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Possio's Subsonic Derivative Theory and its Application to Flexural-Torsional Wing Flutter

PART I

Possio's Derivative Theory for an Infinite Aerofoil Moving at Subsonic Speeds

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PART II

Influence of Compressibility on the Flexural-Torsional Flutter of a Tapered Cantilever Wing Moving at Subsonic Speed

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PART I

Possio's Derivative Theory for an Infinite Aerofoil Moving at Subsonic Speeds

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Summary.—Range of Investigation.—The derivative theory due to C. Possio for an infinite aerofoil moving at subsonic speeds is reviewed, and certain modifications are proposed. Derivative values are calculated for a Mach number of 0·7, and for values of the frequency parameter λ ranging from 0 to 5·0.

Conclusions and Further Developments.—For $\lambda < 1$ the derivative values based on a three-point collocation method are in fair agreement with those given by Possio. For the range $1 \cdot 0 < \lambda < 2 \cdot 0$ five-point collocation is necessary, while for $\lambda = 5 \cdot 0$ even seven-point collocation may prove unsatisfactory.

The numerical results obtained are applied in Part II to estimate the influence of compressibility and flying height on the critical speed for flutter of a tapered cantilever wing.

1. *Introductory Remarks.*—A method of calculating the aerodynamical derivatives for an aerofoil of infinite aspect ratio moving at subsonic speeds has been proposed by C. Possio[1] (1937). The present paper gives a review of the theory and proposes certain modifications which are referred to in sections (vi) and (vii) of the Appendix. The modified theory is used to calculate the derivatives appropriate to a Mach number of 0·7 and to a frequency parameter range of 0 to 5·0.

The essential features of Possio's theory are a linearisation of the equations of flow of a compressible fluid (on the assumption that the disturbances are small) and the use of the acceleration potential ϕ. An acceleration potential exists when the pressure is a function of the density (as is assumed), and it is then defined in the usual hydrodynamical notation by

$$\frac{Du}{Dt} = \frac{\partial \phi}{\partial x} \, ; \; \frac{Dv}{Dt} = \frac{\partial \phi}{\partial y} \, . \qquad (1)$$

The corresponding equations defining the velocity potential Φ are, of course

$$u = \frac{\partial \Phi}{\partial x} \, ; \quad v = \frac{\partial \Phi}{\partial y} \, . \qquad (2)$$

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The exact theory of a compressible fluid does not lead to any simple differential equation for ϕ. However, when the equations are linearized according to Possio's assumptions, ϕ is found to satisfy a relatively simple partial differential equation of the second order, and elementary solutions of the source and doublet types can be constructed and superposed. For the solution of the problem of the oscillating aerofoil Possio adopts a distribution of special doublets over the chord. The variation of intensity is represented by a series, and the free coefficients in the series are determined by making the induced normal velocity agree with the actual velocity of the aerofoil at an appropriate number of points of the chord. When the intensity distribution for ϕ is known, the pressures along the chord can be calculated, and the derivatives can then be deduced. The conception of free or bound vortices is not introduced into the theory.

The principal formulae to be used are summarized in section 2, and the omitted details are supplied in the Appendix. The account differs in some respects from that given by Possio. For instance, some of the formulae, which are merely stated in the original paper, are proved and generalized. Moreover, the notation has been modified to accord with standard usage in this country.

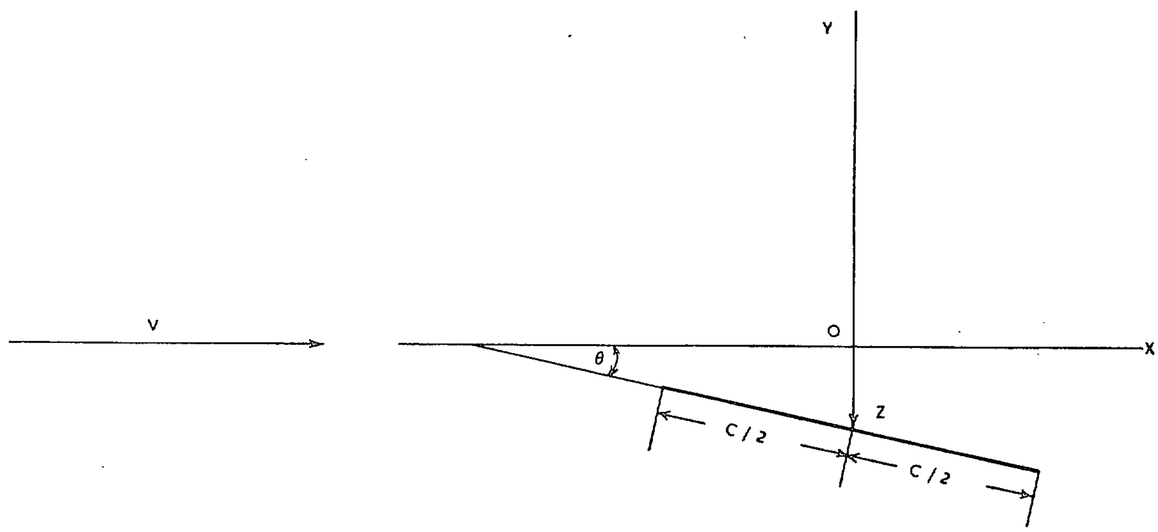

FIG. 1. Diagram of Oscillating Wing.

2. *Summary of the Theory*.—The axis OX is directed downstream, and it coincides with the chord when the aerofoil is in its mean position: the origin O lies at mid-chord (*see* Fig. 1). At great distances upstream of the aerofoil the constant airspeed is V and the local speed of sound is V_{s0}. The Mach number V/V_{s0} is denoted by μ, and λ denotes the frequency parameter $\omega c/V$, where $\omega/2\pi$ is the frequency of oscillation and c the chord.*

Equations (1), when linearized according to Possio's assumptions, become

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = \frac{\partial \phi}{\partial y}; \quad \frac{\partial v}{\partial t} + V\frac{\partial v}{\partial x} = \frac{\partial \phi}{\partial y},$$

and these, in conjunction with (2) give a relation connecting ϕ and Φ, namely

$$\phi = \frac{\partial \Phi}{\partial t} + V\frac{\partial \Phi}{\partial x} - V^2. \qquad (3)$$

This accords with the condition, arbitrarily assigned, that $\phi = 0$ at $x = -\infty$.

*Possio denotes the Mach number by λ, the chord by l, and the reduced frequency (one-half of the frequency parameter by Ω in his text and by ω in his diagrams. He also denotes the actual frequency by $\omega/2\pi$.

Integration of (3) yields

$$\Phi = Vx + \frac{1}{V}\int_{-\infty}^{x} \phi\left(\xi, y, t - \frac{x-\xi}{V}\right) d\xi ,$$

whence

$$v = \frac{\partial \Phi}{\partial y} = \frac{1}{V}\int_{-\infty}^{x} \frac{\partial}{\partial y}\phi\left(\xi, y, t - \frac{x-\xi}{V}\right) d\xi . \qquad (4)$$

By means of (4) the normal velocity v can be calculated, when ϕ is known.

The differential equation satisfied by ϕ is obtained by the use of (3) and differentiation of the equation of continuity. It is *

$$(1-\mu^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \frac{2\mu^2}{V}\frac{\partial^2\phi}{\partial x \partial t} - \frac{\mu^2}{V^2}\frac{\partial^2\phi}{\partial t^2} = 0 . \qquad (5)$$

A typical source solution, appropriate to subsonic speeds, is

$$1/(V^2t^2 - \mu^2(x - Vt)^2 - \mu^2y^2)^{1/2}, \qquad (6)$$

and a doublet can be deduced in the usual way by differentiation. For an incompressible fluid $\mu = 0$, and (5) then becomes independent of t and reduces to Laplace's equation $\nabla^2\phi = 0$.

By superposing sources Possio builds up a more complex solution representing a doublet of special type at the origin, with its axis along OY. His formulae can be generalized, and at the same time simplified by the use of the complex notation. Write

$$z = \lambda\left(\frac{x}{c}\right); \quad w = \frac{\lambda\mu}{1-\mu^2}\left(\frac{x}{c}\right); \quad w' = \frac{\lambda\mu}{1-\mu^2}\left[\left(\frac{x}{c}\right)^2 + (1-\mu^2)\left(\frac{y}{c}\right)^2\right]^{1/2}. \qquad (7)$$

Also let

$$\left.\begin{aligned} B(w) &\equiv N(w) + iJ(w) , \\ B_1(w) &\equiv N_1(w) + iJ_1(w) = -\frac{dB}{dw} , \end{aligned}\right\} \qquad (8)$$

where N and J are the usual cylinder (Neumann and Bessel) functions of *zero* order, N_1, J_1 are the corresponding functions of order unity, and i denotes $\sqrt{-1}$. Then Possio's real doublet is represented by the coefficient of $\sqrt{-1}$ in the complex potential

$$\phi = -a\mu^2\omega^2 y B_1(w')\, e^{i(\omega t + \mu w)} / 4w'(1-\mu^2)^{3/2}, \qquad (9)$$

where a is a real constant denoting the strength of the doublet.

The normal velocity induced at the point x of the chord, as calculated by substitution of (9) in (4), can be expressed as †

$$v = a\omega e^{i\omega t} h(z) \equiv a\omega e^{i\omega t}\{f(z) + ig(z)\} , \qquad (10)$$

where

$$4h(z)\sqrt{(1-\mu^2)} = e^{i\mu w}\{\mu B_1(w) + iB(w)\}$$
$$+ (1-\mu^2)\, e^{-iz}\int_{-\infty}^{w/\mu} e^{iu} B(\mu u)\, du . \qquad (11)$$

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*See Section (ii) of the Appendix.

†The functions $f(z)$, $g(z)$ correspond to Possio's $v_f/a\omega$ and $Vg/a\omega$. Explicit expressions for f and g are given in section (iv) of the Appendix.

To calculate the total pressure effect due to ϕ at points of the chord, the relation

$$p - p_0 = -\rho\dot{\phi} \qquad (12)$$

is applied. This formula follows from the hydrodynamical equations

$$\frac{\partial \dot{\phi}}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \text{ and } \frac{\partial \dot{\phi}}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

and the condition that the pressure disturbances are infinitesimally small. The calculations are simplified by the fact that the value of ϕ given by (9) is zero at all points of the chord except those in the immediate vicinity of the origin, where ϕ changes discontinuously on passage from the upper to the lower surface of the profile. When allowance is made for this discontinuity, the total (upward) force exerted on an element of chord dx due to a doublet of intensity $a(x)dx$ situated at a *general* point x of the chord, is found to reduce to

$$dP = \rho V^2 a(x)\, e^{i\omega t} dx \,. \qquad (13)$$

A complete system of such doublets is next assumed to be distributed along the chord. The intensity at position x can be represented most conveniently by the familiar form of series*

$$(a)x = A_0 \cot\left(\frac{\vartheta}{2}\right) + \sum_1^\infty A_n \sin n\vartheta\,, \qquad (14)$$

where $x = -\frac{c}{2}\cos\vartheta$, and A_0, A_1, etc. are complex constants left free for choice.

From (13) and (14) it follows that the normal force coefficient per unit span (Z reckoned positively downwards) is given by

$$Ze^{-i\omega t}/\pi\rho c V^2 = -\frac{1}{2\pi}\int_0^\pi a(x)\sin\vartheta\, d\vartheta = -\tfrac{1}{2}\left(A_0 + \tfrac{1}{2}A_1\right). \qquad (15)$$

Similarly, the pitching-moment coefficient per unit span about the quarter-chord point (M reckoned positively when it depresses the trailing edge) is given by

$$Me^{-i\omega t}/\pi\rho c^2 V^2 = \frac{1}{4\pi}\int_0^\pi a(x)\,(\cos\vartheta - \tfrac{1}{2})\sin\vartheta\, d\vartheta$$

$$= -\tfrac{1}{16}(A_1 - A_2)\,. \qquad (16)$$

Expressions for the aerodynamical derivatives can now be deduced. Suppose the displacements of the aerofoil (*see* Fig. 1) to be specified in terms of complex amplitudes z_0, θ_0 by

$$z/c = z_0 e^{i\omega t}\,; \qquad \theta = \theta_0 e^{i\omega t}\,. \qquad (17)$$

Then

$$Ze^{i\omega t} = cz_0\,(Z_z - \omega^2 Z_{\ddot{z}} + i\omega Z_{\dot{z}}) + \theta_0\,(Z_\theta - \omega^2 Z_{\ddot{\theta}} + i\omega Z_{\dot{\theta}})\,,$$

and a similar formula applies for M. Hence

$$-Ze^{-i\omega t}/\pi\rho c V^2 = z_0\,(Z_1 + iZ_2) + \theta_0\,(Z_3 + iZ_4) = \tfrac{1}{2}\,(A_0 + \tfrac{1}{2}A_1)\,, \qquad (18)$$

$$-Me^{-i\omega t}/\pi\rho c^2 V^2 = z_0\,(M_1 + iM_2) + \theta_0\,(M_3 + iM_4) = \tfrac{1}{16}\,(A_1 - A_2)\,, \qquad (19)$$

* Possio adopts an alternative series. His formulae require some correction (*see* Appendix section (vi).

where

$$Z_1 \equiv -Z_z/\pi\rho V^2 + \lambda^2 Z_{\ddot{z}}/\pi\rho c^2 \qquad M_1 \equiv -M_z/\pi\rho c V^2 + \lambda^2 M_{\ddot{z}}/\pi\rho c^3$$
$$Z_2 \equiv -\lambda Z_{\dot{z}}/\pi\rho c V \qquad M_2 \equiv -\lambda M_{\dot{z}}/\pi\rho c^2 V$$
$$Z_3 \equiv -Z_\theta/\pi\rho c V^2 + \lambda^2 Z_{\ddot{\theta}}/\pi\rho c^3 \qquad M_3 \equiv -M_\theta/\pi\rho c^2 V^2 + \lambda^2 M_{\ddot{\theta}}/\pi\rho c^4$$
$$Z_4 \equiv -\lambda Z_{\dot{\theta}}/\pi\rho c^2 V \qquad M_4 \equiv -\lambda M_{\dot{\theta}}/\pi\rho c^3 V\,.$$

The derivatives Z_1, Z_2/λ, Z_3, Z_4/λ correspond respectively to Possio's

$$-\frac{l}{\pi}\frac{\partial c_p}{\partial \eta}\,,\quad -\frac{V_0}{\pi}\frac{\partial c_p}{\partial \dot{\eta}}\,,\quad \frac{1}{\pi}\frac{\partial c_p}{\partial \alpha}\,,\quad \frac{V_0}{\pi l}\frac{\partial c_p}{\partial \dot{\alpha}}\,;$$

while M_1, M_2/λ, M_3, M_4/λ correspond to

$$-\frac{l}{\pi}\frac{\partial c_m}{\partial \eta}\,,\quad -\frac{V_0}{\pi}\frac{\partial c_m}{\partial \dot{\eta}}\,,\quad \frac{1}{\pi}\frac{\partial c_m}{\partial \alpha}\,,\quad \frac{V_0}{\pi l}\frac{\partial c_m}{\partial \dot{\alpha}}\,.$$

Lastly, the unknown coefficients A_n in (14) must be determined by the usual method of assigning the velocity v at an appropriate number of points of the chord. If the first m coefficients are retained, and if the m stations for collocation are $x_1, x_2, \ldots x_m$, then from (10) and (14) it readily follows that the equations for the coefficients are

$$\int_0^\pi a(\vartheta) \sin \vartheta\, h(z_n - z)\, d\vartheta = -2iz_0 + \left(-\frac{2}{\lambda} + i \cos \vartheta_n\right)\theta_0\,, \qquad \ldots \quad (20)$$

where $z = -\frac{\lambda}{2}\cos\vartheta$ and $n = 1, 2, \ldots m$. The graphical evaluation of the integral on the left of (20) is complicated by the presence of a singularity in the integrand at $z = z_n$. A modification of the equation, which removes this singularity, is proposed in section (vii) of the Appendix.

3. *Derivative Results.*—Possio includes in his paper a tabulation of the function $h(z)$ to three significant figures as well as graphs of the aerodynamic derivatives, appropriate to values of λ ranging from 0 to 0·7. He uses the series (28) referred to in section (vi) of the Appendix, retaining only the first three coefficients and choosing the mid-point and two ends of the chord as the collocating positions. He remarks that check calculations with only the first two coefficients retained showed maximum differences of about 5 per cent. for $\lambda = 1{\cdot}2$.

The derivatives appropriate to a Mach number 0·7 and to values of the frequency parameter ranging from 0 to 5·0 were calculated independently at the N.P.L. For these calculations use was made of the series (14), and the function $h(z)$ was replaced by the regular function $\bar{h}(z) = h(z) - h_0(z)$ as proposed in section (vii) of the Appendix. The results are summarised in the tables and diagrams at the end of Part I. In all cases the mid-chord point is adopted as the reference centre for the definition of normal displacement and normal force, but in Tables 1 (a) and 1 (b) and in Fig. 2 the pitching moment is evaluated (as by Possio) about the quarter-chord point, whereas in Table 2 it is evaluated about the mid-chord point, in accordance with standard derivative practice in this country. It will be seen that for $\lambda < 1$ the N.P.L. derivative results are in fair agreement with Possio's, and that three-point collocation is adequate for this range of λ. However, five-point collocation is required for the range $1{\cdot}0 < \lambda < 2{\cdot}0$, while results given in Table 1 (a) suggest that for $\lambda = 5{\cdot}0$ even seven-point collocation may prove unsatisfactory.

The results are applied in Part II to estimate the influence of compressibility and flying height on the critical speed for flutter of a tapered cantilever wing.

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No.	Author	Title
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2	W. P. Jones	Summary of Formulae and Notations Used in Two-dimensional Derivative Theory. R. & M. 1958. (August, 1941).

APPENDIX

Supplementary Notes on the Theory

(i) *Preliminary Statement of the Exact Equations.*—If $P \equiv \int dp/\rho$ and if $V_s \equiv \sqrt{(dp/d\rho)}$ denotes the local velocity of sound, the usual differential equations of flow of a compressible fluid are

$$\frac{Du}{Dt} = -\frac{\partial P}{\partial x}; \qquad \frac{Dv}{Dt} = -\frac{\partial P}{\partial y}, \quad \ldots \quad (21)$$

$$\frac{1}{V_s^2}\frac{DP}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad \ldots \quad (22)$$

Let ϕ denote the acceleration potential and Φ the velocity potential, these functions being defined by

$$\frac{Du}{Dt} = \frac{\partial \phi}{\partial x}; \quad \frac{Dv}{Dt} = \frac{\partial \phi}{\partial y}, \quad \ldots \quad (23)$$

$$u = \frac{\partial \Phi}{\partial x}; \qquad v = \frac{\partial \Phi}{\partial y}. \quad \ldots \quad (24)$$

By direct integration of equations (21)

$$\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}(u^2 + v^2) + P = \text{const.}, \quad \ldots \quad (25)$$

and by comparison of (21) and (24)

$$\phi + P = \text{const.} \quad \ldots \quad (26)$$

The arbitrary functions of time t, which should in general appear on the right-hand sides of (25) and (26), are here replaced by absolute constants, since the flow is assumed to be uniform at great distances upstream of the aerofoil. If the axes are chosen as in Fig. 1, then by (25) and (26)

$$\phi = \frac{\partial \Phi}{\partial t} + \tfrac{1}{2}(u^2 + v^2) - \tfrac{1}{2}V^2, \quad \ldots \quad (27)$$

where the constants are chosen such that $\phi = 0$ at $x = -\infty$.

Next, by equations (21) and (26),

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y} = -\frac{\partial \phi}{\partial t} - u\frac{Du}{Du} - v\frac{Dv}{Dt}.$$

On substitution of this expression for DP/Dt in (22) and use of (27) the following equation is obtained for the velocity potential

$$\left(1-\frac{u^2}{V_s^2}\right)\frac{\partial^2\Phi}{\partial x^2}+\left(1-\frac{v^2}{V_s^2}\right)\frac{\partial^2\Phi}{\partial y^2}-\frac{2uv}{V_s^2}\frac{\partial^2\Phi}{\partial x\partial y}-\frac{2u}{V_s^2}\frac{\partial^2\Phi}{\partial x\partial t}$$
$$-\frac{2v}{V_s^2}\frac{\partial^2\Phi}{\partial y\partial t}-\frac{1}{V_s^2}\frac{\partial^2\Phi}{\partial t^2}=0. \qquad (28)$$

A simple differential equation for ϕ does not exist.

(ii) *Possio's Linearized Form of the Equations.*—Possio linearizes (28) by replacing the values of u, v, V_s, in the coefficients of the derivatives by their values V, 0, V_{s0} at $x = -\infty$. This leads to

$$(1-\mu^2)\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}-\frac{2\mu^2}{V}\frac{\partial^2\Phi}{\partial x\,\partial t}-\frac{\mu^2}{V^2}\frac{\partial^2\Phi}{\partial t^2}=0\,, \qquad (29)$$

where $\mu \equiv V/V_{s0}$.

Similarly, the linearized form of (27) is

$$\phi=\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}(u+V)\,(u-V)=\frac{\partial\Phi}{\partial t}+V\frac{\partial\Phi}{\partial x}-V^2$$

in agreement with equation (3). In view of this relation it is clear that (29) is also satisfied by the acceleration potential ϕ (*see* equation (5)). Possio remarks that when a solution of (29) is adopted as the acceleration potential, it is necessary to ensure that equation (29) itself, and not merely a differentiated form of the equation of continuity, is satisfied at a certain instant at all points of a surface which cuts all the lines of flow. This condition, however, is evidently satisfied at $x = -\infty$.

(iii) *Construction of Possio's Doublet Solution.*—The equation for the acceleration potential is reducible to the familiar Laplacian form by a change of the independent variables. Thus, if new variables X, T are introduced defined by the substitution $X = x - Vt$, $T = t$, the equation reduces to

$$\frac{\partial^2\phi}{\partial X^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{1}{V_{s0}^2}\frac{\partial^2\phi}{\partial T^2}.$$

A typical simple source solution, appropriate to subsonic speeds and referred to an arbitrary origin of time, is

$$\phi=\frac{1}{[V_{s0}^2\,(T-\tau)^2-X^2-y^2]^{1/2}}=\frac{1}{[V_{s0}^2\,(t-\tau)^2-\{x-V\,(t-\tau)\}^2-y^2]^{1/2}}.$$

A more general solution is

$$\phi=-\frac{aV_0^2V_{s0}}{2\pi}\int_{-\infty}^{g}\frac{e^{i\omega\tau}d\tau}{[V_{s0}^2\,(t-\tau)^2-\{x-V\,(t-\tau)\}^2-y^2]^{1/2}} \qquad (30)$$

where a is a real constant and g denotes the lesser of the values of τ which annul the denominator.

Now change the variables of integration in (30) from τ to u where

$$\omega\,(t-\tau)=w'u-\mu w\,,$$

and w, w' are as defined by (7). Then (30) is found to reduce to

$$\phi_0=-\frac{aV^2}{2\pi\sqrt{(1-\mu^2)}}\,e^{i(\omega t+\mu w)}\int_1^{\infty}\frac{e^{-iw'u}}{\sqrt{(u^2-1)}}\,du,$$
$$=aV^2B\,(w')\,e^{i(\omega t+\mu w)}/4\sqrt{(1-\mu^2)}\,.$$

Differentiation of this solution with respect to y yields the complex potential (9).

Possio simplifies the above analysis in his paper by assuming $y = 0$ $(w = w')$, but the generalized solution $(y \neq 0)$ is actually required.

An alternative procedure is to assume for equation (5) a trial solution of the type

$$\phi = F(w')\, e^{i(\omega t + \mu w')} .$$

On substitution of this expression in (5), the equation for F is readily found to be

$$\frac{d^2F}{dw'^2} + \frac{1}{w'}\frac{dF}{dw'} + F = 0 ,$$

so that F is a cylinder function of order zero. The solution so obtained represents a source, and the doublet (9) actually required can be deduced by differentiation.

(iv) *Explicit Expressions for Functions* $f(z)$, $g(z)$.—The explicit expressions for the components of the normal induced velocity given by equation (11) are as follows

$$\begin{aligned} 4\sqrt{(1-\mu^2)}\, f(z) &= \cos \mu w\, \{\mu N_1(w) - J(w)\} - \sin \mu w\, \{\mu J_1(w) + N(w)\} \\ &\quad + (1-\mu^2) \cos z \int_{-\infty}^{w/\mu} \{\cos uN(\mu u) - \sin uJ(\mu u)\}\, du \\ &\quad + (1-\mu^2) \sin z \int_{-\infty}^{w/\mu} \{\cos uJ(\mu u) + \sin uN(\mu u)\}\, du . \qquad (31) \end{aligned}$$

$$\begin{aligned} 4\sqrt{(1-\mu^2)}\, g(z) &= \cos \mu w\, \{\mu J_1(w) + N(w)\} + \sin \mu w\, \{\mu N_1(w) - J(w)\} \\ &\quad + (1-\mu^2) \cos z \int_{-\infty}^{w/\mu} \{\cos uJ(\mu u) + \sin uN(\mu u)\}\, du \\ &\quad + (1-\mu^2) \sin z \int_{-\infty}^{w/\mu} \{\cos uN(\mu u) - \sin uJ(\mu u)\}\, du . \qquad (32) \end{aligned}$$

The integrals in (31) and (32) must be calculated numerically, but the limits 0 and w/μ can be used in view of the known relations

$$\int_{-\infty}^{0} \{\cos uN(\mu u) - \sin uJ(\mu u)\}\, du = 0 , \qquad (33)$$

$$\int_{-\infty}^{0} \{\cos uJ(\mu u) + \sin uN(\mu u)\}\, du = \frac{2}{\pi\sqrt{(1-\mu^2)}} \log_e \left(\frac{\mu}{1-\sqrt{(1-\mu^2)}}\right) . \qquad (34)$$

Since w may be either positive or negative, it should be noted that $J(-y^2) = J(y^2)$ and $N(-y^2) = N(y^2)$.

When z is small the following approximations are valid,

$$-f(z) = \frac{\sqrt{(1-\mu^2)}}{2\pi z} + \frac{1}{4\sqrt{(1-\mu^2)}} , \qquad (35)$$

$$g(z) = \frac{\log|z|}{2\pi\sqrt{(1-\mu^2)}} + \frac{1}{2\pi\sqrt{(1-\mu^2)}}\left[\sqrt{(1-\mu^2)} \log \frac{\mu}{1-\sqrt{(1-\mu^2)}} + \log \frac{\gamma\mu}{2\,(1-\mu^2)} - \mu^2\right] , \qquad (36)$$

where $\log \gamma$ denotes Euler's constant $0 \cdot 5772$.

(v) *Limiting Forms of $f(z)$, $g(z)$ for Incompressible Fluid.*—To obtain the limiting forms of (31) and (32) when $\mu \to 0$ (with $w \to \mu z \to 0$) use should be made of (33) and (34) and of the approximations

$$J(w) \to 1\,; \quad \mu J(w) \to 0\,; \quad N(w) \to \frac{2}{\pi} \log\left(\frac{\gamma w}{2}\right); \quad N_1(w) \to -\frac{2}{\pi w}\,.$$

Note also that

$$\int_0^z \sin u \log u \, du = \int_0^z \log u \, d(1 - \cos u) = (1 - \cos z) \log z + Ci(z) - \log \gamma z\,,$$

$$\int_0^z \cos u \log u \, du = \int_0^z \log u \, d(\sin u) = \sin z \log z - Si(z)\,,$$

where $$Si(z) \equiv \int_0^z \frac{\sin u \, du}{u}\,;$$

$$Ci(z) = -\int_z^\infty \frac{\cos u \, du}{u} = \log \gamma z - \int_0^z \frac{(1 - \cos u)\, du}{u}\,.$$

In the limit, the functions $f(z)$, $g(z)$ take the familiar forms

$$\left.\begin{aligned} -f(z) &= \frac{\cos z}{4} + \frac{1}{2\pi z} - \frac{\sin z \, Ci(z)}{2\pi} + \frac{\cos z \, Si(z)}{2\pi}\,, \\ g(z) &= \frac{\sin z}{4} + \frac{\sin z \, Si(z)}{2\pi} + \frac{\cos z \, Ci(z)}{2\pi}\,. \end{aligned}\right\} \quad \ldots \quad (37)$$

(vi) *The Series for Intensity Distribution.*—Possio adopts a series of the Birnbaum type for the intensity distribution. His series is

$$a(\xi) = A_0' \sqrt{\left(\frac{1 - \xi}{1 + \xi}\right)} + \sqrt{(1 - \xi^2)} \sum_1^m A_n' \xi^n\,,$$

but accented coefficients are here introduced in place of his unaccented ones, to avoid confusion with the series (14) of the present paper. He denotes the chord as l, and defines ξ in terms of x by the relation $\xi = lx/2$, stating that the ends of the profile correspond to $\xi = -1$ and $+1$.

Two misprints have evidently occurred in his paper. The series should be corrected to

$$a(\xi) = A_0' \sqrt{\left(\frac{1 - \xi}{1 + \xi}\right)} + \sqrt{(1 - \xi^2)} \sum_1^m A_n' \xi^{n-1}\,, \quad \ldots \quad (38)$$

and ξ should be defined by $x = l\xi/2$. Thus Possio's ξ is equivalent to $-\cos \vartheta$ of the present paper.

The advantage of the trigonometrical series (14) is that the exact expressions for the force and moment coefficients involve merely the first three coefficients A_0, A_1, A_2. The corresponding expressions when (38) is used involve *all* the coefficients A_0', A_1', etc.; this disadvantage does not emerge from Possio's analysis, because he happens to restrict his approximation to three terms only.

(vii) *Removal of the Singularity from the Collocation Equations.*—The solution of the collocation equations (20) involves a graphical evaluation of the integral

$$I = \int_0^\pi a(\vartheta) \sin \vartheta \, h(z_n - z) \, d\vartheta\,.$$

From equations (35) and (36) it is seen that in the vicinity of $z = z_n$ the function $h(z_n - z)$ is represented approximately by

$$h_0(z_n - z) \equiv \frac{A}{z_n - z} + B \log |z_n - z| + C ,$$

where A, B, C are known (real or complex) constants dependent on μ only. Hence the integrand in I has a singularity at $z = z_n$. To overcome this difficulty, it may prove convenient to express the integral in the form

$$I = \int_0^{\pi} a(\vartheta) \sin \vartheta \, \bar{h}(z_n - z) \, d\vartheta - \frac{2A}{\lambda} I_0 + BI_1 + \left(B \log \frac{\lambda}{2} + C\right) I_2 ,$$

where

$$\bar{h}(z_n - z) = h(z_n - z) - h_0(z_n - z) = \bar{f}(z_n - z) + i\bar{g}(z_n - z) , \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

and I_0, I_1, I_2 are integrals defined by

$$I_0 \equiv \int_0^{\pi} \frac{a(\vartheta) \sin \vartheta \, d\vartheta}{\cos \vartheta_n - \cos \vartheta} ;$$

$$I_1 \equiv \int_0^{\pi} a(\vartheta) \sin \vartheta \log |(\cos \vartheta_n - \cos \vartheta)| \, d\vartheta ;$$

$$I_2 \equiv \int_0^{\pi} a(\vartheta) \sin \vartheta \, d\vartheta .$$

The first integrand in I now remains finite throughout the range $\vartheta = 0$ to π and vanishes at $\vartheta = \vartheta_n$. The remaining integrals I_0, I_1, I_2 can be evaluated independently as follows :—

Integrals I_0 and I_2.—It is known that

$$\int_0^{\pi} \frac{\cos s\vartheta}{\cos \vartheta_n - \cos \vartheta} = - \frac{\pi \sin s\vartheta_n}{\sin \vartheta_n} .$$

Also

$$a(\vartheta) \sin \vartheta = A_0 (1 + \cos \vartheta) + \tfrac{1}{2} \sum_{s=1}^{m-1} A_s \{\cos (s - 1) \vartheta - \cos (s + 1) \vartheta\} .$$

It readily follows that

$$I_0 = - \pi A_0 + \pi \sum_{s=1}^{m-1} A_s \cos s\vartheta_n ,$$

$$I_2 = \pi \left(A_0 + \frac{A_1}{2}\right) .$$

Integral I_1.—The method of reduction will be indicated without entry into details.

$$I_1 = \int_0^{\vartheta_n - \varepsilon} a(\vartheta) \sin \vartheta \log (\cos \vartheta - \cos \vartheta_n) \, d\vartheta$$

$$+ \int_{\vartheta_n + \varepsilon}^{\pi} a(\vartheta) \sin \vartheta \log (\cos \vartheta_n - \cos \vartheta) \, d\vartheta .$$

Hence

$$\frac{\partial I_1}{\partial \vartheta_n} = [a(\vartheta) \sin \vartheta \log (\cos \vartheta - \cos \vartheta_n)]_{\vartheta=\vartheta_n-\varepsilon}$$

$$- [a(\vartheta) \sin \vartheta \log (\cos \vartheta_n - \cos \vartheta)]_{\vartheta=\vartheta_n+\varepsilon}$$

$$+ \int_0^{\vartheta_n-\varepsilon} \frac{a(\vartheta) \sin \vartheta \sin \vartheta_n \, d\vartheta}{\cos \vartheta - \cos \vartheta_n} - \int_{\vartheta_n+\varepsilon}^{\pi} \frac{a(\vartheta) \sin \vartheta \sin \vartheta_n \, d\vartheta}{\cos \vartheta_n - \cos \vartheta},$$

$$= - I_0 \sin \vartheta_n \text{ when } \varepsilon \to 0 ,$$

$$= \pi A_0 \sin \vartheta_n - \frac{\pi}{2} \sum_{s=1}^{m-1} A_s \{\sin (s+1) \vartheta_n - \sin (s-1) \vartheta_n\} .$$

Accordingly

$$I_1 = K - \pi A_0 \cos \vartheta_n + \frac{\pi}{4} A_1 \cos 2\vartheta_n$$

$$+ \frac{\pi}{2} \sum_{s=2}^{m-1} A_s \left\{ \frac{\cos (s+1) \vartheta_n}{s+1} - \frac{\cos (s-1) \vartheta_n}{s-1} \right\},$$

where K is a constant, independent of ϑ_n.

To determine K, substitute $\vartheta_n = \pi/2$. The integral I_1 then reduces to

$$\int_0^{\pi} a(\vartheta) \sin \vartheta \log |\cos \vartheta| \, d\vartheta ,$$

and this can be evaluated on application of the known results

$$\int_0^{\pi/2} \cos 2s\vartheta \log \cos \vartheta \, d\vartheta = \frac{\pi}{4s} (-1)^{s-1} ,$$

$$\int_0^{\pi/2} \log \cos \vartheta \, d\vartheta = - \frac{\pi}{2} \log 2 .$$

It is found that

$$K = - \pi (A_0 + \tfrac{1}{2} A_1) \log 2.$$

TABLE 1 (a)

N.P.L. Calculated Two-dimensional Derivatives Referred to Possio's Axes

Three-point collocation except where indicated: Mach number 0·7

λ	Z_1	Z_2	Z_3	Z_4	M_1	M_2	M_3	M_4
0	0	0	1·400	0	0	0	0	0
0·2	0·05883	0·19445	0·9922	—0·2470	—0·00531	0·00135	0·00633	0·04445
0·4	0·09456	0·3188	0·8390	—0·1628	—0·01862	0·00573	0·01356	0·08330
0·6	0·09923	0·4326	0·7842	—0·07154	—0·03900	0·01334	0·02192	0·1214
0·8	0·08302	0·5502	0·7730	0·00656	—0·06603	0·02521	0·03300	0·1591
1·0	0·05319	0·6766	0·7856	0·06978	—0·09885	0·04274	0·04780	0·1954
1·0§	0·05359	0·6818	0·7962	0·07183	—0·1011	0·04480	0·05072	0·2010
2·0	—0·1539	1·434	0·9431	0·19565	—0·2865	0·2275	0·1552	0·3128
2·0§	—0·10225	1·4395	0·9709	0·1472	—0·2890	0·2571	0·18625	0·3289
2·0*	—0·08949	1·450	0·9837	0·1359	—0·28155	0·2697	0·1999	0·32385
3·0	—0·39295	2·311	1·059	0·1956	—0·4400	0·4896	0·2183	0·3415
4·0	—0·6521	3·306	1·137	0·1722	—0·5740	0·77185	0·2243	0·3499
5·0	—0·9534	4·433	1·171	0·1640	—0·6998	1·065	0·1877	0·3548
5·0§	—0·5606	4·613	1·252	0·07630	—0·2587	1·207	0·3782	0·26785
5·0*	—0·6083	4·241	1·172	—0·02899	—0·4547	1·218	0·3147	0·3260

§ Five-point collocation.   * Seven-point collocation.

*Incompressible Fluid*** (*Possio's Axes*)

λ	Z_1	Z_2	Z_3	Z_4	M_1	M_2	M_3	M_4
0	0	0	1·0	0	0	0	0	0
0·2	0·02446	0·1664	0·8405	—0·08071	—0·00251	0	—0·00032	0·02500
0·4	0·03545	0·2910	0·7464	—0·01587	—0·01000	0	—0·00125	0·05000
0·6	0·01759	0·3990	0·6919	0·07043	—0·02250	0	—0·00281	0·07500
0·8	—0·02801	0·5000	0·6580	0·1600	—0·04000	0	—0·00500	0·1000
1·0	—0·09929	0·5979	0·6356	0·2488	—0·06250	0	—0·00782	0·1250
2·0	—0·79945	1·079	0·5896	0·6694	—0·2500	0	—0·03125	0·2500
3·0	—2·029	1·563	0·5762	1·067	—0·5625	0	—0·07031	0·3750
4·0	—3·769	2·052	0·57065	1·455	—1·000	0	—0·1250	0·5000
5·0	—6·0135	2·544	0·5679	1·839	—1·5625	0	—0·1953	0·6250

**Standard derivatives from Table 5 of R. & M. 1958[2], converted to Possio's axes.

TABLE 1 (b)

Possio's Results for Mach Number 0·7

Three-point collocation

λ	Z_1	Z_2	Z_3	Z_4	M_1	M_2	M_3	M_4
0	0	0	1·360	—	0	0	0	0
0·12	0·035	0·142	1·200	—	—	0·0001	0·0005	—
0·2	0·065	0·220	1·125	—	—0·0062	0·0003	0·0015	0·046
0·4	0·105	0·360	0·953	—0·148	—0·0217	0·0026	0·0060	0·085
0·6	0·108	0·470	0·840	—0·072	—0·0450	0·0086	0·0135	0·120
0·8	0·090	0·568	0·772	0·010	—0·0692	0·0205	0·0240	0·152
1·0	0·052	0·660	0·730	0·080	—0·0982	0·0400	0·0375	0·182
1·2	0·005	0·739	0·698	0·146	—	0·0691	0·0540	0·211

TABLE 2

N.P.L. Calculated Two-dimensional Derivatives Referred to Standard Mid-Chord Axes

(Pitching moment here evaluated about mid-chord point. Values of Z_1, Z_2, Z_3, Z_4, as in Table 1(a))

λ	Compressible Fluid: Mach number 0·7. three-point collocation except where indicated.				Incompressible Fluid.			
	M_1	M_2	M_3	M_4	M_1	M_2	M_3	M_4
0	0	0	−0·3501	0	0	0	−0·25	0
0·2	−0·02002	−0·04726	−0·2417	0·1062	−0·00862	−0·04160	−0·21045	0·04518
0·4	−0 04226	−0 07396	−0·1962	0 1240	−0 01886	−0·07276	−0·1879	0·05397
0 6	−0·06381	−0·09482	−0·1741	0·1393	−0·02690	−0·09975	−0·1758	0·05739
0·8	−0·08679	−0·11235	−0·16025	0·1574	−0·03300	−0·1250	−0·1695	0·06000
1·0	−0·11215	−0·1264	−0 1486	0 17795	−0·03768	−0·1495	−0·1667	0·06281
1·0§	−0·1145	−0·1257	−0·1483	0·1830	—	—	—	—
2·0	−0·2481	−0·1310	−0·08057	0·26385	−0·05014	−0·2697	−0·1786	0·08264
2·0§	−0·2634	−0·1028	−0·05649	0·29215	—	—	—	—
2·0*	−0·2592	−0·09270	−0·04599	0·2899	—	—	—	—
3·0	−0·3415	−0·08824	−0·04652	0·2926	−0·05517	−0·3908	−0·2144	0·1082
4·0	−0·4108	−0·05466	−0·06005	0·3068	−0·05769	−0·5130	−0·2677	0·1362
5·0	−0·4615	−0·04319	−0·1051	0·3138	−0·05912	−0·6359	−0·3373	0·1653
5·0§	−0·1185	0·05355	0·06531	0·2488	—	—	—	—
5·0*	−0·3026	0·1576	0·02159	0·3332	—	—	—	—

§Five-point collocation. *Seven-point collocation.

TABLE 3

Values of Function $\bar{h}(\lambda x/c) \equiv \bar{f} + i\bar{g}$ *for* $\mu = 0·7$

$\lambda x/c$	$\bar{f}$	$\bar{g}$	$\lambda x/c$	$\bar{f}$	$\bar{g}$
5·10000	0·16270	−0·90056	−5·10000	0·33369	−0·440905
5·02714	0·19674	−0·91036	−5·02714	0·33350	−0·43707
4·95429	0·231625	−0·91761	−4·95429	0·33317	−0·433165
4·88143	0·26714	−0·92225	−4·88143	0·332685	−0·42922
4·80857	0·30313	−0·92425	−4·80857	0·33205	−0·42524
4·73571	0·33937	−0·92358	−4·73571	0·33126	−0·42126
4·66286	0·37570	−0·92022	−4·66286	0·33033	−0·41730
4·59000	0·41189	−0·91418	−4·59000	0·32926	−0·413365
4·51714	0·44778	−0·90547	−4·51714	0·32807	−0·40949
4·44429	0·48316	−0·894125	−4·44429	0·32677	−0·40568
4·37143	0·51786	−0·88018	−4·37143	0·32539	−0·401945
4·29857	0·551675	−0·86370	−4·29857	0·32395	−0·398305
4·22571	0·58445	−0·84475	−4·22571	0·32248	−0·39476
4·15286	0·615985	−0·82342	−4·15286	0·32099	−0·391315
4·08000	0·65026	−0·80391	−4·08000	0·31952	−0·38796
4·00714	0·67473	−0·77398	−4·00714	0·31810	−0·38469
3·93429	0·70163	−0·74610	−3·93429	0·31676	−0·38150
3·86143	0·72667	−0·71629	−3·86143	0·31552	−0·37836
3·78857	0·74975	−0·68468	−3·78857	0·31440	−0·37525
3·71571	0·77073	−0·65142	−3·71571	0·31343	−0·37215
3·642855	0·789495	−0·61666	−3·642855	0·31263	−0·36902
3·57000	0·80596	−0·58058	−3·57000	0·31200	−0·36583
3·49714	0·82003	−0·54333	−3·49714	0·31157	−0·36253
3·42429	0·83164	−0·50511	−3·42429	0·31132	−0·35910
3·35143	0·840725	−0·46607	−3·35143	0·311265	−0·35548
3·27857	0·847245	−0·42643	−3·27857	0·31139	−0·35164
3·20571	0·85116	−0·38635	−3·20571	0·311675	−0·34752

TABLE 3 (*contd.*)

$\lambda x/c$	$\bar{f}$	$\bar{g}$	$\lambda x/c$	$\bar{f}$	$\bar{g}$
3·13286	0·85246	—0·34603	—3·13286	0·31210	—0·34310
3·06000	0·85113	—0·30566	—3·06000	0·31264	—0·33832
2·98714	0·84719	—0·26544	—2·98714	0·31325	—0·33316
2·91428	0·840665	—0·22554	—2·91428	0·31389	—0·32757
2·84143	0·83159	—0·18616	—2·84143	0·31452	—0·32155
2·76857	0·82001	—0·14747	—2·76857	0·31509	—0·31505
2·69571	0·80601	—0·10966	—2·69571	0·31554	—0·30808
2·62286	0·789645	—0·07290	—2·62286	0·31581	—0·30062
2·55000	0·77103	—0·03736	—2·55000	0·315835	—0·29267
2·47714	0·74989	—0·003195	—2·47714	0·315565	—0·28425
2·40429	0·72743	0·02944	—2·40429	0·31493	—0·27537
2·33143	0·70270	0·06041	—2·33143	0·31388	—0·266055
2·25857	0·67620	0·08958	—2·25857	0·31234	—0·256345
2·18571	0·648065	0·11682	—2·18571	0·31028	—0·24628
2·11286	0·61847	0·14203	—2·11286	0·30764	—0·23591
2·04000	0·58756	0·16510	—2·04000	0·30438	—0·22529
1·96714	0·555535	0·18596	—1·96714	0·30047	—0·21448
1·89429	0·52255	0·20452	—1·89429	0·295875	—0·20356
1·82143	0·48881	0·22072	—1·82143	0·29058	—0·19258
1·74857	0·45450	0·23454	—1·74857	0·28459	—0·18163
1·67571	0·41982	0·24591	—1·67371	0·27791	—0·17077
1·60286	0·38495	0·25485	—1·60286	0·27050	—0·16007
1·53000	0·35012	0·26134	—1·53000	0·26244	—0·14961
1·45714	0·31551	0·26540	—1·45714	0·25375	—0·13946
1·384285	0·28134	0·26706	—1·384285	0·24446	—0·129665
1·31143	0·24780	0·26636	—1·31143	0·23462	—0·12028
1·23857	0·21510	0·26335	—1·23857	0·22429	—0·11136
1·16571	0·18345	0·25813	—1·16571	0·21354	—0·10293
1·09286	0·15304	0·25079	—1·09286	0·20242	—0·09500
1·02000	0·12406	0·24140	—1·02000	0·19100	—0·08762
0·94714	0·09671	0·23013	—0·94714	0·17937	—0·08072
0·874285	0·07121	0·21709	—0·874285	0·16756	—0·07433
0·80143	0·04772	0·20245	—0·80143	0·15567	—0·06839
0·72857	0·02643	0·18570	—0·72857	0·14375	—0·06349
0·69214	0·01668	0·17784	—0·69214	0·13832	—0·06085
0·65571	0·00757	0·16903	—0·65571	0·13181	—0·05762
0·619285	—0·00092	0·15995	—0·619285	0·12635	—0·05574
0·58286	—0·00872	0·15064	—0·58286	0·11990	—0·05261
0·54643	—0·01583	0·14113	—0·54643	0·11440	—0·05087
0·51000	—0·02218	0·13142	—0·51000	0·10800	—0·04771
0·47357	—0·02780	0·12161	—0·47357	0·10244	—0·04600
0·43714	—0·03262	0·11165	—0·43714	0·09607	—0·04276
0·40071	—0·03663	0·10165	—0·40071	0·09057	—0·04138
0·364285	—0·03974	0·09157	—0·364285	0·08461	—0·03917
0·32796	—0·04198	0·08152	—0·32786	0·07844	—0·03646
0·29143	—0·04325	0·07149	—0·29143	0·07213	—0·03363
0·25500	—0·04351	0·06166	—0·25500	0·06567	—0·03050
0·21857	—0·04265	0·05177	—0·21857	0·05897	—0·02735
0·18214	—0·04062	0·04217	—0·18214	0·05197	—0·02387
0·14571	—0·03723	0·03284	—0·14571	0·04451	—0·02007
0·10929	—0·03230	0·02383	—0·10929	0·03540	—0·01587
0·07286	—0·02542	0·01526	—0·07286	0·02724	—0·01123
0·06557	—0·02377	0·01359	—0·06557	0·02524	—0·01025
0·05829	—0·02200	0·01197	—0·05829	0·02317	—0·00923
0·05100	—0·02011	0·01036	—0·05100	0·02100	—0·00815
0·04371	—0·01809	0·00877	—0·04371	0·01875	—0·00713
0·03643	—0·01588	0·00723	—0·03643	0·01633	—0·00601
0·02914	—0·01351	0·00569	—0·02914	0·01380	—0·00491
0·02186	—0·01088	0·00422	—0·02186	0·01105	—0·00373
0·01457	—0·00795	0·00276	—0·01457	0·00803	—0·00253
0·00729	—0·00468	0·00135	—0·00719	0·00470	—0·00130
0	0	0	0	0	0

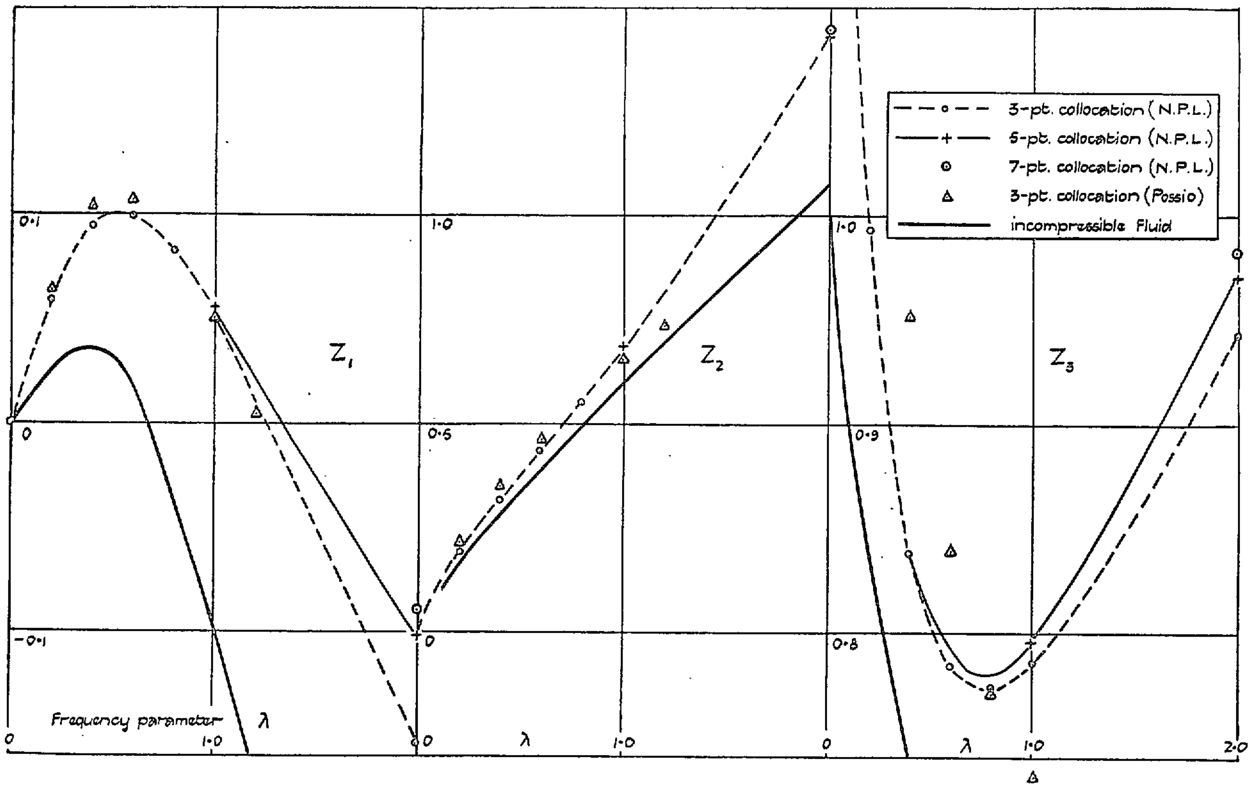

FIG. 2. Comparison of N.P.L. and Possio's Derivative Results.
(Possio's Axes : Mach Number 0·7.)

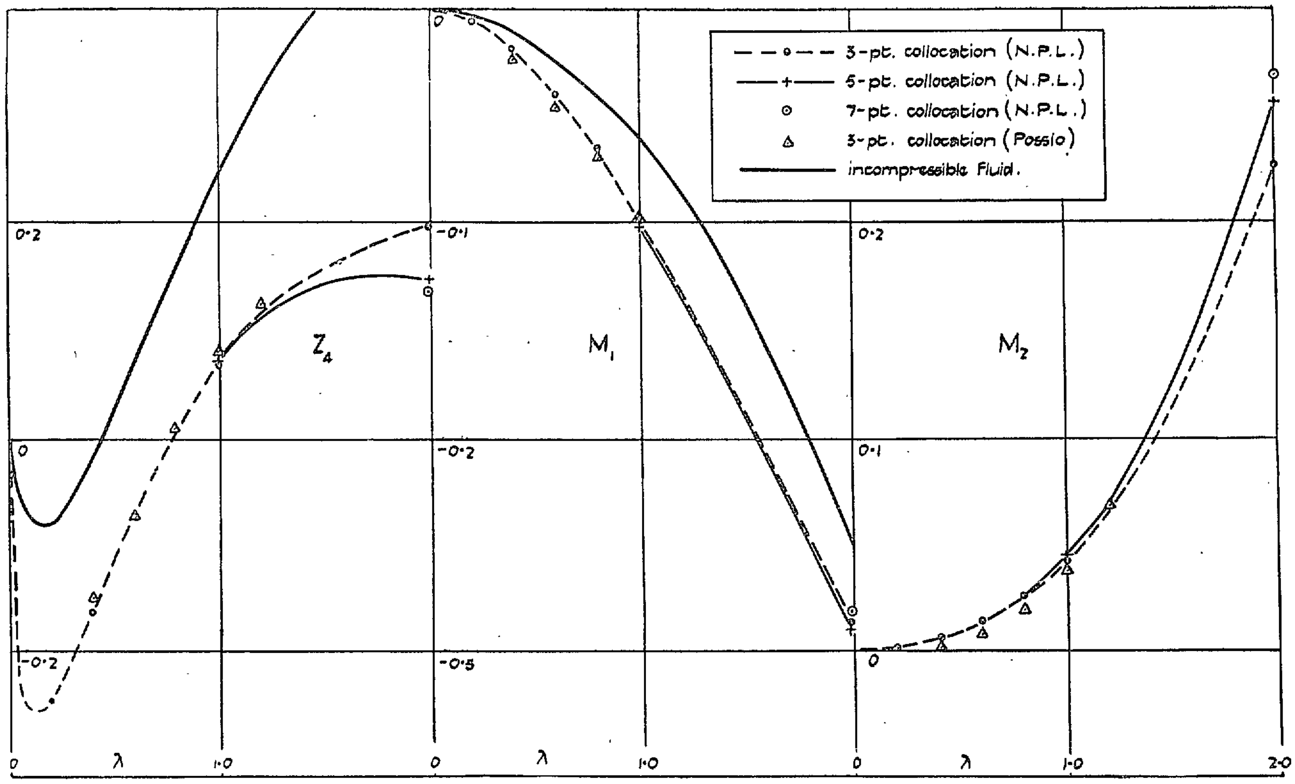

FIG. 2 (*contd.*). Comparison of N.P.L. and Possio's Derivative Results (Possio's Axes : Mach Number 0·7).

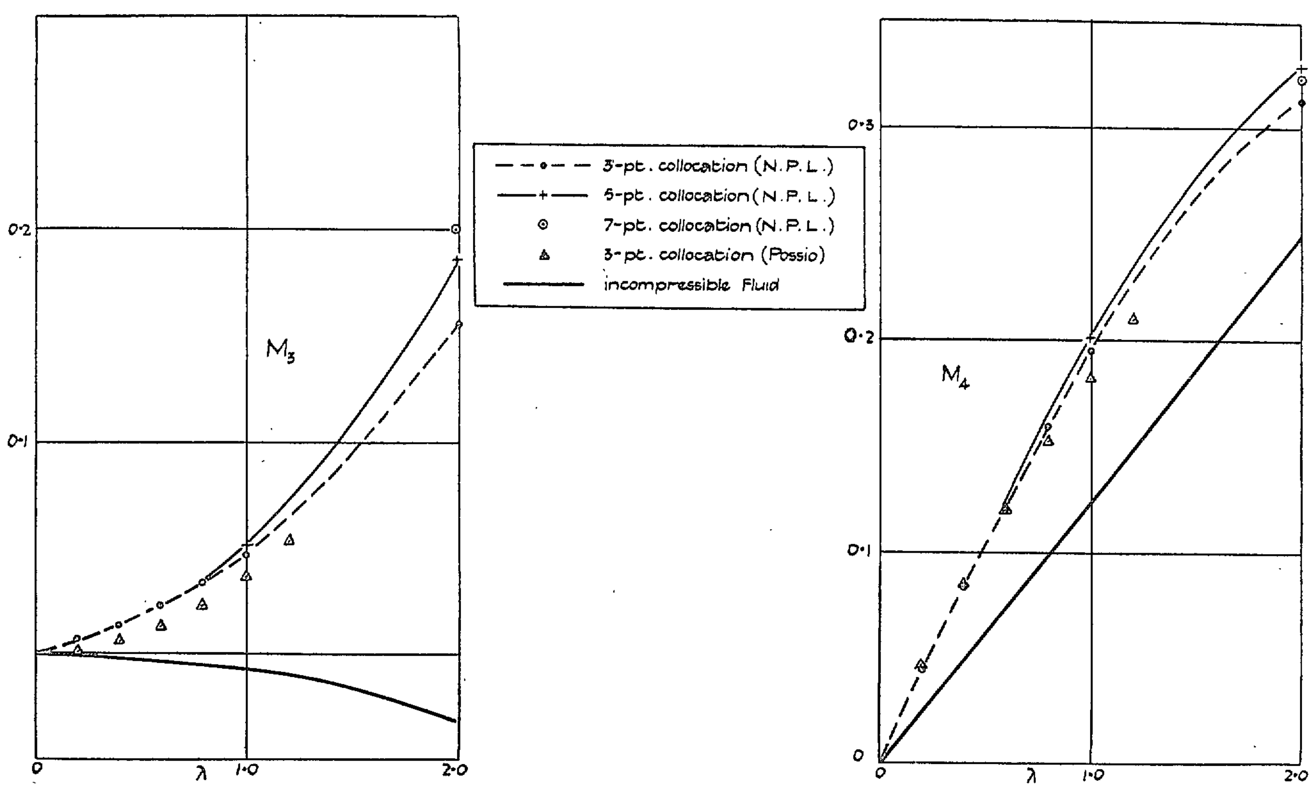

FIG. 2 (*contd.*). Comparison of N.P.L. and Possio's Derivative Results (Possio's Axes : Mach Number 0·7.)

PART II

Influence of Compressibility on the Flexural-Torsional Flutter of a Tapered Cantilever Wing Moving at Subsonic Speed

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Summary.—Calculations based on Possio's subsonic derivative theory and on vortex strip theory were made to obtain preliminary information on the influence of compressibility and flying height on the critical speed for flexural-torsional flutter. The results are summarised by curves corresponding to constant altitude H, which show the variation of N with wing stiffness ratio r, where N denotes the ratio of the critical speed for flutter of the wing in compressible air at a Mach number of $0 \cdot 7$ to the critical speed for flutter of the same wing in incompressible air. The results indicate that for $1 \leqslant r \leqslant 3$ the compressibility correction is insignificant at sea level, and that N is of the order $0 \cdot 95$ to $0 \cdot 92$ at $H = 30{,}000$ ft. More extensive test calculations are very desirable.

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1. *Introduction and Summary.*—The purpose of the calculations was to obtain preliminary information on the influence of compressibility and altitude on the critical speed for flexural-torsional flutter of a tapered cantilever wing moving at subsonic speeds. The derivatives appropriate to the wing were estimated by vortex strip theory, the basic two-dimensional values being chosen to accord with the curves of Fig. 2 of Part I corresponding to five-point collocation. As these curves only apply for a Mach number μ of $0 \cdot 7$, the calculations are necessarily very restricted in scope.

The results are shown in Fig. 1 of Part II, where the ordinate N denotes the ratio of the critical speed V_{cc} for flutter of the wing in compressible air at $\mu = 0 \cdot 7$ to the critical speed V_{ci} for flutter of the same wing in incompressible air. The results indicate that with normal stiffness ratios $(1 \leqslant r \leqslant 3)$ the compressibility correction is insignificant at sea level, and that N is of the order $0 \cdot 95$ to $0 \cdot 92$ at altitude $H = 30{,}000$ ft. The corresponding divergence speed ratio is given by $V_{dc}/V_{di} = (1 - \mu^2)^{1/4}$, which accords with a simple formula applied by Theodorsen and Garrick[1] (19) for critical flutter speeds.

A noteworthy feature of the results is the fact that, despite the pronounced differences between the derivatives appropriate to the compressible and incompressible fluids, the corresponding changes of critical speeds are so small—provided, of course, the sets of derivatives used are themselves self-consistent and accurate. More extensive test calculations are very desirable. A valuable advance in the theory would be the development of a direct formal solution of the fundamental integral equation (equation (20) of Part I) in terms of known functions, by a generalisation of the methods which have been applied successfully for the incompressible fluid.

2. *Method of Calculation.*—The formulæ used for the calculation of the derivatives and critical speeds are taken with slight modifications from R. & M. 1943[2].

(*a*) *Notation.*—The essential notation is as follows:—

s	span (root to tip)
c, c_0	root and tip chords, respectively
$\beta\ (= (c_0 - c_t)/c_0)$	taper ratio
l	distance of reference section from root

$\xi\ (= y/s)$	spanwise co-ordinate
$f_1(\xi),\ F_2(\xi)$	flexural and torsional distortion modes
$\lambda_0\ (= wc_0/V)$	frequency parameter at root
mc^2	wing mass per unit span at section ξ
jc	asbcissa of centre of mass at section ξ
$d\ (= 0{\cdot}9s)$	distance from root to " equivalent tip section "
Kc	radius of gyration at section ξ
ρ	air density at height H
$\rho_0\ (= 0{\cdot}002378)$	standard air density at sea level
σ_w	wing density (slugs/ft^3)
$\sigma = \rho/\rho_0$	
$\lambda/\lambda_0 = c/c_0 = 1 - \beta\xi$	
V_c	critical speed
$\bar{V}_c\{= V_c\sqrt{\rho}/\sqrt{(m_\theta/dc_m^2)}\}$	critical speed coefficient
$r = (l_\phi/d^3)/(m_\theta/dc_m^2)$.	

(*b*) *Derivatives.*—The integrals for the air-load coefficients are

$$L_1 + iL_2 = -\frac{L_\phi}{\rho V^2 l^3} + \frac{\lambda_0^2 L_{\ddot\phi}}{\rho l^3 c_0^2} - \frac{i\lambda_0 L_{\dot\phi}}{\rho V l^3 c_0} = \frac{\pi s}{l}\int_0^1 (\bar{Z}_1 + i\bar{Z}_2)\, f_1^2 d\xi\ ,$$

$$L_3 + iL_4 = -\frac{L_\theta}{\rho V^2 l^2 c_0} + \frac{\lambda_0^2 L_{\ddot\theta}}{\rho l^2 c_0^3} - \frac{i\lambda_0 L_{\dot\theta}}{\rho V l^2 c_0^2} = \frac{\pi s}{l}\int_0^1 (\bar{Z}_3 + i\bar{Z}_4)\left(\frac{c}{c_0}\right) f_1 F_2 d\xi\ ,$$

$$M_1 + iM_2 = -\frac{M_\phi}{\rho V^2 l^2 c_0} + \frac{\lambda_0^2 M_{\ddot\phi}}{\rho l^2 c_0^3} - \frac{i\lambda_0 M_{\dot\phi}}{\rho V l^2 c_0^2} = \left(\frac{\pi s}{l}\right)\int_0^1 (\bar{M}_1 + i\bar{M}_2)\left(\frac{c}{c_0}\right) f_1 F_2 d\xi\ ,$$

$$M_3 + iM_4 = -\frac{M_\theta}{\rho V^2 l c_0^2} + \frac{\lambda_0^2 M_{\ddot\theta}}{\rho l c_0^3} - \frac{i\lambda_0 M_{\dot\theta}}{\rho V l c_0^3} = \left(\frac{\pi s}{l}\right)\int_0^1 (\bar{M}_3 + \bar{M}i_4)\left(\frac{c}{c_0}\right)^2 F_2^2 d\xi_1\ .$$

In these formulae the barred terms in the integrands denote the two-dimensional air-load coefficients appropriate to the wing reference axis. The values for these coefficients were chosen to accord with the curves of Fig. 2 of Part I corresponding to five-point collocation.

(*c*) *Inertial Coefficients.*—These are given by

$$a_1 = \frac{A_1}{\rho l^3 c_0^2} = \left(\frac{s}{l}\right)\left(\frac{m}{\rho}\right)\int_0^1 \left(\frac{c}{c_0}\right)^2 f_1^2 d\xi\ ,$$

$$p = \frac{P}{\rho l^2 c_0^3} = \left(\frac{s}{l}\right)\left(\frac{m}{\rho}\right)\int_0^1 \left(\frac{c}{c_0}\right)^3 f_1 F_2 d\xi\ ,$$

$$g_3 = \frac{G_3}{\rho l c_0^3} = \left(\frac{s}{l}\right)\left(\frac{m}{\rho}\right) K^2 \int_0^1 \left(\frac{c}{c_0}\right)^4 F_2^2 d\xi\ .$$

For the particular type of wing under consideration

$$m = \frac{3\sigma_w}{4}\left(\frac{4 - 4\beta + \beta^2}{3 - 3\beta + \beta^2}\right).$$

If a_{10}, p_0, g_{30} denote the inertial coefficients appropriate to $\rho = \rho_0$, and if $\sigma \equiv \rho/\rho_0$, then

$$a_1 = a_{10}/\sigma\,; \quad p = p_0/\sigma\,; \quad g_3 = g_{30}/\sigma\,.$$

(*d*) *Critical Speeds.*—The critical conditions for the wing are determined by solution of the two real equations contained in

$$\begin{vmatrix} \dfrac{rY'}{0\cdot1512\,(2-\beta)^2} - \dfrac{a_{10}\lambda_0^2}{\sigma} + L_1 + iL_2, & -\dfrac{p_0\lambda_0^2}{\sigma} + L_3 + iL_4 \\ -\dfrac{p_0\lambda_0^2}{\sigma} + M_1 + iM_2\,, & Y' - \dfrac{g_{30}\lambda_0^2}{\sigma} + M_3 + iM_4 \end{vmatrix} = 0\,, \quad .. \qquad (1)$$

where

$$\bar{V}_c = V_c\sqrt{\rho}/\sqrt{(m_\theta/dc_m^2)} = 0\cdot567\,(2-\beta)/\sqrt{Y'}\,. \quad .. \quad .. \quad .. \qquad (2)$$

In the case of the compressible fluid the simplest procedure in calculation is to assign (i) the Mach number μ, (ii) the values of ρ, σ and V_{s0} (velocity of sound) appropriate to a chosen altitude H, and (iii) the critical frequency parameter λ_0. Then the critical speed, say V_{cc} $(= \mu V_{s0})$ is also assigned. The two unknowns in (1) are accordingly regarded as r and Y', and if the calculations lead to real and positive values for both of these, the corresponding elastic stiffnesses l_ϕ, m_θ can be derived. Thus in brief, the method is to calculate the stiffnesses which will lead to flutter with a given frequency at a given altitude and Mach number. The critical speed V_{ci} and critical frequency appropriate to the incompressible fluid and to the same altitude and stiffnesses can then be calculated in the usual way.

The ratio of the *divergence* speeds appropriate to the compressible and incompressible fluids is readily seen to be given by

$$V_{dc}/V_{di} = \sqrt{\left(\frac{-M_3\,(\lambda_0 = 0, \mu = 0)}{-M_3\,(\lambda_0 = 0, \mu = \mu)}\right)} = (1 - \mu^2)^{1/4}\,. \quad .. \quad .. \quad .. \quad .. \qquad (3)$$

This accords with Theodorsen and Garrick's proposed correction for critical speeds for flutter.

(*e*) *Numerical Application.*—The values adopted for the wing constants were

$$\beta = 1/2\cdot1\,; \quad l = 0\cdot7s\,; \quad \sigma_w = 0\cdot02485\,; \quad j = 0\cdot1\,; \quad K = 0\cdot296\,.$$

The flexural axis was assumed to lie at distance $0\cdot3c$ behind the leading edge, and the distortion modes were taken for simplicity to be

$$f_1(\xi) = (s/l)^2\xi^2\,; \quad F_2(\xi) = (s/l)\xi\,.$$

These data yield

$$a_{10} = 4\cdot436\,; \quad p_0 = 0\cdot2623\,; \quad g_{30} = 0\cdot1670\,.$$

For the variation of ρ and V_{s0} with altitude reference was made to the tables given by Pankhurst and Conn[3]. The following figures are representative.

Altitude (ft)	ϱ (slugs/ft³)	σ	V_{so} (ft/sec)	V_{cc} ($\mu = 0 \cdot 7$)
0	0·002378	1·000	1117	781·9
10,000	0·001756	0·7385	1078	754·6
20,000	0·001267	0·5328	1037	725·9
30,000	0·0008896	0·3741	995	696·5
40,000	0·0005857	0·2463	968	677·6

The calculated air-load coefficients for the wing appropriate to $\mu = 0 \cdot 7$ and $\mu = 0$ are given in Table 1, and the results of the flutter calculations are summarized in Table 2 and Fig. 1.

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1	T. Theodorsen and I. E. Garrick ..	Mechanism of Flutter. N.A.C.A. Report No. 685.
2	R. A. Frazer and S. W. Skan	A Comparison of the Observed and Predicted Flexture-Torsion Flutter Characteristics of a Tapered Model Wing. R. & M. 1943. August, 1941.
3.	R. C. Pankhurst and J. F. C. Conn ..	Physical Properties of the Standard Atmosphere. R. & M. 1891. January, 1941.

TABLE 1

N.P.L. Calculated Derivatives for Tapered Wing

(Derivatives referred to flexural axis at $0 \cdot 3c$ behind leading edge. Flexural distortion mode parabolic and torsional mode linear. Mach number $0 \cdot 7$.)

λ	L_1	L_2	L_3	L_4	M_1	M_2	M_3	M_4
0	0	0	2·8355	0	0	0	−0·09015	0
0·6	0·3335	1·1005	1·762	−0·2223	−0·04357	−0·01989	−0·04305	0·1148
1·0	0·3662	1·630	1·636	0·05849	−0·09709	−0·01492	−0·03068	0·1751
1·2	0·3353	1·904	1·616	0·1913	−0·13045	−0·00693	−0·02400	0·20755
1·6	0·2189	2·462	1·638	0·4179	−0·2108	0·02630	−0·00614	0·2741
1·8	0·1480	2·789	1·665	0·51095	−0·2546	0·05168	0·00513	0·3053
2·0	0·07589	3·111	1·695	0·5791	−0·2983	0·08204	0·01804	0·3345

Incompressible Fluid

λ	L_1	L_2	L_3	L_4	M_1	M_2	M_3	M_4
0	0	0	2·025	0	0	0	−0·06438	0
0·6	0·1269	0·9968	1·549	0·06449	−0·02160	−0·02815	−0·05307	0·06194
1·0	0·05333	1·486	1·398	0·3329	−0·05209	−0·04176	−0·05586	0·09550
1·2	−0·03999	1·716	1·342	0·4710	−0·07133	−0·04829	−0·05928	0·1124
1·6	−0·3325	2·165	1·244	0·74535	−0·12055	−0·06077	−0·07002	0·1467
1·8	−0·5477	2·381	1·196	0·8755	−0·1515	−0·06694	−0·07662	0·1640
2·0	−0·7978	2·599	1·150	1·010	−0·18585	−0·07301	−0·08466	0·1812

TABLE 2

Influence of Compressibility and Altitude on the Critical Speed Coefficients of a Family of Tapered Wings

Explanation.—For the compressible fluid the critical speed V_{cc} at each altitude H is assigned to correspond to a constant Mach number $\mu = 0 \cdot 7$. The corresponding critical speed coefficient $\bar{V}_{cc}(\mu = 0 \cdot 7) = V_{cc} \sqrt{\rho}/\sqrt{(m_\theta/dc_m{}^2)}$ effectively determines the torsional stiffness m_θ associated with each value of H and r. The influence of the compressibility on the critical speeds of the different wings is shown by the ratio $N = V_{cc}/V_{ci}$.

H (ft)	r	Compressible Fluid			Incompressible Fluid		$N = \dfrac{V_{cc}}{V_{ci}}$	Divergence Ratio $\dfrac{V_{dc}}{V_{di}}$
		V_{cc}	$\bar{V}_{cc}$	$m_\theta/dc_m{}^2$	$\bar{V}_{ci}$	V_{ci}		
0	0	781·9	1·995	365·3	1·948	763·6	1·024	0·845
	1	781·9	1·756	471·5	1·751	779·6	1·003	
	2	781·9	1·591	574·3	1·572	772·6	1·012	(constant for all H and r)
	3	781·9	1·457	684·8	1·426	765·1	1·022	
	4	781·9	1·349	798·9	1·296	751·1	1·041	
	5	781·9	1·263	911·4	1·182	732·1	1·068	
	6	781·9	1·194	1020	1·087	712·1	1·098	
	7	781·9	1·162	1077	1·008	678·1	1·153	
10,000	1	754·6	1·680	479·8	1·707	766·9	0·984	
	2	754·6	1·488	611·6	1·532	777·1	0·971	
	3	754·6	1·360	732·1	1·373	761·5	0·991	
	4	754·6	1·255	859·7	1·250	751·6	1·004	
	5	754·6	1·172	985·8	1·143	736·2	1·025	
	6	754·6	1·115	1089	1·051	711·2	1·061	
	7	754·6	1·065	1194	0·970	687·2	1·098	
20,000	1	725·9	1·611	482·8	1·667	751·4	0·966	
	2	725·9	1·432	611·1	1·496	758·5	0·957	
	3	725·9	1·286	757·7	1·342	757·7	0·958	
	4	725·9	1·175	907·6	1·211	748·4	0·970	
	5	725·9	1·093	1049	1·104	733·2	0·990	
	6	725·9	1·041	1156	1·017	708·9	1·024	
	7	725·9	1·020	1204	0·948	674·6	1·076	
30,000	2	696·5	1·357	626·5	1·460	749·7	0·929	
	3	696·5	1·212	785·3	1·314	757·1	0·920	
	4	696·5	1·106	943·1	1·185	746·5	0·933	
	5	696·5	1·026	1096	1·076	730·1	0·954	
	6	696·5	0·978	1206	0·993	707·1	0·985	
40,000	3	677·6	1·149	827·0	1·271	749·6	0·904	
	4	677·6	1·047	996·0	1·156	747·9	0·906	
	5	677·6	0·963	1177	1·057	743·8	0·911	
	6	677·6	0·920	1290	0·973	716·3	0·946	

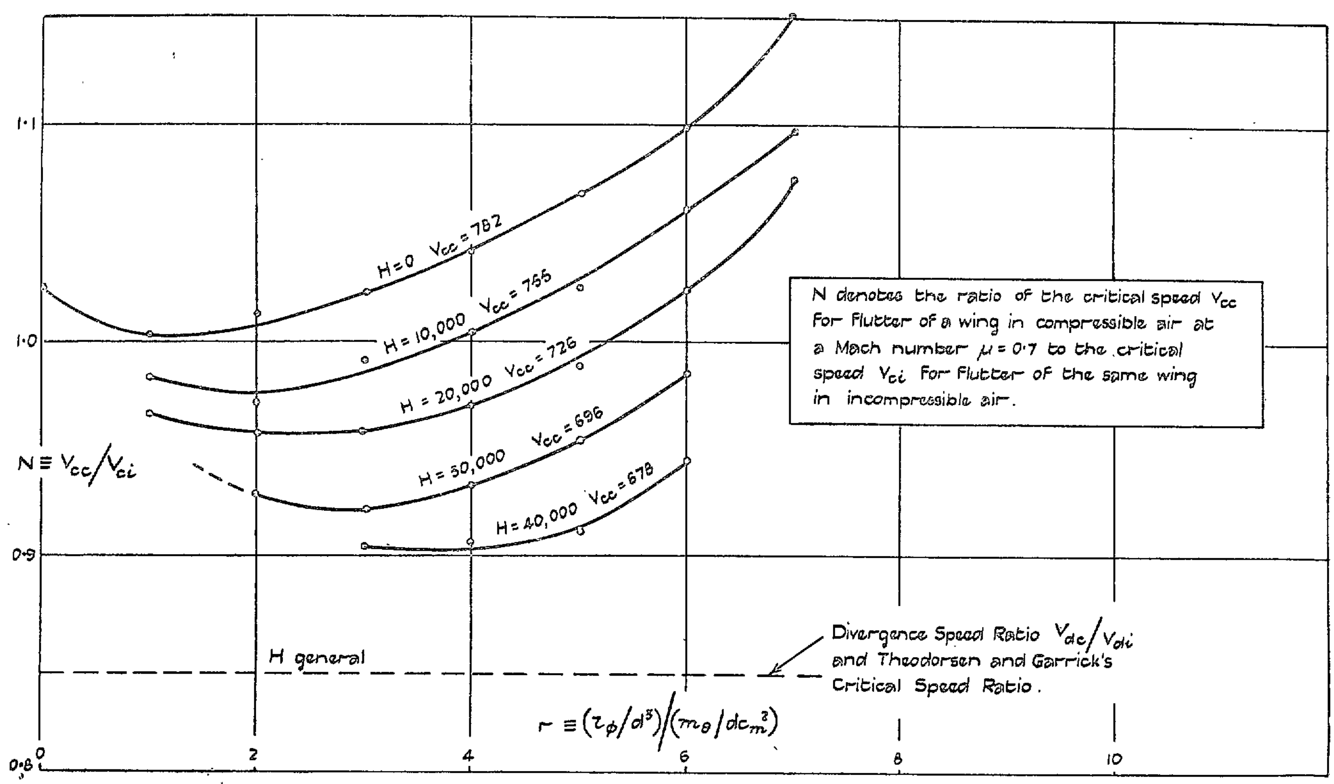

FIG. 1. Influence of Compressibility on Critical Speed.

(94984) Wt. 14/806 K.5 12/51 Hw.

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