R. & M. No. 3363

MINISTRY OF AVIATION

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

On the Vibrations of Cylindrically Curved Elastic Sandwich Plates

Part I.—With the Solution for Flat Plates

Part II.—The Solution for Cylindrical Plates

By D. J. MEAD, Ph.D., D.C.Ae. and A. J. PRETLOVE, B.Sc., Ph.D.

LONDON: HER MAJESTY'S STATIONERY OFFICE
1964
PRICE £1 1s. 0d. NET

On the Vibrations of Cylindrically Curved Elastic Sandwich Plates

Part I.—With the Solution for Flat Plates

Part II.—The Solution for Cylindrical Plates

By D. J. Mead, Ph.D., D.C.Ae. and A. J. Pretlove, B.Sc., Ph.D.

*Reports and Memoranda No. 3363**

February, 1962

Summary.

In Part I, consideration of the equilibrium of the anisotropic core of a cylindrically curved sandwich plate leads to the three simultaneous differential equations for the three orthogonal deformations. Boundary conditions on two sides are found by considering the equilibrium at the core/face-plate interface. A separable solution leads to the two sets (symmetric and anti-symmetric) of three homogeneous equations for the flat sandwich plate with a honeycomb core. These two sets lead to non-dimensional determinantal frequency equations for the flexural and bubbling modes. The calculation of mode shapes is indicated and the non-dimensional frequency is plotted against other variables for all likely aircraft panel configurations.

In Part II, a solution is found for curved sandwich plates by assuming that a flat-plate core solution bounded by curved-plate edge conditions holds true. An indication, though not a proof, of the validity of this assumption is given. A determinantal frequency equation is thus found for curved plates and an expected anomalous frequency variation with circumferential wavelength becomes evident.

The variation of frequency with wavelength for bubbling modes is only very slight for both flat and curved plates.

LIST OF CONTENTS

* Replaces A.A.S.U. Reports Nos. 186 and 187—A.R.C. 23 942 and 23 943.

LIST OF CONTENTS—*continued*

Section

LIST OF ILLUSTRATIONS

Figure

PART I

With the Solution for Flat Plates

By D. J. Mead, Ph.D., D.C.Ae. and A. J. Pretlove, B.Sc., Ph.D.

1. *Introduction.*

Much work has been carried out over the last ten or fifteen years on the analysis of elastic sandwich structures. Bending theories were followed by buckling theories and these eventually led to vibration theories which have only been developed in recent years. Most of the work published to date on the vibrations of elastic sandwich plates has been produced by Professor Yi-Yuan Yu of the Polytechnic Institute of Brooklyn. The aim of this paper is to analyse the vibrations of single-curvature elastic sandwich plates using the exact three-dimensional theory of elasticity with special reference to aircraft structures. Because the usual aircraft sandwich-panel cores are neither homogeneous nor isotropic (honeycomb cores) the core has been regarded as anisotropic. However, the core has had to be regarded as homogeneous, and this places limitations on the size of wavelength which can be considered (*see* Section 3.7).

It is hoped to extend this work in a later report (by including external forcing terms which vary in time and space) to the analysis of the response of honeycomb sandwich panels when subject to acoustic excitation such as might be found on aircraft. The analysis will be carried out by representing the response of the panel by an infinite sum of principal modes in the usual way. The r.m.s. strains in the core close to the faces are of interest as fatigue failures in honeycomb sandwich plates under acoustic excitation have occurred here. This suggests that the symmetric 'bubbling' mode (*see* Section 2.2.1) could be excited to a greater extent than might at first be expected. For this reason a detailed analysis of this mode is included in this paper.

Initially, the least number of simplifying assumptions compatible with obtaining an analytical solution is made, so that further simplifications can be made at a later stage and their effect on the solutions can then be judged. It has been found possible to compute almost exact solutions for some of the modes of simply-supported flat plates. Only the simply-supported case has been considered here. Although this boundary condition is rather unrealistic, an acoustic fatigue analysis for simply-supported sandwich plates can lead to certain conclusions which will be equally applicable to plates with other boundary conditions.

In Section 2 of this report the nature of the possible modes of a simply-supported sandwich plate are described in order to define terminology. Also, in order to clarify the position concerning the simplifying assumptions used in this and other theories, Section 3 is devoted to a description of the various assumptions often made and to their significance. In Section 4 a concise review of past work on sandwich structures is presented. Section 4 also indicates the assumptions of the work carried out so far by Yu (and to a lesser extent by Mindlin) in the U.S.A.

The principal differences between this analysis and the work of Yu are as follows: The approach to the problem here is an extension of a buckling theory[5] and starts with basic equilibrium equations, whereas Yu makes use of a variational approach. The application of this work differs also in that it deals rather specifically with honeycomb cores whereas Yu considered isotropic cores. A more basic difference is that, here, no assumption of planar or linear deformation or plane strain of the core is made. However, it is assumed that there is no shear deformation in the face plates (for reasons, *see* next paragraph) whereas Yu does not make this assumption. A difference between Yu's work on the vibration of cylinders[14] and this work is that this analysis is carried out for the vibrations of segments of cylinders simply-supported at their edges whilst Yu's analysis was concerned with the axially symmetric and torsional types of modes of complete infinite cylinders. The reason for the solution of this type of vibration problem is that this work is concerned with applications to aircraft whilst Yu's paper is probably concerned with applications to space vehicles.

The basic approach to the problem here is to obtain the equilibrium equations of the core and faces, and then to apply the boundary conditions given by the equilibrium at the core/face-plate interface. A separable solution is assumed for the deformation with harmonic time variation. For the flat plate two uncoupled sets of three homogeneous equations are thus derived in the six arbitrary constants which constitute the mode shapes of the symmetric and anti-symmetric modes (*see* Section 2). From these equations the frequency equation is derived and substituting the appropriate frequency back into the homogeneous equations, the appropriate mode shape can be found. The basic equations are derived in a similar manner (and with a similar notation) to that employed by Tod[5]. The present paper neglects shear effects in the faces of the sandwich because only longer wavelengths can be considered due to the honeycomb core (*see* Section 3.7). For longer-wavelength modes Fig. 1 of Ref. 11 indicates that this assumption will be valid even though materials of different mass density and elastic moduli are being considered. The full equations of motion are derived for the flat plate under these assumptions. For the flat-plate case with simply-supported edges the frequency equations and six first-order modes result. However, for the cases under consideration the modes can be safely decoupled. The anti-symmetric 'flexural' mode and the symmetric 'bubbling' mode are of the most interest. By assuming that the inertia forces in the plane of the plate are negligible, solutions are restricted to those which correspond to flexural and bubbling modes. These modes are analysed in detail for the flat plate. Non-dimensional forms of solution for frequency are obtained and these are shown in the graphs at the end of the report.

2. *The Nature of the Possible First-Order Modes of Vibration of a Flat Finite Rectangular Elastic Sandwich Plate, Simply-Supported at its Edges.*

For a flat sandwich plate with symmetry about its middle plane there are two distinct sets of modes of vibration. The first, and perhaps the more important, is the anti-symmetric set. For this set the displacements in the plane of one of the face plates are equal but opposite in direction to those in the other face. Therefore the middle plane of the plate is a plane of 'anti-symmetry'. Clearly, the classical flexural modes of the plate are anti-symmetric. There is also a set of symmetric modes in which the displacements in the plane of one of the face plates are equal to, and in the same direction as, those in the other face. It is found that there are three anti-symmetric first-order modes and three symmetric first-order modes. However, two of the anti-symmetric modes are of the same type as also are two of the symmetric modes. Thus there are only four types of mode. These modes are now described in some detail and are shown in Fig. 1.

2.1. *Anti-Symmetric Modes.*

2.1.1. *The flexural mode.*—In this mode, the displacement normal to the middle plane of the plate (z-direction) varies with x and y (the orthogonal co-ordinates in the plane of the plate), but for most purposes it can be assumed not to vary with z (Fig. 1a). Energies associated with this mode are lower than those associated with other modes when the plate depth is small compared with its length and width, and therefore the first-order flexural mode will have a comparatively low frequency.

2.1.2. *Thickness-shear modes.*—For the plate described above there are two thickness-shear modes, one associated with the x-direction and the other associated with the y-direction (Fig. 1b). The displacements in the zero-order thickness-shear modes are independent of the x and y co-ordinates and are parallel to the plane of the plate. Displacements perpendicular to the plate are zero. The displacements of the first-order thickness-shear modes are also parallel to the plane of the plate but they now vary with x and y. For most purposes the first-order thickness-shear modes can be regarded as a linear variation of u and v with respect to the normal co-ordinate z. There will now be some displacement perpendicular to the plate surface and superficially the mode appears to be of a flexural type. These modes have a higher frequency than the flexural modes for the sandwiches considered here.

2.2. *Symmetric Modes.*

For the flat plate described, first-order symmetric modes are not coupled in any way with first-order anti-symmetric modes.

2.2.1. *The wrinkling or bubbling mode.*—The motion in these modes produces direct tension and compression of the core in the z-direction, the faces moving in opposite directions (Fig. 1c). Most of the strain energy is in the core. There is a variation of the z-wise displacement w with x and y however, and consequently there is some flexure of the faces. This mode usually has a higher-natural-frequency than its anti-symmetric counterpart, the flexural mode; the frequencies of the bubbling modes are possibly of the same order as those of the thickness-shear modes. It is probable that this mode is lightly damped, and owing to its high natural frequency it might become important where fatigue is being considered.

2.2.2. *Longitudinal modes.*—These modes are the symmetric counterpart of the thickness-shear modes. Again, two such first-order modes exist for the plate under consideration, one with motion predominantly in the x-direction and the other with motion predominantly in the y-direction. Only displacements of second order of magnitude take place normal to the plate, and so all displacements can be assumed parallel to the middle plane of the plate. In this mode the top and bottom faces move by the same amount in the same direction (whereas in the thickness-shear mode they moved by the same amount in opposite directions). In the first-order mode u and v vary with x and y. The natural frequencies of these modes are comparatively high, and therefore they are not of much interest acoustically as they would be difficult to excite.

3. *A Summary of Assumptions Commonly Made in Elastic-Sandwich-Plate Theories.*

3.1. *The Assumption of Plane Strain.*

A simplification of the exact three-dimensional theory of thin plates is possible when one of the characteristic dimensions is large. It can be assumed that strain in this long-dimension direction is zero[16]. Thus for a plate which has a long side, the bending and vibration characteristics will depend almost entirely on the short dimension, and on the boundary conditions on the long edges. Only strains in the plane defined by the normal to the plate and the lines parallel to the shorter dimension need be considered, i.e. plane strain analysis is sufficient. It is often found convenient to make this assumption when the effect of other approximations is to be judged. Most sandwich-plate analyses to date have assumed plane strain and have been called 'one-dimensional' even though displacements, strains and stresses occur in two dimensions.

3.2. *The Assumption of Planar and Linear Deformation.*

Three-dimensional bending and vibration theories of thin plates often consider the in-plane deflections, u and v, to vary linearly with the co-ordinate, z, i.e. straight lines which are normal to the plane of the plate in the unstrained state remain straight after loading. This is the linear deformation assumption.

In two-dimensional theories a similar assumption can be made whereby plane sections remain plane. This is a particular case of linear deformation where no deformation occurs in the direction of one of the in-plate co-ordinates. This is known as planar deformation.

As far as long-wavelength low-frequency vibration theories of sandwich plates are concerned these assumptions are accurate enough. Exact elasticity theory shows the relationship between in-plate deformation and normal co-ordinate to be:

$$u = U \sin \beta z .$$

For long-wavelength low-frequency vibrations, the maximum value of the argument βz is small, and thus the sine can be replaced approximately by the argument and the assumption of planar deformation is seen to hold true. However, for higher-frequency modes of vibration the argument βz is no longer small, and the planar-deformation assumption is likely to give rise to inaccurate results. For thickness-shear modes Yu[10] has had to apply corrections to his planar-deformation theory.

3.3. *The Assumption of Zero Transverse Shear Strain in the Face Plates.*

Bending theories of thin plates have shown that the effect of transverse shear strains is negligible unless the characteristic length (e.g. the wavelength of vibration) is of the order of the plate depth (*see*, for example, Ref. 7). Likewise, if the faces of a sandwich plate are thin compared with the characteristic length then the inclusion of transverse shear effects is unnecessary.

3.4. *The Assumption of Zero Rotatory Inertia.*

The kinetic energy of a plate vibrating in flexure is commonly assumed to consist of the transverse translational energy only. Another component of kinetic energy exists due to rotational velocity of plate elements about an axis in the middle plane of the plate. Its origin lies in the so-called 'Rotatory Inertia' of the plate section. As with transverse shear, this effect becomes important when the characteristic plate dimension becomes of the order of the total plate depth.

3.5. *The Assumption of Zero Flexural Rigidity of the Face Plates about Their Own Middle Plane.*

In a deformed stiff-cored sandwich most of the strain energy in the faces is extensional energy, i.e. the strain in the face is almost constant throughout the thickness of the face. This, however, is not strictly true and the effect of the true variation of strain can be included by superimposing a flexural strain (with the middle plane of the face plate as origin) upon the extensional strain. Flexural rigidity of the faces about their own middle plane is included in this way. The flexural effect is much smaller than the extensional effect when the face thickness is small compared with the core depth and when the wavelength is large compared with the plate depth. When the face is very thin, its flexural rigidity can be neglected.

3.6. *The Assumption that Different Modes of Vibration of the Same Order are Not Coupled.*

It can be shown that there is no coupling at any time between the symmetric and anti-symmetric modes of vibration of a symmetrically arranged flat sandwich plate. However, coupling exists between the different symmetric and the different anti-symmetric modes. Ekstein[8] has pointed out that weakly coupled vibrations of two zero-order modes are likely to become strongly coupled when the natural frequencies of the two modes approach each other. A similar effect can be expected for first- and higher-order modes. Usually the natural frequency of the first-order flexural mode is far removed from the natural frequencies of the first-order thickness-shear modes and also the natural frequency of the first-order bubbling mode is far removed from the natural frequencies of first-order longitudinal modes. In this case, therefore, the four types of mode can be treated separately with a reasonable degree of accuracy. However, as the plate thickness-wavelength ratio becomes larger and as the thickness of the faces becomes larger the coupling will become stronger and it may be necessary to take it into account.

3.7. *The Assumption of a Homogeneous Core.*

The cores of sandwich plates used by the aircraft industry are often of the honeycomb type. This type of core is certainly not homogeneous as it consists of many small hexagonal cells bounded by metal foil. However, if the cell size is small compared with the characteristic dimension under consideration, then the assumption of homogeneity is a reasonable one. Care must be taken, however, when vibrational wavelengths become small.

3.8. *The Assumption of an Isotropic Core.*

The cores of composite sandwich plates are often assumed to be isotropic. Where practical applications are concerned this is not usually true and therefore here the differential equations are derived for an anisotropic core. For honeycomb cores, three of the six elastic moduli can usually be assumed to be zero. These are the direct moduli in the plane of the plate, and the shear modulus G_{xy}. For British honeycombs the two non-zero shear moduli are usually in the ratio of 2 to 3.

4. *A Review of Previous Work on the Analysis of Elastic Sandwich Plates.*

4.1. *Bending and Buckling Theories.*

Papers concerned with the bending and buckling of elastic sandwich structures started to appear soon after the war. Hemp's theory[1] of sandwich construction was published in 1948, and other papers by Reissner[2], Hoff[3], Eringen[4] and Tod[5] appeared soon afterwards.

Reissner's paper was the first to derive the equilibrium equations for the bending deformation of a normal type of sandwich-construction shell. He has first derived the equilibrium equations of the core and faces. Then after finding the appropriate expression for the strain energy of the composite plate Castigliano's principle has been applied to minimise the complementary energy. The minimum of complementary energy gives the equilibrium stress system. The assumptions made in this work are that the faces are thin and thus behave like membranes (i.e. constant direct stress across the thickness and no shear stresses normal to the face), that the core behaves as a homogeneous elastic material in which only transverse stresses occur (two shear stresses, one direct stress), and that deflections are small. The equations have been obtained from the three-dimensional theory of elasticity, subject to the restrictions above, and for the ordinary type of sandwich plate (in which the skins are thin compared with the total depth of the sandwich) these assumptions hold good.

4.2. *Vibration Theories.*

If Reissner's paper is to form the basis of work on the vibrations of sandwich plates then the restrictions of his theory, as stated in Section 4.1 above, must be borne in mind. For longer-wavelength low-frequency flexural vibrations of sandwich plates, assumptions similar to Reissner's can be made. Mindlin has derived the equations of flexural motion of elastic sandwich plates with thin faces by neglecting the flexural rigidity of the faces about their own middle plane, and by neglecting the transverse shear deformation and rotatory inertia of the faces. These assumptions are equivalent to those made by Reissner in his bending theory, and they have given accurate results for low-order flexural modes of vibration of ordinary sandwich plates.

The analysis of short-wavelength flexural vibrations of sandwich plates with, perhaps, thick skins, therefore becomes very complicated when a solution derived from the exact theory of elasticity is contemplated. The additional factors which must now be included are:

(i) The effect of transverse shear deformation in the face plates.

(ii) The effect of rotatory inertia of the faces.

(iii) The coupling which might exist between the three anti-symmetric modes.

(iv) For honeycombs, the fact that the cell dimensions are likely to be of the order of size of the wavelength.

For a discussion of these factors and when they become significant *see* Section 3.

In a series of six recent papers[9, 10, 11, 12, 13, 14] Yu has developed a new theory of bending and vibration of elastic sandwich plates including transverse shear effects in the faces and based on the three-dimensional theory of elasticity.

In his first paper[9] he has considered the bending of an infinite elastic sandwich plate in the so-called one-dimensional case, under the assumption of planar deformation, as described in Section 3.2, in each of the three components of the sandwich, although planar deformation of the cross-section as a whole has not been assumed. Thus he has been able to include the effects of transverse shear deformation in the faces. One advantage of this theory over previous bending theories of sandwich plates is that no restriction has been imposed on the thickness of the faces and the solution is therefore more general. As a result of including transverse shear effects in the faces, subsequent application of the theory of this paper to vibration problems applies up to higher-frequency modes than other papers have done. The frequency and wavelength at which departure from planar deformation becomes important is dependent on the skin thickness and density and the core mean density, and the departure becomes larger as the wavelength decreases and the frequency increases. And so at very high frequencies accuracy is gained by including transverse shear effects in the skin but at the same time it is lost by assuming planar deformation. The theory of this paper is only concerned with small deflections, as is usual.

The second paper by Yu[10] is a short note on the simple one-dimensional thickness-shear modes of an infinite plate. The modes discussed are the zero-order modes of the type described in Section 2.1.2. Only free vibrations have been considered. For the case of infinite homogeneous isotropic plates, such modes have been discussed by Mindlin[15] firstly on the basis of a plate theory which takes account of transverse shear deformation and rotatory inertia but assumes plane strain and planar deformation, and secondly on the basis of exact elasticity theory (plane strain being assumed again but not planar deformation). The discrepancy between the lowest frequency values obtained by the two methods is removed by introducing to the former a factor K whose value is found to be $\pi^2/12$ for a homogeneous isotropic plate. Yu's paper has investigated the one-dimensional zero-order thickness-shear modes of sandwich plates in exactly the same manner as Mindlin has done for homogeneous isotropic plates using the theory of Ref. 9 and matching the approximate frequencies to the frequencies derived from the exact elasticity theory with the factor K. However, for sandwich plates, it is found that K varies between $\pi^2/12$ and 1. $K = \pi^2/12$ corresponds to the case of a sandwich plate with relatively thick faces whereas $K = 1$ corresponds to the case of a sandwich with relatively thin faces. As $K = 1$ implies perfect matching of frequencies between the exact and approximate theories, it is seen that the assumption of planar deformation is valid for thin faces. (*Note.*—This is true only in a broad sense because other factors do enter the argument. However, the relative thickness of the faces is the predominant factor.) The frequency equation for these modes derived from the theory of Ref. 9, has been given in this report, as has the equation obtained from exact elasticity theory.

In Yu's third paper[11] the equations of motion of flexural vibrations of sandwich plates have been developed using the theory of Ref. 9 (one-dimensional case). All of the usual effects are included in this analysis, the important assumptions being those of plane strain and planar deformation in each of the plate components. The theory therefore applies to sandwich plates with thick faces because transverse shear effects in the faces have been included, as pointed out previously. In the Introduction the need to include transverse shear and rotatory inertia in the theory has been emphasised again so that higher-frequency modes may be computed more accurately. The flexural vibrations of sandwich plates have also been analysed using exact elasticity theory which has been restricted to the so-called one-dimensional case at the outset so that results thus obtained can be compared with the results obtained from the approximate analysis. On the basis of these theories the flexural vibrations have been investigated with the emphasis on ordinary sandwich plates. It is shown, by a numerical example, that the two methods described above give almost identical results; which indeed they should, since the assumption of planar deformation holds good for the flexural vibrations of ordinary sandwich plates at the lower frequencies. It is found that the neglect of shear deformation in the core gives rise to inaccurate results and therefore this effect should be included at all times. It is also found that the joint flexural-extensional rigidity of the faces must be included in the analysis if accurate results are to be expected, unless the faces are exceptionally thin, in which case, the flexural rigidity of the faces can be neglected. For plates with very thin faces and vibration modes of a low frequency, the possibility of some considerable simplification is forecast.

This simplification has been carried out in the next paper to be published by Yu[12]. The flexural-vibration equations derived in Ref. 11 are complicated, and they hold good for a wide frequency range. For the low-frequency ranges and ordinary sandwich plates with thin faces a considerable simplification of the problem can be effected. In this paper these simpler equations have been introduced and their accuracy is determined by a comparison with the more complete equations of Ref. 11.

In the theory developed in Refs. 9 and 11, it has been assumed that the transverse displacement is constant across the depth of the plate, and that the in-plate displacements are proportional to the plate normal co-ordinate, the derivatives of in-plate displacements in the core and faces, with respect to the normal co-ordinate, being different from each other. In this way all of the deformation energies have been taken into account. For sandwich plates with thin heavy faces the important factors have been found to be shear in the core, rotational and translational inertia of the core, translational inertia of the faces, and the joint effect of flexural and extensional rigidity of the faces. These effects have been included in this simpler analysis[12]. Two sets of equations have been derived.

The derivation of the first set of simplified equations consists of a two-dimensional analysis with transverse shear deformation in the faces not being taken into account. The equations have been derived from three-dimensional elasticity theory using a variational procedure as in Ref. 9. This set of equations has been found to yield the same frequency equation as a certain degeneration of equations derived in Ref. 11.

The form of deformation used for these two simplified sets is as described above except that the in-plate deformation of the faces has been regarded as constant over the face depth. For the first set these displacements have been derived, in the faces, by considering the core as extending to the interface whereas for the second set the core has been considered as extending to the middle plane of the faces. The second set of equations thus obtained has also been found to be a degenerate version of the equations of Ref. 11, but the restriction necessary for the simplified frequency equation to hold is not as great as the restriction for the first set, and therefore the second set yields more accurate solutions. In this way a simplified analysis for the flexural vibrations of thin-skin sandwich plates in the lower frequency range has been found which yields solutions of good accuracy.

In a paper concerned with the forced flexural vibrations of sandwich plates in plane strain[13] Yu has started again with the displacement equations of motion derived in Ref. 9. Treating the case of a simply-supported plate he has expanded the response as a series of principal modes which had been found previously[11] and the orthogonality condition of the principal modes has been derived. A method due to Mindlin and Goodman[18] has been used to analyse the response to time-dependent boundary conditions and an example has been worked out for this type of forced motion. The assumptions of Ref. 9 have been made for this work.

Recently, Yu has also written a paper on the vibrations of sandwich cylindrical shells[14]. This analysis is almost identical to the analysis for flat plates in Ref. 9 except, of course, that the cylindrical quantities are introduced. The equations of motion so obtained have been simplified by making the assumption that the faces of the sandwich are very thin. The assumption of planar deformation has been made, and therefore as described previously in this section, shear coefficients K had to be calculated to match the thickness-shear frequencies in order to correct errors brought about by this assumption. The simplified equations have been used to analyse the axially symmetric and torsional vibrations of an infinite cylinder.

Yu's papers are a useful set of works on the vibrations of sandwich structures, especially for one-dimensional analysis of flat plates and two-dimensional analysis for low-frequency thin-skinned flat plates.

5. *The Analysis of the Vibrations of Cylindrically Curved Sandwich Plates.*

5.1. *Sandwich-Plate Core Equations.*

5.1.1. *Stress-strain equations for an anisotropic homogeneous medium.*—The convention for co-ordinates and face stress resultants is given in Fig. 2. These co-ordinates are much the same as those used by Tod[5] and are of the cylindrical polar type. The usual θ co-ordinate is replaced by y where $y = R\theta$ and R is the cylindrical curvature radius. R is assumed to be large compared with the thickness of the panel. The co-ordinate z is used with the middle plane of the plate as its origin and with the R direction as its direction. Measurement of x is parallel to the generator of the cylinder and in the middle plane of the plate. As mentioned in the Introduction, the core has had to be assumed homogeneous in order to obtain a solution. This limits the frequency and wavelength down to which it is possible to obtain an accurate solution to the vibration problem for honeycomb cores (*see* Section 3.7). The stress-strain equation is:

$$\begin{Bmatrix} \\ e \\ \\ \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_y} & -\frac{\nu_{xz}}{E_z} \\ -\frac{\nu_{yx}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{yz}}{E_z} \\ -\frac{\nu_{zx}}{E_x} & -\frac{\nu_{zy}}{E_y} & \frac{1}{E_z} \end{bmatrix} \begin{Bmatrix} \\ \sigma \\ \\ \end{Bmatrix} \tag{1}$$

where

$$\begin{Bmatrix} \\ e \\ \\ \end{Bmatrix} = \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \\ \sigma \\ \\ \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}$$

and the matrix of coefficients of σ must be symmetric for a linear structure due to Clerk-Maxwell's reciprocity theorem. The inversion of equation (1) will give:

$$\begin{Bmatrix} \\ \sigma \\ \\ \end{Bmatrix} = \begin{bmatrix} A & E & F \\ E & B & H \\ F & H & C \end{bmatrix} \begin{Bmatrix} \\ e \\ \\ \end{Bmatrix} \tag{2}$$

where the matrix of coefficients must again by symmetric. For the particular type of core considered in this report (a honeycomb core) it can be assumed that $E_x = E_y = 0$, and therefore examination of equations (1) and (2) gives:

$A, B, E, F, H = 0$, $C = E_z$ and the only component of equation (2) which remains is

$$\sigma_{zz} = Ce_{zz} \tag{3}$$

As far as shear stresses and strains are concerned it can be assumed that $G_{xy} = 0$ for the honeycomb material and we are therefore left with:

$$\begin{aligned} \sigma_{zx} &= G_{zx}e_{zx} \\ \sigma_{zy} &= G_{zy}e_{zy}\,. \end{aligned} \tag{4}$$

However, we will solve for the more general case of an anisotropic core (assuming there to be no coupling between shear strains and direct stresses) where the stress-strain relationships can be written:

(i) as equation (2) for direct stresses, and

(ii)
$$\begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{bmatrix} L & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & N \end{bmatrix} \begin{Bmatrix} e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix} \tag{5}$$

and the special case of honeycomb cores will be brought into the solution later.

5.1.2. *Equilibrium relationships.*—The well-known equilibrium equations in cylindrical polar co-ordinates (*see*, for example, Ref. 16, p. 306), when body forces are included, and conversion to our form of axes has been effected, are:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} - \frac{1}{R}\left\{z\frac{\partial \sigma_{xy}}{\partial y} - \sigma_{xz}\right\} + p_x = 0 \tag{6}$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} - \frac{1}{R}\left\{z\frac{\partial \sigma_{yy}}{\partial y} - 2\sigma_{yz}\right\} + p_y = 0 \tag{7}$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} - \frac{1}{R}\left\{z\frac{\partial \sigma_{zy}}{\partial y} - \sigma_{zz} + \sigma_{yy}\right\} + p_z = 0 \tag{8}$$

where p_i is the body force per unit volume in the positive i direction.

5.1.3. *Strain-displacement relationships.*—The equations connecting strain and displacement in cylindrical co-ordinates are also well known (*see* Ref. 16, p. 305). These also have been suitably converted to the system of this paper (neglecting terms in z/R^2), thus:

$$\begin{aligned}
e_{xx} &= \frac{\partial u}{\partial x} \\
e_{yy} &= \frac{\partial v}{\partial y} - \frac{1}{R}\left\{z\frac{\partial v}{\partial y} - w\right\}^* \\
e_{zz} &= \frac{\partial w}{\partial z} \\
e_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \frac{z}{R}\frac{\partial u}{\partial y} \\
e_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \frac{1}{R}\left\{z\frac{\partial w}{\partial y} + v\right\} \\
e_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}.
\end{aligned} \tag{9}$$

5.1.4. *Overall core equilibrium equations.*—Substituting equations (2), (5) and (9) in equation (6) we obtain:

$$\begin{aligned}
A\frac{\partial^2 u}{\partial x^2} + L\frac{\partial^2 u}{\partial y^2} + N\frac{\partial^2 u}{\partial z^2} + (E+L)\frac{\partial^2 v}{\partial x \partial y} + (F+N)\frac{\partial^2 w}{\partial x \partial z} + p_x \\
= \frac{1}{R}\left\{z\frac{\partial}{\partial y}\left[L\left(2\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + E\frac{\partial v}{\partial x}\right] - (E+N)\frac{\partial w}{\partial x} - N\frac{\partial u}{\partial z}\right\}
\end{aligned} \tag{10}$$

and substituting equations (2), (5) and (9) in equation (7) we obtain:

$$\begin{aligned}
(L+E)\frac{\partial^2 u}{\partial x \partial y} + L\frac{\partial^2 v}{\partial x^2} + B\frac{\partial^2 v}{\partial y^2} + M\frac{\partial^2 v}{\partial z^2} + (G+M)\frac{\partial^2 w}{\partial y \partial z} + p_y \\
= \frac{1}{R}\left\{z\frac{\partial}{\partial y}\left[(L+E)\frac{\partial u}{\partial x} + 2B\frac{\partial u}{\partial y} + (G+M)\frac{\partial w}{\partial z}\right] - B\frac{\partial w}{\partial y} - M\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right]\right\}
\end{aligned} \tag{11}$$

---

* A term $-wz/R^2$ has also been neglected in the equation. It has thus been assumed that the radial deflection w is of the order of the sandwich thickness or less.

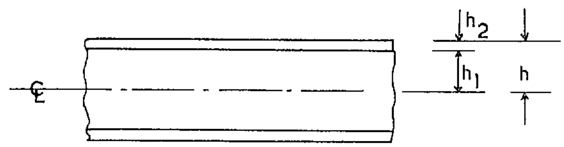
and substituting equations (2), (5) and (9) in equation (8) we obtain:

$$N\frac{\partial^2 w}{\partial x^2}+M\frac{\partial^2 w}{\partial y^2}+C\frac{\partial^2 w}{\partial z^2}+(G+M)\frac{\partial^2 v}{\partial y\partial z}+(F+N)\frac{\partial^2 u}{\partial x\partial z}+p_z$$

$$=\frac{1}{R}\left\{z\frac{\partial}{\partial y}\left[(G+M)\frac{\partial v}{\partial z}+2M\frac{\partial w}{\partial y}\right]+(E-F)\frac{\partial u}{\partial x}+(M+B)\frac{\partial v}{\partial y}-C\frac{\partial w}{\partial z}\right\}. \quad (12)$$

5.2. *The Equilibrium of the Face Plates.*

5.2.1. *Equilibrium parallel to the face surface.*—The assumption of plane stress is made for the faces, i.e. $\sigma_{zz}=\sigma_{yz}=\sigma_{xz}=0$ (*see* Ref. 16, p. 30). This is another assumption which will limit the frequency and wavelength to which this work will apply, but it is reasonable to make this assumption having once limited the wavelength by assuming a homogeneous core.



For plane stresses in the faces we have:

$$\sigma_{xx}=\frac{E_m}{1-\nu^2}(e_{xx}+\nu e_{yy})$$

$$\sigma_{yy}=\frac{E_m}{1-\nu^2}(\nu e_{xx}+e_{yy})$$

$$\sigma_{xy}=Ge_{xy}=\frac{E_m}{2(1+\nu)}e_{xy}. \quad (13)$$

If quantities at the middle surfaces of the face plates {i.e. at $z=\pm(h_1+h_2/2)$} are denoted by a dash and if quantities at the interface ($z=\pm h_1$) are denoted by a subscript $_0$ then deflections in the face plate are given approximately by:

$$u=u_0\mp\left(\frac{h_2}{2}+\zeta\right)\left(\frac{\partial w}{\partial x}\right)_0$$

$$v=v_0\mp\left(\frac{h_2}{2}+\zeta\right)\left(\frac{\partial w}{\partial y}\right)_0$$

$$w=w_0 \quad (14)$$

noting that $e_{xz\,0}=(\partial u/dz)_0+(\partial w/\partial x)_0=0$, etc., and where ζ is a plate normal co-ordinate measured from the middle surface of the face plate (i.e. $z=z'+\zeta$ where $z'=h_1+h_2/2$).

At the middle surface of the face plate $\zeta=0$ and we have:

$$u'=u_0\mp\frac{h_2}{2}\left(\frac{\partial w}{\partial x}\right)_0$$

$$v'=v_0\mp\frac{h_2}{2}\left(\frac{\partial w}{\partial y}\right)_0$$

$$w'=w_0. \quad (15)$$

Neglecting terms in h_2/R and $1/R^2$, the resulting equations are:

$$M_x = -D\left[\left(\frac{\partial^2 w}{\partial x^2}\right)_0 + \nu\left(\frac{\partial^2 w}{\partial y^2}\right)_0 \mp \frac{\nu}{R}\left\{h_1\left(\frac{\partial^2 w}{\partial y^2}\right)_0 \mp \left(\frac{\partial v}{\partial y}\right)_0\right\}\right]$$

$$M_y = -D\left[\nu\left(\frac{\partial^2 w}{\partial x^2}\right)_0 + \left(\frac{\partial^2 w}{\partial y^2}\right)_0 \mp \frac{1}{R}\left\{h_1\left(\frac{\partial^2 w}{\partial y^2}\right)_0 \mp \left(\frac{\partial v}{\partial y}\right)_0\right\}\right]$$

$$M_{xy} = -(1-\nu)D\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)_0 \mp \frac{1}{R}\left\{\frac{h_1}{2}\left(\frac{\partial^2 w}{\partial x\partial y}\right)_0 \mp \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)_0\right\}\right] \qquad (21)$$

where $D = E_m h_2^3/12(1-\nu^2)$. Equilibrium of the transverse loads on a skin element gives:

$$\frac{\partial Q_x}{\partial x} + \left(1 \mp \frac{h_1}{R}\right)\frac{\partial Q_y}{\partial y} + f_z h_2 = \frac{N_y}{R} \pm \sigma_{zz\,0} \qquad (22)$$

and by considering also the equilibrium of shears and couples on the element, we have:

$$Q_x = \frac{\partial M_x}{\partial x} + \left(1 \mp \frac{h_1}{R}\right)\frac{\partial M_{xy}}{\partial y} + \frac{h_2}{2}\sigma_{zx\,0} \qquad (23)$$

$$Q_y = \frac{\partial M_{xy}}{\partial x} + \left(1 \mp \frac{h_1}{R}\right)\frac{\partial M_y}{\partial y} + \frac{h_2}{2}\sigma_{zy\,0}. \qquad (24)$$

By substituting equations (23) and (24) into (22) to eliminate Q_x and Q_y we obtain:

$$\frac{\partial^2 M_x}{\partial x^2} + \left(1 \mp \frac{2h_1}{R}\right)\frac{\partial^2 M_y}{\partial y^2} + 2\left(1 \mp \frac{h_1}{R}\right)\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{h_2}{2}\left[\left(\frac{\partial \sigma_{zx}}{\partial x}\right)_0 + \left(1 \mp \frac{h_1}{R}\right)\left(\frac{\partial \sigma_{zy}}{\partial y}\right)_0\right] +$$

$$+ f_z h_2 = \frac{N_y}{R} \pm \sigma_{zz\,0}. \qquad (25)$$

By using equations (5) and (9) to obtain σ in terms of displacement, and by substituting equations (21) into (25) to eliminate M_x, M_y, and M_{xy} we obtain at the interface:

$$D\nabla^4 w - \frac{h_2}{2}\left[N\left\{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x\partial z}\right\} + M\left\{\frac{\partial^2 v}{\partial z\partial y} + \frac{\partial^2 w}{\partial y^2}\right\}\right] \pm C\frac{dw}{dz} \pm \left[F\frac{\partial u}{\partial x} + G\frac{\partial v}{\partial y}\right] - f_z h_2$$

$$= \frac{D}{R}\left[\pm 3h_1\nabla^2\frac{\partial^2 w}{\partial y^2} - \nabla^2\frac{\partial v}{\partial y}\right] \mp \frac{Mh_2}{2R}\left\{2h_1\frac{\partial^2 w}{\partial y^2} + h_1\frac{\partial^2 v}{\partial z\partial y} \pm \frac{\partial v}{\partial y}\right\} + \frac{G}{R}\left[h_1\frac{\partial v}{\partial y} \mp w\right] -$$

$$- \frac{E_m h_2}{R(1-\nu^2)}\left[\nu\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{w}{R}\right]. \qquad (26)$$

It will be seen that the last term is of order h_2/R^2. This term has not been neglected because it is not multiplying a derivative and because E_m is assumed to be much greater than any of the core moduli.

Equations (10), (11), (12), (18), (19) and (26) constitute the total equilibrium equations of the composite plate when under the influence of the body force system p and f. (*N.B.* No external loading is included in these equations.) For free motion the body forces can be expressed:

$$p_i = -\frac{\rho_1}{g}\frac{\partial^2 u_i}{\partial t^2} \qquad (i = x, y, z)$$

$$f_i = -\left[\frac{\rho_2}{g}\frac{\partial^2 u_i}{\partial t^2} + \frac{\tau}{g h_2}\frac{\partial^2 u_i}{\partial t^2}\right]$$

ρ_1 = weight density of the core

u = displacement

where the adhesive mass is assumed to be spread uniformly over the face.

ρ_2 = weight density of the faces

τ = weight of bonding per unit area of face.

If harmonic motion of frequency ω is a dynamic solution of interest:

$$\frac{\partial^2 u_i}{\partial t^2} = -\omega^2 u_i$$

so

$$p_i = \rho_1 \frac{\omega^2}{g} u_i$$

$$f_i = \frac{\omega^2}{g} u_i \left[\rho_2 + \frac{\tau}{h_2}\right] = \rho_f \frac{\omega^2}{g} u_i. \tag{27}$$

Equations (10), (11), (12), (18), (19), (26) and (27) now constitute the equations of harmonic motion of an elastic sandwich plate with a general anisotropic core, under the assumptions:

(i) $\frac{h_2}{R}$ and $\frac{1}{R^2} \ll 1$

(ii) Core homogeneity

(iii) No shear deformation in the faces.

5.3. *The Differential Equations of Harmonic Vibration of Cylindrically Curved Honeycomb-Cored Elastic Sandwich Plates.*

For honeycomb cores we have seen that we can assume that there are only three non-zero elastic moduli. In our notation these three moduli are C, M, and N. Some of the equations of motion can thus be simplified, and for honeycomb cores the six equations (10), (11), (12), (18), (19) and (26) can be rewritten:

$$N\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] + p_x = -\frac{N}{R}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \tag{28}$$

$$M\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] + p_y = \frac{M}{R}\left[z\frac{\partial^2 w}{\partial y \partial z} - \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \tag{29}$$

$$N\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] + M\frac{\partial}{\partial y}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] + C\frac{\partial^2 w}{\partial z^2} + p_z = \frac{1}{R}\left[z\frac{\partial}{\partial y}\left\{M\left[\frac{\partial v}{\partial z} + 2\frac{\partial w}{\partial y}\right]\right\} + M\frac{\partial v}{\partial y} - C\frac{\partial w}{\partial z}\right] \tag{30}$$

for core equilibrium in the three directions x, y and z. The three boundary-condition equations at the interface are:

$$\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial y}\mp\frac{h_2}{2}\frac{\partial}{\partial x}\nabla^2 w+\frac{1-\nu^2}{E_m}f_x$$
$$=\pm K\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\pm\frac{1}{R}\left[h_1\frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial y}\mp\nu\frac{\partial w}{\partial x}+h_1(1-\nu)\frac{\partial^2 u}{\partial y^2}\right] \quad (31)$$

$$\frac{\partial^2 v}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial y}\mp\frac{h_2}{2}\frac{\partial}{\partial y}\nabla^2 w+\frac{1-\nu^2}{E_m}f_y$$
$$=\pm K^*\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)-\frac{K^*}{R}\left(h_1\frac{\partial w}{\partial y}\pm v\right)\pm\frac{1}{R}\left[h_1\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial y}\mp\frac{\partial w}{\partial y}+2h_2\frac{\partial^2 v}{\partial y^2}\mp\frac{K^*}{M}D\frac{\partial}{\partial y}\nabla^2 w\right] \quad (32)$$

$$D\nabla^4 w-\frac{h_2}{2}\left[N\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial z}\right)+M\left(\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial y^2}\right)\right]\pm C\frac{\partial w}{\partial z}-f_z h_2$$
$$=\frac{D}{R}\left[\pm 3h_1\nabla^2\frac{\partial^2 w}{\partial y^2}-\nabla^2\frac{\partial v}{\partial y}\right]\mp\frac{Mh_2}{2R}\left[2h_1\frac{\partial^2 w}{\partial y^2}+h_1\frac{\partial^2 v}{\partial y\partial z}\pm\frac{\partial v}{\partial y}\right]-\frac{E_m h_2}{R(1-\nu^2)}\left[\nu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{w}{R}\right] \quad (33)$$

and it will be observed that equations (31) and (32) are identical to equations (18) and (19) as no simplification of these is possible.

6. *Solutions of the Equations Listed in Section* 5.3.

6.1. *A Separable Form of Solution for Curved Panels.*

Let us assume a separable solution of the form:

$$w = W(z)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \quad (34)$$

which can only hold for simply-supported edges. The in-plate deflections u and v must then take the form:

$$u = U(z)\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \quad (35)$$

and

$$v = V(z)\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}. \quad (36)$$

If equations (34), (35) and (36) are now substituted into the core equilibrium equations (28), (29) and (30), we obtain:

(i) Equation (28).

$$\left[N\delta^2+\frac{N\delta}{R}+\rho_1\frac{\omega^2}{g}\right]U+N\frac{n\pi}{a}\left[\delta+\frac{1}{R}\right]W=0 \quad (37)$$

where δ is an operator denoting differentiation with respect to z;

(ii) Equation (29).

$$\left[M\delta^2+\frac{M\delta}{R}+\rho_1\frac{\omega^2}{g}\right]V+M\frac{m\pi}{b}\left[\left(1-\frac{z}{R}\right)\delta+\frac{1}{R}\right]W=0; \quad (38)$$

(iii) Equation (30).

$$-N\frac{n\pi}{a}\,\delta U - M\frac{m\pi}{b}\left[\left(1-\frac{z}{R}\right)\delta - \frac{1}{R}\right]V + \left[C\delta^2 + \frac{C\delta}{R} - \left\{N\left(\frac{n\pi}{a}\right)^2 + \right.\right.$$
$$\left.\left. + M\left(\frac{m\pi}{b}\right)^2 - 2\frac{Mz}{R}\left(\frac{m\pi}{b}\right)^2 - \rho_1\frac{\omega^2}{g}\right\}\right] W = 0. \qquad (39)$$

Equations (37), (38) and (39) are three simultaneous differential equations in U, V, and W. The solution of these equations is complicated because the coefficients are, in some cases, variable (with z). It was thought that it might be easy to find a series type of solution to this set of equations, but it proved to be immensely cumbersome and was not completed. It was found that the six modes for the curved plate corresponding to the three symmetric and the three anti-symmetric modes for the flat plate (*N.B.* these modes for the curved plate are in fact neither symmetric nor anti-symmetric) were coupled together, whereas for the flat plate, the symmetric and anti-symmetric modes were uncoupled. The degenerate case of the flat plate can be solved by putting $1/R = 0$.

6.2. *A Solution for Flat Panels.*

It is now assumed that the coupling between symmetric modes with the same nodal pattern is negligible and similarly that the coupling between anti-symmetric modes with the same nodal pattern negligible. This coupling exists by virtue of x-wise and y-wise inertia forces in the core, but is only significant when the core is very heavy or the face plates are very thick. Hence we ignore the in-plate inertia terms of the core for flexural and bubbling modes. Equations (37), (38) and (39) now become:

(i) Equation (37).

$$\delta^2 U + \frac{n\pi}{a}\,\delta W = 0\,; \qquad (40)$$

(ii) Equation (38).

$$\delta^2 V + \frac{m\pi}{b}\,\delta W = 0\,; \qquad (41)$$

(iii) Equation (39).

$$-N\frac{n\pi}{a}\,\delta U - M\frac{m\pi}{b}\,\delta V + \left[C\delta^2 - \left(N\left(\frac{n\pi}{a}\right)^2 + M\left(\frac{m\pi}{b}\right)^2 - \rho_1\frac{\omega^2}{g}\right)\right] W = 0. \qquad (42)$$

Integration of equations (40) and (41) gives:

$$\frac{dU}{dz} = -\frac{n\pi}{a}\,W + \Gamma \qquad (43)$$

and

$$\frac{dV}{dz} = -\frac{m\pi}{b}\,W + \Delta \qquad (44)$$

and substituting (43), (44) in (42) gives:

$$C\frac{d^2W}{dz^2} + \rho_1\frac{\omega^2}{g}\,W = M\frac{m\pi}{b}\,\Delta + N\frac{n\pi}{a}\,\Gamma. \qquad (45)$$

The solution of the differential equation (45) is:

$$W = \Theta\sin\beta z + \Lambda\cos\beta z + \frac{g}{\rho_1\omega^2}\left\{M\frac{m\pi}{b}\,\Delta + N\frac{n\pi}{a}\,\Gamma\right\}. \qquad (46)$$

Substituting equation (46) into (43) and (44) we find the corresponding modal shapes of U and V, thus:

$$U = -\frac{n\pi}{a}\left[-\frac{\Theta}{\beta}\cos\beta z + \frac{\Lambda}{\beta}\sin\beta z + \frac{zg}{\rho_1\omega^2}\left\{M\frac{m\pi}{b}\Delta + N\frac{n\pi}{a}\Gamma\right\}\right] + \Gamma z + \Xi \qquad (47)$$

$$V = -\frac{m\pi}{b}\left[-\frac{\Theta}{\beta}\cos\beta z + \frac{\Lambda}{\beta}\sin\beta z + \frac{zg}{\rho_1\omega^2}\left\{M\frac{m\pi}{b}\Delta + N\frac{n\pi}{a}\Gamma\right\}\right] + \Delta z + \Pi . \qquad (48)$$

The solutions for U, V and W {equations (46), (47) and (48)} have one part corresponding to the symmetric modes, the other corresponding to the anti-symmetric modes. It can easily be shown, by substituting these equations into the boundary-condition equations (31), (32) and (33), that the symmetric and anti-symmetric parts are not coupled in any way. They can therefore be treated separately.

6.2.1. *A solution for flexural modes of flat panels ignoring rotatory inertia.*—Extracting the anti-symmetric parts of equations (46), (47) and (48) we have:

$$W = \Lambda\cos\beta z + \frac{g}{\rho_1\omega^2}\left[M\frac{m\pi}{b}\Delta + N\frac{n\pi}{a}\Gamma\right]$$

$$U = -\frac{n\pi}{a}\left[\frac{\Lambda}{\beta}\sin\beta z + \frac{zg}{\rho_1\omega^2}\left\{M\frac{m\pi}{b}\Delta + N\frac{n\pi}{a}\Gamma\right\}\right] + \Gamma z$$

$$V = -\frac{m\pi}{b}\left[\frac{\Lambda}{\beta}\sin\beta z + \frac{zg}{\rho_1\omega^2}\left\{M\frac{m\pi}{b}\Delta + N\frac{n\pi}{a}\Gamma\right\}\right] + \Delta z . \qquad (49)$$

If these equations are substituted into the boundary-condition equations (31), (32) and (33) with $1/R = 0$, we then obtain a set of three homogeneous equations in Λ, Γ and Δ. Thus a determinantal frequency equation is obtained. For long-wavelength flexural modes the in-plate inertia forces of the faces can be regarded as small and the f_x and f_y terms in equations (31) and (32) are thus neglected. This assumption is effectively that of ignoring rotatory inertia for ordinary sandwich plates and has been shown to hold true both by an analysis of the authors and by Yu[11].

The resulting frequency equation in matrix form is then:

$$\begin{bmatrix} \dfrac{n\pi l^2}{a}\left[\dfrac{\sin\phi}{\beta} + \dfrac{h_2}{2}\cos\phi\right] & M\left[\dfrac{mn\pi^2}{ab}\dfrac{gl^2}{\rho_1\omega^2}\left(h_1+\dfrac{h_2}{2}\right) - \dfrac{mn\pi^2}{ab}\dfrac{h_1}{M}\dfrac{(1+\nu)}{2}\right] & N\left[\left(\dfrac{n\pi}{a}\right)^2\dfrac{gl^2}{\rho_1\omega^2}\left(h_1+\dfrac{h_2}{2}\right) - \dfrac{h_1\epsilon}{N} - \dfrac{K}{N}\right] \\ \dfrac{m\pi l^2}{b}\left[\dfrac{\sin\phi}{\beta} + \dfrac{h_2}{2}\cos\phi\right] & M\left[\left(\dfrac{m\pi}{b}\right)^2\dfrac{gl^2}{\rho_1\omega^2}\left(h_1+\dfrac{h_2}{2}\right) - \dfrac{K^*}{M} - h_1\dfrac{\epsilon^*}{M}\right] & N\left[\dfrac{mn\pi^2}{ab}\dfrac{gl^2}{\rho_1\omega^2}\left(h_1+\dfrac{h_2}{2}\right) - \dfrac{mn\pi^2}{ab}\dfrac{h_1}{N}\dfrac{(1+\nu)}{2}\right] \\ \left(Dl^4 - h_2\rho_f\dfrac{\omega^2}{g}\right)\times\cos\phi - C\beta\sin\phi & M\left[\left(Dl^4 - h_2\rho_f\dfrac{\omega^2}{g}\right)\times\dfrac{g}{\rho_1\omega^2}\dfrac{m\pi}{b} + \dfrac{h_2}{2}\dfrac{m\pi}{b}\right] & N\left[\left(Dl^4 - h_2\rho_f\dfrac{\omega^2}{g}\right)\times\dfrac{g}{\rho_1\omega^2}\dfrac{n\pi}{a} + \dfrac{h_2}{2}\dfrac{n\pi}{a}\right] \end{bmatrix} \begin{Bmatrix} \Lambda \\ \Delta \\ \Gamma \end{Bmatrix} = 0 \qquad (50)$$

where

$$\beta^2 = \frac{\omega^2 \rho_1}{Cg}, \qquad \phi = \beta h_1, \qquad l^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2,$$

$$\epsilon = \left(\frac{n\pi}{a}\right)^2 + \frac{1-\nu}{2}\left(\frac{m\pi}{b}\right)^2, \qquad \epsilon^* = \left(\frac{m\pi}{b}\right)^2 + \frac{1-\nu}{2}\left(\frac{n\pi}{a}\right)^2.$$

The frequency equation is obtained by putting the determinant of the matrix equal to zero. This equation is then easily non-dimensionalised by substituting the following non-dimensional quantities:

$$\mu = \frac{h_2}{h_1}, \qquad A_n = \frac{a}{nh_1}, \qquad A_m = \frac{b}{mh_1},$$

$$\lambda = \frac{E}{C} = \frac{\rho_2}{\rho_1}, \qquad \bar{\epsilon} = h_1^2 \epsilon, \qquad \bar{\epsilon}^* = h_1^2 \epsilon^*,$$

$$\gamma = \frac{N}{C}, \qquad \bar{\psi} = h_1^2 l^2.$$

$$\frac{3\gamma}{2} = \frac{M}{C},$$

and by suitable rearrangement to obtain zeros the determinantal frequency equation becomes:

$$\begin{vmatrix} 0 & \left[A_m^2\left(\bar{\epsilon}^* + \frac{3\gamma}{2\mu\lambda}(1-\nu^2)\right) - \pi^2\frac{(1+\nu)}{2}\right] & \left[\frac{5}{4}\pi^2(1+\nu) - \frac{3}{2}A_n^2\bar{\epsilon} - A_m^2\bar{\epsilon}^* - \frac{3\gamma}{2\mu\lambda}(1-\nu^2)(A_n^2+A_m^2)\right] \\ \bar{\psi}\pi\left[\frac{\sin\phi}{\phi} + \frac{\mu}{2}\cos\phi\right] & A_m^2\left[\left(1+\frac{\mu}{2}\right)\frac{3\gamma\bar{\psi}\pi^2}{2\phi^2 A_m^2} - \bar{\epsilon}^* - \frac{3\gamma}{2\mu\lambda}(1-\nu^2)\right] & \left[A_m^2\left(\bar{\epsilon}^* + \frac{3\gamma}{2\mu\lambda}(1-\nu^2)\right) - \frac{3\pi^2}{2}\frac{1+\nu}{2}\right] \\ \lambda\mu\cos\phi\left(\frac{\mu^2\bar{\psi}^2}{12(1-\nu^2)} - \phi^2\right) - \phi\sin\phi & \frac{3\pi\gamma}{2}\left[\frac{\mu}{2} - \lambda\mu + \frac{\lambda\mu^3\bar{\psi}^2}{12(1-\nu^2)\phi^2}\right] & 0 \end{vmatrix} = 0. \quad (51)$$

It will be observed that ρ_f/ρ_1 has been replaced by λ in some places and that in this way the weight of the bonding has been omitted. This has been done so that non-dimensional graphs of frequency can be plotted against the variables A_n, A_m and μ (γ and λ are regarded as constants for one type of honeycomb). The results of computations using equation (51) are shown in Fig. 3.

All of the graphs at the end of this report were computed using quantities associated with British CIBA (ARL) 'Aeroweb' honeycomb type 142. This aluminium-foil honeycomb, with a cell size of 0·25 inches, is commonly used by the British Aircraft Industry. Other design graphs, like those given in this report, however, can easily be constructed for other materials by solving the equations given herein. Values of constants for 'Aeroweb' 142 are as follows:

$$M = 31{,}900 \text{ Lb/in}^2$$

$$N = 21{,}300 \text{ Lb/in}^2$$

$$C = 238{,}640 \text{ Lb/in}^2$$

$$\rho_1 = 4 \text{ Lb/ft}^3$$

and the constants used were:

$$E_m = 10^7 \text{ Lb/in}^2 \text{ (Aluminium)}$$

$$c = 1123 \text{ ft/sec}$$

$$g = 32 \cdot 2 \text{ ft/sec}^2$$

$$\rho_2 = 167 \cdot 5 \text{ Lb/ft}^3$$

$$\tau = 0 \cdot 09228 \text{ Lb/ft}^2$$

$$\nu = 0 \cdot 34 \text{ (Aluminium).}$$

The range of values of parameters for which these computations were carried out is as follows:

(i) Non-dimensional wavelength

$$\text{from } A_{n,m} = 16 \text{ to } A_{n,m} = 96.$$

(ii) Core/face-plate thickness ratio

$$\text{from } \mu = 0 \cdot 05 \text{ to } \mu = 0 \cdot 20.$$

For a particular honeycomb panel the apparent natural frequency, ω_0, is computed from the value of ϕ thus:

$$\omega_0 = \frac{\phi}{h_1} \sqrt{\frac{Cg}{\rho_1}} \tag{52}$$

and this must then be corrected for the mass of the interface adhesive using the equation

$$\left(\frac{\omega}{\omega_0}\right)^2 = \frac{\rho_1 + \mu\rho_2}{\rho_1 + \mu\rho_2 + \dfrac{12\tau}{h_1}} \tag{53}$$

where h_1 is measured in inches and τ is in Lb/ft^2.

This equation has been derived assuming that w does not vary with z in the core (flexural modes only).

This method of correction does not give exact answers, but the maximum error for the range of values of the variables used in the computations is small (not greater than 4 per cent). Equations (52) and (53) are plotted in Figs. 4 and 5 and these graphs can be used in conjunction with Fig. 3 to determine the natural frequency of flexural vibration of any flat honeycomb panel within the limits of the assumptions of this report (*N.B.* The graphs only apply to CIBA honeycomb 142 with Aluminium face plates but similar graphs for other sandwiches can easily be constructed).

6.2.2. *A solution for bubbling modes of flat panels.*—Again, the in-plane inertia of the core is neglected as it will be small. The symmetric mode of principal interest is the bubbling mode because its frequency is lower than the frequencies of the corresponding longitudinal modes (to which it is only very lightly coupled, *see* Section 6.2).

Extracting the symmetric parts of equations (46), (47) and (48) we find for symmetric modes that:

$$W = \Theta \sin \beta z$$

$$U = \frac{n\pi}{a}\frac{\Theta}{\beta} \cos \beta z + \Xi$$

$$V = \frac{m\pi}{b}\frac{\Theta}{\beta} \cos \beta z + \Pi . \tag{54}$$

If this set of equations is now substituted into the interface boundary-condition equations (31), (32) and (33) with $1/R = 0$ (as before in Section 6.2.1) a set of homogeneous equations in Θ, Ξ and Π is found, thus:

$$\begin{bmatrix} l^2 \dfrac{n\pi}{a}\left[\dfrac{\cos \beta h_1}{\beta} - \dfrac{h_2}{2}\sin \beta h_1\right] & \epsilon & \dfrac{1+\nu}{2}\dfrac{mn\pi^2}{ab} \\ l^2 \dfrac{m\pi}{b}\left[\dfrac{\cos \beta h_1}{\beta} - \dfrac{h_2}{2}\sin \beta h_1\right] & \dfrac{1+\nu}{2}\dfrac{mn\pi^2}{ab} & \epsilon^* \\ \left[Dl^4 - h_2\rho_f \dfrac{\omega^2}{g}\right]\sin \beta h_1 + C\beta \cos \beta h_1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Theta \\ \Xi \\ \Pi \end{Bmatrix} = 0 . \tag{55}$$

The only permissible solution to this equation is:

$$\tan \beta h_1 = -\frac{C\beta}{\left[Dl^4 - h_2\rho_f \dfrac{\omega^2}{g}\right]} \tag{56}$$

or, by using the non-dimensional form of Section 6.2.1:

$$\tan \phi = \frac{\phi}{\mu\lambda\phi^2 - \dfrac{\lambda\mu^3\bar{\psi}^2}{12(1-\nu^2)}} . \tag{57}$$

It has been found that the second term in the denominator of the right-hand side of equation (57) is much smaller ($10^{-4} \times$ at least) than the first term, for ordinary sandwich plates, and it can therefore be neglected. Equation (57) then becomes:

$$\mu\lambda\phi \tan \phi - 1 = 0 . \tag{58}$$

Thus the bubbling-mode frequency is virtually independent of wavelength. This would be expected because the contribution of the flexural strain energy of the face plates to the total strain energy is clearly very small.

Bubbling-mode frequency parameter is shown plotted against μ in Fig. 6.

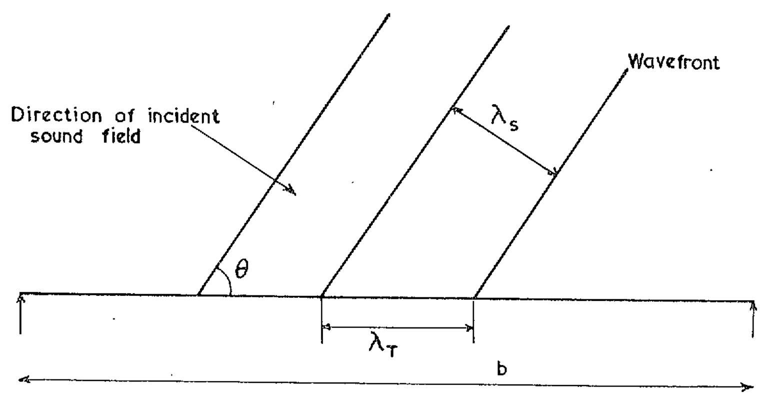
Here again for non-dimensional plotting λ has been used in place of ρ_f/ρ_1. However, for the bubbling mode, w varies considerably with z and the mass ratio of equation (53) will not apply. The generalised mass in this mode will depend on an integration of mass times displacement squared throughout the depth of the plate. If we assume the face and bonding to move as point masses, and if we assume a linear deformation of the core (this latter should be sufficiently accurate, as the core is usually much lighter than the faces) then the frequency correction equation corresponding to equation (53) becomes:

$$\left(\frac{\omega}{\omega_0}\right)^2 = \frac{\rho_1 + 3\rho_2\mu(1+\mu)}{\rho_1 + 3\rho_2\mu(1+\mu) + 36\tau/h_1} \tag{59}$$

for bubbling modes. This is plotted in Fig. 7.

6.2.3. *Flat-panel thickness-shear and longitudinal modes.*—These vibration modes are of less interest than the flexural and bubbling modes because they are usually of a higher natural frequency and because they would not be directly excited by normal pressures. However, in certain cases, it may be desirable to calculate their natural frequencies and this can be done by including the in-plate inertias in the equilibrium and boundary condition equations. The frequency equations can be found as before but the solution (i.e. finding the roots of the determinantal equation) is rather more tedious.

7. *The Possibility of Excitation of Flat Finite Sandwich Plates by the Acoustical Coincidence Effect.*



If we have a sound field of harmonic plane waves impinging on a flat plate as shown in the figure above, at an angle θ to the plate, then there is a unique type of excitation possible, known as the coincidence effect. When the effect occurs with higher-order modes, the generalised force exciting the mode is very much larger than that occurring when waves of normal incidence impinge on the plate. Plate stress can consequently become high. If there is no structural damping the panel then becomes virtually 'transparent' to the acoustic pressure field (i.e. the panel surface is vibrating at the particle velocity of the incident acoustic field). λ_T is the projected sound wavelength in the plate and is known as the 'trace' wavelength. We have:

$$\lambda_T \sin\theta = \lambda_S \tag{60}$$

where λ_S is the sound wavelength perpendicular to the sound wavefront. The coincidence effect occurs when the sound frequency is the same as the plate natural frequency and when the natural wavelength of the plate is also equal to the trace wavelength, i.e.

$$2b = \lambda_T \tag{61}$$

(b is the plate semi-wavelength). A necessary condition for the coincidence effect to occur is that $\lambda_S < 2b$ as can be seen by considering the figure. If the natural frequency of the panel is ω rad/sec then for the coincidence effect to occur:

$$\lambda_S = \frac{2\pi c}{\omega} \tag{62}$$

where c is the speed of sound in air. Also:

$$\lambda_S = \lambda_T \sin\theta = 2b \sin\theta \tag{63}$$

so that:

$$\sin\theta = \frac{\pi c}{\omega b} \tag{64}$$

or in non-dimensional form, using equation (52):

$$\sin\theta_{m,n} = \frac{\pi c}{A_{m,n}\phi}\sqrt{\frac{\rho_1}{Cg}}. \tag{65}$$

Calculations have been carried out taking a particular honeycomb-sandwich configuration (*see* Section 6.2.1). In consequence a specific (constant) value of $\sqrt{(\rho_1/Cg)}$ has been used, together with a value for c of 1123 ft/sec.

Equation (65) then reduces to:

$$\sin\theta_{m,n} = \frac{0\cdot2125}{A_{m,n}\phi}. \tag{66}$$

This equation can be superimposed on the frequency vs. wavelength graphs (Fig. 3) by plotting lines of constant sound-field incidence angle θ and are shown in Fig. 8. However, owing to the nature of Fig. 3, there will clearly be two sets of constant θ lines, one set for coincidence in the x-direction and the other for coincidence in the y-direction. As stated above, a necessary condition for the coincidence effect to occur is that $\lambda_S < 2b$ or alternatively, the right-hand side of equation (66) must be less than unity. For certain small regions of Fig. 8 (or Fig. 3) this condition is not fulfilled and in these regions no coincidence effect can occur. These regions are edged by a shaded line in Fig. 8.

No coincidence effect is considered in detail for bubbling modes as the frequencies of these modes are comparatively high. To illustrate this we will consider two examples:

(i) For a plate 24 in. × 24 in., skin thickness 0·036 in., core depth 0·5 in., $\phi = 0\cdot448$ (bubbling mode, frequency = 56,900 c/s), $A_n = 96$ and thus from equation (66) $\theta_n = 0\cdot283°$.

(ii) For a plate 24 in. × ∞, skin thickness 0·024 in., core depth 6 in. (corresponding approximately to a control surface constructed from a honeycomb wedge), $\phi = 1\cdot189$ (frequency = 12,600 c/s), $A_n = 8$ and from equation (66) $\theta_n = 1\cdot278°$.

From these examples it can be seen that (*a*) the frequency of bubbling modes of normal sandwich plates is well outside the expected excitation range (0 to 20 Kc/s), and (*b*) that the bubbling modes of all sandwich plates are only excited by the acoustical coincidence effect when the sound-field wavefronts are nearly parallel to the plate surface.

In fact, normal aircraft sandwich panels have thinner face plates than those quoted above, and therefore their frequencies will be even higher.

8. *Discussion.*

This analysis has assumed that a state of plane stress exists in the face plates of the sandwich and an exact deformation solution has been found for the core under this assumption. On the other hand Yu, in his recent work, has made the assumption of planar deformation in the core but has included shear effects in the face plate. Yu's analysis therefore applies to vibrations of sandwich plates with all thicknesses of face plates but is frequency limited because of the breakdown of the planar deformation assumption at high frequency. Thus, his work applies perfectly adequately to the low-order flexural vibrations of most sandwich plates. This paper is necessarily limited in application to sandwiches with thinner face plates, but will treat high-frequency modes (bubbling modes, for example) with accuracy. As this work is being carried out with aircraft sandwich structures specifically in mind, the assumption that face plates are thin is reasonable. The analysis of this paper applies down to wavelengths of approximately four inches (for a core depth of 0·5 inches and a cell size of 0·25 inches). The cell size is the critical parameter for the lower limit of wavelength because the assumption of a homogeneous core becomes doubtful when the plate half-wavelength is less than about 16 times the cell size. This length, however, should be adequate enough for the analysis of first-order modes of typical aircraft sandwich panels.

The vibration solution for sandwich plates has been found by solving the core differential equations under the boundary conditions imposed on them by the face plates. The determinantal frequency equations for the flexural and bubbling modes (predominantly z-wise motions) have been obtained by neglecting the in-plane-of-plate inertia of the core and faces. This assumption, which is, in effect, the neglect of rotatory inertia, has been found to be sufficiently accurate for all of the configurations of sandwich plate under consideration. (Yu also draws the same conclusion in Ref. 11.) These equations are in non-dimensional form for ease of display, and, because the bonding weight has been included in the analysis, a correction has been applied to the frequencies obtained, as this parameter could not be included in non-dimensional work. This correction is not exact, but it does not give rise to errors in excess of 4% over the ranges of variables considered. The error is due, of course, to the assumed form of displacement used in the calculation of the generalised mass. For flat-plates, flexural-mode frequencies in their non-dimensional form are shown in Fig. 1 for various values of μ, the ratio of skin to core thickness. Bubbling-mode frequencies are shown in Fig. 6. Neither the cross-sectional mode shapes for the flexural modes nor those for the bubbling modes have been computed. For flat plates these ratios can easily be found however by substituting the appropriate frequency back into either equation (50) or equation (55). Figs. 3 and 6 for frequencies of flexural and bubbling modes respectively, were computed for CIBA (A.R.L.) 'Aeroweb' Honeycomb type number 142, which is a honeycomb commonly used by the British Aircraft Industry.

9. *Conclusions to Part I.*

Fig. 3 shows that the variation of frequency with wavelength takes the usual form. It was found that the bubbling-mode frequencies were virtually independent of wavelength (*see* Fig. 6), for the range of wavelengths considered. This is due to the relatively small amount of strain energy stored by flexure of the face plates. Computations carried out by the authors, but not listed in this report, have shown that Rotatory Inertia of the cross-sections can be safely regarded as negligible for the range of values described in Section 6.2.1 but the shear deformation effects in the core must be included for an accurate analysis of the flexural and bubbling vibrations of sandwich plates. This confirms the identical conclusion reached by Yu in Ref. 11.

The acoustical coincidence effect was found to occur over a wide range of sound-field incidence angles for the wavelength range considered. The cut-off areas on the graphs (Fig. 8), where it is impossible for the coincidence effect to occur, were a very small part of the total range. This area coincided with longer plate wavelength vibrations. Thus it is concluded that the coincidence effect can occur for sandwich plates under acoustic excitation, and, because of the wide range of sound-field incidence angles at which the effect can occur, the effect may be caused by any of the types of sound fields normally associated with aircraft (*viz.* jet noise, boundary-layer pressure fluctuations, etc.).

PART II

The Solution for Cylindrical Plates

By A. J. Pretlove, B.Sc., Ph.D.

10. *Introduction to Part II.*

In Part I of this report the differential equations of motion of cylindrically curved sandwich plates were derived. Only the flat-plate frequency solutions were obtained, under the assumption that the inertia forces in the plane of the plate were negligible. For the curved plate the same assumption is made and thereby frequencies are restricted to those which correspond to flexural and bubbling modes, these modes consisting predominantly of motion normal to the plate surface. In the problem of acoustic excitation of sandwich structures (to which this work is ultimately to be directed) the only modes likely to be excited are these flexural and bubbling modes.

An exact solution of the differential equations of motion with curvature included cannot be obtained analytically. In the early work on this problem a perturbation technique was attempted in order to obtain a solution. In this, the solution for curved-plate core deflection was assumed to be a perturbed flat-plate deflection. The problem formulated in this way was not solvable. However, it is argued that the use of a flat-plate core-deflection solution together with curved-plate boundary conditions gives sufficiently accurate values for natural frequencies. Hence, by using the same technique as in Part I, six homogeneous equations are obtained in six variables (arbitrary constants) and the determinant of coefficients is equated to zero to find the natural frequencies. The frequency solutions have been obtained using a numerical technique and a digital computer.

In the 1940's Arnold and Warburton noticed an unexpected variation of natural frequency with variation of flexural wavelength around the circumference of thin cylinders. It appeared that the natural frequency, for a certain configuration, was dropping as the circumferential wavelength decreased. Subsequently, they propounded a theory[19] which showed that this effect could occur, and which agreed remarkably well with their experimental results. A physical explanation of the mechanism of this surprising effect is to be found in the types of strain energy involved in a deflection from the equilibrium state. The type of strain energy associated with the small deflections of thin flat plates (*viz.* flexural or bending strain energy) is present. Also, however, it is now possible for in-plane direct strain of the mid-surface of the plate to occur involving an energy which is comparable in magnitude to the flexural strain energy. For a given central deflection and radius of curvature this in-plane stretching energy intrinsically decreases as the wavelength decreases, because the sector angle of the plate is becoming smaller (i.e. nearer the flat-plate condition). Thus, it is this variation of the in-plane mid-surface stretching energy which causes the unusual variation of curved-plate natural frequency. This effect has been noticed for some configurations of the curved sandwich plates, and Fig. 9 shows such a variation.

In a paper by Yu[14] on the vibrations of cylindrically curved sandwich plates the axially symmetric and torsional vibrations of complete cylinders are considered. It is interesting to note that he uses the same type of core deflection in this paper as he had previously used for flat plates in an earlier paper. This core deflection was assumed to be linear (*see* Part I, Section 3.2).

11. *A Vibration Solution for Cylindrically Curved Panels.*

In Part I of this report a separable form of solution was assumed for a cylindrically curved panel and equations (37), (38) and (39) were derived for the z-wise variation of the three orthogonal deflections in the core. Inertia terms in the plane of the plate are now ignored for flexural and bubbling vibrations and these core equilibrium equations become:

$$\left[\delta^2 + \frac{\hat{r}}{h_1}\,\delta^{\dagger}\right] U + \frac{n\pi}{a}\left[\delta + \frac{\hat{r}}{h_1}{}^{\dagger}\right] W = 0 \tag{67}$$

$$\left[\delta^2 + \frac{\hat{r}}{h_1}\,\delta^{\dagger}\right] V + \frac{m\pi}{b}\left[\left(1 - \frac{z}{h_1}\hat{r}\right)\delta + \frac{\hat{r}}{h_1}{}^{\dagger}\right] W = 0 \tag{68}$$

$$C\delta^2 W = N\frac{n\pi}{a}\,\delta U + M\frac{m\pi}{b}\left[\left(1 - \frac{z}{h_1}\hat{r}\right)\delta - \frac{\hat{r}}{h_1}\right] V +$$

$$+\left[-C\frac{\hat{r}\delta}{h_1} + \left\{N\left(\frac{n\pi}{a}\right)^2 + M\left(\frac{m\pi}{b}\right)^2 - \rho_1\frac{\bar{\omega}_n{}^2}{g}\right\} - 2M\frac{z}{h_1}\hat{r}\left(\frac{m\pi}{b}\right)^2\right] W. \tag{69}$$

Now an exact solution of these core equilibrium equations is not feasible and an alternative approach must be sought. If the terms in the core equations with $\hat{r}$ as a factor are neglected the general solution can be found for the core {*viz.* the flat-plate core deflection solution given in equations (47), (48) and (49)}. Applying the curved boundary conditions of equations (31), (32) and (33) to this core solution we can obtain a set of frequencies and mode shapes for this hypothetical condition which will be denoted by a zero subscript. An attempt was made to perturb this solution for deflection of the core in order to obtain a more accurate solution for the complete curved problem. In fact we will argue that the frequency obtained using the flat-plate core deflection is sufficiently accurate for this particular core configuration. The hypothetical deflection solution is perturbed in order to fit it to the full core equilibrium equations (67), (68) and (69) above, thus:

$$\begin{aligned} U_n &= U_{n\,0} + \hat{r}U_{n\,0}{}^* \\ V_n &= V_{n\,0} + \hat{r}V_{n\,0}{}^* \\ W_n &= W_{n\,0} + \hat{r}W_{n\,0}{}^* \end{aligned} \tag{70}$$

and a perturbation of frequency must also be allowed:

$$\bar{\omega}_n = \omega_{n\,0} + \hat{r}\omega_{n\,0}{}^* \tag{71}$$

† These terms will be referred to later.

In this way, only a first-order perturbation is allowed, but this will be sufficient for sufficiently small $\hat{r}$. If equations (70) are now substituted in equations (67), (68) and (69), the terms with different powers of $\hat{r}$ can be separately equated to zero because the equations are valid for all $\hat{r}$. The equations of order one are:

$$\delta^2 U_{n0} + \frac{n\pi}{a}\,\delta W_{n0} = 0$$

$$\delta^2 V_{n0} + \frac{m\pi}{b}\,\delta W_{n0} = 0$$

$$-N\frac{n\pi}{a}\,\delta U_{n0} - M\frac{m\pi}{b}\,\delta V_{n0} + \left[C\delta^2 - \left\{N\left(\frac{n\pi}{a}\right)^2 + M\left(\frac{m\pi}{b}\right)^2 - \rho_1\frac{\omega_{n0}^2}{g}\right\}\right] W_{n0} = 0. \quad (72)$$

These equations correspond exactly to equations (40), (41) and (42) of Part I. The solution of equations (72) is exactly similar to the flat-plate solution obtained in Part I, equations (46), (47) and (48), but the constants will be slightly different (they will be denoted Γ′, Δ′, Θ′, Λ′, Ξ′, Π′) because curvature terms will be retained in the boundary conditions {equations (31), (32) and (33)}.

Now the equations of order $\hat{r}$ derived from the substitution of the perturbed deflection are as follows:

$$\delta^2 U_{n0}^* + \frac{n\pi}{a}\,\delta W_{n0}^* = -\left[\delta U_{n0} + \frac{n\pi}{a}\,W_{n0}\right]$$

$$\delta^2 V_{n0}^* + \frac{m\pi}{b}\,\delta W_{n0}^* = -\left[\delta V_{n0} + \frac{m\pi}{b}(1-z\delta)W_{n0}\right]$$

$$-N\frac{n\pi}{a}\,\delta U_{n0}^* - M\frac{m\pi}{b}\,\delta V_{n0}^* + \left[C\delta^2 - \left\{N\left(\frac{n\pi}{a}\right)^2 + M\left(\frac{m\pi}{b}\right)^2 - \rho_1\frac{\omega_{n0}^2}{g}\right\}\right] W_{n0}^*$$
$$= -M\frac{m\pi}{b}(1+z\delta)V_{n0} - \left[C\delta + 2Mz\left(\frac{m\pi}{b}\right)^2\right] W_{n0}. \quad (73)$$

This set of equations gives the differential relationships for the perturbation deflections U_{n0}^*, etc., and because these functions must also satisfy the identical boundary conditions to the solutions of equations (72) we can expand them as an infinite series of the natural modes of equations (72) but omitting the mode with which we are concerned, i.e.

$$U_{r0}^* = \sum_{\substack{n=1\\ n\neq r}}^{\infty} a_n U_{n0}$$

$$V_{r0}^* = \sum_{\substack{n=1\\ n\neq r}}^{\infty} a_n V_{n0}$$

$$W_{r0}^* = \sum_{\substack{n=1\\ n\neq r}}^{\infty} a_n W_{n0}. \quad (74)$$

Note that a_n is the same for all series as the combination U, V, W makes up the natural mode. Substitution of equation (74) into equations (73) gives:

(*a*) For the first two equations, some trivial equation because inertia terms have been dropped.

(*b*) For the third equation:

$$\frac{\rho_1}{g}\sum_{\substack{n=1\\ n\neq r}}^{\infty} a_n(\omega_{r0}^2 - \omega_{n0}^2)W_{n0} = -M\frac{m\pi}{b}(1+z\delta)V_{r0} - \left[C\delta + \frac{\rho_1}{g}2\omega_{r0}\omega_{r0}^* + 2Mz\left(\frac{m\pi}{b}\right)^2\right] W_{r0}. \quad (75)$$

If equation (75) is differentiated with respect to z, multiplied by $\delta W_{r\,0}$, and integrated over the z-domain, we get:

$$\frac{\rho_1}{g}\sum_{\substack{n=1\\n\neq r}}^{\infty}\left\{a_n(\omega_{r\,0}^2-\omega_{n\,0}^2)\int_{-h_1}^{h_1}\delta W_{n\,0}\delta W_{r\,0}dz\right\} = -M\frac{m\pi}{b}\int_{-h_1}^{h_1}(2\delta+z\delta^2)V_{r\,0}\delta W_{r\,0}dz -$$

$$-\int_{-h_1}^{h_1}\left(C\delta^2+2M\left(\frac{m\pi}{b}\right)^2+2Mz\left(\frac{m\pi}{b}\right)^2\delta\right)W_{r\,0}\delta W_{r\,0}dz -$$

$$-\frac{\rho_1}{g}2\omega_{r\,0}\omega_{r\,0}^*\int_{-h_1}^{h_1}\delta W_{r\,0}\delta W_{r\,0}dz. \tag{76}$$

Now the first two terms on the right-hand side of equation (76) can be neglected. If the combination $U_{r\,0}$, $V_{r\,0}$, $W_{r\,0}$ was either a purely symmetrical or a purely anti-symmetrical mode these two terms would be identically zero because the integrands would be odd functions of z (the z-domain being from $z = -h_1$ to $z = h_1$). Now this would be true for flat-plate core-deflection solutions when bounded by flat face plates (the problem in Part I). However, for curved face plates bounding a flat-plate core deflection, small cross-coupling terms exist between the symmetrical and anti-symmetrical core modes. The modes previously called 'symmetrical' now have a very small anti-symmetric component and *vice-versa*. Thus, these two integrals have non-zero, but small values derived from the cross-coupling between the now wrongly called symmetric and anti-symmetric modes. These terms, being small, will be neglected. The equation (76) then reduces to:

$$\frac{\omega_{r\,0}^*}{\omega_{r\,0}} = \sum_{\substack{n=1\\n\neq r}}^{\infty}\frac{a_n}{2}\left(\frac{\omega_{n\,0}^2}{\omega_{r\,0}^2}-1\right)\frac{\int_{-h_1}^{h_1}\delta W_{r\,0}\delta W_{n\,0}dz}{\int_{-h_1}^{h_1}\delta W_{r\,0}\delta W_{r\,0}dz}. \tag{77}$$

If the flat-plate core-deflection solution is to be a sufficiently accurate solution for this problem then it must be shown that $\omega_{r\,0}^*/\omega_{r\,0}$ is a small quantity ($\ll 1$). If this can be proved then we can say that $\bar{\omega}_r$ is sufficiently represented by $\omega_{r\,0}$, the error term being of second order of small magnitude, and only a flat-plate core-deflection solution is needed to find this frequency. In order to show that $\omega_{r\,0}^*/\omega_{r\,0}$ is small we must:

(*a*) Find some value for a_n, using an orthogonality condition.

(*b*) Show that the product of the large term $(\omega_{n\,0}^2/\omega_{r\,0}^2 - 1)a_n$, and the ratio of integrals is small.

(*c*) Show that the series converges, and that it converges on to a small value.

These tasks are complicated and have not, as yet, been completed. However, this hypothetical solution can be shown intuitively to be a good approximation. The application of the face-plate boundary conditions to a core solution of deflection for honeycomb sandwiches is, in essence, a matching of the surface impedances of the face plates to the core. Now following from the explanation in the Introduction, the surface impedances of a curved plate to flexural-type deflection are considerably different from those of a flat plate because in-plane direct strain occurs. On the other hand, the surface impedance of a cylindrical honeycomb core of normal proportions is likely to be little different from that of a flat core since direct in-plane strain effects are negligible (A and B are very small) and curvature shear effects {terms indicated by † in equations (67) and (68)} are also likely to be negligible. Therefore, the use of a flat-core deflection solution (and hence of a flat-core surface impedance) should not give rise to large errors in the natural frequencies of curved sandwich plates. The effect of curvature on the assembled plate is restricted to the face plates, therefore, which will be stiffer when curved. Thus, the matching of impedances which occurs when the full face-plate boundary conditions are applied to the flat-plate core-deflection solution will ensure that the major effects of curvature are included in the analysis. ω_{n0} will therefore give a good approximation to the true natural frequency.

The solution for ω_{n0} is easily found by substituting a flat-plate-type core solution for deflection {*see* equations (46), (47) and (48) of Part I} into the curved-plate boundary-condition equations which are given in full in Part I {*see* equations (31), (32) and (33)}.

Six homogeneous equations result thus:

$$[A]\{\Gamma'\} = 0 \tag{78}$$

where $\{\Gamma'\}$ is the column vector $\{\Gamma', \Delta', \Theta', \Lambda', \Xi', \Pi'\}$.

The characteristic determinant of equation (78) is equated to zero to give natural frequencies. A simplified and non-dimensionalised form of this determinant is shown in Table 1. The characteristic equation has been solved numerically for flexural and bubbling frequencies using a digital computer. The numerical method used to obtain the solution of the characteristic equation was a successive application of the rule of false position. Results are shown in Tables 2 and 3 and specimen graphs show typical variation of frequency with the various parameters involved. The non-dimensionalised parameters used are the same as those of Part I. The frequencies were evaluated for the set of constants and the range of wavelength given in Part I, Section 6.2.1. The range of values of $\hat{r}$ for which frequencies were calculated is from $\hat{r}$ = zero to $\hat{r} = 0 \cdot 020$. This range is likely to cover the curvatures of fuselage, wing, and control-surface panels. A correction to be applied to frequencies to take account of the bonding mass is, to the same degree of approximation used in the solution, the same as that given in Part I, Sections 6.2.1 and 6.2.2 for flexural and bubbling modes of flat panels.

12. *Concluding Remarks.*

The frequencies of flexural and bubbling modes have been found for cylindrically curved sandwich plates. They are shown in their non-dimensional form in Tables 2 and 3. The anomalous variation of frequency with circumferential wavelength shown in Fig. 9 was not unexpected. This effect, *viz.* a decrease in frequency with decrease in circumferential wavelength for some plate configurations, is associated with the relative proportions of strain energy stored by flexure and in-plane stretching, and was first found and explained by Arnold and Warburton[19]. It should be noted, however, that this effect is only sufficiently significant to show this decrease in frequency with a decrease in wavelength at higher curvatures. An inspection of Table 2 will confirm this statement (i.e. Fig. 9 is not a typical variation of frequency with circumferential wavelength). Figs. 10, 11 and 12 show the typical variations of flexural frequency with axial wavelength, skin-core thickness ratio, and curvature respectively.

Fig. 13 shows the variation of bubbling-mode frequency with skin-core thickness ratio. For the ranges of parameters chosen it was found that the variation of bubbling-mode frequency with both wavelengths and curvature was only very slight. The maximum variation of this frequency with wavelength was less than $0 \cdot 1\%$, and with curvature less than $0 \cdot 2\%$. This is due, in exactly the same way as it was for flat plates, to the high proportion of strain energy stored in direct stretching of the core, compared with the strain energy stored in flexure and stretching of the faces.

As in Part I a correction must be applied to the computed natural frequencies to take account of the bonding mass. Now the curved-plate-core mode shape has been assumed, on reasonable grounds, to be of the same form as the flat-plate-core mode shape, to the first order of magnitude. Therefore the correction equations (53) and (59) of Part I will apply to the flexural- and bubbling-mode frequencies of the curved panels of this report.

The ranges of parameters chosen for the computations were the same as those used in Part I. These ranges were considered to be representative of aircraft structural elements. The values of curvature used in the computations covered the range which included low-curvature wing panels and the higher-curvature fuselage panels.

The modal shapes have not been computed for the curved-plate core because of the complexity of the terms involved. However, these mode shapes can be found by substituting the frequency back into the equation (78) in the usual way.

The problem has not yet been solved of showing analytically that the flat-plate core deflections applied to the curved boundary conditions do give satisfactory values for curved-plate natural frequencies. The difficulties involved in showing that the perturbation of frequency will only be of second order of small magnitude compared with the first-order perturbation of deflection have been described in the text. However, it has been possible to explain why this simplification can be made, on physical grounds. It appears to give reasonable results which conform to established patterns.

Acknowledgements (Part II).

The author thanks Mr. D. O. N. James who has given invaluable help with the many attempts to justify the solution of the curved-plate equations which has been given in this report.

NOTATION

a, b		Plate lengths in x and y directions
$A, B, C, E, F, G, H, L, M$ and N		Elastic moduli
c		The speed of sound in air
D		The plate flexural rigidity $(= E_m h_2^3/12(1-\nu^2))$
e		Strain
f		Face-plate body force per unit volume
g		Gravity
$2h$		Sandwich-plate depth
$2h_1$		Sandwich-core depth
h_2		Face-plate thickness
K	=	$N(1-\nu^2)/E_m h_2$
K^*	=	$M(1-\nu^2)/E_m h_2$
l^2	=	$\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$
n, m		Number of half-waves in the x and y directions
p		Core body force per unit volume
N_i, M_j, Q_k		Force and moment resultants as shown in Fig. 2
R		Cylindrical radius of the plate
t		Time
x, y, z—u, v, w (U, V, W)		Cylindrical orthogonal co-ordinates as shown in Fig. 2 and their respective displacements
β^2	=	$\frac{\rho_1 \omega^2}{Cg}$
δ	=	d/dz
ϵ	=	$\left(\frac{n\pi}{a}\right)^2 + \frac{(1-\nu)}{2}\left(\frac{m\pi}{b}\right)^2$
ϵ^*	=	$\left(\frac{m\pi}{b}\right)^2 + \frac{(1-\nu)}{2}\left(\frac{n\pi}{a}\right)^2$
ζ		z-wise co-ordinate with origin at the middle plane of the face plates
θ		Incidence angle of plane waves on a plate

λ_S		Wavelength of sound
λ_T		Trace wavelength (*see* Section 7)
ρ_1		Core density
ρ_2		Face-plate density
ρ_f	=	$\rho_2 + \tau/h_2$
σ		Stress
τ		Bonding weight per unit area
ω		Frequency of vibration (rad/sec)
∇^2		Laplace's operator, $\partial^2/\partial x^2 + \partial^2/\partial y^2$
Γ, Δ, Θ, Λ, Ξ, Π		Mode arbitrary constants

Non-Dimensional Quantities

A_n	=	$a/h_1 n$
A_m	=	$b/h_1 m$
γ	=	N/C
$3\gamma/2$	=	M/C (For British honeycombs)
$\bar{\epsilon}$	=	$h_1{}^2\epsilon$
$\bar{\epsilon}^*$	=	$h_1{}^2\epsilon^*$
λ	=	$E_m/C = \rho_2/\rho_1$
μ	=	h_2/h_1
ν		Poisson's ratio
ϕ	=	βh_1
$\bar{\psi}$	=	$h_1{}^2 l^2$

Note.—The incidence of a $\pm$ sign or $\mp$ sign does not indicate a choice. This is simply a shorthand method of writing two equations as one. The upper signs give the equation connected with the upper face plate and *vice versa.*

REFERENCES

No.	Author(s)	Title, etc.
1	W. S. Hemp	On a theory of sandwich construction. A.R.C. R. & M. 2672. March, 1948.
2	E. Reissner	Small bending and stretching of sandwich type shells. N.A.C.A. Report 975. 1950.
3	N. J. Hoff	Bending and buckling of rectangular sandwich plates. N.A.C.A. Tech. Note 2225. November, 1950.
4	A. C. Eringen	Bending and buckling of rectangular sandwich plates. Proc. 1st U.S. nat. Congr. app. Mech. 1951.
5	J. H. Hunter-Tod	The elastic stability of sandwich plates. A.R.C. R. & M. 2778. March, 1949.
6	R. D. Mindlin	Flexural vibrations of elastic sandwich plates. Columbia University Tech. Report 35. 1959.
7	S. Timoshenko	On the transverse vibrations of bars of uniform cross-section. *Phil. Mag.*, Vol. 43, p. 125. 1922.
8	H. Ekstein	High frequency vibrations of thin crystal plates. *Phys. Rev.*, Vol. 68, p. 11. July, 1945.
9	Y. Yu	A new theory of elastic sandwich plates—one-dimensional case. *J. App. Mech.*, Vol. 26, p. 415. 1959.
10	Y. Yu	Simple thickness-shear modes of vibration of infinite sandwich plates. *J. App. Mech.*, Vol. 26, p. 679. 1959.
11	Y. Yu	Flexural vibrations of elastic sandwich plates. *J. Aero/Space Sci.*, Vol. 27, p. 272. April, 1960.
12	Y. Yu	Simplified vibration analysis of elastic sandwich plates. *J. Aero/Space Sci.*, Vol. 27, p. 894. December, 1960.
13	Y. Yu	Forced flexural vibrations of sandwich plates in plane strain. *J. App. Mech.*, Vol. 27, p. 535. 1960.
14	Y. Yu	Vibrations of elastic sandwich cylindrical shells. *J. App. Mech.*, Vol. 27, p. 653. December, 1960.
15	R. D. Mindlin	Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates. *J. App. Mech.*, Vol. 18, p. 31. 1951.
16	S. Timoshenko and J. N. Goodier ..	*Theory of Elasticity.* McGraw-Hill. 1953.
17	C. Wang	*Applied Elasticity.* McGraw-Hill. 1953.
18	R. D. Mindlin and L. E. Goodman	Beam vibrations with time-dependent boundary conditions. *J. App. Mech.*, Vol. 17, p. 377. December, 1950.
19	R. N. Arnold and G. B. Warburton	Flexural vibrations of the walls of thin cylindrical shells having freely supported ends. *Proc. Roy. Soc.* A, Vol. 197, p. 238. 1949.

TABLE 1

Simplified Form of the Non-Dimensional Characteristic Determinant

$\frac{\pi\bar{\psi}}{A_n}\left[\frac{\mu}{2}\sin\phi - \frac{1}{\phi}\cos\phi\right]$	$-\bar{\epsilon}$	$-\frac{A_m}{A_n}\bar{\psi}$
0	$\frac{1-\nu}{2}\bar{\psi}\frac{A_n}{A_m}$	0
$\sin\phi\left[\frac{\lambda\mu^3\bar{\psi}^2}{12(1-\nu^2)} - \lambda\mu\phi^2 + \frac{\lambda\hat{r}^2\mu}{(1-\nu^2)}\right] + \phi\cos\phi$	0	0
$\frac{\pi\hat{r}}{A_n}\left[\frac{\cos\phi}{\phi}\frac{\pi^2}{A_m^2}\frac{(3-\nu)}{2} + \nu\sin\phi\right]$	$\hat{r}\frac{\pi^2}{A_m^2}(1-\nu)$	$\hat{r}\frac{(3-\nu)}{2}\frac{\pi^2}{A_nA_m}$
$\frac{\pi\hat{r}}{A_m}\left[\frac{\cos\phi}{\phi}\left\{\frac{3\gamma}{2\lambda\mu}(1-\nu^2) + \frac{2\mu\pi^2}{A_m^2} + \frac{1+\nu}{2}\frac{\pi^2}{A_m^2} - \frac{(3-\nu)}{2}\frac{\pi^2}{A_m^2}\right\} + \sin\phi\left\{1-\nu+\frac{3\gamma}{2\mu\lambda}(1-\nu^2) - \frac{\mu^2\bar{\psi}}{12}\right\}\right]$	$\frac{\hat{r}\pi^2}{A_nA_m}\left[\frac{1+\nu}{2} - (1-\nu)\frac{A_n^2}{A_m^2}\right]$	$\hat{r}\left[\frac{3\gamma}{2\mu\lambda}(1-\nu^2) + \frac{\pi^2}{A_m^2}\left\{2\mu - \frac{(3-\nu)}{2}\right\} + \frac{\pi^2}{A_n^2}\frac{1+\nu}{2}\right]$
$\hat{r}\left[\frac{\cos\phi}{\phi}\left\{\frac{\bar{\psi}\pi^2\lambda\mu^3}{12A_m^2(1-\nu^2)} - \frac{3\gamma\mu\pi^2}{4A_m^2} - \frac{\lambda\mu\zeta}{(1-\nu^2)}\right\} - \frac{\pi^2}{A_m^2}\sin\phi\left\{\frac{\lambda\bar{\psi}\mu^3}{4(1-\nu^2)} + \frac{3\mu\gamma}{4}\right\}\right]$	$-\hat{r}\frac{\nu}{1-\nu^2}\lambda\mu\frac{\pi}{A_n}$	$\frac{\hat{r}\pi}{A_m}\left[\frac{\lambda\mu^3\bar{\psi}}{12(1-\nu^2)} - \frac{3\mu\gamma}{4} - \frac{A_m^2\mu\lambda\zeta}{\pi^2(1-\nu^2)}\right]$

TABLE 1—*continued*

$\frac{\hat{r}\pi}{A_n}\left[\nu\cos\phi - \frac{(3-\nu)}{2}\frac{\sin\phi}{\phi}\frac{\pi^2}{A_m^2}\right]$	$\frac{\hat{r}\pi^2}{A_nA_m}\left[\frac{3\gamma}{2\phi^2}\left\{\nu - \frac{3-\nu}{2}\frac{\pi^2}{A_m^2}\right\} + \frac{1+\nu}{2}\right]$	$\hat{r}\left[\frac{\pi^2}{A_m^2}(1-\nu) - \frac{\pi^2}{A_n^2}\frac{(1+\nu)}{3}\right]$
$\hat{r}\frac{\pi}{A_m}\left[\cos\phi\left\{\frac{3\gamma}{2\mu\lambda}(1-\nu^2) + (1-\nu) - \frac{\mu^2\bar{\psi}}{12}\right\} - \frac{\sin\phi}{\phi}\left\{\frac{3\gamma}{2\mu\lambda}(1-\nu^2) + \frac{2\mu\pi^2}{A_m^2} + \frac{1+\nu}{2}\frac{\pi^2}{A_n^2} - \frac{(3-\nu)}{2}\frac{\pi^2}{A_m^2}\right\}\right]$	$\hat{r}\left[\frac{3\gamma}{2\phi^2}\frac{\pi^2}{A_m^2}\left\{(1-\nu) + \frac{\pi^2}{A_m^2}\left(\frac{(3-\nu)}{2} - 2\mu\right) - \frac{1+\nu}{2}\frac{\pi^2}{A_n^2} - \frac{\mu^2\bar{\psi}}{12}\right\} + \frac{3\gamma}{2\mu\lambda}(1-\nu^2) + \frac{\pi^2}{A_m^2}\left\{2\mu - \frac{1+\nu}{2}\right\}\right]$	$\hat{r}\frac{\pi^2}{A_nA_m}\left[\frac{5}{6}(1+\nu) - \frac{A_n^2}{A_m^2}(1-\nu) - \frac{2}{3}\left\{2\mu + \frac{3\gamma}{2\mu\lambda}\frac{A_m^2}{\pi^2}(1-\nu^2)\right\}\right]$
$\hat{r}\left[\frac{\sin\phi}{\phi}\left\{\frac{\lambda\mu\bar{\zeta}}{1-\nu^2} + \frac{\pi^2}{A_m^2}\left(\frac{3\mu\gamma}{4} - \frac{\lambda\mu^3\bar{\psi}}{12(1-\nu^2)}\right)\right\} - \cos\phi\frac{\pi^2}{A_m^2}\left\{\frac{\lambda\mu^3\bar{\psi}}{4(1-\nu^2)} + \frac{3}{4}\mu\gamma\right\}\right]$	$\hat{r}\frac{\lambda\mu}{(1-\nu^2)}\frac{\pi}{A_m}\left[\frac{1}{\phi^2}\left\{\frac{3\gamma\bar{\zeta}}{2} - \frac{\bar{\psi}\mu^2\gamma}{2}\frac{\pi^2}{A_m^2}\right\} - \left\{1 - \frac{\bar{\psi}\mu^2}{12}\right\}\right] - \hat{r}\frac{3\gamma\mu}{2}\frac{\pi}{A_m}$	$\hat{r}\frac{\lambda\mu}{(1-\nu^2)}\frac{\pi}{A_n}\left[\frac{2}{3}\left\{1 - \frac{\bar{\psi}\mu^2}{12}\right\} - \nu\right] - \hat{r}\gamma\mu\frac{\pi}{A_n}$
$\frac{\pi}{A_n}\bar{\psi}\left[\frac{\sin\phi}{\phi} + \frac{\mu}{2}\cos\phi\right]$	$\frac{\pi^2}{A_nA_m}\left[\frac{3\gamma\bar{\psi}}{2\phi^2}\left(1 + \frac{\mu}{2}\right) - \frac{1+\nu}{2}\right]$	$\frac{2}{3}\frac{\pi^2}{A_n^2}\frac{1+\nu}{2} - \bar{\epsilon} - \frac{\gamma(1-\nu^2)}{\mu\lambda}$
0	$\frac{\pi^2}{A_m^2}\frac{1+\nu}{2} - \bar{\epsilon}^* - \frac{3}{2}\frac{\gamma(1-\nu^2)}{\lambda\mu}$	$\frac{2}{3}\frac{A_m}{A_n}\left\{\bar{\epsilon}^* + \frac{3}{2}\frac{\gamma(1-\nu^2)}{\lambda\mu}\right\} + \frac{A_n}{A_m}\left\{\bar{\epsilon} + \frac{\gamma(1-\nu^2)}{\lambda\mu}\right\} - \frac{5}{3}\frac{\pi^2}{A_nA_m}\frac{1+\nu}{2}$
$\cos\phi\left[\frac{\lambda\mu^3\bar{\psi}^2}{12(1-\nu^2)} + \frac{\lambda\mu\hat{r}^2}{(1-\nu^2)} - \lambda\mu\phi^2\right] - \phi\sin\phi$	$\frac{3\gamma}{2}\frac{\pi}{A_m}\left[\frac{\mu}{2} - \lambda\mu + \frac{1}{\phi^2}\frac{\lambda\mu}{(1-\nu^2)}\left\{\hat{r}^2 + \frac{\mu^2\bar{\psi}^2}{12}\right\}\right]$	0

TABLE 2

Curved Honeycomb-Sandwich Plate. Non-Dimensional Natural Frequencies for Flexural-Type Modes

(a) Values of ϕ for $\mu = 0{\cdot}05$

A_n	A_m	$\hat{r} = 0$	$\hat{r} = 0{\cdot}002$	$\hat{r} = 0{\cdot}005$	$\hat{r} = 0{\cdot}008$	$\hat{r} = 0{\cdot}020$
96	96	0·001877	0·002082	0·002931	0·004054	0·008983
70	96	0·002672	0·002898	0·003874	0·005219	0·011428
50	96	0·004246	0·004447	0·005379	0·006778	0·013838
25	96	0·012612	0·012707	0·013193	0·014051	0·019965
17	96	0·022769	0·022825	0·023117	0·023650	0·027820
96	70	0·002682	0·002770	0·003188	0·003841	0·007213
70	70	0·003462	0·003577	0·004126	0·004985	0·009515
50	70	0·005007	0·005134	0·005754	0·006754	0·012344
25	70	0·013243	0·013324	0·013744	0·014492	0·019794
17	70	0·023282	0·023334	0·023606	0·024103	0·028022
96	50	0·004295	0·004327	0·004488	0·004770	0·006636
70	50	0·005045	0·005090	0·005315	0·005708	0·008296
50	50	0·006535	0·006595	0·006904	0·007443	0·011007
25	50	0·014515	0·014576	0·014895	0·015469	0·019731
17	50	0·024322	0·024367	0·024603	0·025036	0·028495
96	25	0·013225	0·013230	0·013251	0·013291	0·013622
70	25	0·013835	0·013840	0·013864	0·013910	0·014286
50	25	0·015054	0·015061	0·015095	0·015158	0·015686
25	25	0·021766	0·021783	0·021872	0·022037	0·023403
17	25	0·030365	0·030386	0·030499	0·030708	0·032452
96	17	0·024725	0·024726	0·024731	0·024741	0·024822
70	17	0·025199	0·025200	0·025206	0·025216	0·025300
50	17	0·026154	0·026155	0·026162	0·026175	0·026280
25	17	0·031562	0·031566	0·031591	0·031636	0·032021
17	17	0·038807	0·038816	0·038863	0·038950	0·039690

TABLE 2—*continued*

(b) Values of ϕ for $\mu = 0\cdot10$

A_n	A_m	$\hat{r} = 0$	$\hat{r} = 0\cdot002$	$\hat{r} = 0\cdot005$	$\hat{r} = 0\cdot008$	$\hat{r} = 0\cdot020$
96	96	0·002055	0·002260	0·003121	0·004279	0·009496
70	96	0·002894	0·003132	0·004165	0·005596	0·012266
50	96	0·004511	0·004731	0·005751	0·007275	0·014947
25	96	0·012363	0·012477	0·013062	0·014083	0·020908
17	96	0·020955	0·021028	0·021404	0·022086	0·027262
96	70	0·002915	0·002995	0·003382	0·004000	0·007345
70	70	0·003725	0·003841	0·004400	0·005281	0·010019
50	70	0·005289	0·005426	0·006095	0·007173	0·013194
25	70	0·012944	0·013042	0·013548	0·014439	0·020575
17	70	0·021400	0·021468	0·021818	0·022454	0·027327
96	50	0·004606	0·004630	0·004754	0·004976	0·006528
70	50	0·005361	0·005402	0·005611	0·005979	0·008472
50	50	0·006826	0·006889	0·007212	0·007774	0·011514
25	50	0·014108	0·014182	0·014564	0·015249	0·020205
17	50	0·022298	0·022357	0·022661	0·023215	0·027532
96	25	0·013321	0·013323	0·013332	0·013349	0·013491
70	25	0·013858	0·013861	0·013874	0·013900	0·014110
50	25	0·014920	0·014925	0·014952	0·015003	0·015426
25	25	0·020554	0·020574	0·020679	0·020872	0·022465
17	25	0·027445	0·027473	0·027618	0·027885	0·030089
96	17	0·023502	0·023502	0·023502	0·023503	0·023508
70	17	0·023879	0·023880	0·023881	0·023882	0·023895
50	17	0·024637	0·024637	0·024641	0·024646	0·024693
25	17	0·028873	0·028878	0·028906	0·028958	0·029399
17	17	0·034459	0·034471	0·034531	0·034643	0·035591

TABLE 2—*continued*

(c) Values of ϕ for $\mu = 0 \cdot 15$

A_n	A_m	$\hat{r} = 0$	$\hat{r} = 0 \cdot 002$	$\hat{r} = 0 \cdot 005$	$\hat{r} = 0 \cdot 008$	$\hat{r} = 0 \cdot 020$
96	96	0·002130	0·002335	0·003202	0·004375	0·009710
70	96	0·002972	0·003217	0·004275	0·005743	0·012604
50	96	0·004557	0·004788	0·005855	0·007439	0·015377
25	96	0·011797	0·011925	0·012575	0·013701	0·021036
17	96	0·019311	0·019395	0·019830	0·020613	0·026414
96	70	0·003003	0·003081	0·003458	0·004064	0·007406
70	70	0·003805	0·003922	0·004489	0·005385	0·010220
50	70	0·005319	0·005462	0·006161	0·007282	0·013509
25	70	0·012332	0·012442	0·013004	0·013989	0·020605
17	70	0·019710	0·019788	0·020193	0·020924	0·026395
96	50	0·004689	0·004711	0·004824	0·005026	0·006469
70	50	0·005417	0·005458	0·005664	0·006027	0·008514
50	50	0·006806	0·006872	0·007207	0·007790	0·011650
25	50	0·013397	0·013480	0·013906	0·014665	0·020045
17	50	0·020514	0·020582	0·020934	0·021571	0·026435
96	25	0·012937	0·012938	0·012944	0·012955	0·013046
70	25	0·013409	0·013411	0·013423	0·013443	0·013616
50	25	0·014340	0·014345	0·014372	0·014421	0·014839
25	25	0·019207	0·019230	0·019348	0·019564	0·021339
17	25	0·025089	0·025121	0·025289	0·025599	0·028123
96	17	0·022028	0·022027	0·022027	0·022027	0·022020
70	17	0·022345	0·022345	0·022345	0·022346	0·022350
50	17	0·022982	0·022982	0·022985	0·022991	0·023036
25	17	0·026543	0·026550	0·026582	0·026641	0·027149
17	17	0·031254	0·031267	0·031338	0·031469	0·032576

TABLE 2—*continued*

(d) Values of ϕ for $\mu = 0 \cdot 20$

A_n	A_m	$\hat{r} = 0$	$\hat{r} = 0 \cdot 002$	$\hat{r} = 0 \cdot 005$	$\hat{r} = 0 \cdot 008$	$\hat{r} = 0 \cdot 020$
96	96	0·002175	0·002380	0·003249	0·004429	0·009829
70	96	0·003010	0·003258	0·004332	0·005821	0·012788
50	96	0·004550	0·004789	0·005885	0·007508	0·015600
25	96	0·011288	0·011427	0·012126	0·013329	0·021007
17	96	0·018085	0·018177	0·018657	0·019515	0·025753
96	70	0·003049	0·003126	0·003498	0·004099	0·007443
70	70	0·003833	0·003952	0·004526	0·005432	0·010328
50	70	0·005291	0·005439	0·006159	0·007310	0·013664
25	70	0·011787	0·011906	0·012512	0·013566	0·020512
17	70	0·018453	0·018539	0·018986	0·019788	0·025680
96	50	0·004712	0·004733	0·004841	0·005034	0·006431
70	50	0·005410	0·005450	0·005657	0·006022	0·008521
50	50	0·006724	0·006792	0·007137	0·007738	0·011689
25	50	0·012779	0·012869	0·013329	0·014144	0·019822
17	50	0·019195	0·019270	0·019658	0·020358	0·025612
96	25	0·012529	0·012530	0·012535	0·012544	0·012620
70	25	0·012954	0·012956	0·012967	0·012986	0·013152
50	25	0·013789	0·013795	0·013822	0·013874	0·014306
25	25	0·018141	0·018165	0·018293	0·018529	0·020448
17	25	0·023400	0·023435	0·023621	0·023963	0·026726
96	17	0·020835	0·020835	0·020835	0·020834	0·020829
70	17	0·021115	0·021115	0·021115	0·021116	0·021123
50	17	0·021677	0·021677	0·021681	0·021687	0·021741
25	17	0·024831	0·024838	0·024873	0·024940	0·025505
17	17	0·029033	0·029048	0·029127	0·029273	0·030503

TABLE 3

Curved Honeycomb-Sandwich Plate. Non-Dimensional Natural Frequencies for Bubbling-Type Modes

(*N.B.* This natural frequency is almost independent of wavelength and curvature.)

μ	ϕ
0·05	0·640–0·641
0·10	0·470
0·15	0·389
0·20	0·339

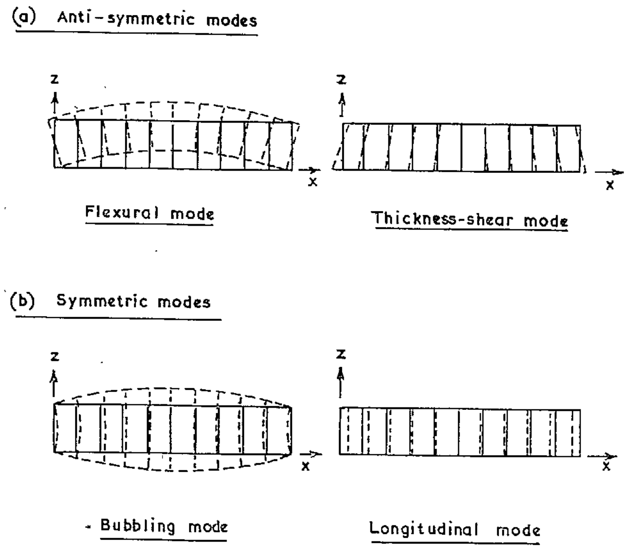


FIG. 1. First-order mode shapes of a simply-supported sandwich plate.

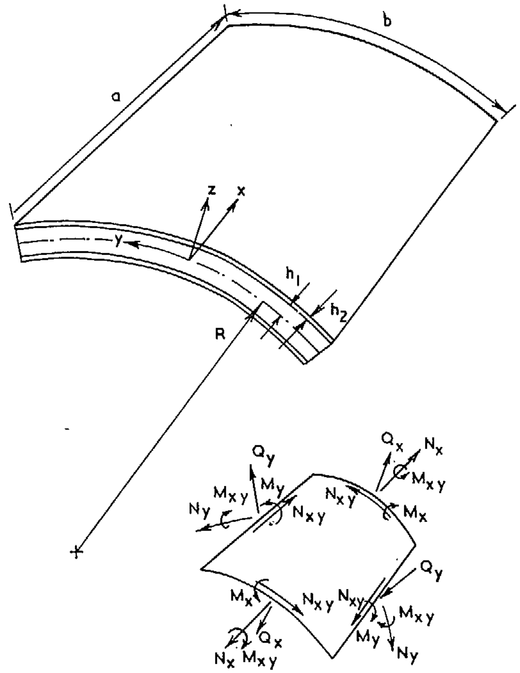


FIG. 2. Sketch showing plate and skin conventions.

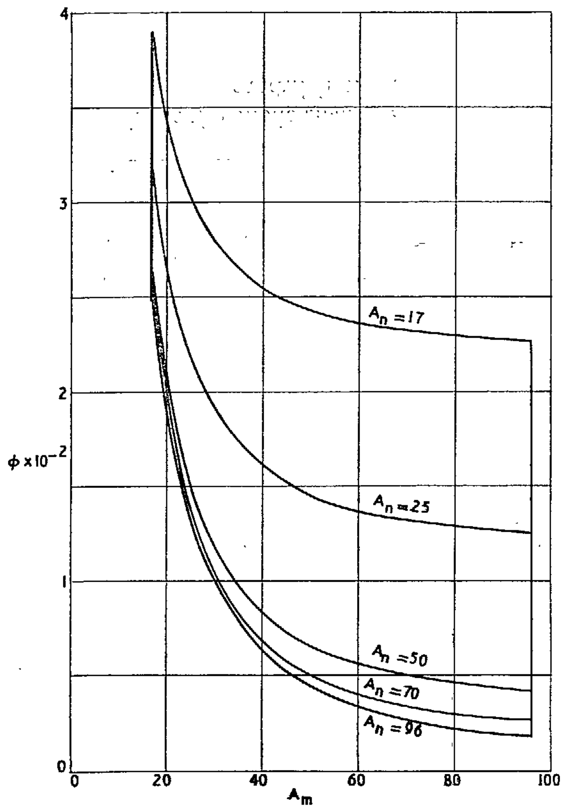


FIG. 3a. Flat-plate flexural-mode frequencies (non-dimensional): $\mu = 0 \cdot 05$.

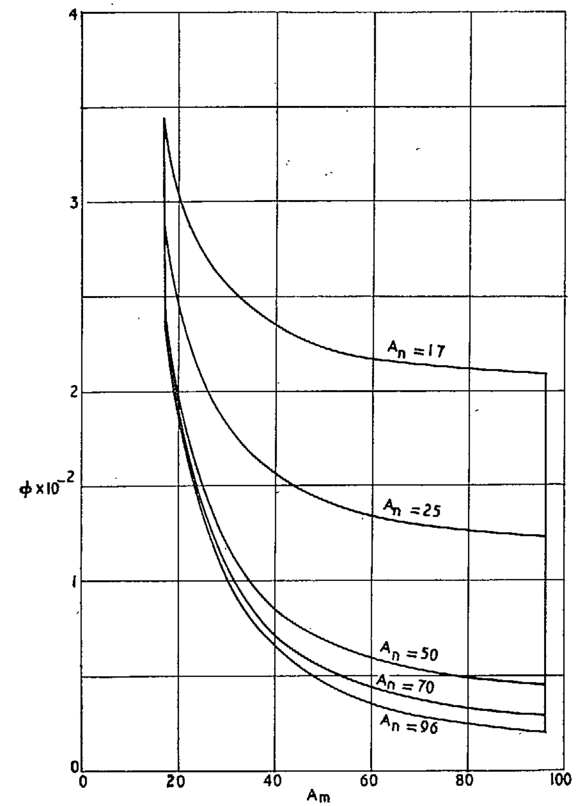


FIG. 3b. Flat-plate flexural-mode frequencies (non-dimensional): $\mu = 0 \cdot 10$.

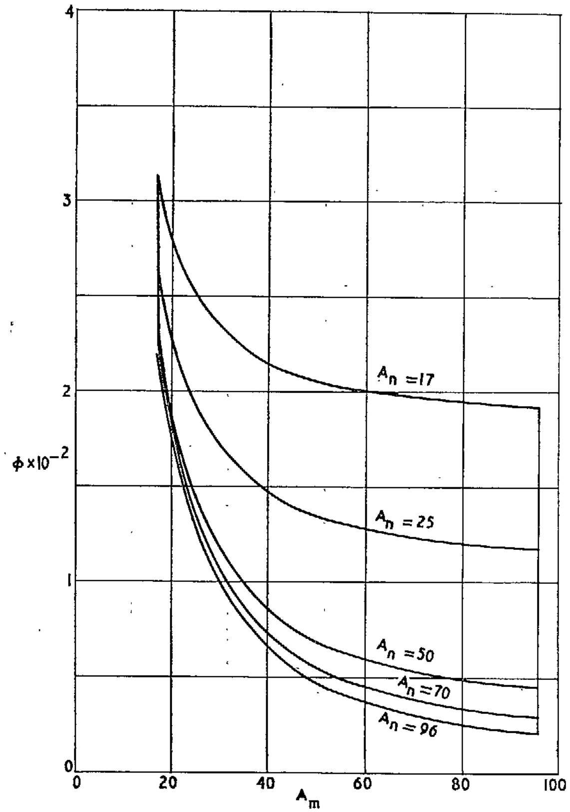


FIG. 3c. Flat-plate flexural-mode frequencies (non-dimensional): $\mu = 0 \cdot 15$.

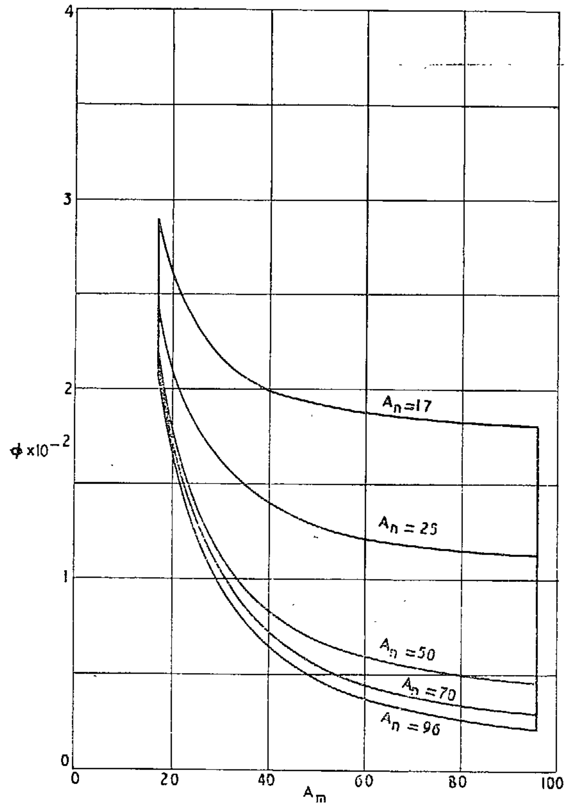


FIG. 3d. Flat-plate flexural-mode frequencies (non-dimensional): $\mu = 0 \cdot 20$.

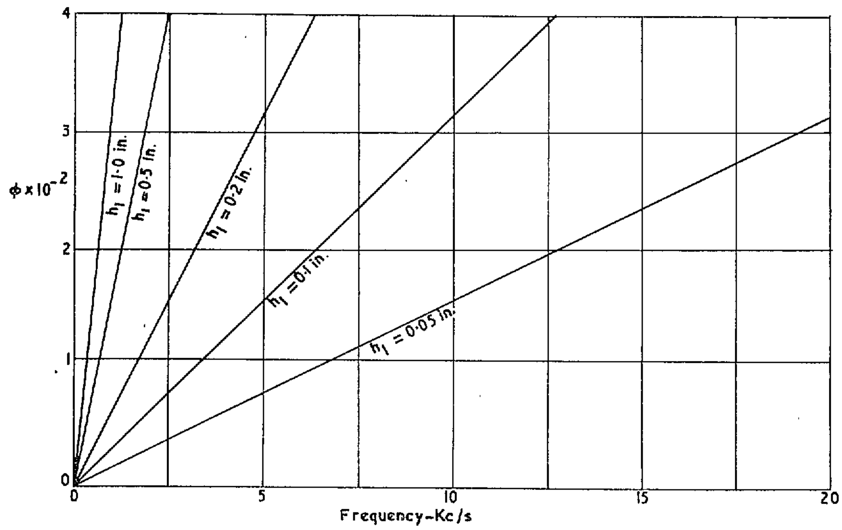


FIG. 4. Non-dimensional frequency vs. frequency in Kc/s.

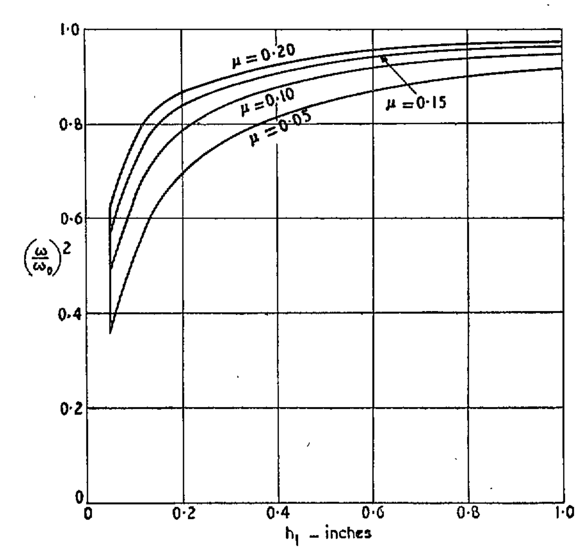


FIG. 5. Flat-plate flexural-mode. Mass ratio frequency correction.

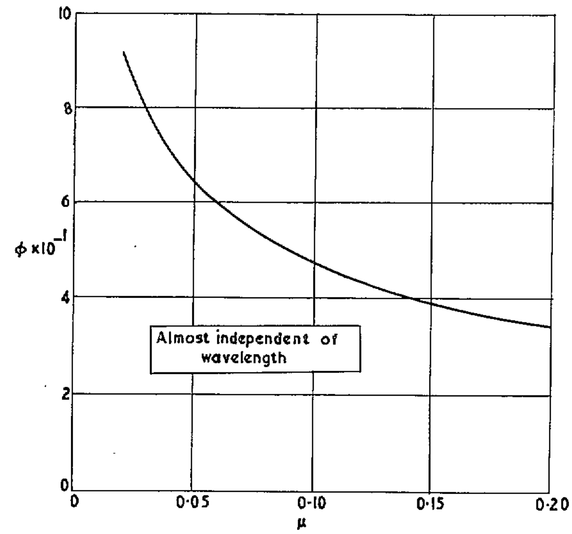


FIG. 6. Flat-plate bubbling-mode frequencies (non-dimensional).

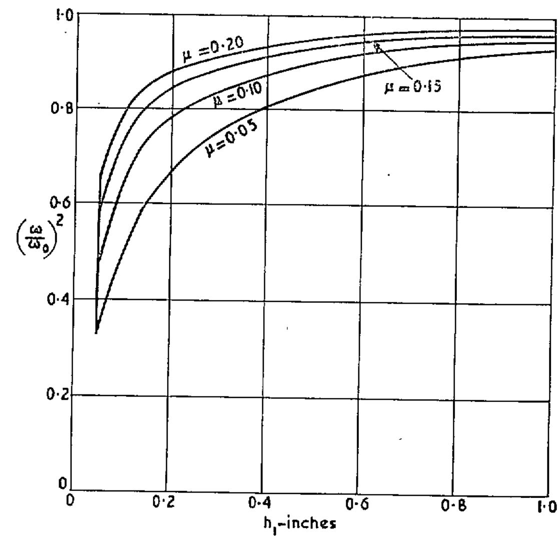


FIG. 7. Flat-plate bubbling-mode. Mass-ratio frequency correction.

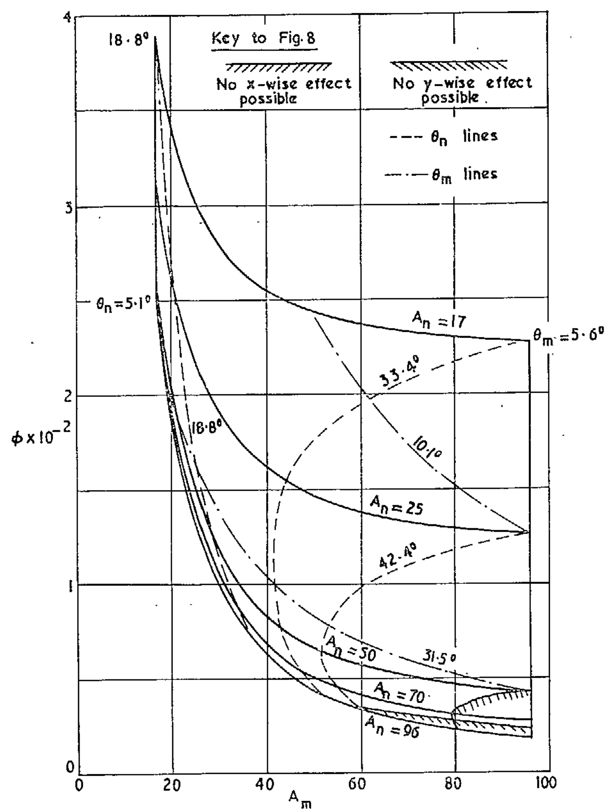


FIG. 8a. Flat-plate flexural-mode frequencies with incidence angles for the coincidence effect: $\mu = 0{\cdot}05$.

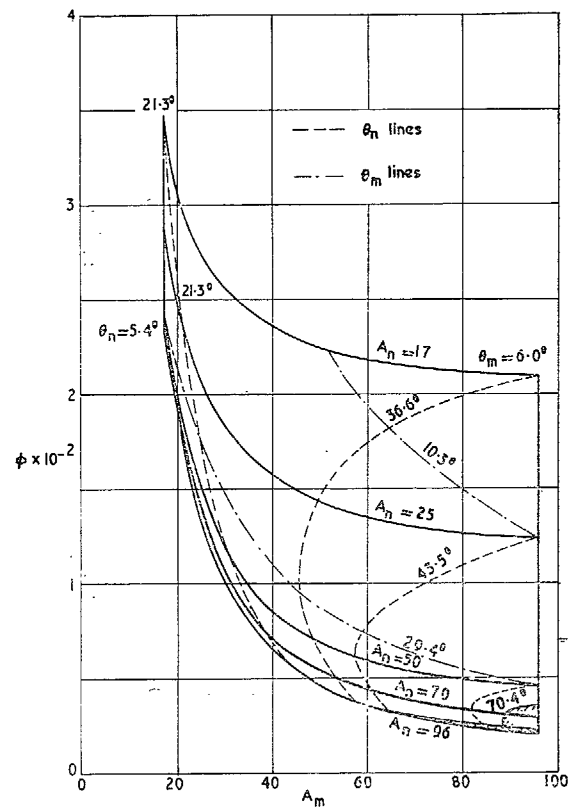


FIG. 8b. Flat-plate flexural-mode frequencies with incidence angles for the coincidence effect: $\mu = 0{\cdot}10$.

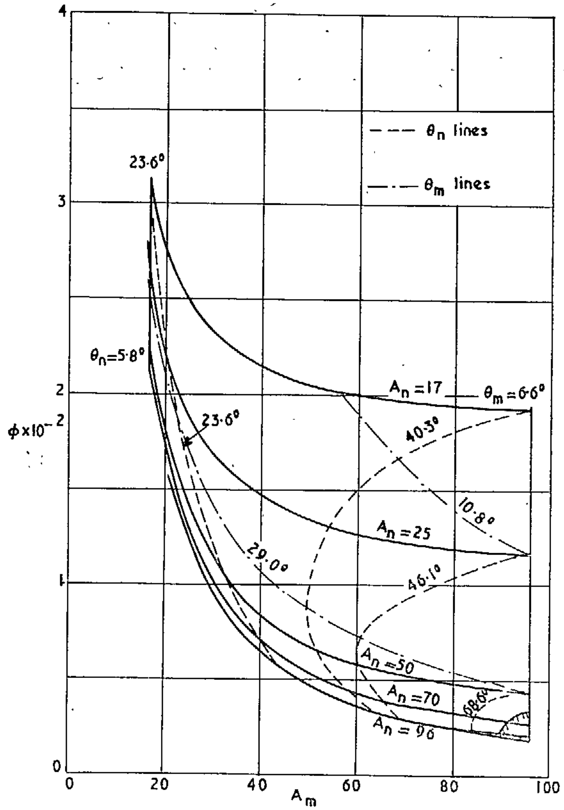


FIG. 8c. Flat-plate flexural-mode frequencies with incidence angle for the coincidence effect: $\mu = 0{\cdot}15$.

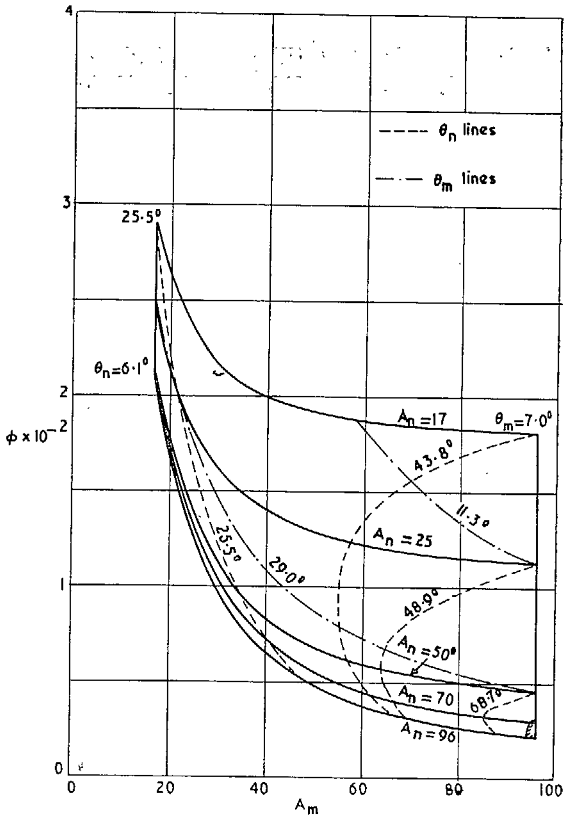


FIG. 8d. Flat-plate flexural-mode frequencies with incidence angle for the coincidence effect: $\mu = 0{\cdot}20$.

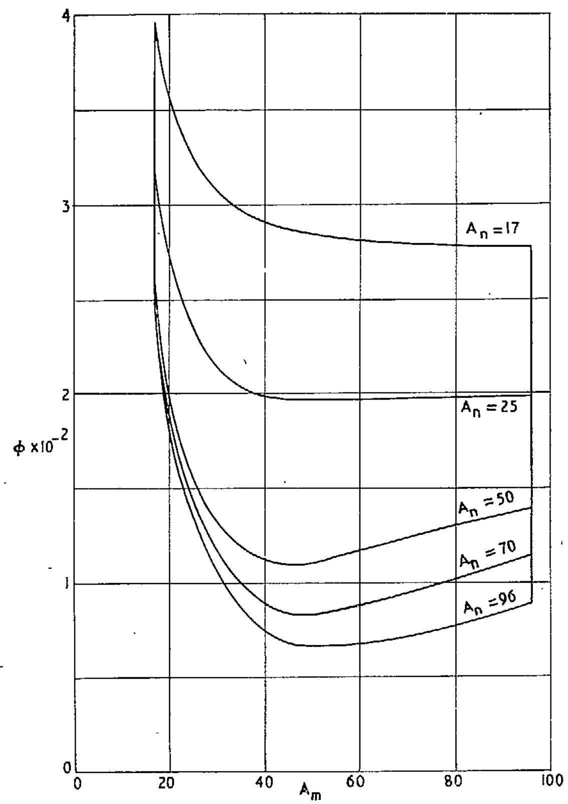


FIG. 9. Curved-plate flexural-mode frequencies vs. circumferential wavelength: $\mu = 0 \cdot 05$, $\hat{r} = 0 \cdot 020$.

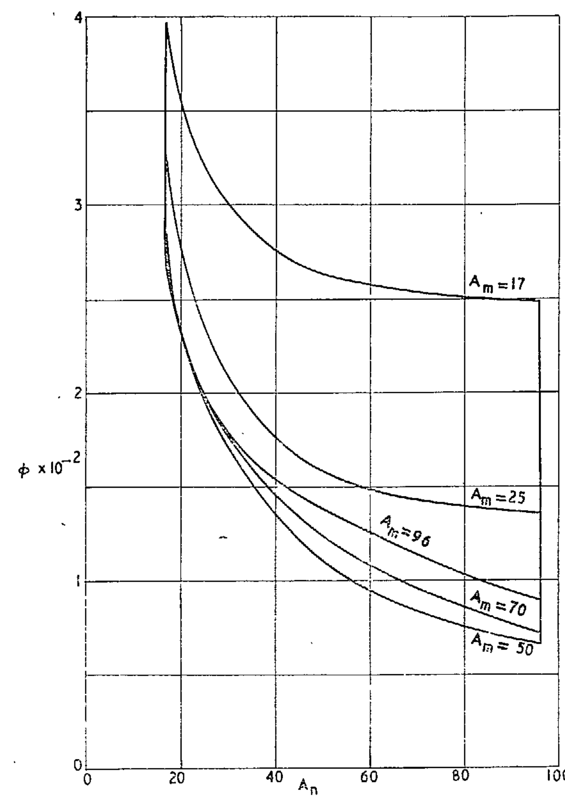


FIG. 10. Curved-plate flexural-mode frequencies vs. axial wavelength: $\mu = 0 \cdot 05$, $\hat{r} = 0 \cdot 020$.

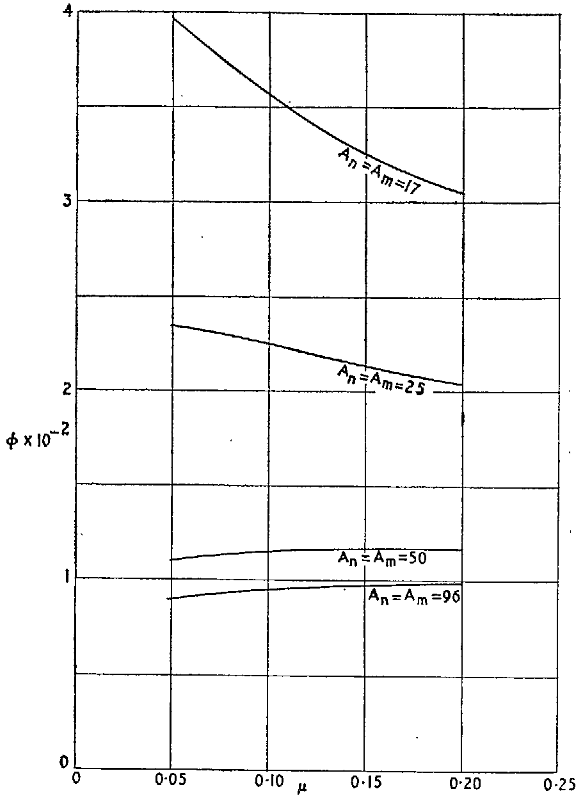


FIG. 11. Curved-plate flexural-mode frequencies vs. μ; $\hat{r} = 0 \cdot 020$.

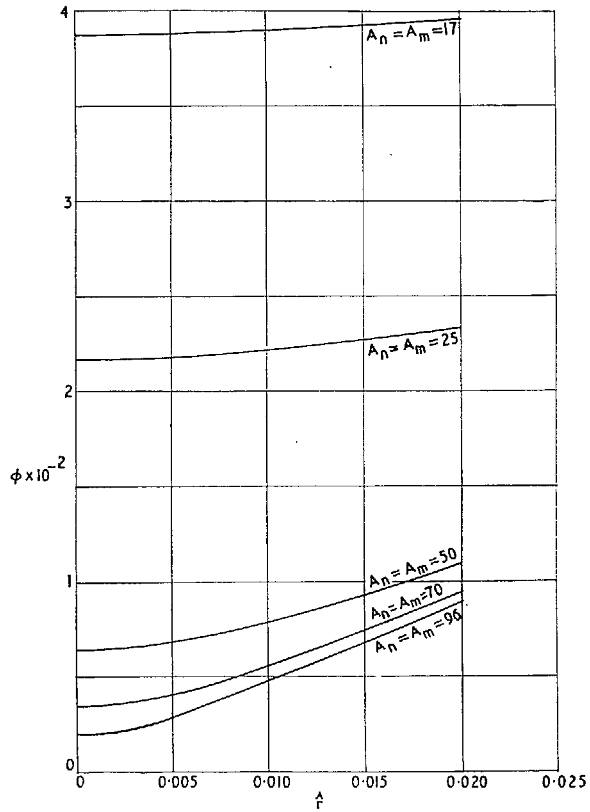


FIG. 12. Curved-plate flexural-mode frequencies vs. curvature ($\hat{r}$): $\mu = 0 \cdot 05$.

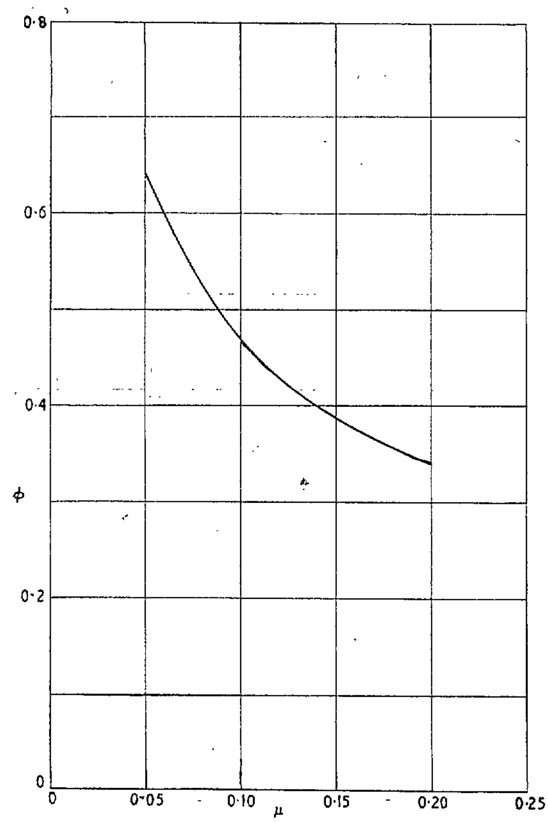


FIG. 13. Curved-plate bubbling-mode frequencies vs. μ (almost independent of wavelength and $\hat{r}$).

(89020) Wt. 65/1418 K.5 4/64 H .

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)

1942 Vol. I. Aero and Hydrodynamics, Aerofoils, Airscrews, Engines. 75*s.* (post 2*s.* 9*d.*)
Vol. II. Noise, Parachutes, Stability and Control, Structures, Vibration, Wind Tunnels. 47*s.* 6*d.* (post 2*s.* 3*d.*)

1943 Vol. I. Aerodynamics, Aerofoils, Airscrews. 80*s.* (post 2*s.* 6*d.*)
Vol. II. Engines, Flutter, Materials, Parachutes, Performance, Stability and Control, Structures. 90*s.* (post 2*s.* 9*d.*)

1944 Vol. I. Aero and Hydrodynamics, Aerofoils, Aircraft, Airscrews, Controls. 84*s.* (post 3*s.*)
Vol. II. Flutter and Vibration, Materials, Miscellaneous, Navigation, Parachutes, Performance, Plates and Panels, Stability, Structures, Test Equipment, Wind Tunnels. 84*s.* (post 3*s.*)

1945 Vol. I. Aero and Hydrodynamics, Aerofoils. 130*s.* (post 3*s.* 6*d.*)
Vol. II. Aircraft, Airscrews, Controls. 130*s.* (post 3*s.* 6*d.*)
Vol. III. Flutter and Vibration, Instruments, Miscellaneous, Parachutes, Plates and Panels, Propulsion. 130*s.* (post 3*s.* 3*d.*)
Vol. IV. Stability, Structures, Wind Tunnels, Wind Tunnel Technique. 130*s.* (post 3*s.* 3*d.*)

1946 Vol. I. Accidents, Aerodynamics, Aerofoils and Hydrofoils. 168*s.* (post 3*s.* 9*d.*)
Vol. II. Airscrews, Cabin Cooling, Chemical Hazards, Controls, Flames, Flutter, Helicopters, Instruments and Instrumentation, Interference, Jets, Miscellaneous, Parachutes. 168*s.* (post 3*s.* 3*d.*)
Vol. III. Performance, Propulsion, Seaplanes, Stability, Structures, Wind Tunnels. 168*s.* (post 3*s.* 6*d.*)

1947 Vol. I. Aerodynamics, Aerofoils, Aircraft. 168*s.* (post 3*s.* 9*d.*)
Vol. II. Airscrews and Rotors, Controls, Flutter, Materials, Miscellaneous, Parachutes, Propulsion, Seaplanes, Stability, Structures, Take-off and Landing. 168*s.* (post 3*s.* 9*d.*)

1948 Vol. I. Aerodynamics, Aerofoils, Aircraft, Airscrews, Controls, Flutter and Vibration, Helicopters, Instruments, Propulsion, Seaplane, Stability, Structures, Wind Tunnels. 130*s.* (post 3*s.* 3*d.*)
Vol. II. Aerodynamics, Aerofoils, Aircraft, Airscrews, Controls, Flutter and Vibration, Helicopters, Instruments, Propulsion, Seaplane, Stability, Structures, Wind Tunnels. 110*s.* (post 3*s.* 3*d.*)

Special Volumes

Vol. I. Aero and Hydrodynamics, Aerofoils, Controls, Flutter, Kites, Parachutes, Performance, Propulsion, Stability. 126*s.* (post 3*s.*)
Vol. II. Aero and Hydrodynamics, Aerofoils, Airscrews, Controls, Flutter, Materials, Miscellaneous, Parachutes, Propulsion, Stability, Structures. 147*s.* (post 3*s.*)
Vol. III. Aero and Hydrodynamics, Aerofoils, Airscrews, Controls, Flutter, Kites, Miscellaneous, Parachutes, Propulsion, Seaplanes, Stability, Structures, Test Equipment. 189*s.* (post 3*s.* 9*d.*)

Reviews of the Aeronautical Research Council

1939–48 3*s.* (post 6*d.*) 1949–54 5*s.* (post 5*d.*)

Index to all Reports and Memoranda published in the Annual Technical Reports

1909–1947 R. & M. 2600 (out of print)

Indexes to the Reports and Memoranda of the Aeronautical Research Council

Between Nos. 2351–2449	R. & M. No. 2450	2*s.* (post 3*d.*)
Between Nos. 2451–2549	R. & M. No. 2550	2*s.* 6*d.* (post 3*d.*)
Between Nos. 2551–2649	R. & M. No. 2650	2*s.* 6*d.* (post 3*d.*)
Between Nos. 2651–2749	R. & M. No. 2750	2*s.* 6*d.* (post 3*d.*)
Between Nos. 2751–2849	R. & M. No. 2850	2*s.* 6*d.* (post 3*d.*)
Between Nos. 2851–2949	R. & M. No. 2950	3*s.* (post 3*d.*)
Between Nos. 2951–3049	R. & M. No. 3050	3*s.* 6*d.* (post 3*d.*)
Between Nos. 3051–3149	R. & M. No. 3150	3*s.* 6*d.* (post 3*d.*)

HER MAJESTY'S STATIONERY OFFICE

from the addresses overleaf

Printed and published by
HER MAJESTY'S STATIONERY OFFICE

To be purchased from
York House, Kingsway, London W.C.2
423 Oxford Street, London W.1
13A Castle Street, Edinburgh 2
109 St. Mary Street, Cardiff
39 King Street, Manchester 2
50 Fairfax Street, Bristol 1
35 Smallbrook, Ringway, Birmingham 5
80 Chichester Street, Belfast 1
or through any bookseller

Printed in England

R. & M. No. 3363

S.O. Code No. 23-3363