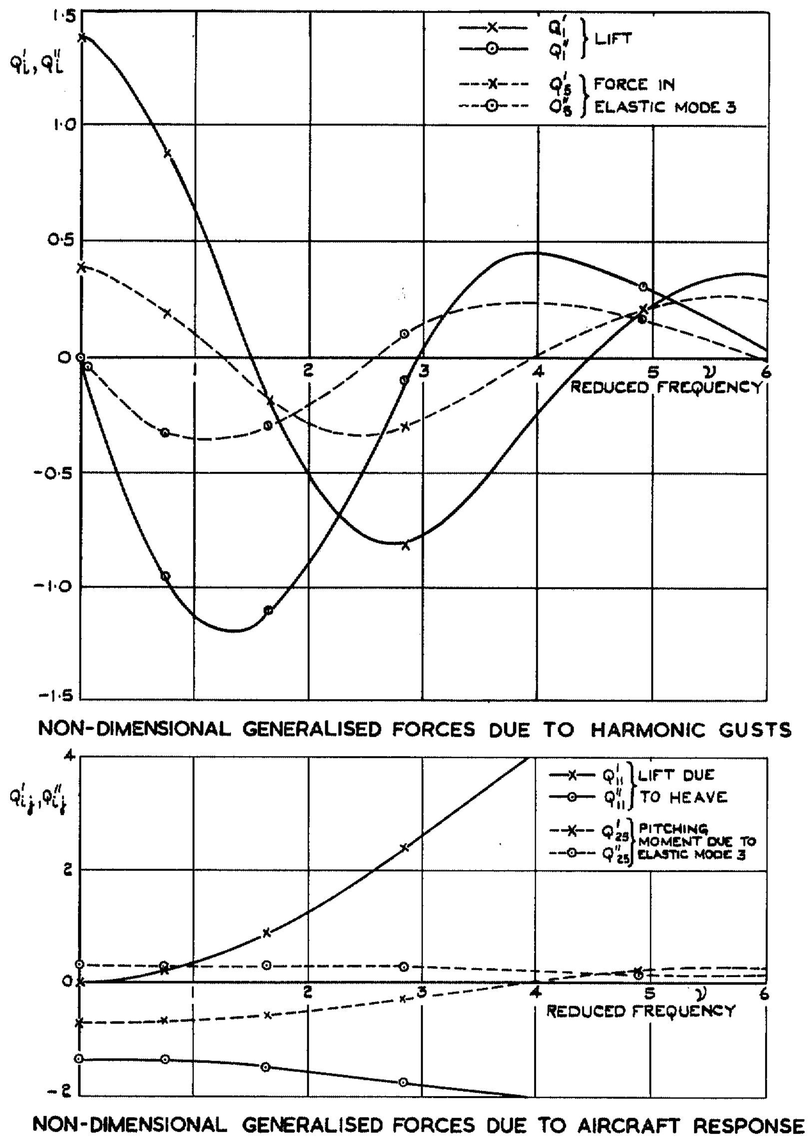


FIG. 5. Typical oscillatory aerodynamic generalised forces on the slender-wing aircraft.

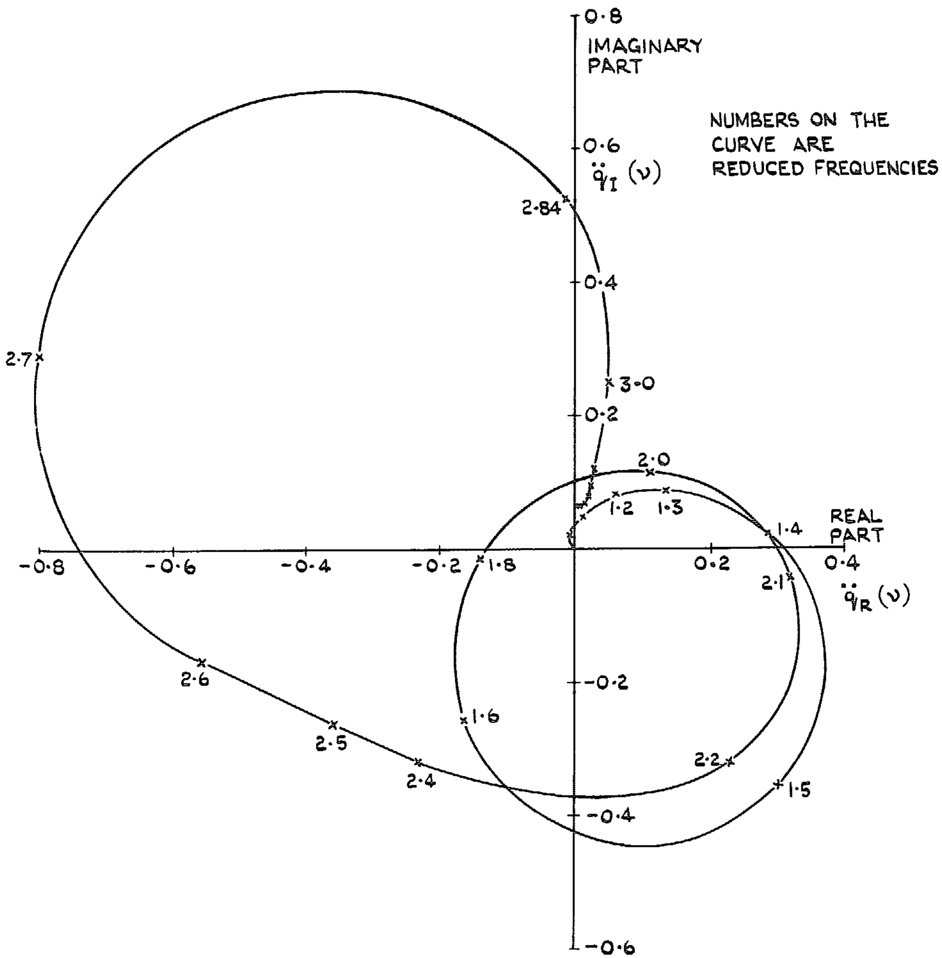


FIG. 6. Non-dimensional transfer function for acceleration in the first elastic mode, drawn as a vector plot.

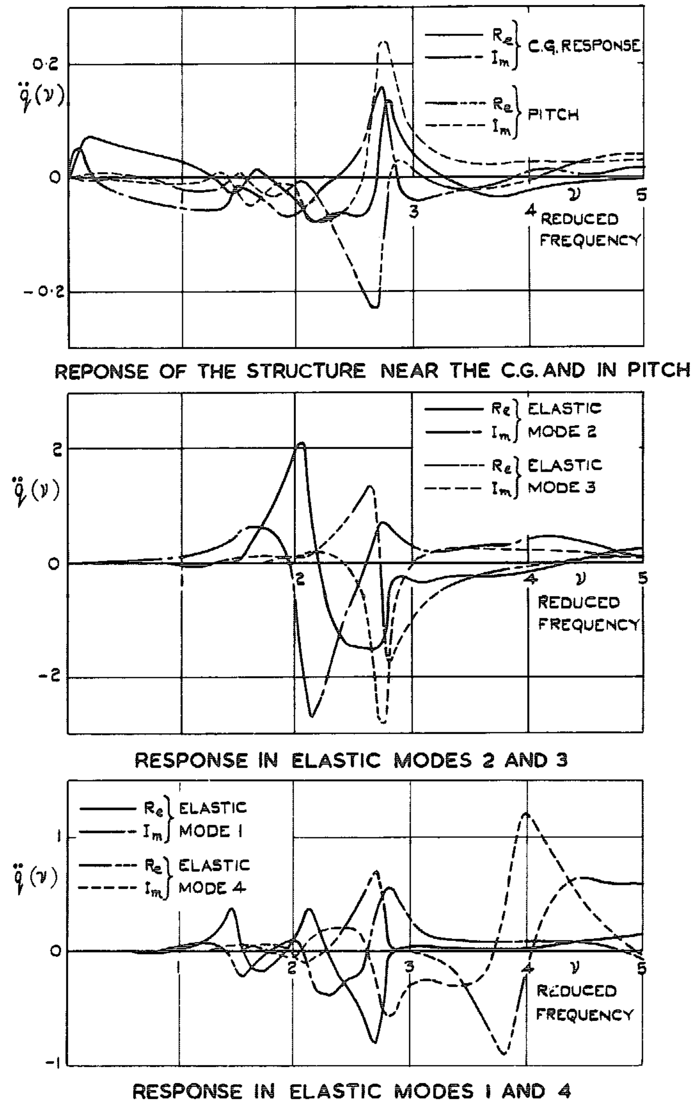


FIG. 7. Non-dimensional transfer functions for the acceleration responses of a flexible slender-wing aircraft to harmonic excitation.

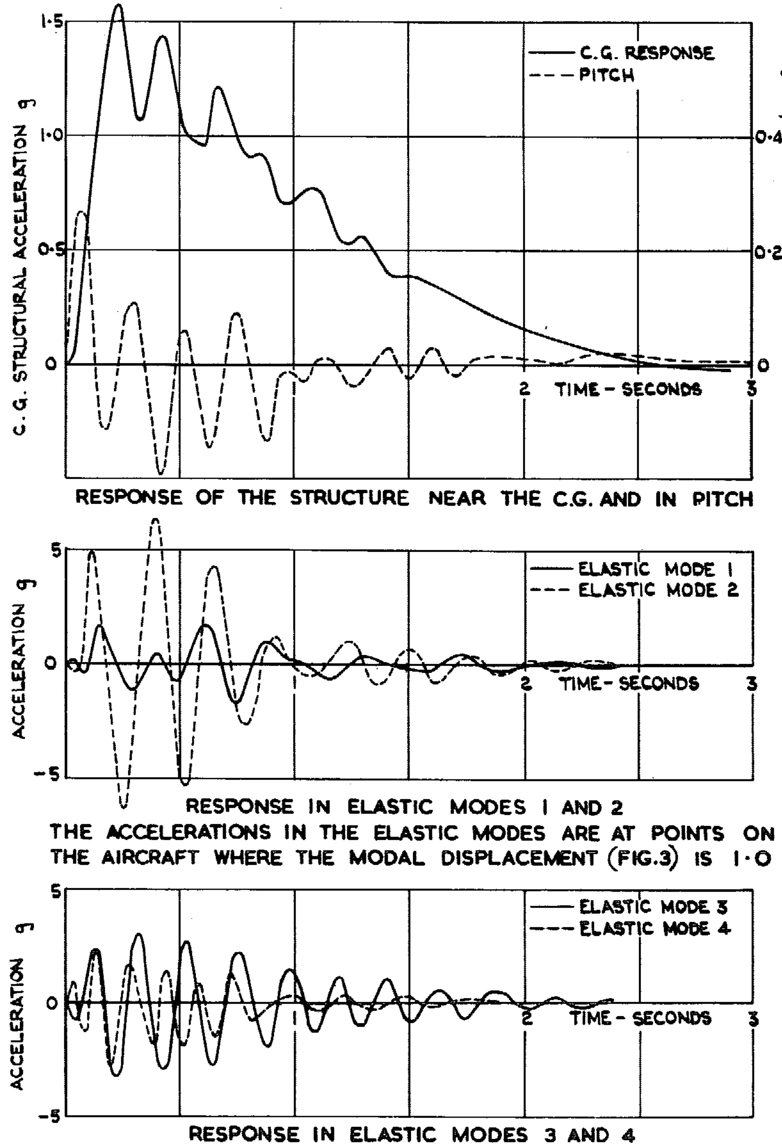


FIG. 8. Transient response of the slender-wing aircraft to a 50 ft/sec. E.A.S. step gust.

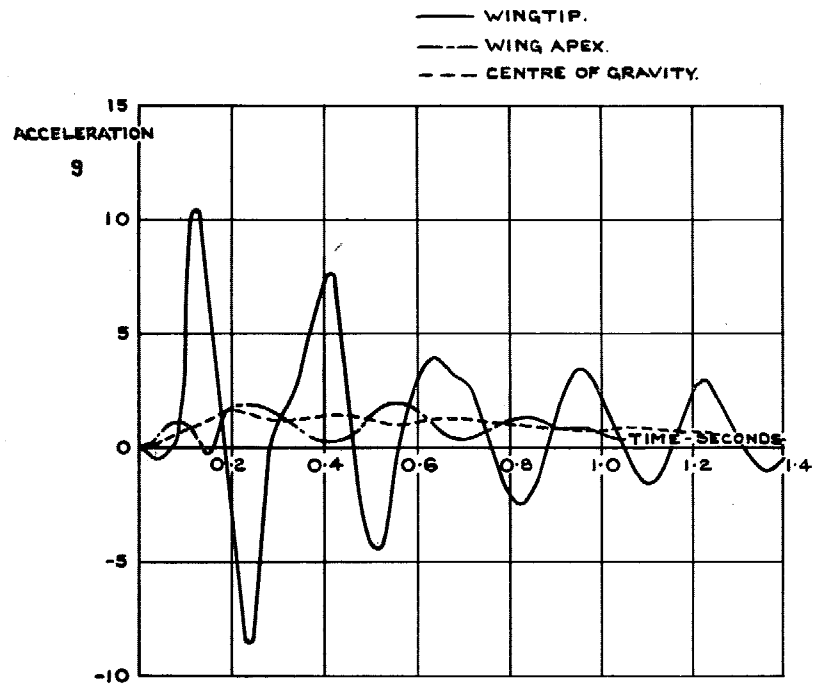


FIG. 9. Transient responses of three points on the structure of the slender-wing aircraft to a 50 ft/sec. E.A.S. step gust.

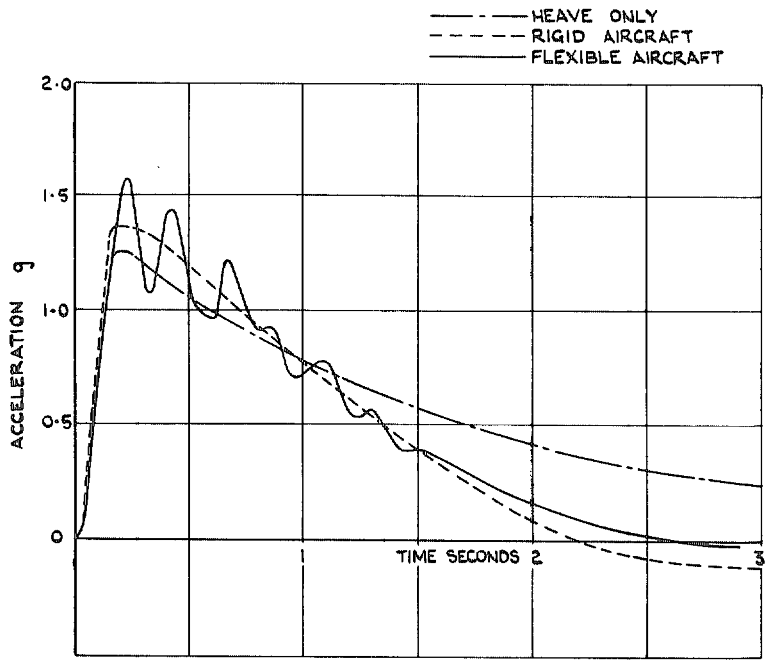


FIG. 10. The effects of pitching and flexibility on the response of the structure near the centre of gravity for the slender-wing aircraft entering a 50 ft/sec. E.A.S. step gust.

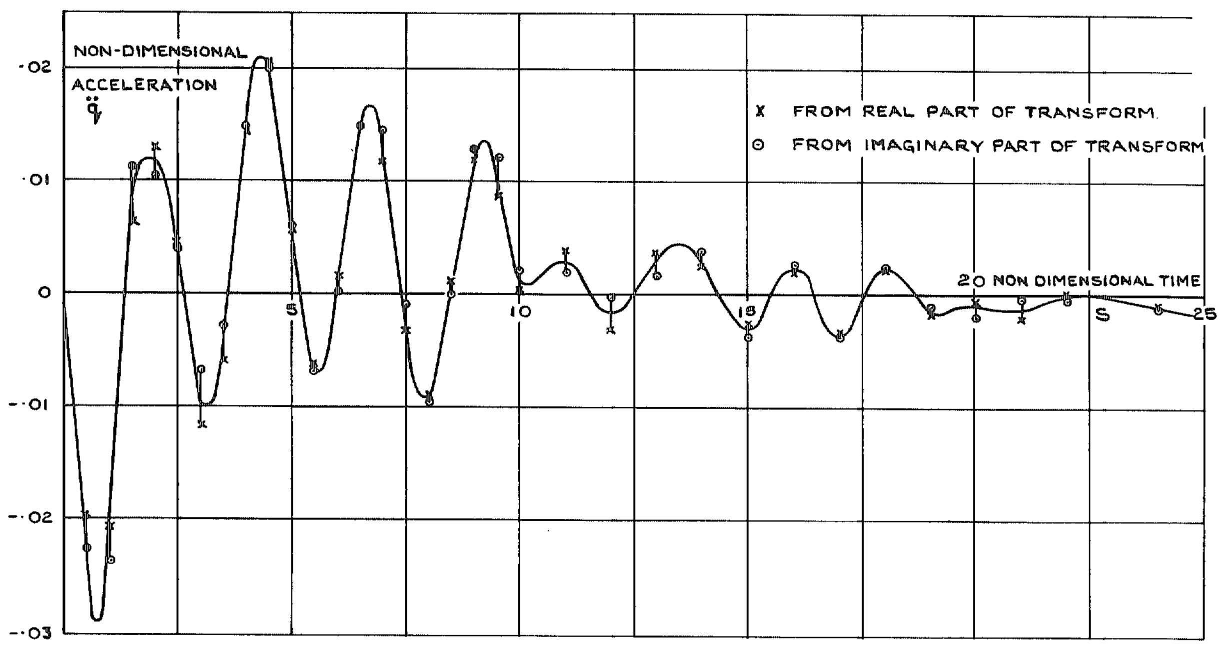


FIG. 11. Comparison of the pitch acceleration of the slender-wing aircraft obtained from the real and imaginary parts of the transform.

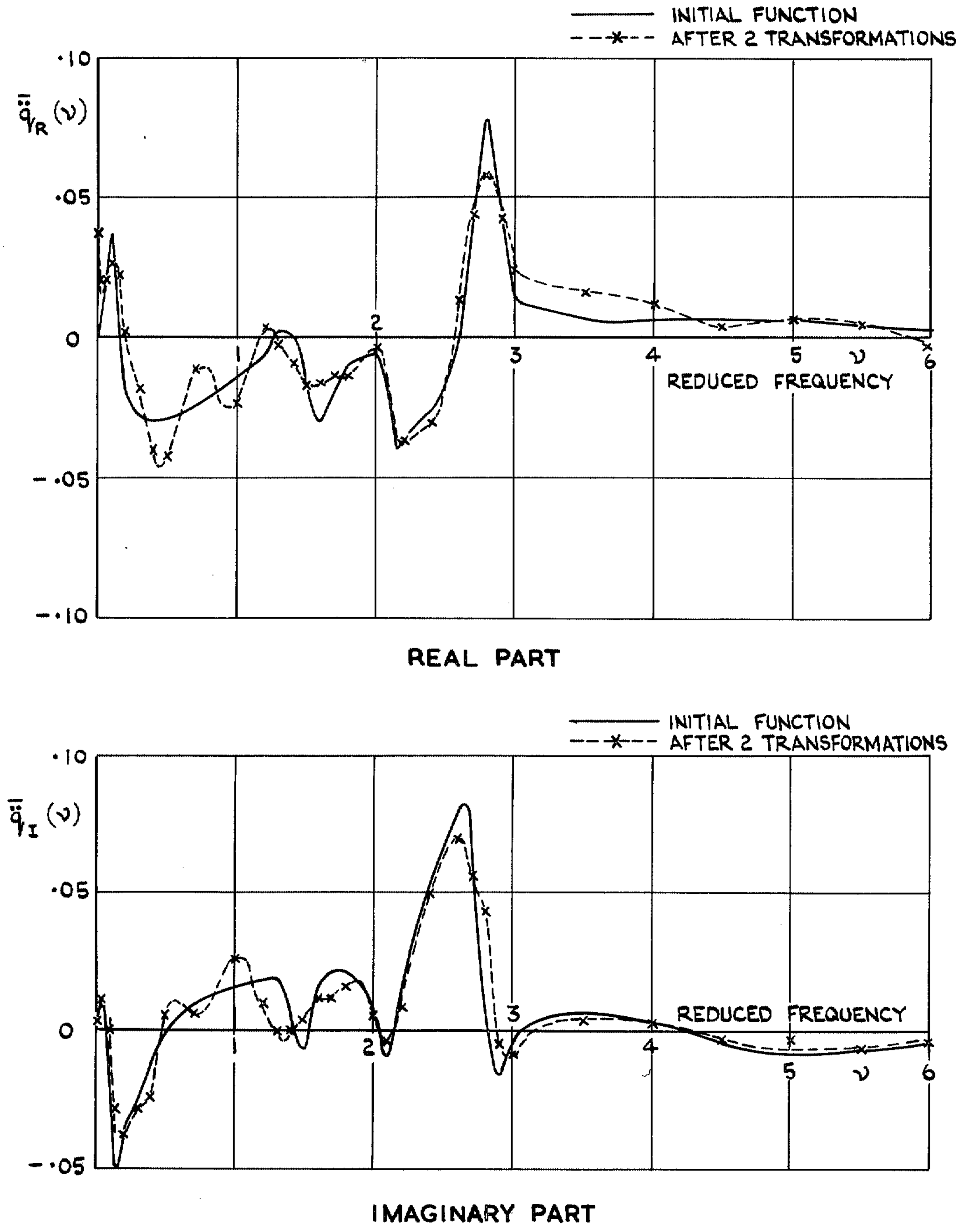


FIG. 12. The accuracy of the transformation program demonstrated by the transform of the pitch acceleration after transformation to the time domain (*see* Fig. 11) and back.

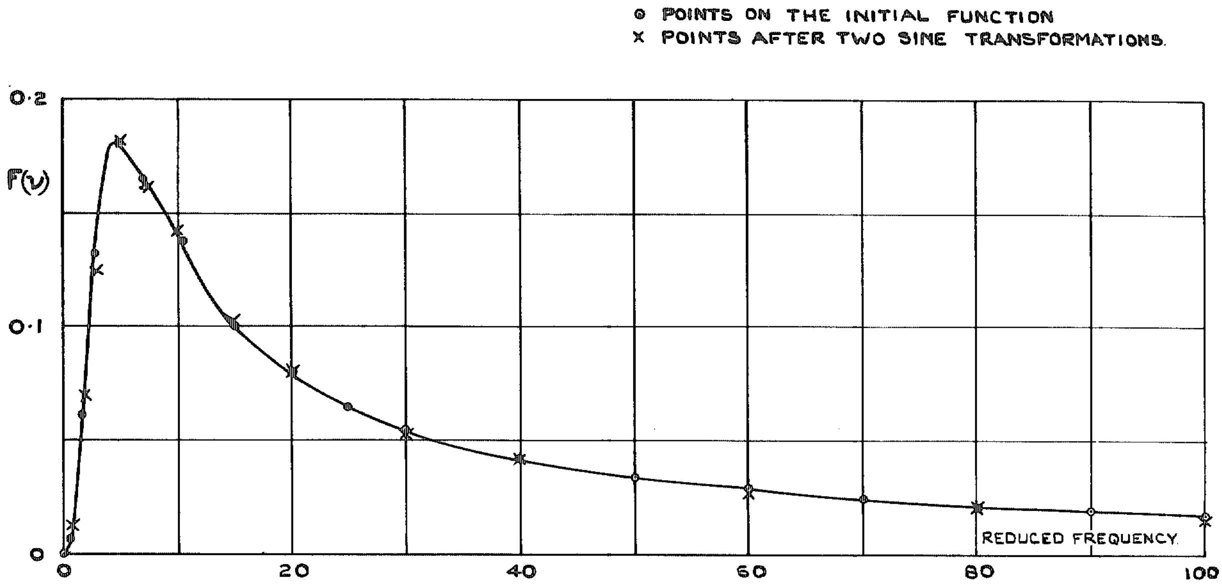


FIG. 13. The accuracy of the transformation program demonstrated by the double transformation of a smooth function.

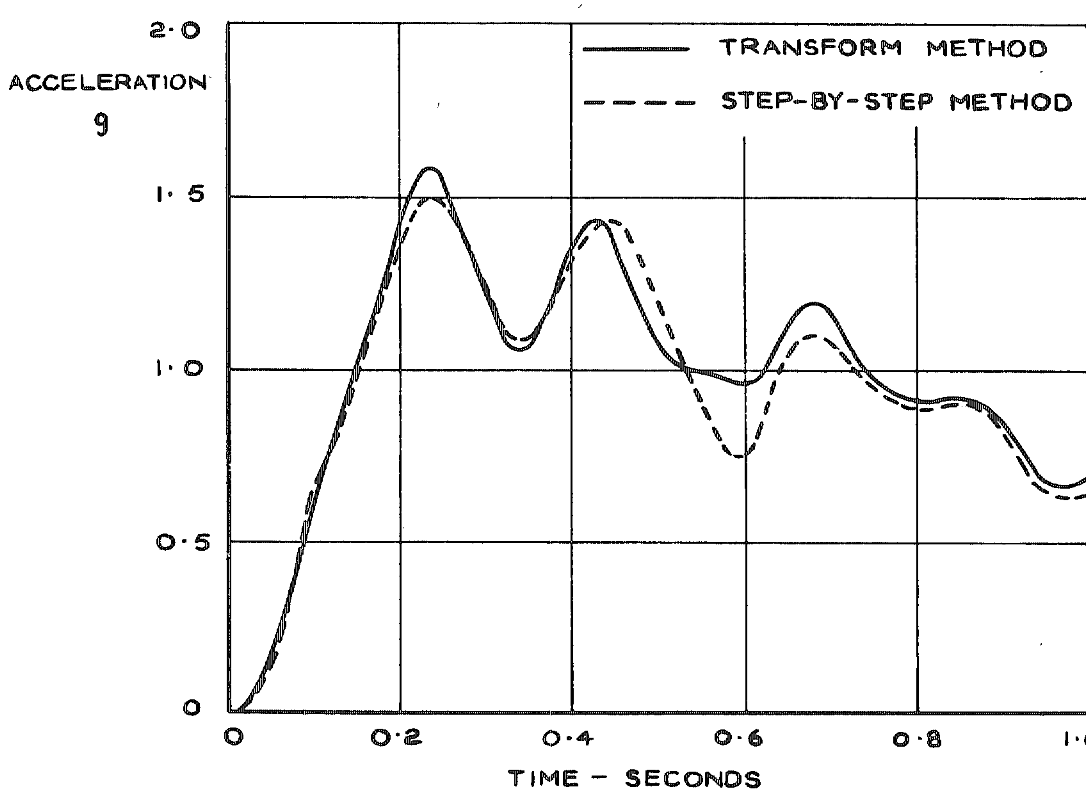
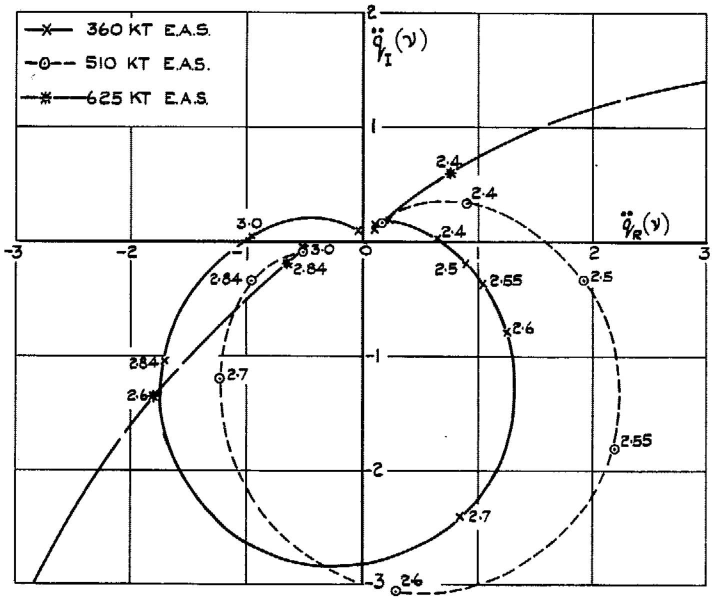
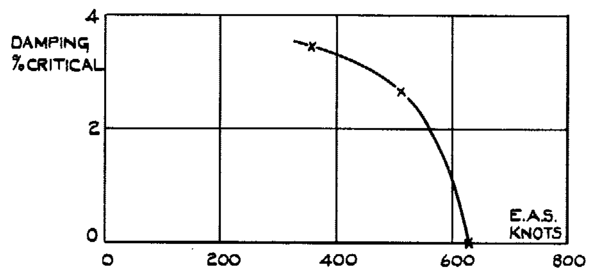


FIG. 14. Transient response of the structure near the centre of gravity for the slender-wing aircraft entering a 50 ft/sec. E.A.S. step gust.

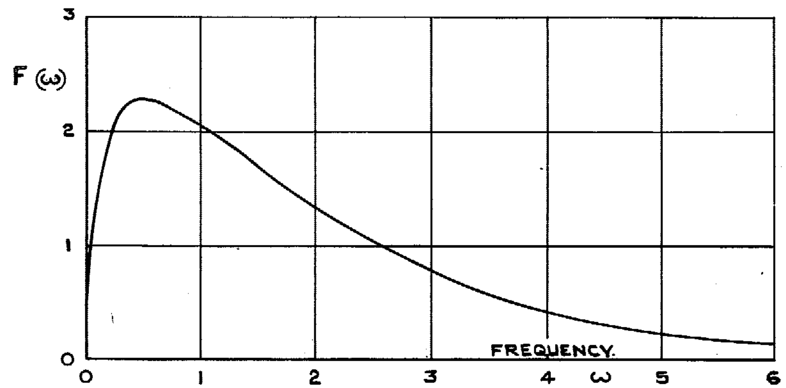


(a) VARIATION OF THE NON-DIMENSIONAL TRANSFER FUNCTION WITH KINETIC PRESSURE AT M=0·80

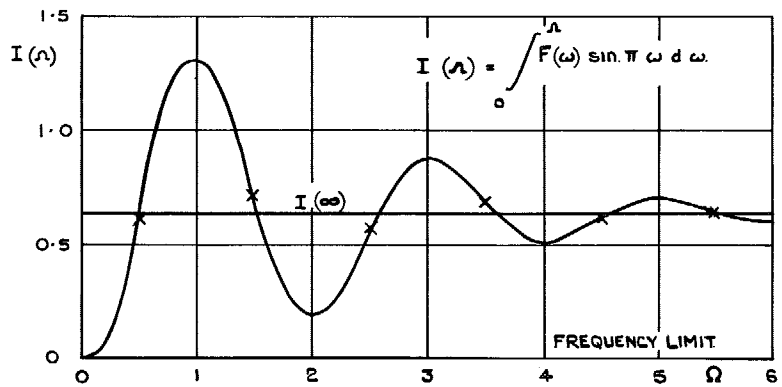


(b) VARIATION OF DAMPING WITH E.A.S. AT M=0·80

FIG. 15 (a & b). The effect of variation of kinetic pressure at constant Mach number on the response of elastic mode 3 to harmonic gust excitation.



(a) A TYPICAL FUNCTION OF FREQUENCY.



(b) VARIATION OF THE INTEGRAL $\int_0^{\Omega} F(\omega) \sin \pi \omega \, d\omega$ WITH THE UPPER LIMIT Ω

FIG. 16 (a & b). Variation of the Fourier transform integral as the upper limit is increased towards infinity.

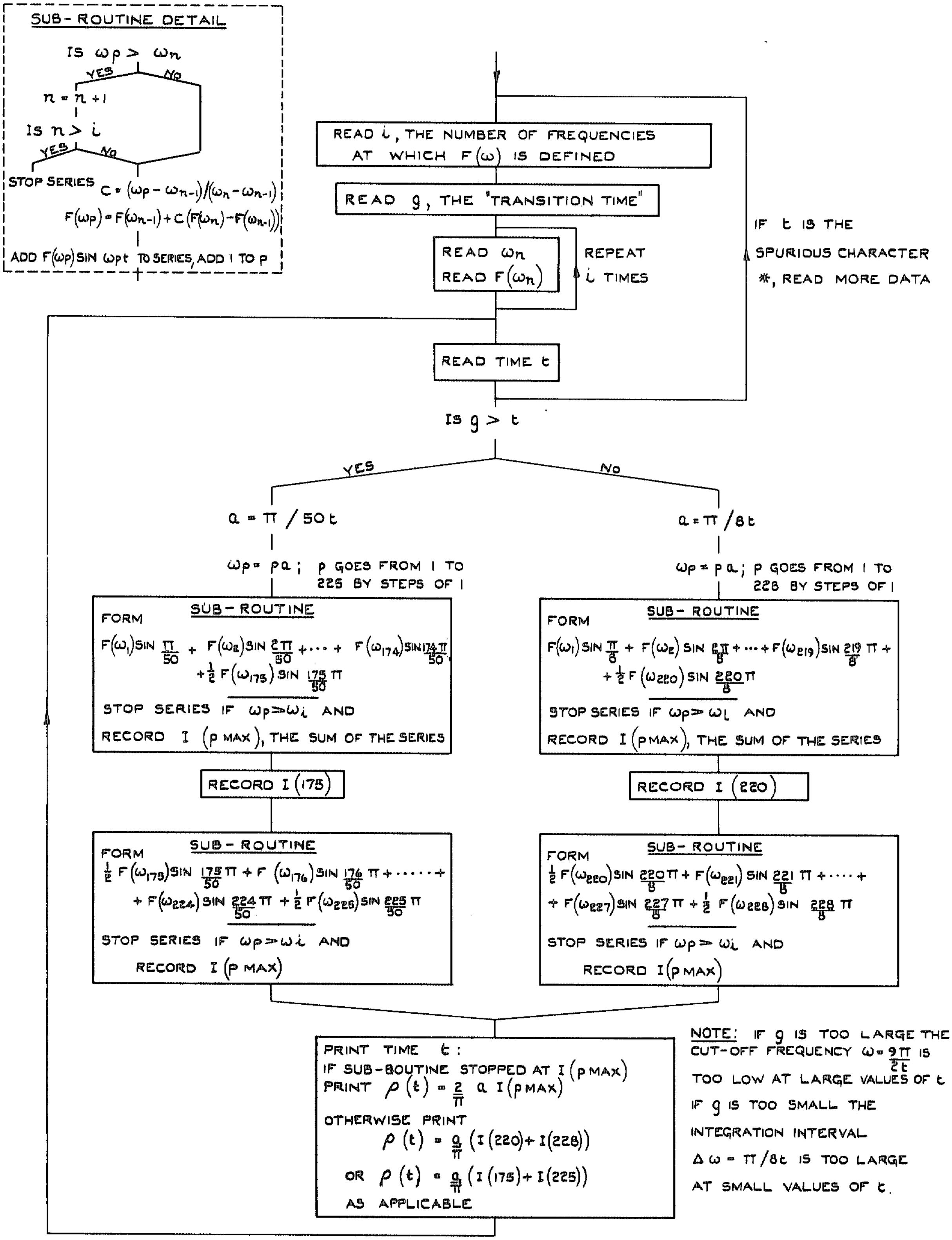


FIG. 17. Flow diagram of Fourier inverse transform program.

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