R. & M. No. 3407

MINISTRY OF AVIATION

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

# Stress-Concentration Factors for Rounded Rectangular Holes in Infinite Sheets

By A. J. SOBEY, M.A.

LONDON: HER MAJESTY'S STATIONERY OFFICE
1965
PRICE 6*s*. 6*d*. NET

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By A. J. Sobey, M.A.

Communicated by the Deputy Controller Aircraft (Research and Development),
Ministry of Aviation

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*Reports and Memoranda No. 3407**
*November, 1963*

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*Summary.*

Comprehensive numerical estimates of the stress concentration around unreinforced rounded rectangular holes in infinite sheets are presented. Both tensile and shear loading are discussed.

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* Replaces R.A.E. Report No. Structures 292—A.R.C. 25 629.

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## 1. *Introduction.*

The introduction of a hole in a stressed elastic sheet causes a redistribution of the stress field. This effect is greatest on the boundary of the hole and diminishes with distance until, in regions remote from the hole, it is negligible. Around the boundary of the hole the variation of edge or hoop stress may be considerable and the largest values of the stress are associated with regions of large curvature (i.e. corners). The value of the maximum hoop stress is an important design parameter and its estimation is, in general, very difficult.

The exceptional case of the ellipse (and circle) has been analysed by Inglis[1] who used the results to provide an empirical estimate of the peak stress for a general hole. In default of more accurate information, his concept of an 'equivalent ellipse' has been found useful. Cox[2] has confirmed the hypothesis for a family of polygonal holes with rounded, but not circular arc, corners. He showed also that for deep narrow grooves the equivalent ellipse is very useful.

More accurate methods of stress estimation applicable to practical holes (such as those depicted in Fig. 1) are required. A general procedure for the solution of such problems has been presented by the author[3] using the complex-variable methods of Mushkhelishvili[4]. In this analysis the given hole is replaced by one differing only slightly in profile from the given one and which admits of exact analysis. This approximate profile can be made as close as desired to the given one by taking a sufficiently large number of terms in the mapping function[3], so that the stresses can be derived with sufficient accuracy. This procedure has been applied in this report to a family of rounded rectangular holes (Fig. 1) in sheets subjected to biaxial stress or shear.

## 2. *Summary of Numerical Investigations.*

The shape of the hole is defined by the ratios $a/\rho$ and $b/\rho$ (Fig. 1), which range over the values 0, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 5, 10 and 20. The symbol $[f_1, f_2; q]$ is used to denote the loading conditions at infinity where $f_1$ and $f_2$ are tensions applied along and normal to the $a$-side and $q$ is the applied shear. Thus [0, 1; 0] is a unit tensile loading tending to open the slot of Fig. 1b. All holes are examined for [1, 0; 0], [0, 1; 0] and [0, 0; 1] loadings separately.

### 2.1. *Approximate Mapping of Holes.*

The mapping of these profiles is accomplished using 40-term mapping-functions of polynomial type[3] although for $a/\rho$ or $b/\rho$ of 20 more terms are necessary. A typical profile, which corresponds to $a = 5\rho$, $b = 2\rho$ is presented in detail. The use of a finite number of terms in the mapping function represents imperfectly the required curvature distribution. The variation of curvature for the example realized by a 40-term mapping-function is shown in Fig. 3 using as abscissa the variable $\theta$ (which is the argument of the auxiliary variable of Ref. 3 and whose variation with position on the boundary is shown in Fig. 2). It is noteworthy that about 30% of the working range of $\theta$ corresponds to points within the quadrant and illustrates the natural weighting of the corner region implicit in the mapping technique used.

### 2.2. *Smoothing of Computed Stresses.*

Owing to the induced oscillation of the local curvature about the nominal corner value, the stress distributions are in error. Where, within the quadrant, the local curvature is high, so is the local stress and *vice versa.* The stress distributions for the three basic loadings all contain an oscillatory variation. This perturbation of stress is due to errors in the local curvature and is exactly in phase with the perturbation of curvature. The computed stresses (Fig. 3, solid line) can be smoothed (Fig. 3, dotted line) and the inferred maximum of the stress variation, which would correspond to a perfectly represented profile, can be quoted with confidence.

If the local curvature is perturbed from an approximately constant value by a sinusoidal component of small amplitude and wavelength the influence of this component on the stress field can be regarded as local. Thus the perturbation can be interpreted as affecting the local stress field in exactly the same way as it would affect a constant stress field and this factor is known. (See Section 3.2 of Ref. 2 or Appendix I of Ref. 3.) Hence the stress for an ideal, unperturbed profile can be deduced. In practice (Fig. 3) graphical methods are quite satisfactory for smoothing the results.

In the remainder of this report all reference to computed stress implies that the perturbation error has been smoothed out.

## 3. *Results.*

### 3.1. *Tensile Loading.*

Hoop-stress distributions are presented in Fig. 4 for the representative example ($a/\rho = 5$, $b/\rho = 2$) in various biaxial loading conditions. These are converted into stress-concentration factors using the usual critical stress combination[5, 6, 7, 8]:

$$Q = [\sigma_n^2 + \sigma_t^2 - \sigma_n \sigma_t + 3\tau^2]^{1/2} .$$

Along the boundary of the hole $Q = \sigma_t$ so that the stress-concentration factor is

$$F = \frac{[\sigma_t]_{\max} \text{ on boundary}}{[Q]_\infty} .$$

For the various biaxial cases of Fig. 4 the following values of the stress-concentration factor apply:

4·30 in [0, 1; 0]

5·03 in [1, 2; 0]

4·68 in [1, 1; 0]

3·96 in [2, 1; 0]

and 3·09 in [1, 0; 0]

The corresponding value for shear [0, 0; 1] is 4·37. This example serves to illustrate the effects of applying biaxial tension to a typical profile.

On the basis of the equivalent ellipse, the stress-concentration factor for a profile in [1, 0; 0] loading should vary linearly with $(1+b/\rho)^{1/2}$, but only for the slot ($a = 0$) is the relationship effectively linear. Fig. 5 shows how the variation changes for different values of $a/\rho$ and in Fig. 6 the results are cross-plotted for constant values of $b/\rho$. The limiting case of a slot ($a/\rho \rightarrow \infty$, $b/\rho \rightarrow 0$) in [1, 0; 0] loading appears to have a stress-concentration factor of 2.

Figs. 7 to 9 present variations of maximum hoop stress in 1:½ and 1:1 biaxial loading conditions. In the former case, the maximum hoop stress is $\sqrt{3}/2$ times the stress-concentration factor whilst in the latter the two factors are identical. In Fig. 10 the rounded square in tension is compared with the slot in [0, 1; 0] and [1, 0; 0] and is seen to be approximately midway between these cases. Note that for the square in uniaxial tension there is a minimum stress-concentration factor of about 2·81 which corresponds to a value of $a/\rho$ of about 1/3 and represents a slight improvement over the circular hole.

### 3.2. *Shear Loading.*

In Fig. 11 the variation of hoop stress with $a/\rho$ and $b/\rho$ is shown for the sheet in shear. For small $b/\rho$ the peak stress rises steadily with increasing $a/\rho$ but as $b/\rho$ increases, to about 10, this trend is checked and reversed and it is possible to improve on the performance of the slot. (See Fig. 12) For values of $b/\rho$ of 20 and above this diminution in stress is notable and some relief of stress by the introduction of a 'side' is apparent.

## 4. *Comparison with Previous Work.*

Approximations to the rounded-square profile based on the Schwarz-Christoffel transformation of the sharp-cornered square have been presented many times[4, 5, 6, 7]. In the range of $a/\rho$ considered, only two approximations corresponding to the first two and first three terms respectively of the square transformation are obtained. In both cases the profile differs considerably in local curvature variation from the ideal profile of Fig. 1c so that the stress distributions are not very accurate (Figs. 10, 12). The local curvature in the neighbourhood of the stress concentration is smaller than the nominal corner value, which tends to reduce the estimated stress-concentration factor.

Wittrick[5] has introduced a 3-term transformation of the 'rounded-square' of the form

$$Z = R\{\zeta + a_4\zeta^{-3} + a_8\zeta^{-7}\}$$

where $a_4$ and $a_8$ are real constants so chosen that the curvature at the 45° point is correct and also that there is zero curvature at the mid-points of the sides. (Fig. 13). For small $a/\rho$ the profile is a poor estimate of the rounded square of Fig. 1c and the estimated stresses are in error. As $a/\rho$ decreases the Wittrick profile becomes less and less comparable with Fig. 1c as the centre of curvature at the corner moves outside the square. {For $a/\rho$ of 100, the centre of curvature at the corner has co-ordinates of about $1\cdot 1$ $(a+\rho)$.} This incutting of the corner (Fig. 10) increases the loading on the corner and so raises the apparent stress concentration which compensates for the previous effect. For $a/\rho$ of about 20 the results for the Wittrick profile are superficially accurate due to these errors being approximately equal and opposite.

## 5. *Conclusion.*

Detailed stress-concentration factors for the unreinforced rounded-rectangular hole in an infinite sheet in tension or shear have been evaluated and comprehensive charts presented from which the peak stress for any hole of this type may be interpolated.

## SYMBOLS

| | |
|---|---|
| $a$, $b$ | Geometric parameters specifying the rounded rectangle (Fig. 1) |
| $\rho$ | Radius of curvature of the quadrants forming corners of the rounded rectangle |
| $f_1$, $f_2$ | Tensions applied to the sheet at a great distance from the hole in the $a$ and $b$ directions |
| $q$ | The shear stress applied to the sheet at a great distance from the hole |
| $\theta$ | An auxiliary parameter from Ref. 3 |
| $\sigma_n$, $\sigma_t$ | Stresses associated with arbitrary directions $n$ and $t$ at a point in the sheet |
| $\tau$ | The corresponding shear stress |
| $Q$ | A stress combination (*see* Section 3.1) |
| $F$ | A stress-concentration factor |
| $a_4$, $a_8$ | Mapping coefficients associated with Wittrick's profile (*See* Section 4) |

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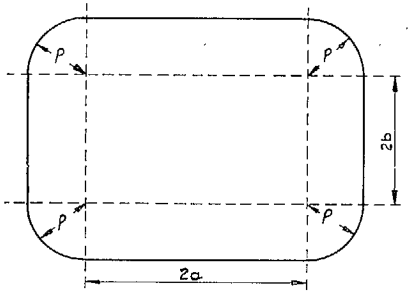
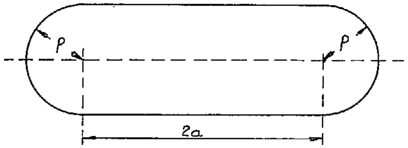
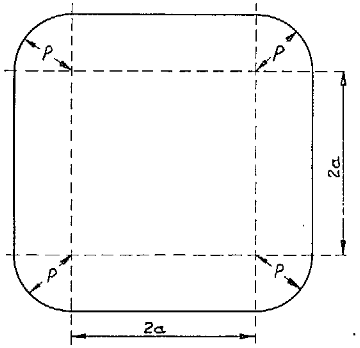


(a) ROUNDED RECTANGLE.



(b) ROUNDED SLOT.



(c) ROUNDED SQUARE.

FIG. 1a to c. Typical profiles and notation.

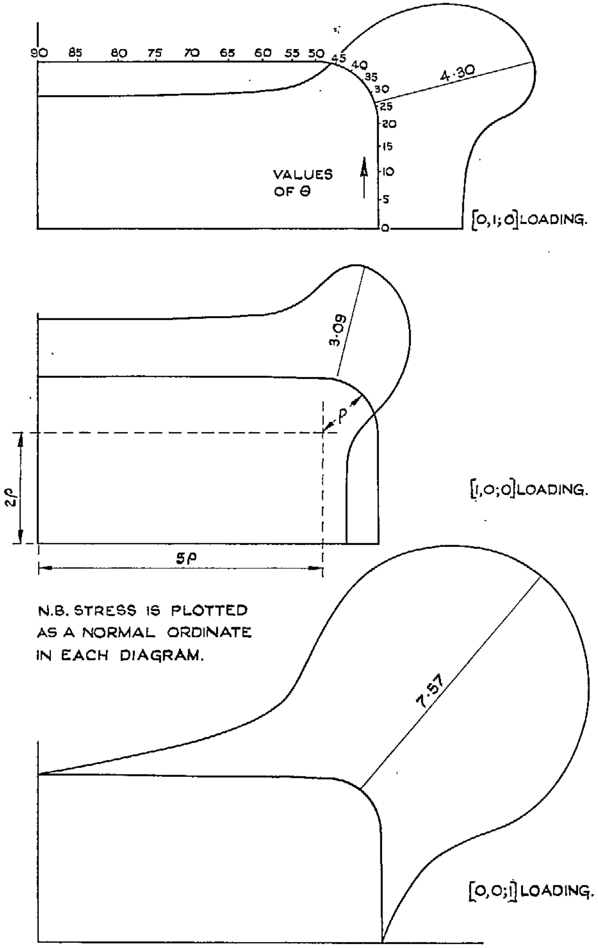


FIG. 2. Variation of hoop stress around boundary for typical hole.

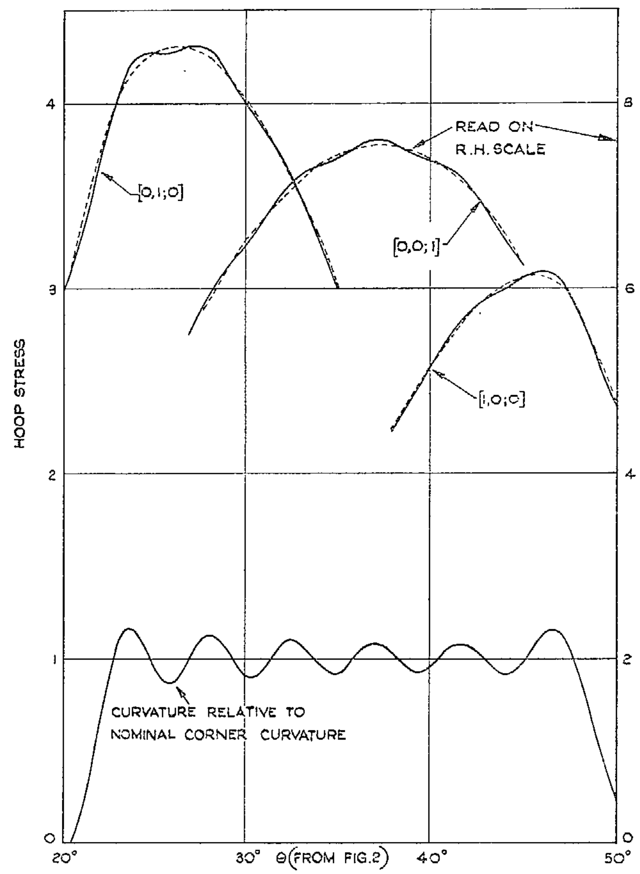


FIG. 3. Perturbation of stress distributions due to small irregularities in the mapped profile of the hole.

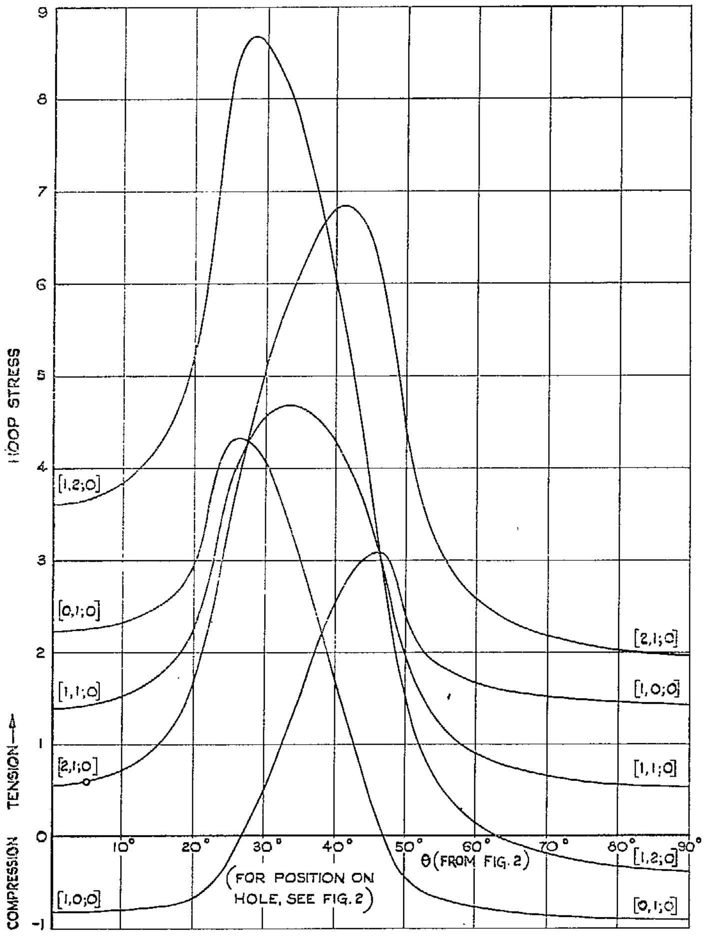


FIG. 4. Hoop-stress variation in various biaxial loading conditions.

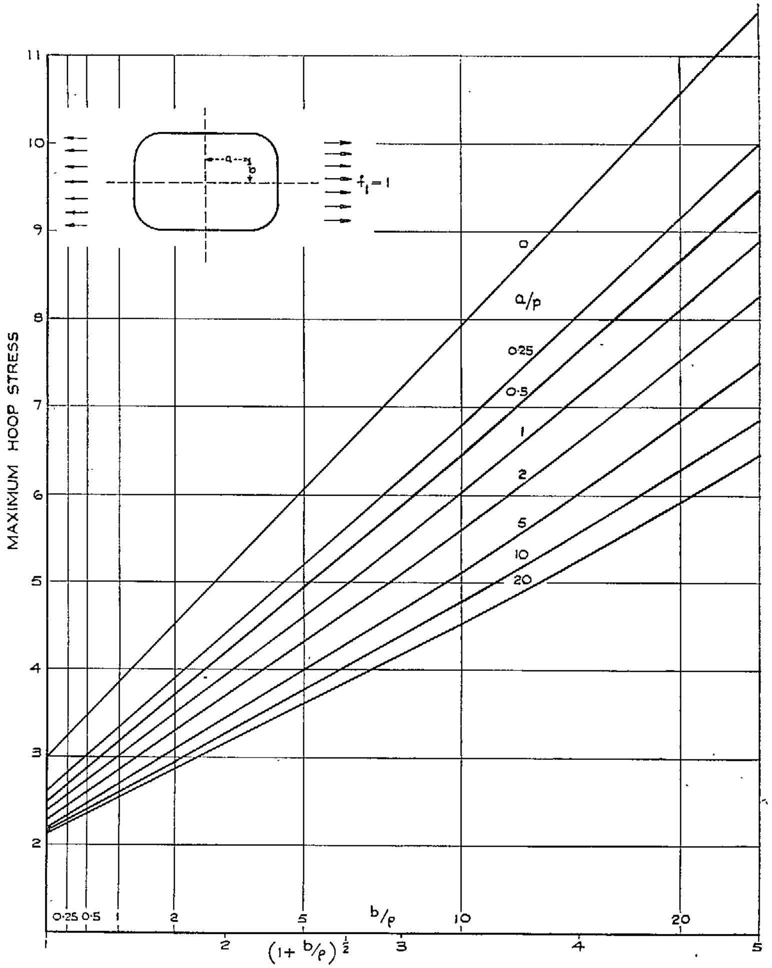


FIG. 5. Variation of maximum hoop stress with $(b/\rho)$ for constant $(a/\rho)$ for [1, 0; 0] loading.

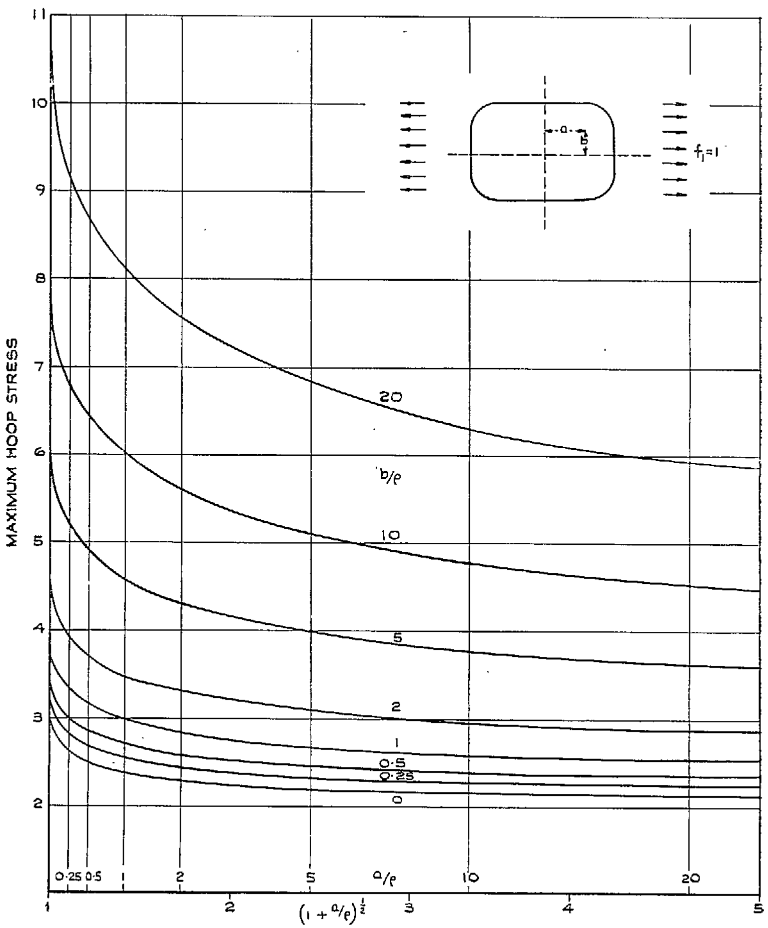


FIG. 6. Variation of maximum hoop stress with $(a/\rho)$ for constant $(b/\rho)$ for [1, 0; 0] loading.

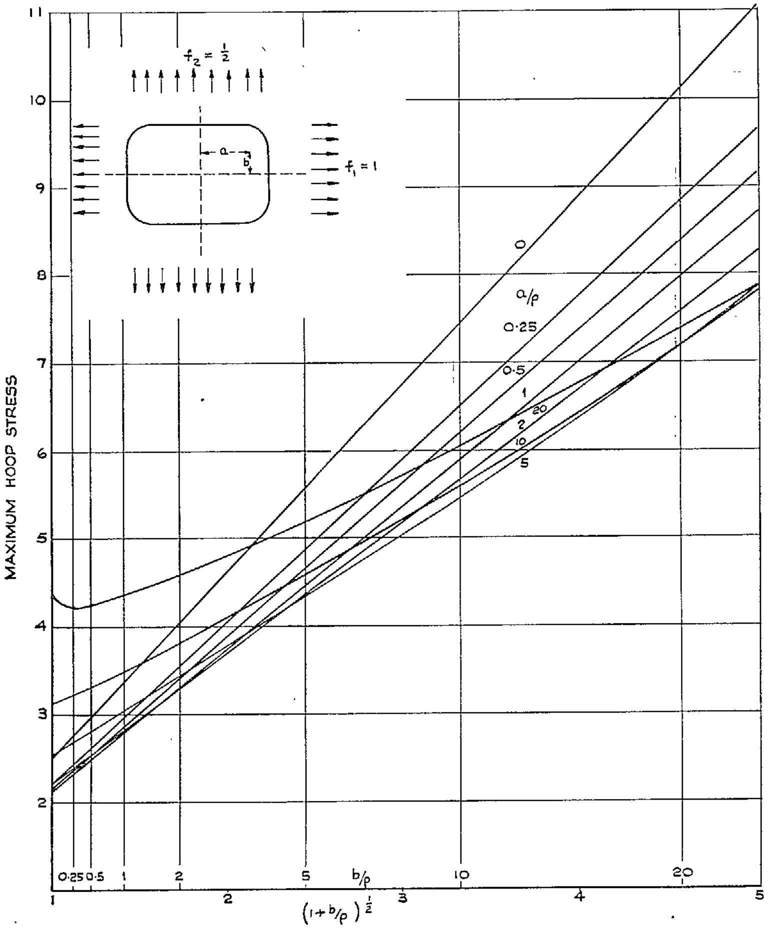


FIG. 7. Variation of maximum hoop stress with $(b/\rho)$ for constant $(a/\rho)$ for $[1, \frac{1}{2}; 0]$ loading.

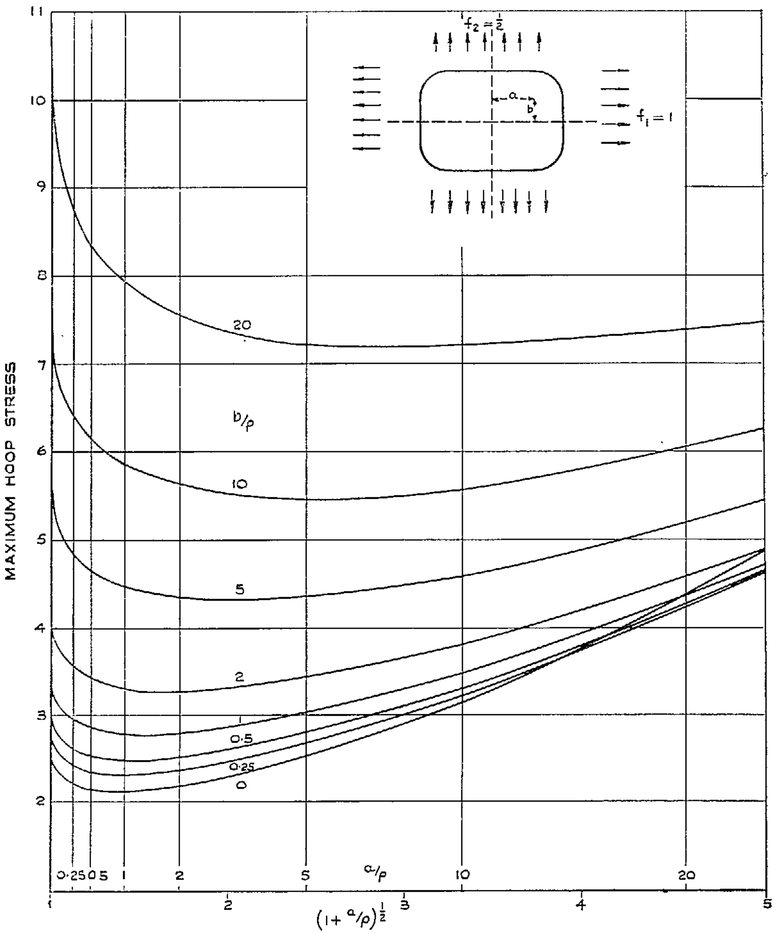


FIG. 8. Variation of maximum hoop stress with $(a/\rho)$ for constant $(b/\rho)$ for $[1, \frac{1}{2}; 0]$ loading.

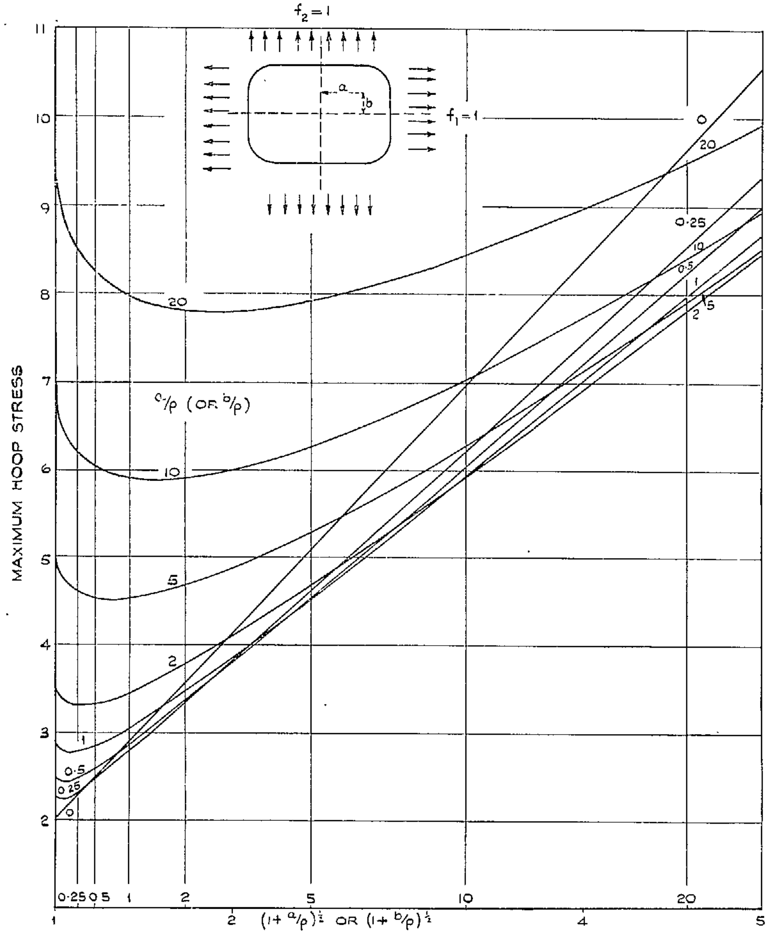


FIG. 9. Variation of maximum hoop stress with $(a/\rho)$ and $(b/\rho)$ for [1, 1; 0] loading.

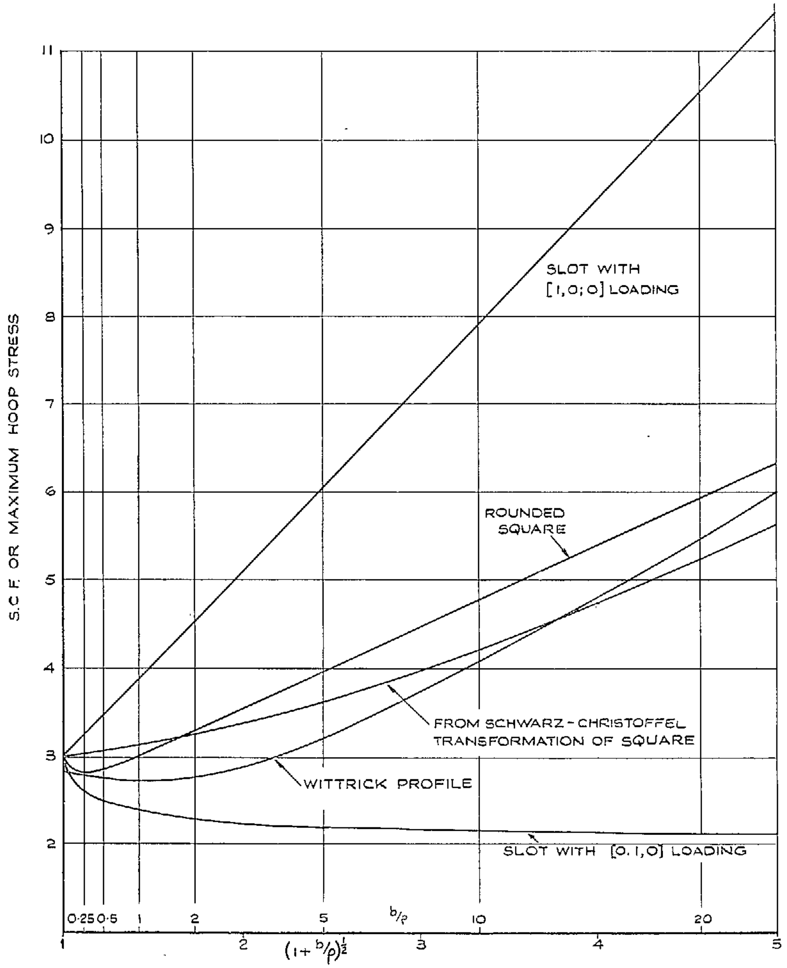


FIG. 10. Stress-concentration factors for rounded squares and rounded slots in tension.

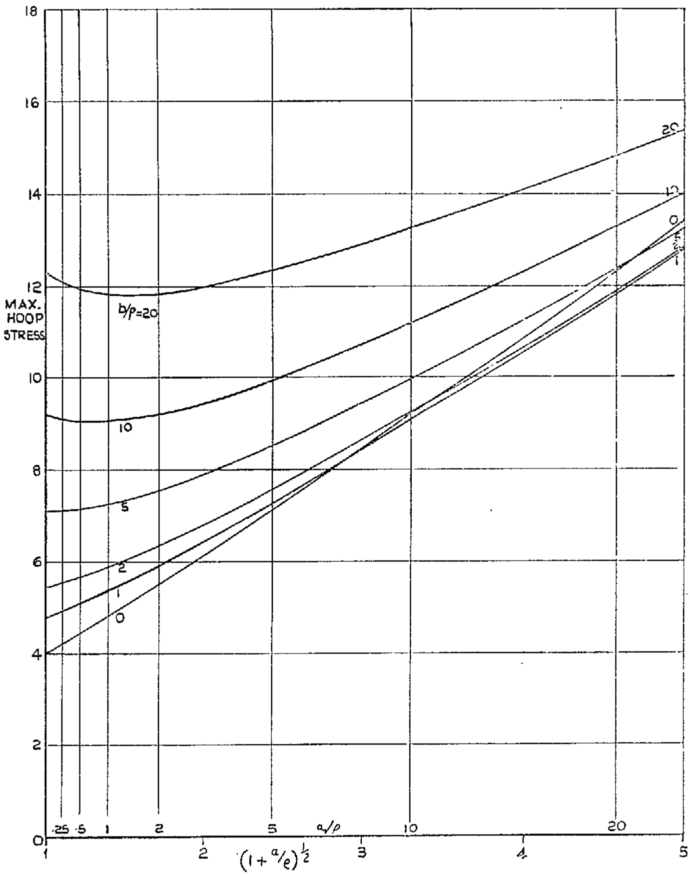


FIG. 11. Maximum hoop stress for shear loading.

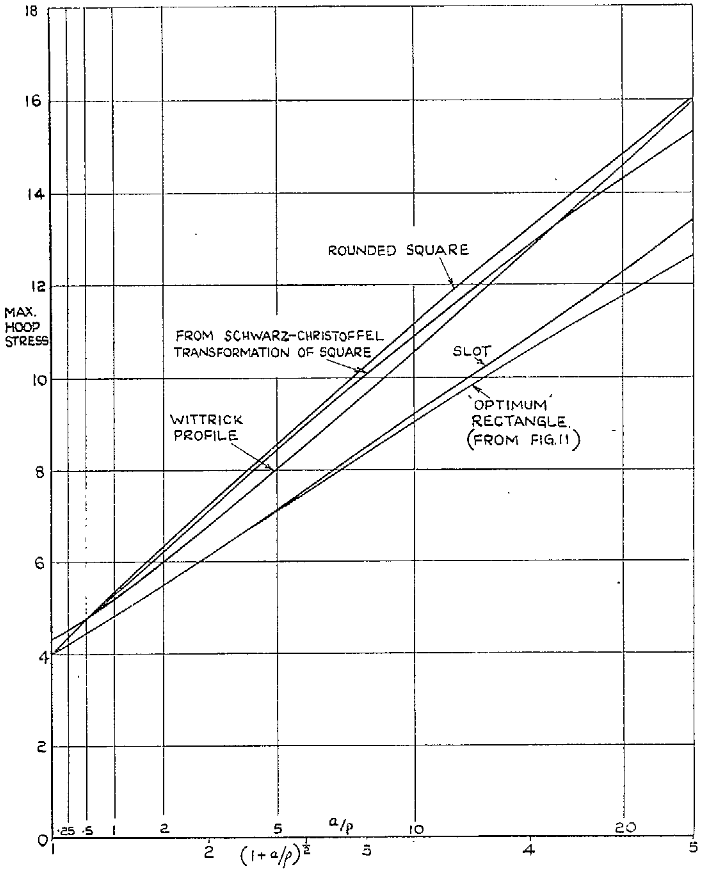


FIG. 12. Extreme profiles in shear.

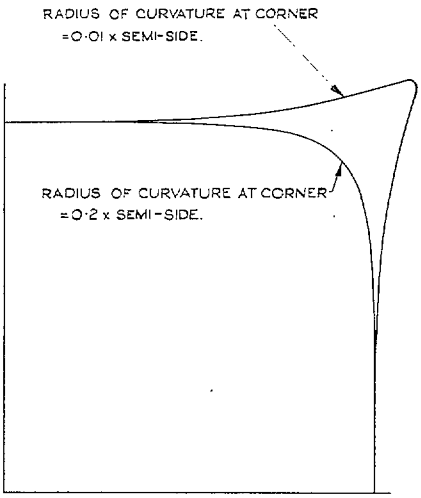


FIG. 13. Wittrick profiles.

(91390) Wt. 66/2301 K5 4/65 Hw.

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