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On the Numerical Evaluation of the Drag Integral

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On the Numerical Evaluation of the Drag Integral

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Summary.

By means of a Fourier analysis a function is derived which gives the minimum value of the integral

$$I = -\frac{1}{2\pi}\int_0^1\int_0^1 S''(x)S''(y)\log|x-y|\,dx\,dy$$

for any function $S(x)$ which has a continuous first derivative and given values at a number of discrete points. This minimal function suggests a method for the numerical evaluation of I from a graph or table of $S(x)$. An example is constructed to illustrate the method and to give some indication of its accuracy.

LIST OF CONTENTS

* Replaces A.R.C. 24,063.

LIST OF TABLES

LIST OF ILLUSTRATIONS

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1. *Introduction.*

Integrals of the form

$$I = -\frac{1}{2\pi}\int_0^1\int_0^1 S''(x)S''(y)\log|x-y|\,dx\,dy \tag{1}$$

{in which $S''(x)$ denotes $d^2S(x)/dx^2$} occur in several contexts in theoretical aerodynamics and many attempts have been made at their numerical evaluation. Two difficulties underlie the problem confounding any direct and simple approach: the loss of accuracy attendant on any kind of numerical differentiation, and the troubles inherent in the singular nature of the integrand.

In this paper we consider a limited class of functions $S(x)$, namely those which satisfy the conditions:

$$S'(x) \text{ continuous for } 0 \leqslant x \leqslant 1$$
$$S'(0) = S'(1) = 0. \tag{2}$$

We suppose $S(x)$ to be known at $x = 0, 1$ and at n other stations and evaluate I for that particular $S(x)$ which makes it a minimum under these conditions. The calculation is simple and the value of I that results is then the exact value of I for some $S(x)$ which approximates to the given function.

The restrictions we impose on $S(x)$ are those which must be satisfied by the area distribution of a slender body in order that the integral I shall represent its wave drag at zero lift. It was to calculate this that the method was originally devised. In Section 6 we shall collect together other expressions in which integrals of this form occur, indicating to which the method is immediately applicable and which violate the assumptions on which the method is based.

2. *Fourier Analysis of the Function $S(x)$ and the Integral I.*

Since $S'(x)$ is a continuous function of x for $0 \leqslant x \leqslant 1$ with $S'(0) = S'(1) = 0$, it may be expressed by the transformation

$$x = \tfrac{1}{2}(1-\cos\theta) \tag{3}$$

as a function of θ for $0 \leqslant \theta \leqslant \pi$ which has a convergent Fourier sine series. So we may write

$$S'(x) = \sum_{r=1}^{\infty} a_r \sin r\theta, \quad 0 \leqslant x \leqslant 1, \tag{4}$$

where

$$a_r = \frac{2}{\pi}\int_0^{\pi} S'(x)\sin r\theta\, d\theta, \quad r = 1, 2 \ldots . \tag{5}$$

A single integration gives $S(x)$ as a function of θ and of the Fourier coefficients a_r:

$$\begin{aligned} S(x) &= \sum_{r=1}^{\infty} a_r \int \sin r\theta \, dx \\ &= \tfrac{1}{2} \sum_{r=1}^{\infty} a_r \int \sin r\theta \sin\theta \, d\theta \\ &= a + \tfrac{1}{4} a_1(\theta - \tfrac{1}{2}\sin 2\theta) + \tfrac{1}{4} \sum_{r=2}^{\infty} a_r \left[\frac{\sin(r-1)\theta}{r-1} - \frac{\sin(r+1)\theta}{r+1}\right] \\ &= a + \tfrac{1}{4} a_1\theta + \tfrac{1}{4} \sum_{r=1}^{\infty} \frac{1}{r}(a_{r+1} - a_{r-1}) \sin r\theta \qquad (6) \end{aligned}$$

where a_0 is defined to be zero and the constant of integration $a = S(0)$. (7)

A double integration gives the integral I as a function of the Fourier coefficients a_r. Integrations of this form have often been published and the result is now well known*. In the notation of this paper, with the transformation $x = \frac{1}{2}(1 - \cos\theta)$, $y = \frac{1}{2}(1 - \cos\phi)$, the main steps in the argument are:

$$\begin{aligned} I &= -\frac{1}{2\pi} \int_0^1 \int_0^1 S''(x) S''(y) \log|x - y| \, dx\, dy \\ &= \frac{1}{2\pi} \int_0^1 S'(x) \; {}_P\!\!\int_0^1 \frac{S''(y)}{x - y} \, dy\, dx \qquad (8) \end{aligned}$$

where ${}_P\!\int$ denotes the Cauchy principal value of the integral; but

$$\begin{aligned} {}_P\!\!\int \frac{S''(y)}{x-y} \, dy &= 2 \sum_{r=1}^{\infty} r a_r \; {}_P\!\!\int_0^{\pi} \frac{\cos r\phi \, d\phi}{\cos\phi - \cos\theta} \\ &= 2\pi \sum_{r=1}^{\infty} r a_r \frac{\sin r\theta}{\sin\theta} \qquad (9) \end{aligned}$$

so that

$$\begin{aligned} I &= \tfrac{1}{2} \int_0^{\pi} \sum_{r=1}^{\infty} r a_r \sin r\theta \sum_{s=1}^{\infty} a_s \sin s\theta \, d\theta \\ &= \tfrac{1}{2} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} r a_r a_s \int_0^{\pi} \sin r\theta \sin s\theta \, d\theta \\ &= \frac{\pi}{4} \sum_{r=1}^{\infty} r a_r^2 . \end{aligned}$$

We therefore have the result: if $S'(x)$ is a continuous function of x for $0 \leqslant x \leqslant 1$ with $S'(0) = S'(1) = 0$,

and

$$x = \tfrac{1}{2}(1 - \cos\theta) \qquad (3)$$

then

$$S'(x) = \sum_{r=1}^{\infty} a_r \sin r\theta \qquad (4)$$

$$S(x) = a + \tfrac{1}{4} a_1\theta + \tfrac{1}{4} \sum_{r=1}^{\infty} \frac{1}{r}(a_{r+1} - a_{r-1}) \sin r\theta \qquad (6)$$

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* *See*, for example, Glauert[1] chapter XI.

and

$$I = \frac{\pi}{4}\sum_{r=1}^{\infty} r a_r^2 \tag{10}$$

where

$$a = S(0) \tag{7}$$

$$a_r = \frac{2}{\pi}\int_0^{\pi} S'(x)\sin r\theta \, d\theta\,, \quad r = 0, 1 \ldots . \tag{5}$$

3. *Minimal Functions for the Integral I.*

The Fourier analysis of Section 2 provides a method of deriving functions which minimise the integral I under certain specified conditions. We are confined to smooth functions because of the nature of the analysis of Section 2 but within this limitation we shall consider all functions which satisfy the conditions we impose.

Since both the function $S(x)$ and the integral I can be expressed as functions of the coefficients of the Fourier sine series of $S'(x)$ we simply require the values of the coefficients which make I a minimum subject to the conditions on $S(x)$ which relate them. If these conditions can be written

$$\xi_i\{S(x)\} = 0\,, \quad i = 1, 2 \ldots m\,, \tag{11}$$

then a necessary condition for I to be a minimum subject to $\xi_i = 0$, $i = 1, 2, \ldots m$, is

$$\frac{\partial I}{\partial a_r} + \sum_{i=1}^{m} \lambda_i \frac{\partial \xi_i}{\partial a_r} = 0 \tag{12}$$

for each a_r and some constants λ_i, $i = 1, 2, \ldots m$. If the values of a_r which satisfy this condition yield convergent series for $S(x)$ and I, then these give the required minimal function and the corresponding minimum value of the integral for the specified conditions.

4. *The Minimal Function for which* $S(0) = N$, $S(1) = B$ *and* $S(k_i) = A_i$, $i = 1, 2, \ldots n$.

The function which minimises the integral I when $S(0) = N$, $S(1) = B$ and $S(k_i) = A_i$, $i = 1, 2, \ldots n$, may be deduced from the series for $S(x)$ and I given in Section 2 by writing them in the form

$$S(x) = a + \tfrac{1}{4}a_1(\theta - \sin\theta\cos\theta) + \sum_{r=2}^{\infty} \tfrac{1}{4}\left[\frac{\sin(r-1)\theta}{r-1} - \frac{\sin(r+1)\theta}{r+1}\right] a_r \tag{13}$$

$$I = \frac{\pi}{4}\left\{a_1^2 + \sum_{r=2}^{\infty} r a_r^2\right\} \tag{14}$$

where

$$a = N \tag{15}$$

$$a_r = \frac{2}{\pi}\int_0^{\pi} S'(x)\sin r\theta\, d\theta\,, \quad r = 1, 2 \ldots \tag{16}$$

and in particular

$$a_1 = \frac{2}{\pi}\int_0^{\pi} S'(x)\sin\theta\, d\theta = \frac{4}{\pi}\int_0^{1} S'(x)dx = \frac{4}{\pi}(B-N). \tag{17}$$

Since N and B are given, a and a_1 are uniquely defined. Hence to find the minimal function for given N and B which also satisfies the n conditions $S(k_i) = A_i$, $i = 1, 2, \ldots n$, we require the values of $a_2, a_3 \ldots$ which give a minimum of $\sum_{r=2}^{\infty} r a_r^2$ subject to

$$\sum_{r=2}^{\infty} \tfrac{1}{4}\left[\frac{\sin(r-1)\kappa_i}{r-1} - \frac{\sin(r+1)\kappa_i}{r+1}\right] a_r = A_i - a - \tfrac{1}{4}a_1(\kappa_i - \sin\kappa_i\cos\kappa_i) \tag{18}$$

for $i = 1, 2, \ldots n$, where $k_i = \frac{1}{2}(1-\cos\kappa_i)$. A necessary condition for this is

$$r a_r - \sum_{i=1}^{n} \lambda_i \left[\frac{\sin(r-1)\kappa_i}{r-1} - \frac{\sin(r+1)\kappa_i}{r+1}\right] = 0 \tag{19}$$

for each $r \geqslant 2$, and some constants λ_i, $i = 1, 2, \ldots n$.

The constants λ_i, $i = 1, 2, \ldots n$, are found by substituting the a_r from equation (19) into equation (18). This gives

$$\sum_{r=2}^{\infty} \tfrac{1}{4}\left[\frac{\sin(r-1)\kappa_j}{r-1} - \frac{\sin(r+1)\kappa_j}{r+1}\right] \frac{1}{r}\sum_{i=1}^{n} \lambda_i \left[\frac{\sin(r-1)\kappa_i}{r-1} - \frac{\sin(r+1)\kappa_i}{r+1}\right]$$

$$= A_j - a - \tfrac{1}{4}a_1(\kappa_j - \sin\kappa_j\cos\kappa_j) \tag{20}$$

for $j = 1, 2, \ldots n$. By changing the order of summation and using the result of the Appendix we may write this

$$\sum_{i=1}^{n} \lambda_i \sigma(\kappa_i, \kappa_j) = A_j - a - \tfrac{1}{4}a_1(\kappa_j - \sin\kappa_j\cos\kappa_j), \tag{21}$$

for $j = 1, 2, \ldots n$, where

$$\sigma(\kappa_i, \kappa_j) \equiv \sum_{r=2}^{\infty} \frac{1}{4r}\left[\frac{\sin(r-1)\kappa_i}{r-1} - \frac{\sin(r+1)\kappa_i}{r+1}\right]\left[\frac{\sin(r-1)\kappa_j}{r-1} - \frac{\sin(r+1)\kappa_j}{r+1}\right]$$

$$= -\tfrac{1}{8}(\cos\kappa_i - \cos\kappa_j)^2 \log\frac{1-\cos(\kappa_i+\kappa_j)}{1-\cos(\kappa_i-\kappa_j)} +$$

$$+ \tfrac{1}{4}\sin\kappa_i\sin\kappa_j(1-\cos\kappa_i\cos\kappa_j). \tag{22}$$

With these values of λ_i, $i = 1, 2, \ldots n$, the condition (19) now gives the coefficients

$$a_r = \frac{1}{r}\sum_{i=1}^{n} \lambda_i \left[\frac{\sin(r-1)\kappa_i}{r-1} - \frac{\sin(r+1)\kappa_i}{r+1}\right], \quad r \geqslant 3. \tag{23}$$

Substituting for a_r in equation (13) we find

$$S(x) = a + \tfrac{1}{4}a_1(\theta - \sin\theta\cos\theta) +$$

$$+ \sum_{r=2}^{\infty} \tfrac{1}{4}\left[\frac{\sin(r-1)\theta}{r-1} - \frac{\sin(r+1)\theta}{r+1}\right] \frac{1}{r}\sum_{i=1}^{n} \lambda_i \left[\frac{\sin(r-1)\kappa_i}{r-1} - \frac{\sin(r+1)\kappa_i}{r+1}\right]$$

$$= a + \tfrac{1}{4}a_1(\theta - \sin\theta\cos\theta) + \sum_{i=1}^{n} \lambda_i \sigma(\theta, \kappa_i). \tag{24}$$

Substituting for a_r in equation (14) we find

$$I = \frac{\pi}{4} a_1^2 + \frac{\pi}{4} \sum_{r=2}^{\infty} \frac{1}{r} \left\{ \sum_{i=1}^{n} \lambda_i \left[\frac{\sin (r-1)\kappa_i}{r-1} - \frac{\sin (r+1)\kappa_i}{r+1} \right] \right\}^2$$

$$= \frac{\pi}{4} a_1^2 + \frac{\pi}{4} \sum_{r=2}^{\infty} \frac{1}{r} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \left[\frac{\sin (r-1)\kappa_i}{r-1} - \frac{\sin (r+1)\kappa_i}{r+1} \right] \times$$

$$\times \left[\frac{\sin (r-1)\kappa_j}{r-1} - \frac{\sin (r+1)\kappa_j}{r+1} \right]$$

$$= \frac{\pi}{4} a_1^2 + \pi \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \sigma(\kappa_i, \kappa_j). \tag{25}$$

In terms of x, therefore, the minimal function for which $S(0) = N$, $S(1) = B$ and $S(k_i) = A_i$, $i = 1, 2, \ldots n$, is

$$S(x) = N + (B-N)u(x) + \sum_{i=1}^{n} \lambda_i p(x, k_i) \tag{26}$$

where

$$u(x) = \frac{1}{\pi} [\cos^{-1}(1-2x) - 2(1-2x)\sqrt{\{x(1-x)\}}] \tag{27}$$

$$p(x, y) = -\tfrac{1}{2}(x-y)^2 \log \frac{x+y-2xy+2\sqrt{\{xy(1-x)(1-y)\}}}{x+y-2xy-2\sqrt{\{xy(1-x)(1-y)\}}} +$$

$$+ 2(x+y-2xy)\sqrt{\{xy(1-x)(1-y)\}} \tag{28}$$

The corresponding value of the integral I is

$$I = \frac{4}{\pi}(B-N)^2 + \pi \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j p(k_i, k_j) \tag{29}$$

and the constants λ_i, $i = 1, 2, \ldots n$, are given by the n linear equations

$$\sum_{i=1}^{n} \lambda_i p(k_i, k_j) = (A_j - N) - (B-N)u(k_j), \quad j = 1, 2, \ldots n. \tag{30}$$

5. *A Method for the Numerical Evaluation of the Integral I.*

The result of Section 4 suggests a method for the numerical evaluation of the integral I from a graph or table of the function $S(x)$. As Sections 3 and 4 concern only smooth functions this method is similarly limited in its application.

If the value of $S(x)$ is known at $x = 0$, $x = 1$ and at n intermediate points, the minimal function through these points is uniquely defined and the corresponding value of the integral I can be calculated from the result of Section 4. If n is increased in such a way that each set of n points includes the previous set, the values of the integral for the corresponding minimals form a monotonic non-decreasing sequence bounded above by the value of the integral for the given function. If this is the least upper bound, then by taking n large enough it should be possible in this way to approximate to the integral as accurately as the data allow.

Suppose that we take n arbitrarily spaced points $x = k_i$, $i = 1, 2, \ldots n$. To evaluate I we have first to compute the functions $u_i \equiv u(k_i)$ and $p_{ij} \equiv p(k_i, k_j)$ for $i, j = 1, 2, \ldots n$. Then with

$$c_i = (A_i - N) - (B - N)u_i, \quad i = 1, 2, \ldots n \tag{31}$$

we must solve the n simultaneous equations

$$\sum_{i=1}^{n} \lambda_i p_{ij} = c_j, \quad j = 1, 2, \ldots n. \tag{32}$$

Now with these values of λ_i

$$\begin{aligned} I &= \frac{4}{\pi}(B-N)^2 + \pi \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j p_{ij} \\ &= \frac{4}{\pi}(B-N)^2 + \pi \sum_{i=1}^{n} \lambda_i c_i. \end{aligned} \tag{33}$$

If we anticipate doing a large number of these sums (appreciably more than n) we can save time by taking the same n points in each case and inverting the matrix $[p_{ij}]$ once and for all. Then if the elements of its inverse are f_{ij}, $i, j = 1, 2, \ldots n$,

$$I = \frac{4}{\pi}(B-N)^2 + \pi \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j f_{ij} \tag{34}$$

an expression which takes little time to evaluate once the f_{ij} have been found*.

To test the method a polynomial was chosen to represent $S(x)$,

$$S(x) = 400x^6 - 1176x^5 + 1257x^4 - 588x^3 + 108x^2 \tag{35}$$

and the n points were equally spaced along the axis at $k_i = i/(n+1)$, $i = 1, 2, \ldots n$. This rather odd polynomial was originally chosen for its resemblance to the area distribution of a swept wing on a cylindrical body. The shape of the function is shown in Fig. 1, and the values of the integral I for the function itself and for the minimals which approximate to it in Fig. 2.

During the computation of these results, using the Ferranti Mercury digital computer, one practical limitation of the method became apparent. For small values of n accurate enough results were obtained working with eight significant figures but as n increased the equations (32) became rapidly more ill conditioned until at $n = 19$ the programme failed. A second programme using double precision for the solution of the equations extended the useful range of the method to $n = 35$.

Despite this difficulty the accuracy of the method seems adequate for most practical purposes. In this example an error of not more than 2% occurs with 17 points and the simplest programme. The second programme gives an error of less than 1% with 25 points and ½% with 35 points.

To facilitate an immediate application of the method the functions u_i and f_{ij} have been tabulated in this paper for the 19 equally spaced points $k_i = i/20$, $i = 1, 2, \ldots 19$. Rounding off the elements

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* These results are perhaps more obvious if written entirely in terms of matrices. Equation (32) becomes

$$p\lambda = c, \text{ whence } \lambda = p^{-1}c,$$

where λ and c are column vectors and p a symmetric matrix. The series in I becomes

$$\begin{aligned} \lambda' p \lambda &= \lambda' c, \text{ as in (33)}, \\ &= c' p^{-1} c, \text{ as in (34)}. \end{aligned}$$

of f_{ij} to three decimal places gives, for this particular example, an arithmetical error of less than $0 \cdot 1\%$ in I. This is negligible compared with the 2% difference between the 19 point approximation and the true value of I.

6. *Applications in the Evaluation of Drag Forces.*

Consider any kind of body in a steady uniform stream. To determine the drag forces acting on the body it is convenient to imagine it completely surrounded by a large control surface S and to equate the forces acting on S to the rate of change of fluid momentum through S. If S is a circular cylinder with its axis in the stream direction the drag D is made up of three surface integrals—one over the curved surface of the cylinder and one over each plane end. As the cylinder grows both in length and diameter the integral over the upstream end tends to zero, the integral over the downstream end tends to a limit known as the vortex drag, and the integral over the curved surface tends to a limit known as the wave drag.

If the nature of the stream and the geometry of the body satisfy the assumptions of small-perturbation theory there exists a perturbation velocity potential ϕ which satisfies the linear equation

$$(1-M^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{36}$$

where M is the Mach number of the undisturbed stream. The two components of the drag are given by

$$D_w = \lim_{r \to \infty} -\rho \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} \phi_x \phi_r \, dx \tag{37}$$

$$D_v = \tfrac{1}{2}\rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\phi_y{}^2 + \phi_z{}^2) dy\, dz \tag{38}$$

where x lies in the stream direction, y, z and r, θ are co-ordinates in the transverse plane, and ρ is the density of the undisturbed stream.

From these two basic relations come expressions for the drag of a wide variety of bodies—slender bodies, thin wings, slender thin wings, not-so-slender thin wings—all including somewhere a double integral related to

$$I = -\frac{1}{2\pi} \int_0^1 \int_0^1 S''(x)S''(y) \log|x-y| dx\, dy.$$

Some, but by no means all of these are amenable to evaluation by the method of Section 5.

In the notes that follow the fact that the integrals are taken between other limits is only a superficial difference since a simple transformation or a new choice of the unit of length brings them to the same form. For example

$$\begin{aligned}\int_0^l \int_0^l S''(x)S''(y) \log \left|\frac{x-y}{l}\right| dx\, dy \\ &= \frac{1}{l^2} \int_0^1 \int_0^1 S''(x_1)S''(y_1) \log|x_1 - y_1| dx_1\, dy_1 \\ &= l^2 \int_0^1 \int_0^1 S''_1(x_1)S''_1(y_1) \log|x_1 - y_1| dx_1\, dy_1 \end{aligned} \tag{39}$$

where

$$x_1 = x/l,\ y_1 = y/l \text{ and } S_1 = S/l^2.$$

6.1. *Slender-Body Theory: Wave Drag at Zero Lift.*

Suppose that $S(x)$, $0 \leqslant x \leqslant l$, is the cross-sectional area distribution of a body, a wing or a wing-body combination. If the configuration is slender and has a pointed nose then according to Ward's slender-body theory[2] for supersonic flow its wave drag at zero lift includes the term

$$I = -\frac{1}{2\pi}\int_0^l\int_0^l S''(x_1)S''(x_2)\log\left|\frac{x_1 - x_2}{l}\right| dx_1\,dx_2. \tag{40}$$

A pointed nose implies $S(0) = S'(0) = 0$. If also $S'(l) = 0$ and in between $S'(x)$ is continuous then all other terms vanish and this integral alone determines the wave drag. This particularly simple case is the one for which the method of Section 5 was designed.

When $S'(l) \neq 0$ the method in its present form is not applicable. However, a modification exists, devised by J. Weber[3] to include this case. It requires an extra term in the function c_i {equation (31)} and a couple of extra terms in the final expression for I {equation (33)} but leaves the rest of the calculation unchanged.

With this extension the method can be used to evaluate the double integral in the wave-drag formula for any slender, pointed configuration with a smooth area distribution.

6.2. *Not-so-Slender Thin-Wing Theory: Lift-Dependent Wave Drag.*

The wave drag of a supersonic wing without thickness has been calculated to second order by Adams and Sears[4]. It includes the term

$$\int_0^l\int_0^l L'(x_1)L'(x_2)\log\left|\frac{x_1 - x_2}{l}\right| dx_1\,dx_2 \tag{41}$$

where $L(x)$ is the load on a cross-section of the wing.

If we replace $L(x)$ by $S'(x)$ we can identify this double integral with the integral I of Section 6.1. The conditions

$$S'(x) \text{ continuous for } 0 \leqslant x \leqslant l \text{ and } S'(0) = 0$$

then become

$$L(x) \text{ continuous for } 0 \leqslant x \leqslant l \text{ and } L(0) = 0$$

which are a remarkably innocuous pair of restrictions. Provided therefore we can find $\int_0^x L(x)\,dx$ to replace $S(x)$ the method of Section 5 is again applicable—in its present form if $L(l) = 0$ and through Weber's extension if $L(l) \neq 0$.

6.3. *Small-Perturbation Theory: Wave Drag.*

Sections 6.1 and 6.2 give expressions for the wave drag of two rather special kinds of body. Both are independent of Mach number although, wave drag being a supersonic phenomenon, they imply $M > 1$. For the wave drag of more general bodies there is an expression derived by Lomax[5] which introduces the concept of oblique sections.

In cylindrical polar co-ordinates x, r, θ imagine a family of co-axial cones $x - x_1 = \beta r$ where $\beta^2 = M^2 - 1$, with planes tangent to each member of the family along the generator given by $\theta =$ constant. For certain values of x_1, say $-l_1(\theta) \leqslant x_1 \leqslant l(\theta)$, each plane slices from the body an oblique section. Lomax defines $L(x_1, \theta)$ to be the load on such a section and $S(x_1, \theta)$ to be the projection of its area on the plane x_1 = constant, and shows that according to supersonic small-perturbation theory the wave drag of the body for any $M > 1$ can be written

$$\frac{D}{q} = \frac{1}{2\pi}\int_0^{2\pi} I(\theta)\,d\theta \tag{42}$$

where

$$I(\theta) = -\frac{1}{2\pi}\int_{-l_1}^{l}\int_{-l_1}^{l}\left[S''(x_1, \theta) - \frac{\beta}{2q}L'(x_1, \theta)\right] \times$$
$$\times \left[S''(x_2, \theta) - \frac{\beta}{2q}L'(x_2, \theta)\right] \log|x_1 - x_2|\,dx_1\,dx_2, \tag{43}$$

In practice most oblique area distributions and the corresponding load distributions satisfy the conditions $S'(-l_1, \theta) = L(-l_1, \theta) = 0$, $S'(l, \theta) = L(l, \theta) = 0$ and $S'(x, \theta)$ and $L(x, \theta)$ are continuous for $-l_1 \leqslant x \leqslant l$, so that the method of Section 5 can be used in the manner of Sections 6.1 and 6.2 to evaluate $I(\theta)$. However, great care must be taken over the exceptions.

For example, wherever the trailing edges of a wing are straight and supersonic the oblique planes will lie parallel to the trailing edge for two values of θ. Unless the edges are cusped this will produce a discontinuity in $S'(x, \theta)$ and unless the load vanishes there, a discontinuity in $L(x, \theta)$ as well. For these two values of θ, $I(\theta)$ is infinite but if the nature of the infinity is known, methods can be devised to integrate $I(\theta)$ across the singularity. For examples of this see the papers of Weber[6] and of Cooke and Beasley[7].

For wings that are thin, $L'(x, \theta)$ is an odd function and $S''(x, \theta)$ an even function of θ. This simplifies the expression for $I(\theta)$ into the form

$$I(\theta) = -\frac{1}{2\pi}\int_{-l_1}^{l}\int_{-l_1}^{l} S''(x_1, \theta)S''(x_2, \theta)\log|x_1 - x_2|\,dx_1\,dx_2 -$$
$$-\frac{\beta^2}{4q^2}\frac{1}{2\pi}\int_{-l_1}^{l}\int_{-l_1}^{l} L'(x_1, \theta)L'(x_2, \theta)\log|x_1 - x_2|\,dx_1\,dx_2 \tag{44}$$

and the wave drag becomes the sum of a zero-lift drag and a lift-dependent drag.

6.4. *Vortex Drag.*

Vortex drag as defined in Section 6 is a surface integral taken over a plane far downstream—the Trefftz plane. If it can be assumed that vorticity is shed from a wing only at the trailing edge in a sheet which meets the Trefftz plane in a straight line then the vortex drag of the wing can be written in the form

$$D_v = -\frac{\rho_\infty}{4\pi}\int_{-s}^{s}\int_{-s}^{s} \Gamma'(y_1)\Gamma'(y_2)\log|y_1 - y_2|\,dy_1\,dy_2 \tag{45}$$

where ρ_∞ is the density of the undisturbed stream, s the semi-span of the wing, and $\Gamma(y)$ the total circulation developed by a chordwise section (*see*, for example, Reference 8, Section D14). If $\Gamma(y)$ is continuous across the span of the wing and vanishes at the tips then the method of Section 5 can be used to evaluate this expression as in Section 6.2. However, for a wing of given geometry the vortex drag can usually be more easily determined as a by-product of the calculation of spanwise loading.

LIST OF SYMBOLS

A_i	$=$	$S(k_i)$
a_r		Fourier coefficient in the series $S'(x) = \sum_{r=1}^{\infty} a_r \sin r\theta$
B	$=$	$S(1)$
c_i	$=$	$(A_i - N) - (B - N)u_i$
D		Drag
f_{ij}		Element of the inverse of the matrix $[p_{ij}]$
$\left.\begin{matrix} I \\ I(\theta) \end{matrix}\right\}$		Drag integral in its various forms
k_i		A point on the x-axis
$L(x)$		Load on a cross-section (Section 5.2)
$L(x, \theta)$		Load on an oblique section (Section 6.3)
l, l_1		Lengths of bodies and distributions of oblique sections (Section 6)
M		Mach number of the undisturbed stream
N	$=$	$S(0)$
n		Number of points k_i, $i = 1, 2, \ldots n$
p_{ij}	$\equiv$	$p(k_i, k_j)$, a function defined in equation (28)
q		Kinetic pressure
$S(x)$		In general a function of x, in particular an area distribution (Section 6.1)
$S(x, \theta)$		An oblique area distribution (Section 6.3)
$t_r(\alpha, \beta)$		A function defined in the Appendix
u_i	$\equiv$	$u(k_i)$, a function defined in equation (27)
x, y		Variables of a repeated integral (Sections 1 to 5) or Cartesian co-ordinates in the plane of a wing (Section 6)
β	$=$	$\sqrt{(M^2 - 1)}$ (except in Appendix)
$\Gamma(y)$		Total circulation developed by a chordwise section (Section 6.4)
θ	$=$	$\cos^{-1}(1 - 2x)$ or, with r and x, cylindrical polar co-ordinates (Section 6.3)
κ_i	$=$	$\cos^{-1}(1 - 2k_i)$
λ_i		Constants introduced as undetermined multipliers, $i = 1, 2, \ldots n$
$\sigma(\kappa_i, \kappa_j)$		A function defined in equation (22), and discussed in the Appendix
ϕ	$=$	$\cos^{-1}(1 - 2y)$, or a perturbation velocity potential (Section 1)

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APPENDIX

Summation of the Series of Section 4

The series to be summed is

$$4\sigma(\alpha, \beta) = \Sigma\, t_r(\alpha, \beta) \equiv \sum_{r=2}^{\infty} \frac{1}{r}\left[\frac{\sin(r-1)\alpha}{r-1} - \frac{\sin(r+1)\alpha}{r+1}\right]\left[\frac{\sin(r-1)\beta}{r-1} - \frac{\sin(r+1)\beta}{r+1}\right].$$

The series may be considered as a function of two variables α, β. Double partial differentiation term by term gives the series

$$\Sigma \frac{\partial^2 t_r}{\partial\alpha\partial\beta} = \sum_{r=2}^{\infty} \frac{1}{r}[\cos(r-1)\alpha - \cos(r+1)\alpha][\cos(r-1)\beta - \cos(r+1)\beta]$$

$$= 4 \sin\alpha \sin\beta \sum_{r=2}^{\infty} \frac{1}{r} \sin r\alpha \sin r\beta$$

$$= 2 \sin\alpha \sin\beta \sum_{r=2}^{\infty} \frac{1}{r}[\cos r(\alpha-\beta) - \cos r(\alpha+\beta)]$$

$$= 2 \sin\alpha \sin\beta \left[\tfrac{1}{2} \log \frac{1-\cos(\alpha+\beta)}{1-\cos(\alpha-\beta)} - 2\sin\alpha\sin\beta\right]$$

for all $\alpha \neq \beta$, since $\sum_{r=1}^{\infty} \frac{1}{r} \cos r\theta = -\log(2\sin\theta/2)$ for all $\theta \neq 0 \pmod{\pi}$.

Since $\Sigma\, t_r(\alpha, \beta)$ converges to zero when either $\alpha = 0$ or $\beta = 0$ and $\Sigma\,(\partial^2 t_r/\partial\alpha\,\partial\beta)$ is uniformly convergent in any region excluding the points $\alpha = \beta$, therefore $\Sigma\, t_r(\alpha, \beta)$ is also uniformly convergent in any region excluding the points $\alpha = \beta$, and whenever $\alpha \neq \beta$,

$$\Sigma\,(\partial^2 t_r/\partial\alpha\,\partial\beta) = (\partial^2/\partial\alpha\,\partial\beta)\, \Sigma\, t_r(\alpha, \beta).$$

Hence by a double integration, when $\alpha \neq \beta$

$$\Sigma\, t_r(\alpha, \beta) = -\tfrac{1}{2}(\cos\alpha - \cos\beta)^2 \log\frac{1-\cos(\alpha+\beta)}{1-\cos(\alpha-\beta)} + \sin\alpha\sin\beta(1-\cos\alpha\cos\beta).$$

When $\alpha = \beta$ the series becomes a function of one variable

$$\Sigma\, t_r(\alpha) \equiv \sum_{r=2}^{\infty} \frac{1}{r}\left[\frac{\sin(r-1)\alpha}{r-1} - \frac{\sin(r+1)\alpha}{r+1}\right]^2.$$

Differentiation of this series term by term gives the series

$$\Sigma \frac{dt_r}{d\alpha} = \sum_{r=2}^{\infty} \frac{2}{r}\left[\frac{\sin(r-1)\alpha}{r-1} - \frac{\sin(r+1)\alpha}{r+1}\right][\cos(r-1)\alpha - \cos(r+1)\alpha]$$

$$= 4\sin\alpha \sum_{r=2}^{\infty} \frac{\sin r\alpha}{r}\left[\frac{\sin(r-1)\alpha}{r-1} - \frac{\sin(r+1)\alpha}{r+1}\right]$$

$$= 4\sin\alpha\left[\tfrac{1}{2}\sin\alpha\sin 2\alpha - \lim_{r\to\infty}\frac{\sin r\alpha}{r}\frac{\sin(r+1)\alpha}{r+1}\right]$$

$$= 2\sin^2\alpha \sin 2\alpha \text{ for all } \alpha.$$

Since $\sum t_r(\alpha)$ converges to zero at $\alpha = 0$ and $\sum (dt_r/d\alpha)$ is uniformly convergent, therefore $\sum t_r(\alpha)$ is also uniformly convergent, and $\sum (dt_r/d\alpha) = (d/d\alpha) \sum t_r(\alpha)$. Hence by integration

$$\sum t_r(\alpha) = \sin^4 \alpha .$$

Finally since

$$\left[-\tfrac{1}{2}(\cos\alpha - \cos\beta)^2 \log \frac{1-\cos(\alpha+\beta)}{1-\cos(\alpha-\beta)} + \sin\alpha \sin\beta(1-\cos\alpha\cos\beta)\right]_{\alpha=\beta} = \sin^4\alpha$$

therefore

$$\sum t_r(\alpha, \beta) \equiv \sum_{r=2}^{\infty} \frac{1}{r}\left[\frac{\sin(r-1)\alpha}{r-1} - \frac{\sin(r+1)\alpha}{r+1}\right]\left[\frac{\sin(r-1)\beta}{r-1} - \frac{\sin(r+1)\beta}{r+1}\right]$$

$$= -\tfrac{1}{2}(\cos\alpha - \cos\beta)^2 \log \frac{1-\cos(\alpha+\beta)}{1-\cos(\alpha-\beta)} + \sin\alpha\sin\beta(1-\cos\alpha\cos\beta)$$

for all α, β.

TABLE 1

The Values u_i for Nineteen Equally Spaced Points

i	u_i
1	0·01869
2	0·05204
3	0·09406
4	0·14238
5	0·19550
6	0·25232
7	0·31192
8	0·37353
9	0·43644
10	0·50000
11	0·56356
12	0·62647
13	0·68808
14	0·74768
15	0·80450
16	0·85762
17	0·90594
18	0·94796
19	0·98131

TABLE 2

The Matrix Elements f_{ij} for Nineteen Equally Spaced Points

j \ i	1	2	3	4	5	6	7	8	9	10
1	373·996									
2	−232·710	349·278								
3	59·098	−228·760	348·637							
4	−6·124	58·526	−228·669	348·625						
5	1·604	−6·039	58·512	−228·667	348·623					
6	0·029	1·590	−6·037	58·510	−228·665	348·622				
7	0·119	0·029	1·590	−6·036	58·510	−228·665	348·623			
8	0·047	0·119	0·029	1·590	−6·036	58·510	−228·666	348·623		
9	0·030	0·047	0·118	0·030	1·590	−6·036	58·511	−228·667	348·624	
10	0·019	0·030	0·047	0·118	0·030	1·589	−6·037	58·512	−228·668	348·625
11	0·012	0·019	0·030	0·047	0·117	0·030	1·589	−6·037	58·512	
12	0·009	0·012	0·019	0·030	0·047	0·118	0·030	1·590		
13	0·006	0·008	0·012	0·018	0·031	0·047	0·118			
14	0·004	0·006	0·008	0·013	0·018	0·031				
15	0·003	0·004	0·006	0·008	0·013					
16	0·003	0·003	0·004	0·006						
17	0·001	0·002	0·004							
18	0·002	0·001								
19	0·000									

$f_{ij} = f_{ji} = f_{20-i\ 20-j}$

maximum error $\pm 0{\cdot}002$

The example of Section 5

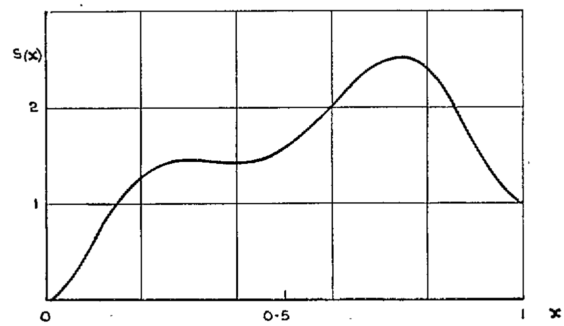

FIG. 1. The function $S(x)$.

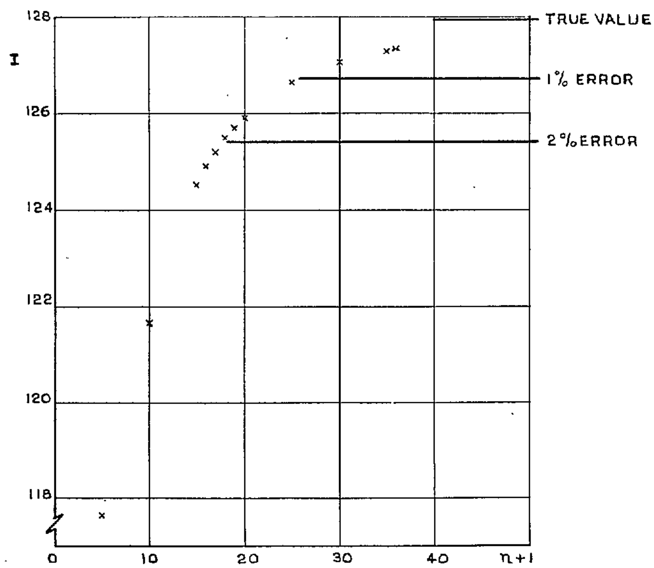

FIG. 2. The integral I and its approximations.

(87839) Wt. 65/1418 K.5 11/63 Hw.

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