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Equilibrium Configurations of Flying Cables of Captive Balloons, and Cable Derivatives for Stability Calculations

By S. NEUMARK, Techn.Sc.D., F.R.Ae.S.

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Summary.

The theory of equilibrium of balloon flying cables is presented in a simplified form most suitable for applications, with numerical tables covering probably the full range of practical cases. A simplified derivation of formulae for longitudinal cable derivatives is presented, and the problem of the lateral cable derivative solved. The normal oscillatory modes of the 'balloon-plus-cable' system (longitudinal and lateral, ignoring balloon aerodynamics) are determined and analysed. All final formulae are expanded into power series which converge well in the important case of a highly tensioned cable.

The report provides the data necessary for studying dynamic stability of captive balloons.

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* Replaces R.A.E. Report No. Aero. 2653—A.R.C. 23,244.

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1. *Introduction.*

The problem of dynamic stability of captive balloons arose initially in connection with observation balloons, for which it was obviously important not only to ensure static stability, but also to have all oscillatory modes sufficiently damped. An early attack on the problem was made in 1915 by Bairstow, Relf and Jones[1] who produced a basic analytical scheme, following broadly the lines of G. H. Bryan's linearised approach to aeroplane dynamic stability, but including also all new elements peculiar to captive balloons, such as apparent inertia, additional degrees of freedom, and cable derivatives. They obtained two basic characteristic equations (both sextic) for the longitudinal and lateral disturbances, respectively. The formidable array of coefficients involving numerous aerodynamic and cable derivatives (most of them presenting at the time insuperable difficulties as regards even rough numerical estimates) resulted in this pioneering work remaining largely unused for nearly half a century. The big experimental and computational efforts required to obtain reliable numerical results, and to establish relationships between the balloon geometry and oscillatory behaviour, could hardly be justified, as some more or less satisfactory designs were arrived at by the empirical trial and error procedure. Also, observation balloons soon lost importance and, as regards barrage balloons, large-amplitude oscillations were not considered a serious inconvenience and were believed unavoidable.

The interest in captive balloons has faded after the last war, but it recurs again and again, with always varying applications in view. Some of these involve carrying frangible instruments, thus a balloon serves as a floating platform, and as such should be as stable and oscillation-free as possible. It appears therefore that the analytical stability investigation should be revived and pursued, the prospects being much more promising now. The main difficulty is still the same as before, viz. estimation of stability derivatives. Of these, the aerodynamic (especially rotary) ones may now be measured easily if only a tiny fraction of the existing tunnel facilities is made available for this purpose. As to the cable derivatives, an analytical approach is more promising, particularly because much preparatory work has been done already[2 to 12], in connection with either balloon performance or with the cognate problems of kites and bodies towed behind aeroplanes (such as radio aerials, aerodynamic instruments, towed gliders, etc.). The existing information, however, is far from complete, difficult of access, and not adapted to designer's needs. The purpose of this paper is to present the full theory, with such computational aids as are available, in the form ready for practical applications.

The first attempts to determine cable derivatives[1, 3] tried to avoid analytical difficulties by neglecting either the action of wind on cable or the latter's weight. Neither of these simplified assumptions is justified, and the need of including both gravity and wind forces has long since been recognised. The first step was made by McLeod[2] who proposed the following simple formula for the normal wind-force component acting on an inclined cable element:

$$P_n\, dl = (C_{Dc}\, \tfrac{1}{2}\rho V^2 d_c) \sin^2\varphi\, dl = n \sin^2\varphi\, dl \qquad (1)$$

(cf. List of Symbols and Fig. 1). The tangential wind-force component is sufficiently small to be reasonably neglected. McLeod's formula was used by almost all later writers*, and it is retained also in the present paper. Originally, it was applied only for determining *cable configurations in equilibrium conditions*, with the sole purpose of performance studies. This was done by Glauert[4] for the case when the wind force acts against the normal component of weight, as is the case with heavy bodies towed behind aircraft, and by Hollingdale and Wild[6] for the inverse case, the one applicable to kite balloons. The results were given in form of graphs in both papers. In an early paper by the present writer[5] (containing references to some even earlier French and German attempts, now obsolete), a solution basically identical with (but differing in form from) that of Ref. 6 was arrived at, with tabulated results. For determining the cable configurations, Refs. 5 and 6 are equally suitable but, for calculating stability derivatives, the tabulated values are naturally more convenient. It has

* Together with the simplifying assumption that ρV^2 does not vary with height, and that the wind direction is also constant, the latter postulation ensuring that the equilibrium configuration lies in a vertical plane.

been decided therefore to give a short summary of Ref. 5 in Section 2 of the present paper and reproduce the full table of that reference which is practically unobtainable now. It may be mentioned that Pode[11] produced voluminous tables for the Glauert's case, which (somewhat surprisingly) take account of the very small tangential wind-force components; they are useless for the investigations of captive balloons. A Russian paper by Kochin[10] is an interesting study of equilibrium configurations (originating from war-time work on barrage balloons). It starts by treating the very general case of 3-dimensional configurations in wind field of varying direction, strength and air density, and ends by more practical simple cases, similar to those considered by the authors mentioned above. The McLeod's $\sin^2\varphi$-law is, curiously, replaced by $\sin\varphi$-law, with reference to some unspecified tunnel tests, this leading to somewhat different, but not simpler, solutions. A few examples were worked out numerically, but no tabulation attempted. Some curious historical references were given. A more recent paper by Longden[12] brings some interesting analysis of balloon performance in connection with a certain special application, the theories of Refs. 2 and 6 being used extensively.

For the *cable derivatives*, Bairstow, Relf and Jones[1] developed formulae applying in the case of 'dragless' balloon cable which then assumes the form of an ordinary catenary. Glauert[3] did a similar work for 'weightless' towing cables (which case, surprisingly, and unnoticed by Glauert, also leads to a catenary, this time with a horizontal axis). These early results are now obsolete and may possibly be used only in some special cases, or for comparison purposes. Brown[7] gave the solution for *longitudinal derivatives* of kite-balloon cables subject to both weight and wind forces, and Mitchell[8] adapted it to glider towing cables (when the wind force may act either in the same or opposite direction to that of the normal component of weight). Both writers noticed grave computational difficulties confronting the user, and made some attempts to overcome them, but these are hardly sufficient now. The present paper brings, in Section 3, a much simplified derivation of the formulae (agreeing with those of Brown but adapted to the basic scheme of Section 2, thus making possible the use of existing tables), and systematic series expansions suitable for practical use, especially for highly tensioned cables (details are treated in Appendices I to III). It may be noticed that O'Hara[9] extended the analysis of cable configuration and derivatives to the more general case of elastic cables. The effect of elasticity may be large for very heavily tensioned and nearly horizontal cables connecting gliders with their tugs; it will be small, however, for balloon cables which are never so highly tensioned, and are nearer to vertical in normal conditions. This effect has been neglected in the present paper.

The *lateral derivative* was treated very superficially by previous writers, none of whom considered the general case including both gravity and wind forces, and the solution of this problem is presented in Section 4, where the final formula is easily adaptable to practical needs, either by a simple numerical integration, or by a suitable power expansion.

The Section 5 deals with what may be considered as a simple example of applying the cable derivatives to a dynamic problem, and also as an introduction to a full study of dynamic stability of captive balloons. This part of the report was stimulated by some remarks in Glauert's paper[3]. It contains an analysis of oscillatory modes of the 'balloon-plus-cable' configuration in the simplest case when the balloon is considered as a floating body subject to constant buoyancy and aerodynamic force, while the aerodynamic forces induced by a longitudinal or lateral disturbance, and the rotation in pitch or yaw, are all ignored. It is shown that, in such a case, there exist two longitudinal modes, in one of which the balloon oscillates rapidly, approximately towards and away from the mooring point, while the other mode may be described very nearly as a slow 'pendulum-like' oscillation. There exists only one, pendulum-like, lateral mode. An important result is that the two pendulum oscillations have nearly equal frequencies, especially in the case of highly tensioned cables. A somewhat more complicated problem, with simple aerodynamic damping included, is briefly treated in Appendix IV. Numerical examples are explained in Section 6 and illustrated by Figs. 2, 3 and 5.

An acknowledgement is due to Mrs. J. Collingbourne and Miss B. Mills who have done the computational work and prepared the illustrations. The name of Dr. W. Wolibner who had produced the table of Ref. 5 (reproduced at the end of this paper), more than 25 years ago, should also be mentioned.

2. *Equilibrium Form of Cable.*

2.1. *General Case.*

The forces acting on cable are shown in Fig. 1 which also explains the notation. Equilibrium equations for an element are:

$$dT = w\,dl \sin\varphi, \tag{2}$$

$$T\,d\varphi = (n\sin^2\varphi + w\cos\varphi)dl, \tag{3}$$

and we have the geometric relationships:

$$dx = dl\cos\varphi \tag{4}$$

$$dz = dl\sin\varphi. \tag{5}$$

The problem consists in integrating the above system of differential equations, so as to express T, x, z and l as functions of φ.

From (2) and (5), we have $dT = w\,dz$, and hence:

$$T = T_0 + wz = T_1 - w(z_1 - z). \tag{6}$$

Eliminating dl from (2) and (3), we obtain:

$$\frac{dT}{T} = \frac{w\sin\varphi\,d\varphi}{n\sin^2\varphi + w\cos\varphi}. \tag{7}$$

Introducing a new constant ψ defined by

$$2\cot 2\psi = w/n, \tag{8}$$

we may write (7) in the more convenient form:

$$\frac{dT}{T} = \frac{2\cot 2\psi\sin\varphi\,d\varphi}{1 + 2\cot 2\psi\cos\varphi - \cos^2\varphi} = \left(\frac{\cos 2\psi}{\tan\psi + \cos\varphi} + \frac{\cos 2\psi}{\cot\psi - \cos\varphi}\right)\sin\varphi\,d\varphi \tag{9}$$

and, integrating, we find that T is proportional to the function:

$$\tau(\varphi) = \left(\frac{\cot\psi - \cos\varphi}{\tan\psi + \cos\varphi}\right)^{\cos 2\psi}, \tag{10}$$

so that we may write:

$$T = T_1\frac{\tau}{\tau_1}, \qquad T_0 = T_1\frac{\tau_0}{\tau_1}, \tag{11}$$

where symbols τ, τ_1, τ_0 have been used for abbreviation, to denote $\tau(\varphi)$, $\tau(\varphi_1)$, $\tau(\varphi_0)$.

From (7) and (11) it follows that the first derivative of $\tau(\varphi)$ is:

$$\tau' = \tau \frac{w \sin \varphi}{n \sin^2\varphi + w \cos \varphi}. \tag{12}$$

We then obtain from (3) and (4), taking into account (11):

$$dl = \frac{T_1}{\tau_1} \frac{\tau \, d\varphi}{n \sin^2\varphi + w \cos \varphi}, \qquad dx = \frac{T_1}{\tau_1} \frac{\tau \cos \varphi \, d\varphi}{n \sin^2\varphi + w \cos \varphi}, \tag{13}$$

so that, introducing the following auxiliary functions:

$$\lambda(\varphi) = \int_0^{\varphi} \frac{n\tau \, d\varphi}{n \sin^2\varphi + w \cos \varphi}, \qquad \sigma(\varphi) = \int_0^{\varphi} \frac{n\tau \cos \varphi \, d\varphi}{n \sin^2\varphi + w \cos \varphi}, \tag{14}$$

we may write:

$$l = \frac{T_1}{n\tau_1}(\lambda - \lambda_0), \tag{15}$$

$$x = \frac{T_1}{n\tau_1}(\sigma - \sigma_0), \tag{16}$$

and we also obtain from (6) and (11):

$$z = \frac{T_1}{w\tau_1}(\tau - \tau_0). \tag{17}$$

The functions τ, σ and λ were tabulated in Ref. 5, for φ varying from 0 to 90° through 1° intervals, for 12 different values of the ratio n/w appropriate for balloon cables. The table is reproduced at the end of this paper. To utilise the table in practice conveniently, without tedious between-columns interpolation, it is advisable not to assume arbitrary values of wind speed, but rather to choose a few appropriate values of the ratio n/w, such as appear in the table, and then deduce the corresponding values of the wind speed V {cf. form. (1)}.

The equations (15 to 17) determine the cable configuration completely, if T_1, φ_1 and φ_0 are known. In practice, T_1 and φ_1 will normally be known ($T_1 \sin \varphi_1 = Z_1$ being equal to the sum of the balloon's reserve buoyancy* and aerodynamic lift if any, and $T_1 \cos \varphi_1 = X_1$ being equal to its drag), but φ_0 will have to be calculated by using our table. Putting $\varphi = \varphi_1$ in (15 to 17), we obtain:

$$n l_1 \tau_1 = T_1(\lambda_1 - \lambda_0), \tag{18}$$

$$n x_1 \tau_1 = T_1(\sigma_1 - \sigma_0), \tag{19}$$

$$w z_1 \tau_1 = T_1(\tau_1 - \tau_0). \tag{20}$$

In practical problems, the height z_1 will usually be known, then (20) will give τ_0, hence φ_0, and so the required cable length from (18), and horizontal displacement of balloon x_1, from (19). It may be mentioned that, sometimes, too big a value may be assigned to z_1, but then (20) will lead to a value of τ_0 less than $\tau(0°)$, and this will show the impossibility of such a configuration.

Alternatively, we may know the cable length l_1, in which case (18) gives λ_0, which must be positive; and then x_1 and z_1 will be obtained from (19) and (20).

---

* Reserve buoyancy is defined as the excess of total buoyancy over the entire weight of balloon (including its envelope, gas and air contents, and rigging, but excluding the cable weight). If the excess buoyancy is considerably greater than the cable weight, then T_0 will be large (though always less than T_1), and the difference $(\varphi_1 - \varphi_0)$ small under all conditions, the cable will then be 'highly tensioned'. In the opposite case T_0 may be small, and $(\varphi_1 - \varphi_0)$ may become quite large.

Fig. 2 illustrates a numerical example, with varying wind speed and constant height. The relevant numerical data and detailed explanations are given in Section 6.

The following relationships, resulting directly from (12) and (14), will be needed in Section 3:

$$\lambda' = \frac{n\tau'}{w \sin \varphi}, \qquad \sigma' = \frac{n\tau' \cos \varphi}{w \sin \varphi}. \tag{21}$$

2.2. *Special Cases.*

Two special cases were often considered in the past, for their simplicity, and may be briefly surveyed:

(A) *Action of wind on cable neglected* ($n \to 0$).—This may be treated as an approximation for a thick heavy cable in a light wind. In this case $\psi \to 0$ and the function τ becomes infinite but, from (10):

$$n\tau(\varphi) \to w \sec \varphi, \tag{22}$$

and we obtain from (14):

$$\left.\begin{aligned} \lambda(\varphi) &\to \int_0^\varphi \sec^2\varphi \, d\varphi = \tan \varphi, \\ \sigma(\varphi) &\to \int_0^\varphi \sec \varphi \, d\varphi = \ln \tan \left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = \mathrm{gd}^{-1}\varphi, \end{aligned}\right\} \tag{23}$$

and from (11):

$$T \cos \varphi = T_1 \cos \varphi_1 = X_1 = \text{const.} \tag{24}$$

The equations (15 to 17) now become:

$$wl = X_1(\tan \varphi - \tan \varphi_0), \tag{25}$$

$$wx = X_1(\mathrm{gd}^{-1}\varphi - \mathrm{gd}^{-1}\varphi_0), \tag{26}$$

$$wz = X_1(\sec \varphi - \sec \varphi_0). \tag{27}$$

These equations, of course, correspond to a *common catenary* with a vertical axis whose Cartesian equation is easily obtained, by eliminating φ from (26) and (27), in the form

$$\frac{wz}{X_1} + \sec \varphi_0 = \cosh \left(\frac{wx}{X_1} + \mathrm{gd}^{-1}\varphi_0\right), \tag{28}$$

and the parameter is seen to be X_1/w.

(B) *Weight of cable neglected* ($w \to 0$).—This may be treated as an approximation for a thin cable in a strong wind. In this case $\psi \to 45°$ and

$$\tau(\varphi) \to 1, \qquad \text{hence} \qquad T = T_1 = \text{const.} \tag{29}$$

The functions $\lambda(\varphi)$ and $\sigma(\varphi)$ both tend to ∞, but

$$\lambda - \lambda_0 \to \int_{\varphi_0}^{\varphi} \operatorname{cosec}^2\varphi \, d\varphi = \cot \varphi_0 - \cot \varphi, \tag{30}$$

$$\sigma - \sigma_0 \to \int_{\varphi_0}^{\varphi} \frac{\cos \varphi}{\sin^2\varphi} \, d\varphi = \operatorname{cosec} \varphi_0 - \operatorname{cosec} \varphi, \tag{31}$$

and it is also found easily:

$$\frac{\tau - \tau_0}{w} \to \frac{1}{n}\left(\ln \tan \frac{\varphi}{2} - \ln \tan \frac{\varphi_0}{2}\right). \tag{32}$$

Equations (15 to 17) now become:

$$nl = T_1(\cot\varphi_0 - \cot\varphi)\,, \tag{33}$$

$$nx = T_1(\operatorname{cosec}\varphi_0 - \operatorname{cosec}\varphi)\,, \tag{34}$$

$$nz = T_1\left(\ln\tan\frac{\varphi}{2} - \ln\tan\frac{\varphi_0}{2}\right), \tag{35}$$

and these equations, somewhat surprisingly, correspond again to a *common catenary*, but this time one *with a horizontal axis*. This can be shown easily by eliminating φ from (34) and (45), whereupon we obtain:

$$\operatorname{cosec}\varphi_0 - \frac{nx}{T_1} = \cosh\left(\frac{nz}{T_1} + \ln\tan\frac{\varphi_0}{2}\right). \tag{36}$$

The parameter is T_1/n, and the tension is constant.

3. *Longitudinal Cable Derivatives.*

3.1. *General Case.*

Let us suppose that a balloon cable is in equilibrium, and that its upper end is displaced in xz-plane from its original position (x_1, z_1) to a slightly different point (x_1+dx_1, z_1+dz_1), while n and w remain unaltered. The quantities φ_0, φ_1, T_1, X_1, Z_1 will then all change, assuming small increments, but the length of the cable l_1 will be considered as constant. The derivatives required are X_x, X_z, Z_x and Z_z, defined by:

$$dX_1 = X_x\,dx_1 + X_z\,dz_1\,, \qquad dZ_1 = Z_x\,dx_1 + Z_z\,dz_1\,, \tag{37}$$

but it will be convenient to consider first the increments of T_1 and φ_1, the quantities appearing in the basic equations (18 to 20). We differentiate these equations, taking into account the relationships (21) which apply both at the upper end of the cable (suffix 1) and at the lower end (suffix 0). We obtain, after some simplification:

$$\left.\begin{aligned}
\frac{l_1\tau_1}{T_1}\,dT_1 + \left(\frac{T_1}{w\sin\varphi_1} - l_1\right)\tau_1'\,d\varphi_1 - \frac{T_1}{w\sin\varphi_0}\,\tau_0'\,d\varphi_0 &= 0\,,\\
\frac{x_1\tau_1}{T_1}\,dT_1 + \left(\frac{T_1\cos\varphi_1}{w\sin\varphi_1} - x_1\right)\tau_1'\,d\varphi_1 - \frac{T_1\cos\varphi_0}{w\sin\varphi_0}\,\tau_0'\,d\varphi_0 &= \tau_1\,dx_1\,,\\
\frac{z_1\tau_1}{T_1}\,dT_1 + \left(\frac{T_1}{w} - z_1\right)\tau_1'\,d\varphi_1 - \frac{T_1}{w}\,\tau_0'\,d\varphi_0 &= \tau_1\,dz_1\,.
\end{aligned}\right\} \tag{38}$$

We now eliminate terms containing differentials $d\varphi_1$ and $d\varphi_0$, by multiplying the above equations by the factors

$$\left.\begin{aligned}
&w\sin\varphi_1(z_1\cos\varphi_0 - x_1\sin\varphi_0) - T_1\sin(\varphi_1-\varphi_0)\,,\\
&T_1(\sin\varphi_1 - \sin\varphi_0) - w\sin\varphi_1(z_1 - l_1\sin\varphi_0)\,,\\
&T_1(\cos\varphi_0 - \cos\varphi_1) - w\sin\varphi_1(l_1\cos\varphi_0 - x_1)\,,
\end{aligned}\right\} \tag{39}$$

respectively, and adding. The result is:

$$dT_1 = \frac{\{T_1(\sin\varphi_1 - \sin\varphi_0) - w\sin\varphi_1(z_1 - l_1\sin\varphi_0)\}dx_1 + \{T_1(\cos\varphi_0 - \cos\varphi_1) - w\sin\varphi_1(l_1\cos\varphi_0 - x_1)\}dz_1}{x_1(\sin\varphi_1 - \sin\varphi_0) + z_1(\cos\varphi_0 - \cos\varphi_1) - l_1\sin(\varphi_1 - \varphi_0)}\,. \tag{40}$$

Similarly, we eliminate terms containing the differentials dT_1 and $d\varphi_0$ from equations (38), by multiplying them by the factors:

$$(z_1 \cos \varphi_0 - x_1 \sin \varphi_0), \qquad -(z_1 - l_1 \sin \varphi_0), \qquad -(l_1 \cos \varphi_0 - x_1), \tag{41}$$

respectively, and adding, taking into account (12), we obtain:

$$T_1\, d\varphi_1 = -(n \sin^2\varphi_1 + w \cos \varphi_1) \frac{(z_1 - l_1 \sin \varphi_0)dx_1 + (l_1 \cos \varphi_0 - x_1)dz_1}{x_1(\sin \varphi_1 - \sin \varphi_0) + z_1(\cos \varphi_0 - \cos \varphi_1) - l_1 \sin(\varphi_1 - \varphi_0)}. \tag{42}$$

To get the four required derivatives, we use the relationships:

$$X_1 = T_1 \cos \varphi_1, \qquad Z_1 = T_1 \sin \varphi_1, \tag{43}$$

which, upon differentiation, become:

$$dX_1 = dT_1 \cos \varphi_1 - T_1 \sin \varphi_1\, d\varphi_1, \qquad dZ_1 = dT_1 \sin \varphi_1 + T_1 \cos \varphi_1\, d\varphi_1, \tag{44}$$

and hence, from (37, 40, 42), we obtain the final formulae:

$$\left.\begin{aligned}
X_x &= \frac{T_1 \cos \varphi_1(\sin \varphi_1 - \sin \varphi_0) + n(z_1 - l_1 \sin \varphi_0) \sin^3\varphi_1}{\delta}, \\
X_z &= \frac{T_1 \cos \varphi_1(\cos \varphi_0 - \cos \varphi_1) + n(l_1 \cos \varphi_0 - x_1) \sin^3\varphi_1}{\delta}, \\
Z_x &= \frac{T_1 \sin \varphi_1(\sin \varphi_1 - \sin \varphi_0) - (w + n \sin^2\varphi_1 \cos \varphi_1)(z_1 - l_1 \sin \varphi_0)}{\delta}, \\
Z_z &= \frac{T_1 \sin \varphi_1(\cos \varphi_0 - \cos \varphi_1) - (w + n \sin^2\varphi_1 \cos \varphi_1)(l_1 \cos \varphi_0 - x_1)}{\delta},
\end{aligned}\right\} \tag{45}$$

where, for abbreviation:

$$\delta = x_1(\sin \varphi_1 - \sin \varphi_0) + z_1(\cos \varphi_0 - \cos \varphi_1) - l_1 \sin(\varphi_1 - \varphi_0). \tag{46}$$

The following relationship, easily derived from (45), is worth noting:

$$X_x Z_z - X_z Z_x = T_1 \frac{n \sin^2\varphi_1 + w \cos \varphi_1}{\delta}. \tag{47}$$

Formulae equivalent to (45) and (47) were derived by Brown[7] and Mitchell[8], in a less simple way. It will be shown in the footnote to Appendix I that δ is always positive. It may also be shown easily that all four derivatives (45) are always positive, and the same applies obviously to the expression (47).

The formulae (45) can, of course, be used directly only in conjunction with (18 to 20) and with the table of functions τ, λ, σ. It may be mentioned that the Table 1 was originally computed for the purpose of estimating the balloon performance, and only a few decimals were deemed sufficient for that purpose. The accuracy is not satisfactory as regards stability derivatives, chiefly because the common denominator δ is obtained from (46) as a small difference of two large numbers. This is found not to be a serious difficulty for moderately tensioned balloons for which the difference $(\varphi_1 - \varphi_0)$ is sufficiently large, and δ not so small, but even then we cannot expect high accuracy of numerical results. The position becomes nearly hopeless for highly tensioned cable when $(\varphi_1 - \varphi_0)$

assumes very small values, and (46) does not give significant results. An appropriate method is then* to expand δ in powers of $(\varphi_1-\varphi_0)$. Such an expansion is derived in Appendix I, the final result being:

$$\delta = \frac{l_1(\varphi_1-\varphi_0)^3}{12}\{1-h_2(\varphi_1-\varphi_0)^2+h_3(\varphi_1-\varphi_0)^3\ldots\}, \tag{48}$$

where:

$$\left.\begin{aligned} h_2 &= \frac{3w^2 - wn\cos\varphi_1 + 3n^2\sin^2\varphi_1}{30(n\sin^2\varphi_1+w\cos\varphi_1)^2}, \\ h_3 &= \sin\varphi_1\frac{6w^3 - 13wn^2\cos\varphi_1 + wn^2(10-3\sin^2\varphi_1) - 6n^3\sin^2\varphi_1\cos\varphi_1}{60(n\sin^2\varphi_1+w\cos\varphi_1)^3}, \end{aligned}\right\} \tag{49}$$

and $(\varphi_1-\varphi_0)$ is measured in radians.

The convergence of the series (48) is excellent for small values of $(\varphi_1-\varphi_0)$, but it is still tolerable for quite high values of this angle, say up to 0·5, the value seldom exceeded in practice.

The numerators in (45) do not suffer from the difficulty encountered in computing δ, and can be calculated with reasonable accuracy by using our table. The entire formulae can, however, be also expanded in powers of $(\varphi_1-\varphi_0)$, and the first two terms of these expansions (as derived in Appendix I) are given below:

$$\left.\begin{aligned} X_x &= \frac{12}{(\varphi_1-\varphi_0)^3}[(n\sin^2\varphi_1+w\cos\varphi_1)\cos^2\varphi_1 + \\ &\quad + \sin\varphi_1\cos\varphi_1(\tfrac{3}{2}w\cos\varphi_1-n\cos 2\varphi_1)(\varphi_1-\varphi_0)\ldots], \\ X_z &= \frac{12}{(\varphi_1-\varphi_0)^3}[(n\sin^2\varphi_1+w\cos\varphi_1)\sin\varphi_1\cos\varphi_1 + \\ &\quad + \tfrac{1}{2}\{w\cos\varphi_1(3\sin^2\varphi_1-1) + n\sin^2\varphi_1(4\sin^2\varphi_1-3)\}(\varphi_1-\varphi_0)\ldots], \\ Z_x &= \frac{12}{(\varphi_1-\varphi_0)^3}[(n\sin^2\varphi_1+w\cos\varphi_1)\sin\varphi_1\cos\varphi_1 + \\ &\quad + \tfrac{1}{2}\{w\cos\varphi_1(3\sin^2\varphi_1-1) + n\sin^2\varphi_1(4\sin^2\varphi_1-3)\}(\varphi_1-\varphi_0)\ldots], \\ Z_z &= \frac{12}{(\varphi_1-\varphi_0)^3}[(n\sin^2\varphi_1+w\cos\varphi_1)\sin^2\varphi_1 + \\ &\quad + \sin\varphi_1\{\tfrac{1}{2}w(3\sin^2\varphi_1-2) - n\sin\varphi_1\sin 2\varphi_1\}(\varphi_1-\varphi_0)\ldots]. \end{aligned}\right\} \tag{50}$$

Coefficients of further terms become very complicated, so they would not be of much use. The approximations (50) may only be used when $(\varphi_1-\varphi_0)$ is really small. It is interesting, however, to see that, to this order of approximation, $X_z = Z_x$. It is also seen that the large first terms of the series are in the ratio $\cos^2\varphi_1 : \sin\varphi_1\cos\varphi_1 : \sin^2\varphi_1$. If, therefore, φ_1 is near to 90°, we may expect Z_z to be very large, X_x small, with X_z and Z_x somewhere in between, and this should obviously be so.

A numerical example is illustrated in Fig. 3 which shows the variation of four longitudinal cable derivatives with the ratio n/w. This corresponds to equilibrium configurations of Fig. 2. Numerical data and other details will be found in Section 6.

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* Unless we embark upon the rather formidable task of computing a new table of τ, φ, σ with some eight decimals.

3.2. *Special Cases.*

(A) *Action of wind on cable neglected* ($n = 0$).—Putting $n = 0$ in (45), and using (24, 27) wherever appropriate for simplification, we obtain:

$$X_x = \frac{X_1(\sin\varphi_1 - \sin\varphi_0)}{\delta}, \quad X_z = Z_x = \frac{X_1(\cos\varphi_0 - \cos\varphi_1)}{\delta}, \quad Z_z = \frac{wx_1 - X_1(\sin\varphi_1 - \sin\varphi_0)}{\delta}, \tag{51}$$

where δ, from (46), becomes:

$$\delta = x_1(\sin\varphi_1 - \sin\varphi_0) - \frac{2X_1}{w}\{1 - \cos(\varphi_1 - \varphi_0)\}, \tag{52}$$

and the relationship (47) assumes a simpler form:

$$X_x Z_z - X_z Z_x = \frac{wX_1}{\delta}. \tag{53}$$

In this case, δ is still a small difference of two large numbers but, as we have to deal with well-known functions whose tables with many decimals are available, all derivatives may be computed with any required accuracy. It is, however, convenient to introduce an auxiliary variable:

$$\mu = \frac{wx_1}{X_1} = \mathrm{gd}^{-1}\varphi_1 - \mathrm{gd}^{-1}\varphi_0, \tag{54}$$

and then the formulae (51) reduce to more convenient forms:

$$\left.\begin{aligned} X_x &= w\frac{\sinh\mu}{\mu\sinh\mu - 2(\cosh\mu - 1)}, \quad X_z = Z_x = w\frac{\sec\varphi_1 - \sec\varphi_0}{\mu\sinh\mu - 2(\cosh\mu - 1)} \\ Z_z &= w\frac{\mu\sec\varphi_1\sec\varphi_0 - \sinh\mu}{\mu\sinh\mu - 2(\cosh\mu - 1)}. \end{aligned}\right\} \tag{55}$$

(B) *Weight of cable neglected* ($w = 0$).—The formulae (45) apply in this case, with only a trivial simplification obtained by omitting terms containing the factor w. The denominator δ, from (47) may be written, using (33, 34):

$$\delta = z_1(\cos\varphi_0 - \cos\varphi_1) - \frac{2T}{n}\{1 - \cos(\varphi_1 - \varphi_0)\}. \tag{56}$$

This is still a small difference of two large numbers, but may be computed with any required accuracy. Alternatively, a transformation analogous to that of 3.2 (A) may be convenient. Introducing

$$\nu = \frac{nz_1}{T} = \ln\tan\frac{\varphi_1}{2} - \ln\tan\frac{\varphi_0}{2}, \tag{57}$$

formulae (45) reduce to:

$$\left.\begin{aligned} X_x &= \frac{n}{\sin\varphi_1}\,\frac{(\cosh\nu - 1)\cos^3\varphi_1 + \sinh\nu\cos 2\varphi_1 + \nu(\cosh\nu + \sinh\nu\cos\varphi_1)\sin^2\varphi_1}{\nu\sinh\nu - 2(\cosh\nu - 1)}, \\ X_z &= n\frac{\sinh\nu\cos\varphi_1 + (\cosh\nu - 1)\sin^2\varphi_1}{\nu\sinh\nu - 2(\cosh\nu - 1)}, \\ Z_x &= n\frac{(\cosh\nu - 1)(1 + \cos^2\varphi_1) + 2\sinh\nu\cos\varphi_1 - \nu(\cosh\nu + \sinh\nu\cos\varphi_1)\cos\varphi_1}{\nu\sinh\nu - 2(\cosh\nu - 1)}, \\ Z_z &= n\sin\varphi_1\frac{\sinh\nu - (\cosh\nu - 1)\cos\varphi_1}{\nu\sinh\nu - 2(\cosh\nu - 1)}. \end{aligned}\right\} \tag{58}$$

4. *Lateral Cable Derivative.*

In equilibrium conditions, if the wind velocity V is parallel to x-axis, there are no lateral forces (in y-direction) and therefore no lateral displacements of the cable which lies entirely in xz-plane. Let us assume, however, that a small lateral force Y_1 is applied to the upper end, then this end will move in the same direction through a small distance y_1. We shall have only one lateral cable derivative:

$$Y_y = \frac{dY_1}{dy_1} = \lim_{y_1 \to 0} \frac{Y_1}{y_1}, \tag{59}$$

and this will play a part in the problem of lateral stability of the balloon.

In the existing literature the derivative Y_y has never been determined for the general case when both gravity and wind forces act on the cable. Bairstow, Relf and Jones[1] neglected the wind forces, and the same assumption appears to have been made by Brown[7]. In this case, a small lateral displacement y_1 of the upper end will result merely in a rotation of the entire system about the vertical axis z through a small angle $\epsilon' = y_1/x_1$. The horizontal component $T_1 \cos \varphi_1$ of the tension at the upper end will also rotate through the same angle, so that the lateral force will be $Y_1 = T_1 \cos \varphi_1 \epsilon'$, and this leads to a very simple formula:

$$Y_y = \frac{T_1 \cos \varphi_1}{x_1}. \tag{60}$$

An alternative assumption was made by Glauert[3] for the case of a body towed by a thin wire whose weight could be neglected in comparison with wind forces. In this case, a small displacement y_1 of the cable upper end will result again in a mere rotation of the entire system, but this time about the axis x, through a small angle $\epsilon'' = y_1/z_1$. The lateral force will arise due to rotation of vertical component $T_1 \sin \varphi_1$ of the tension at upper end, and will be $Y_1 = T_1 \sin \varphi_1 \epsilon''$, so that we obtain another very simple formula:

$$Y_y = \frac{T_1 \sin \varphi_1}{z_1}. \tag{61}$$

However, neither of the two above formulae can be used for captive balloons when the cable is subject to both gravity and wind forces (it is also obvious that the two formulae always give different results). The difficulty in this general case is that a lateral displacement of the upper end results in the cable curve becoming three-dimensional, and it would be very difficult to obtain full equations of that curve. It suffices, however, to consider small displacements from xz-plane, in which case the equations simplify considerably, and this way is followed below.

Let us consider a small element dl of the curve (Fig. 4), and denote by α, β, γ the angles this element makes with x-, y-, z-axes. The element is supposed to be in equilibrium under the tensions at two ends, the weight $(-w\,dl)$ acting along z-axis, and the aerodynamic force due to the wind of velocity V directed along x-axis. This velocity must be resolved in two components, tangential $V \cos \alpha$ and normal $V \sin \alpha$. As in the two-dimensional case, we assume that only the latter produces an aerodynamic force $n \sin^2\alpha \, dl$, and we need expressions for its direction cosines. To find them, we note that the total velocity V has resolutes in x-, y-, z-directions:

$$V, \quad 0, \quad 0, \tag{62}$$

respectively, its tangential component $V \cos \alpha$ has the resolutes:

$$V \cos^2\alpha, \qquad V \cos \alpha \cos \beta, \qquad V \cos \alpha \cos \gamma, \tag{63}$$

and the resolutes of the normal component $V \sin \alpha$, obtained by subtracting (63) from (62), respectively, become:

$$V \sin^2\alpha, \qquad - V \cos \alpha \cos \beta, \qquad - V \cos \alpha \cos \gamma. \tag{64}$$

The direction cosines of the normal component are thus:

$$\cos \alpha' = \sin \alpha, \qquad \cos \beta' = - \frac{\cos \alpha \cos \beta}{\sin \alpha}, \qquad \cos \gamma' = - \frac{\cos \alpha \cos \gamma}{\sin \alpha}, \tag{65}$$

and hence the aerodynamic-force resolutes are obtained as

$$n \sin^3\alpha \, dl, \qquad - n \sin \alpha \cos \alpha \cos \beta \, dl, \qquad - n \sin \alpha \cos \alpha \cos \gamma \, dl, \tag{66}$$

respectively.

To write the equilibrium conditions for the element, we notice that the tension resolutes at the lower end of the element are $T \cos \alpha$, $T \cos \beta$, $T \cos \gamma$, so that the equations are:

$$\left.\begin{aligned} d(T \cos \alpha) + n \sin^3\alpha \, dl &= 0, \\ d(T \cos \beta) - n \sin \alpha \cos \alpha \cos \beta \, dl &= 0, \\ d(T \cos \gamma) - n \sin \alpha \cos \alpha \cos \gamma \, dl - w \, dl &= 0. \end{aligned}\right\} \tag{67}$$

Expanding differentials of products, multiplying equations by $\cos \alpha$, $\cos \beta$, $\cos \gamma$, respectively, and adding, we obtain:

$$dT = w \, dl \cos \gamma = w \, dz, \tag{68}$$

and hence the relationship (6) still holds. Substituting (68) into the first two of equations (67), we may write them as follows:

$$\left.\begin{aligned} (w \cos \alpha \cos \gamma + n \sin^3\alpha) dl &= T \sin \alpha \, d\alpha, \\ (w \cos \gamma - n \sin \alpha \cos \alpha) \cos \beta \, dl &= T \sin \beta \, d\beta, \end{aligned}\right\} \tag{69}$$

and then, eliminating T, we obtain:

$$\frac{\sin \beta \, d\beta}{\cos \beta} = \frac{w \cos \gamma - n \sin \alpha \cos \alpha}{w \cos \alpha \cos \gamma + n \sin^3\alpha} \sin \alpha \, d\alpha. \tag{70}$$

This differential equation contains three variables α, β, γ; we may eliminate γ by using the relationship:

$$\cos \gamma = \sqrt{(\sin^2\alpha - \cos^2\beta)}, \tag{71}$$

but the resulting equation cannot be integrated in an elementary way. If, however, we make use of the assumption that the lateral displacements y are small, then $\cos \beta = dy/dl$ is also small of the same order, and (71) may be written as $\cos \gamma \approx \sin \alpha$, the error being of 2nd order. It will be then convenient to introduce the angle φ (Fig. 3) between the projection of dl on xz-plane and x-axis, and to write:

$$\alpha \approx \varphi, \qquad \gamma \approx \frac{\pi}{2} - \varphi, \tag{72}$$

the errors being still of the 2nd order. The 1st and 3rd of equations (67) then reduce to (2, 3). This means that, to this order of approximation, the projection of the 3-dimensional curve on xz-plane remains the same as the plane curve obtained in Section 2.1. The differential equation (70) then assumes a simple form:

$$\frac{d(\cos \beta)}{\cos \beta} = \frac{w - n \cos \varphi}{n \sin^2\varphi + w \cos \varphi} d(\cos \varphi), \tag{73}$$

and is easily integrable. Introducing the constant ψ from (8) we obtain:

$$\frac{d(\cos\beta)}{\cos\beta} = \frac{(2\cot 2\psi - \cos\varphi)d(\cos\varphi)}{1 + 2\cot 2\psi\cos\varphi - \cos^2\varphi} = \tfrac{1}{2}\left(\frac{1+\cos 2\psi}{\tan\psi + \cos\varphi} - \frac{1-\cos 2\psi}{\cot\psi - \cos\varphi}\right) d(\cos\varphi), \tag{74}$$

and then, integrating, using (10), and simplifying:

$$\cos\beta = K\sqrt{\frac{n\sin^2\varphi + w\cos\varphi}{n\tau(\varphi)}}, \tag{75}$$

where the constant K may be determined from the condition at the upper end:

$$\cos\beta_1 = K\sqrt{\frac{n\sin^2\varphi_1 + w\cos\varphi_1}{n\tau(\varphi_1)}}. \tag{76}$$

We have further:

$$dy = dl\cos\beta, \tag{77}$$

where, according to (7, 15, 21):

$$dl = \frac{T_1}{\tau_1}\frac{\tau}{n\sin^2\varphi + w\cos\varphi}\,d\varphi. \tag{78}$$

Substituting (78) into (77) and simplifying, we get:

$$dy = \frac{T_1\cos\beta_1}{\sqrt{\{n\tau_1(n\sin^2\varphi_1 + w\cos\varphi_1)\}}}\sqrt{\frac{n\tau}{n\sin^2\varphi + w\cos\varphi}}\,d\varphi. \tag{79}$$

Noticing that $T_1\cos\beta_1 = Y_1$, and integrating (79) from φ_0 to φ_1, we obtain:

$$y = \frac{Y_1}{\sqrt{\{n\tau_1(n\sin^2\varphi_1 + w\cos\varphi_1)\}}}\{\vartheta(\varphi) - \vartheta(\varphi_0)\}, \tag{80}$$

where

$$\vartheta(\varphi) = \int_0^{\varphi}\sqrt{\frac{n\tau}{n\sin^2\varphi + w\cos\varphi}}\,d\varphi. \tag{81}$$

(80) is the third parametric equation of our curve, while the first two equations remain (16) and (17).

Putting now $\varphi = \varphi_1$, $y = y_1$ in (80), we obtain finally the lateral cable derivative:

$$Y_y = \frac{Y_1}{y_1} = \frac{\sqrt{\{n\tau_1(n\sin^2\varphi_1 + w\cos\varphi_1)\}}}{\vartheta_1 - \vartheta_0}. \tag{82}$$

The new auxiliary function $\vartheta(\varphi)$, defined by (81), cannot be reduced to elementary integrals, but might be tabulated numerically. There is hardly need for this, however, as $(\vartheta_1 - \vartheta_0)$ may be computed, in each particular case, by applying one of the formulae for approximate integration (such as Simpson's rule), with quite sufficient accuracy. It is also possible to expand (82) in powers of $(\varphi_1 - \varphi_0)$, as previously done for the longitudinal derivatives, with the following result (for derivations, *see* Appendix II):

$$Y_y = \frac{n\sin^2\varphi_1 + w\cos\varphi_1}{\varphi_1 - \varphi_0} + \frac{w - n\cos\varphi_1}{2}\sin\varphi_1 \ldots, \tag{83}$$

and the two first terms often give a sufficiently accurate result (the third term may be found in Appendix II). In practice, formula (82) is quite easy to handle and more accurate, but (83) may be used for quick estimates. Comparing it with (50) it is seen that the lateral derivative is generally much smaller than the longitudinal ones, especially for highly tensioned cables. This, of course, could be expected.

As a final check of the above theory, let us come back to the two particular cases $n = 0$ and $w = 0$:

(A) *Action of wind on cable neglected* ($n = 0$).—Replacing $n\tau$ by $w \sec \varphi$, according to (22), the formula (81) becomes:

$$\vartheta(\varphi) = \int_0^{\varphi} \sec \varphi \, d\varphi = \mathrm{gd}^{-1}\varphi, \tag{84}$$

and hence, taking into account (26), the lateral derivative reduces to:

$$Y_y = \frac{w}{\mathrm{gd}^{-1}\varphi_1 - \mathrm{gd}^{-1}\varphi_0} = \frac{X_1}{x_1}, \tag{85}$$

which is identical with (60).

(B) *Weight of cable neglected* ($w = 0$).—Here τ is to be replaced by 1 {cf. (30)}, so that (81) becomes:

$$\vartheta(\varphi) = \int_0^{\varphi} \operatorname{cosec} \varphi \, d\varphi = \ln \tan \frac{\varphi}{2}, \tag{86}$$

and hence, using (35), the lateral derivative becomes:

$$Y_y = \frac{n \sin \varphi_1}{\ln \tan \dfrac{\varphi_1}{2} - \ln \tan \dfrac{\varphi_0}{2}} = \frac{T_1 \sin \varphi_1}{z_1}, \tag{87}$$

in agreement with (61).

As an illustration to this Section, Fig. 5 shows a few examples of the laterally disturbed cable curves. It should be remembered that they are 3-dimensional curves, but only their projections on yz-plane are shown. For each of the two cases (lightly and highly tensioned cables) three curves are given, corresponding to $n/w = \infty$, 1, and 0. For the first value, the projection is a straight line because then y is proportional to z (*see* 35, 80, 86); for other values, the projections become more curved (convex outward) as n/w decreases, but the curvature is always small, and hence there is little difference between the curves, especially for highly tensioned cables. It must be kept in mind that all lateral displacements y are supposed small but, in the graph, they are magnified to such (arbitrary) scales as to make the differences between curves clearly visible. Numerical data and other details will be found in Section 6.

5. *Normal Oscillatory Modes of Floating Mass at Upper End of Cable.*

5.1. *Longitudinal Oscillations.*

Let us consider a balloon and cable in equilibrium conditions, and suppose that the balloon has been displaced slightly in the xz-plane. In real conditions, this will lead to complicated oscillations with 3 degrees of freedom because, in addition to horizontal and vertical translatory movements, the balloon will also rotate in pitch. The dynamic process will be described by a complex system of differential equations of 6th order. We are going, however, to study only a greatly simplified problem, in which the pitching rotation and all aerodynamic restoring and damping forces caused by the disturbance are ignored. The balloon will therefore be considered as a floating mass particle subject (apart from the constant static forces) only to the forces (37). The theory will apply approximately to a fictitious spherical captive balloon, with all damping forces and cable inertia neglected. The equations of motion will be:

$$\left.\begin{aligned} m\frac{d^2\xi}{dt^2} + X_x\xi + X_z\zeta &= 0\,, \\ m\frac{d^2\zeta}{dt^2} + Z_x\xi + Z_z\zeta &= 0\,, \end{aligned}\right\} \tag{88}$$

where letters ξ, ζ denote small horizontal and vertical displacements. The solution is obtained in the usual way, assuming:

$$\xi = A\sin(\omega t+\epsilon)\,, \qquad \zeta = B\sin(\omega t+\epsilon)\,. \tag{89}$$

Substituting (89) into (88), we obtain

$$\frac{B}{A} = \frac{m\omega^2 - X_x}{X_z} = \frac{Z_x}{m\omega^2 - Z_z}\,, \tag{90}$$

hence

$$(m\omega^2)^2 - (X_x+Z_z)(m\omega^2) + (X_xZ_z - X_zZ_x) = 0 \tag{91}$$

and

$$2m\omega_{1,2}^2 = X_x + Z_z \pm \sqrt{\{(Z_z-X_x)^2 + 4X_zZ_x\}}\,, \tag{92}$$

whence

$$\left(\frac{B}{A}\right)_{1,2} = \frac{Z_z - X_x \pm \sqrt{\{(Z_z-X_x)^2 + 4X_zZ_x\}}}{2X_z}\,. \tag{93}$$

It is seen that there are two simple harmonic oscillatory modes, as both solutions given by (92) are positive. In the mode 1, the balloon oscillates at a high frequency along a straight line of positive slope, while the mode 2 is an oscillation of lower frequency along another straight line, of negative slope. The formulae (92, 93), in conjunction with complicated expressions (45) do not give an immediate insight into the quantitative relations. However, the method of series expansion in powers of $(\varphi_1-\varphi_0)$ proves very helpful again. It requires a very tedious algebra (briefly summarised in Appendix III), but the final formulae are rather simple:

$$m\omega_1^2 = \frac{12(n\sin^2\varphi_1 + w\cos\varphi_1)}{(\varphi_1-\varphi_0)^3}\left\{1 + \frac{\sin\varphi_1}{2}\,\frac{w - 2n\cos\varphi_1}{n\sin^2\varphi_1 + w\cos\varphi_1}(\varphi_1-\varphi_0)\ldots\right\}, \tag{94}$$

$$m\omega_2^2 = \frac{n\sin^2\varphi_1 + w\cos\varphi_1}{\varphi_1 - \varphi_0}\left\{1 + \frac{\frac{1}{2}w\sin\varphi_1}{n\sin^2\varphi_1 + w\cos\varphi_1}(\varphi_1-\varphi_0)\ldots\right\}, \tag{95}$$

(two more terms of each series are given in Appendix III);

$$\left(\frac{B}{A}\right)_1 = s_1 = \tan\varphi_1\left\{1 - \frac{\varphi_1-\varphi_0}{\sin 2\varphi_1} + \left(\frac{1}{4\cos^2\varphi_1} + \frac{1}{12\cos\varphi_1}\,\frac{w - 4n\cos\varphi_1}{n\sin^2\varphi_1 + w\cos\varphi_1}\right)(\varphi_1-\varphi_0)^2\ldots\right\} \tag{96}$$

$$-\left(\frac{B}{A}\right)_2 = s_2 = \cot\varphi_1\left\{1 + \frac{\varphi_1-\varphi_0}{\sin 2\varphi_1} + \left(\frac{1}{4\sin^2\varphi_1} - \frac{1}{12\cos\varphi_1}\,\frac{w}{n\sin^2\varphi_1 + w\cos\varphi_1}\right)(\varphi_1-\varphi_0)^2\ldots\right\}. \tag{97}$$

The latter two expansions may be compared with the following ones (*see* end of Appendix III):

$$\frac{z_1}{x_1} = \tan\varphi_1\left\{1 - \frac{\varphi_1-\varphi_0}{\sin 2\varphi_1} + \left(\frac{1}{4\cos^2\varphi_1} + \frac{1}{6\cos\varphi_1}\,\frac{w - n\cos\varphi_1}{n\sin^2\varphi_1 + w\cos\varphi_1}\right)(\varphi_1-\varphi_0)^2\ldots\right\}, \tag{98}$$

$$\frac{x_1}{z_1} = \cot\varphi_1\left\{1 + \frac{\varphi_1-\varphi_0}{\sin 2\varphi_1} + \left(\frac{1}{4\sin^2\varphi_1} - \frac{1}{6\cos\varphi_1}\,\frac{w - n\cos\varphi_1}{n\sin^2\varphi_1 + w\cos\varphi_1}\right)(\varphi_1-\varphi_0)^2\ldots\right\}. \tag{99}$$

It is seen that, if $(\varphi_1 - \varphi_0)$ is small, the slopes s_1, s_2 are very nearly the same as those of the secant OB (Fig. 1) and of the normal to this secant, respectively. The oscillation 1 thus consists, approximately, in the balloon moving rapidly towards the mooring point O and away from it, so that the cable flexes and unflexes. In the oscillation 2, the system behaves, approximately, as a simple pendulum rotating about the mooring point in xz-plane.

Formulae (94, 95) show that the ratio of two natural frequencies is

$$\frac{\omega_1}{\omega_2} \approx \frac{2\sqrt{3}}{\varphi_1 - \varphi_0} \left\{ 1 - \frac{\frac{1}{2} n \cos \varphi_1 \sin \varphi_1}{n \sin^2 \varphi_1 + w \cos \varphi_1} (\varphi_1 - \varphi_0) \ldots \right\}, \tag{100}$$

so it may become very large when the cable is highly tensioned.

We may still consider the series expansion of the formula (18) for the cable length (*see* Appendix I):

$$l_1 = \frac{T_1(\varphi_1 - \varphi_0)}{n \sin^2 \varphi_1 + w \cos \varphi_1} \left\{ 1 - \sin \varphi_1 \frac{w - n \cos \varphi_1}{n \sin^2 \varphi_1 + w \cos \varphi_1} (\varphi_1 - \varphi_0) \ldots \right\}; \tag{101}$$

multiplying (95) by (101), we obtain:

$$m\omega_2{}^2 l_1 = T_1 \left\{ 1 + \frac{\sin \varphi_1}{2} \frac{2n \cos \varphi_1 - w}{n \sin^2 \varphi_1 + w \cos \varphi_1} (\varphi_1 - \varphi_0) \ldots \right\}, \tag{102}$$

so that

$$\omega_2 = \sqrt{\left(\frac{T_1}{ml_1}\right)} \left\{ 1 + \frac{\sin \varphi_1}{4} \frac{2n \cos \varphi_1 - w}{n \sin^2 \varphi_1 + w \cos \varphi_1} (\varphi_1 - \varphi_0) \ldots \right\}. \tag{103}$$

If the cable is highly tensioned, the frequency ω_2 is thus very nearly given by the ordinary pendulum formula. This result could be expected, and it provides a satisfactory check of the entire theory.

A more realistic approach to the longitudinal stability of a spherical captive balloon will be obtained by introducing damping forces acting on it, proportional to the velocity components $d\xi/dt$ and $d\eta/dt$ in the equations of motion (88). This may also be considered as some sort of approximation for a kite balloon, with disturbance in pitch neglected. A short analysis of this case is given in Appendix IV.

5.2. *Lateral Oscillations.*

This is a very simple case, with a single degree of freedom, and the equation of motion (damping neglected) is:

$$m \frac{d^2\eta}{dt^2} + Y_y \eta = 0, \tag{104}$$

so that the frequency (ω_l) is given by

$$m\omega_l^2 = Y_y \tag{105}$$

or, using the expansions (83, 101):

$$\omega_l = \sqrt{\left(\frac{T_1}{ml_1}\right)} \left\{ 1 + \frac{\sin \varphi_1}{4} \frac{n \cos \varphi_1 - w}{n \sin^2 \varphi_1 + w \cos \varphi_1} (\varphi_1 - \varphi_0) \ldots \right\}. \tag{106}$$

It is seen that, to the first approximation, the frequency ω_l is again given by the ordinary pendulum formula. The second approximations in (103) and (106) differ somewhat but, at least in the case of a highly tensioned cable, the longitudinal and lateral pendulum modes have nearly the same frequency. This conclusion may be of considerable importance for the general problem of kite-balloon oscillations. It is hnown that the existing balloons suffer from insufficient damping in the lateral mode[13,14], so that they are subject to slow oscillations of very large amplitudes*. In these conditions, a (non-linear) coupling between the lateral and longitudinal oscillations may make appearance, and this may contribute to enhancing both, if their natural frequencies are nearly in resonance. The frequencies of the two pendulum modes may, of course, differ from (103) and (106), owing to the complicated effects of balloon aerodynamics, but large changes are unlikely, and the possibility of near-resonance is clearly there.

6. *Numerical Examples.*

(A) *Equilibrium configurations, see Section* 2 *and Fig.* 2.—It is assumed for simplicity that the balloon is set at *zero incidence* at any wind, so that no aerodynamic lift is present; also, that the *balloon height* z_1 *is maintained constant* = 900 ft throughout the range of wind speed, more cable length being paid out as this increases. Other data assumed are: $Z_1 = T_1 \sin \varphi_1$ = reserve buoyancy = 745 lb (constant); $C_D = 0{\cdot}1185$, $\rho = 0{\cdot}00232$ lb. sec²/ft⁴, $d_b = 25{\cdot}2$ ft, $S_b = 499$ ft², so that $X_1 = T_1 \cos \varphi_1 = D = C_D(\frac{1}{2}\rho V^2)(\frac{1}{4}\pi d_b^2) = 0{\cdot}0686\ V^2$ (lb), V being measured in ft/sec; $w = 0{\cdot}1$ lb/ft, $C_{Dc} = 1{\cdot}025$, $d_c = 0{\cdot}0225$ ft, so that {*see* formula (1)} $n = 0{\cdot}0002744\ V^2$ (lb/ft). A number of round values of the ratio n/w appearing in Table 1 have been taken, leading to corresponding values of speed V and other basic parameters, as tabulated below:

n/w	V ft/sec	X_1 lb	T_1 lb	φ_1^0	τ_1	τ_0	φ_0^0	σ_1	σ_0	λ_1	λ_0	x_1 ft	l_1 ft
0·2	27·00	50	746·7	86·16	15·964	14·040	84·07	1·218	1·185	3·054	2·668	77	903
0·4	38·18	100	751·7	82·35	3·849	3·388	78·24	0·753	0·721	1·389	1·201	156	918
0·6	46·76	150	760·0	78·62	1·982	1·747	72·59	0·638	0·602	1·013	0·866	230	939
0·8	53·99	200	771·4	74·97	1·404	1·240	67·11	0·625	0·581	0·913	0·771	302	975
1·0	60·37	250	785·8	71·45	1·159	1·026	62·00	0·652	0·596	0·887	0·741	380	990
1·2	66·13	300	803·1	68·07	1·032	0·916	57·30	0·697	0·626	0·904	0·747	460	1018
1·4	71·43	350	823·1	64·84	0·958	0·853	52·88	0·747	0·660	0·935	0·762	534	1062
1·6	76·36	400	845·6	61·77	0·912	0·815	48·83	0·803	0·695	0·973	0·782	626	1107
2·0	85·37	500	897·2	56·13	0·862	0·776	41·83	0·909	0·762	1·055	0·826	765	1192
2·5	95·45	625	972·4	50·00	0·833	0·756	34·40	1·038	0·821	1·160	0·866	1013	1373

Fig. 2 gives all the corresponding configurations, and it is seen that the curvature is small in all cases, though it increases with wind speed. This is clearly connected with the cable being 'highly tensioned', as could be seen in advance from the fact that the reserve buoyancy is over 8 times greater than the weight of the initial cable length.

A number of similar and somewhat more complicated examples (involving, e.g., constant cable length, aerodynamic lift, etc.) are given in Ref. 5 (calculated by exactly the same method) and in Ref. 6 (using a different but equivalent method).

(B) *Longitudinal derivatives, see Section* 3 *and Fig.* 3.—The same numerical data as under (A) have been used but, of 10 values of n/w and corresponding equilibrium configurations, only 6 (as

* A plausible explanation of this phenomenon is that the damping is negative at small amplitudes where the motions may be analysed by linearised equations, but becomes positive at large amplitudes, where the non-linear effects become significant.

tabulated in Fig. 3) have been selected for computing longitudinal derivatives. This has been found sufficient for drawing curves showing variation of derivatives with n/w and V. The cable being highly tensioned, and the difference $(\varphi_1 - \varphi_0)$ small (especially for low values of n/w), the basic formulae (45, 46), in conjunction with Table 1 containing only few decimals, seem to be inconvenient for computation. In fact, when (46) was used to calculate δ in this way, either 0·000 or minute positive or even negative values were found in most cases. The series expansions are clearly most suitable here and, of several alternatives, series (I.20) proved most convenient. The results are tabulated as inset in Fig. 3. They have been checked by using (47). It should be noted that X_z and Z_x are so nearly (but never exactly) equal that the difference could be shown only in the table but not in the graph. A remark may be not superfluous that X_z, Z_x and Z_z all increase indefinitely when $n/w \to 0$, and only X_x then remains finite, as could be expected.

(C) *Lateral derivative, see Section* 4 *and Fig.* 5.—The derivative Y_y has been calculated for three cases only, of the ten listed above under (A), by using (81, 82), Table 1, and numerical integration by means of Trapezoidal, Simpson's and Weddle's Rules, which all gave identical results, viz.

n/w	0·2	1·0	2·5
Y_y	0·780	0·767	0·737 lb/ft

and it is seen that the variation was too small to warrant a graph. Instead, Fig. 5 gives some examples of laterally distorted cable curves as already discussed at the end of Section 4. It proved sufficient to consider only the extreme values 0, ∞, and one intermediate value 1 of the ratio n/w. The assumed angles φ_1, φ_0 for the highly tensioned cable correspond to the appropriate case of the table given under (A), and kept unaltered for $n/w = 0$ or ∞. For the lightly tensioned cable, arbitrary angles 76° and 30° were chosen.

(87440)

LIST OF SYMBOLS

A		Constant, *see* (89)
a		Lift slope of balloon, *see* Appendix IV
a_1, a_2, a_3, a_4		Coefficients in expansion of l_1, *see* (I.5, 6)
A_1, A_2		Portmanteau symbols, *see* (IV.9, 10)
B		Constant, *see* (89)
B'		Coefficient of λ^3 in characteristic equation of system, *see* (IV.7, 8)
b_1, b_2, b_3, b_4		Coefficients in expansion of longitudinal derivatives, *see* (I.20, 21)
C_D		Drag coefficient of balloon, *see* Section 6 and Appendix IV
C_{Dc}		Cable drag coefficient, *see* (1)
C	=	$1 - \cos\theta$, *see* (I.15)
C'		Coefficient of λ^2 in characteristic equation of system, *see* (IV.7, 8)
c_1, c_2, c_3, c_4		Coefficients in expansion of longitudinal derivatives, *see* (I.20, 21)
D		Balloon drag
D'		Coefficient of λ in characteristic equation of system, *see* (IV.7, 8)
d_b		Diameter of balloon, maximum cross-section
d_c		Cable diameter
d_1, d_2, d_3, d_4		Coefficients in expansion of longitudinal derivatives, *see* (I.20, 21)
E'		Constant term in characteristic equation of system, *see* (IV.7, 8)
$F(\varphi)$		General symbol for any of integrands used in deriving expansions of longitudinal and lateral cable derivatives, *see* Appendices I, II
$f(\lambda)$		Quartic polynomial in characteristic equation, *see* (IV.7)
h_2, h_3, h_4		Coefficients in the expansion of δ, *see* (48, I.18)
j, k		Factors of damping coefficients relating to longitudinal oscillations, *see* (IV.4, 5, 6)
K		Constant, *see* (75)
l		Length of cable from origin to current point, *see* Fig. 1
l_1		Total length of cable
m		Mass of balloon, *see* Section 5 and Appendix IV; apparent mass either ignored or, if included, with differences in its value in x-, y-, z-directions disregarded
N		Normal force per unit length of cable at balloon attachment point, *see* (I.7)

n		Drag per unit length of cable when perpendicular to wind
P_n		Normal wind-force component acting on an inclined cable element, per unit length
p		Portmanteau symbol, *see* (IV.9, 10)
S	=	$\sin \theta$, *see* (I.15)
S_b		Representative balloon area (usually maximum cross-sectional area)
$s_1, (-s_2)$		Slopes of paths of oscillation of balloon in two longitudinal modes
T		Tension at any point of cable
T_0, T_1		Values of T at mooring point and at point of attachment to balloon, respectively
t		Time
V		Wind velocity
w		Weight of cable per unit length
X_1	=	$T_1 \cos \varphi_1$, horizontal component of T_1, equal to balloon drag
X_x, X_z		Horizontal force derivatives due to x- and z-displacements, respectively
x		Horizontal co-ordinate of current point of cable (in wind direction)
x_1		Value of x at upper end of cable
Y_1		Lateral force on upper end of cable
Y_y		Lateral force derivative due to y-displacement
y		Lateral displacement of current point of cable (perpendicular to equilibrium plane)
y_1		Value of y at upper end of cable
Z_1	=	$T_1 \sin \varphi_1$, vertical component of T_1, equal to sum of balloon reserve buoyancy and aerodynamic lift if any (*see* Section 2.1)
Z_x, Z_z		Vertical force derivatives due to x- and z-displacements, respectively
z		Vertical co-ordinate of current point of cable
z_1		Value of z at upper end of cable, or balloon height
α, β, γ		Direction angles of cable element in three-dimensional case, *see* Section 4 (paragraph 3) and Fig. 4
α', β', γ'		Direction angles of normal wind velocity component at cable element in three-dimensional case, *see* Section 4 (paragraph 3)
α_1, α_2		Portmanteau symbols, *see* (IV.9, 10)

δ		Common denominator in formulae (45) for longitudinal derivatives, *see* (46)
ϵ		Phase angle, *see* (89)
ϵ'	=	y_1/x_1, small angle of rotation of entire cable configuration about z-axis in case of small lateral displacement, with wind forces neglected
ϵ''	=	y_1/z_1, small angle of rotation of entire cable configuration about x-axis in case of small lateral displacement, with gravity forces neglected
ζ		Small vertical displacement of balloon, *see* (88)
η		Small horizontal (lateral) displacement of balloon, *see* (88)
$\vartheta(\varphi)$		Auxiliary function, *see* (81)
ϑ, ϑ_0, ϑ_1		Abbreviations of $\vartheta(\varphi)$, $\vartheta(\varphi_0)$, $\vartheta(\varphi_1)$
θ	=	$\varphi_1 - \varphi_0$, difference between cable inclinations at upper and lower ends; basic variable in all power expansions
$\lambda(\varphi)$		Auxiliary tabulated function, *see* (14) and Table 1
λ, λ_0, λ_1		Abbreviations of $\lambda(\varphi)$, $\lambda(\varphi_0)$, $\lambda(\varphi_1)$
μ		Auxiliary variable, *see* (54)
ν		Auxiliary variable, *see* (57)
ξ		Small horizontal (longitudinal) displacement of balloon, *see* (88)
ρ		Air density
$\sigma(\varphi)$		Auxiliary tabulated function, *see* (14) and Table 1
σ, σ_0, σ_1		Abbreviations of $\sigma(\varphi)$, $\sigma(\varphi_0)$, $\sigma(\varphi_1)$
$\tau(\varphi)$		Auxiliary tabulated function, *see* (10) and Table 1
τ, τ_0, τ_1		Abbreviations of $\tau(\varphi)$, $\tau(\varphi_0)$, $\tau(\varphi_1)$
φ		Inclination of cable element to horizontal for equilibrium configuration
φ_0, φ_1		Values of φ at ground and balloon, respectively
χ_1, χ_2		Portmanteau symbols, *see* (IV.9, 10)
ψ		Angle defined by equation (8) (constant for a given cable under given wind speed)
ω		Oscillatory frequency of (balloon + cable) system
ω_1, ω_2		Solutions of frequency equation (91) (frequencies of rapid mode 1, and of slow 'pendulum' mode 2, respectively)
ω_l		Frequency of lateral oscillation of (balloon + cable) system

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APPENDIX I

Power Expansions for Longitudinal Cable Derivatives

The basic idea of the procedure applied in this Appendix and in the two following ones is to assume that the difference $\varphi_1 - \varphi_0 = \theta$ is small, and to expand various quantities appearing in the formulae for cable derivatives as power series in θ. The method was already applied, to a very limited extent, by K. Mitchell[8] and W. S. Brown[7] for the related problem of towed gliders and kites. Here, an attempt is made to go as far as practicable with the procedure, so as to provide most convenient formulae for designer's use. The convergence of all series will be satisfactory only if θ is sufficiently small, but this is believed to be so in most cases. In the case of slack cables θ may be large, but then the unexpanded formulae, such as (45), will give tolerable accuracy.

In what follows, we start by expanding such quantities as l_1, x_1 etc. {which comes to the same as $(\lambda_1-\lambda_0)$, $(\sigma_1-\sigma_0)$, etc.} as power series in θ, with coefficients expressed only in terms of data pertaining to the upper end of the cable (T_1, φ_1) and, of course, of basic parameters w and n. The fundamental series, of which all the subsequent ones are special cases, is:

$$\int_{\varphi_0}^{\varphi_1} F(\varphi)\, d\varphi = F(\varphi_1)\,\theta - \frac{F'(\varphi_1)}{2\,!}\,\theta^2 + \frac{F''(\varphi_1)}{3\,!}\,\theta^3 - \frac{F'''(\varphi_1)}{4\,!}\,\theta^4 + \ldots; \tag{I.1}$$

this is merely one of the many alternative forms of Taylor's expansion, most convenient for our needs. Two simple particular cases (which will be needed later) may be noted:

$$\left.\begin{aligned} \cos\varphi_0 - \cos\varphi_1 &= \int_{\varphi_0}^{\varphi_1} \sin\varphi\, d\varphi = \sin\varphi_1\,\theta - \frac{\cos\varphi_1}{2\,!}\,\theta^2 - \frac{\sin\varphi_1}{3\,!}\,\theta^3 + \frac{\cos\varphi_1}{4\,!}\,\theta^4 + \frac{\sin\varphi_1}{5\,!}\,\theta^5\ldots \\ \sin\varphi_1 - \sin\varphi_0 &= \int_{\varphi_0}^{\varphi_1} \cos\varphi\, d\varphi = \cos\varphi_1\,\theta + \frac{\sin\varphi_1}{2\,!}\,\theta^2 - \frac{\cos\varphi_1}{3\,!}\,\theta^3 - \frac{\sin\varphi_1}{4\,!}\,\theta^4 + \frac{\cos\varphi_1}{5\,!}\,\theta^5\ldots \end{aligned}\right\} \tag{I.2}$$

Let us now consider the expansion of the function {cf. form. (18)}:

$$l_1 = \frac{T_1}{n\tau_1}\,(\lambda_1-\lambda_0) = \frac{T_1}{n\tau_1}\int_{\varphi_0}^{\varphi_1} \lambda'\, d\varphi\,. \tag{I.3}$$

In this case {cf. form. (21, 12)}:

$$F(\varphi) = \lambda' = \frac{n\tau}{n\sin^2\varphi + w\cos\varphi}, \qquad \left[F(\varphi_1) = \lambda_1' = \frac{n\tau_1}{n\sin^2\varphi_1 + w\cos\varphi_1}\right] \tag{I.4}$$

and, differentiating this four times, while making use of (12) each time, and substituting φ_1 for φ, we obtain:

$$F'(\varphi_1) = \lambda_1'' = \lambda_1' a_1, \quad F''(\varphi_1) = \lambda_1''' = \lambda_1' a_2, \quad F'''(\varphi_1) = \lambda_1^{iv} = \lambda_1' a_3, \text{ etc.} \tag{I.5}$$

where:

$$\left.\begin{aligned} a_1 &= \frac{2\sin\varphi_1}{N}\,(w - n\cos\varphi_1), \\ a_2 &= \frac{2}{N^2}\{w^2(1+2\sin^2\varphi_1) - wn\cos\varphi_1(1+4\sin^2\varphi_1) + n^2\sin^2\varphi_1(1+2\cos^2\varphi_1)\}, \\ a_3 &= \frac{2\sin\varphi_1}{N^3}\{4w^3(2+\sin^2\varphi_1) - w^2n\cos\varphi_1(17+12\sin^2\varphi_1) + 3wn^2(4+3\sin^2\varphi_1 - 4\sin^4\varphi_1) - \\ &\quad - 4n^3\sin^2\varphi_1\cos\varphi_1(2+\cos^2\varphi_1)\}, \\ a_4 &= \frac{2}{N^4}\{4w^4(2+11\sin^2\varphi_1+2\sin^4\varphi_1) - w^3n\cos\varphi_1(17+143\sin^2\varphi_1+32\sin^4\varphi_1) + \\ &\quad + w^2n^2(12+194\sin^2\varphi_1 - 84\sin^4\varphi_1 - 48\sin^6\varphi_1) - \\ &\quad - wn^3\cos\varphi_1\sin^2\varphi_1(120+37\sin^2\varphi_1 - 32\sin^4\varphi_1) + 4n^4\sin^4\varphi_1(2+11\cos^2\varphi_1+2\cos^4\varphi_1)\}, \end{aligned}\right\} \tag{I.6}$$

where, for abbreviation:

$$N = n \sin^2\varphi_1 + w \cos \varphi_1 \tag{I.7}$$

{formulae (I.6) look repulsively complicated, but the final formulae will be seen to be much neater}.

Introducing (I.4) into (I.1), and using (I.5), we obtain:

$$l_1 = \frac{T_1\theta}{N}\left(1 - \frac{a_1}{2}\,\theta + \frac{a_2}{6}\,\theta^2 - \frac{a_3}{24}\,\theta^3 + \frac{a_4}{120}\,\theta^4 \ldots\right). \tag{I.8}$$

Similarly, we have:

$$x_1 = \frac{T_1}{n\tau_1}\int_{\varphi_0}^{\varphi_1} \sigma'\,d\varphi\,, \tag{I.9}$$

where

$$\sigma' = \frac{n\tau \cos \varphi}{n \sin^2\varphi + w \cos \varphi} = \lambda' \cos \varphi\,, \tag{I.10}$$

and hence, after differentiating (I.10) several times:

$$x_1 = \frac{T_1\theta}{N}\left(\cos \varphi_1 - \frac{a_1 \cos \varphi_1 - \sin \varphi_1}{2}\,\theta + \frac{a_2 \cos \varphi_1 - 2a_1 \sin \varphi_1 - \cos \varphi_1}{6}\,\theta^2 - \right.$$
$$\left. - \frac{a_3 \cos \varphi_1 - 3a_2 \sin \varphi_1 - 3a_1 \cos \varphi_1 + \sin \varphi_1}{24}\,\theta^3 + \ldots\right). \tag{I.11}$$

An exactly similar procedure yields:

$$z_1 = \frac{T_1\theta}{N}\left(\sin \varphi_1 - \frac{a_1 \sin \varphi_1 + \cos \varphi_1}{2}\,\theta + \frac{a_2 \sin \varphi_1 + 2a_1 \cos \varphi_1 - \sin \varphi_1}{6}\,\theta^2 - \right.$$
$$\left. - \frac{a_3 \sin \varphi_1 + 3a_2 \cos \varphi_1 - 3a_1 \sin \varphi_1 - \cos \varphi_1}{24}\,\theta^3 \ldots\right). \tag{I.12}$$

The next step is to find the expansion of δ {cf. form. (46)}, and this could be done by using (I.2, 8, 11, 12) and the ordinary expansion for $\sin(\varphi_1 - \varphi_0)$. However, the algebra involved is rather heavy, and a more convenient way is to present δ in the easily derived form:

$$\delta = \frac{T_1}{n\tau_1}\int_{\varphi_0}^{\varphi_1} \lambda'\{\sin(\varphi - \varphi_0) + \sin(\varphi_1 - \varphi) - \sin(\varphi_1 - \varphi_0)\}\,d\varphi \tag{I.13}$$

and apply (I.1) directly*. It is easily found that, in this case:

$$\left.\begin{aligned} &F(\varphi_1) = 0\,; \quad F'(\varphi_1) = -\lambda_1' C\,; \quad F''(\varphi_1) = -\lambda_1'(2a_1C + S)\,; \\ &F'''(\varphi_1) = -\lambda_1'\{(3a_2 - 1)C + 3a_1S\}\,; \quad F^{iv}(\varphi_1) = -\lambda_1'\{(4a_3 - 4a_1)C + (6a_2 - 1)S\}\,; \\ &F^{v}(\varphi_1) = -\lambda_1'\{(5a_4 - 10a_2 + 1)C + (10a_3 - 5a_1)S\}\,; \\ &F^{vi}(\varphi_1) = -\lambda_1'\{(6a_5 - 20a_3 + 6a_1)C + (15a_4 - 15a_2 + 1)S\}, \end{aligned}\right\} \tag{I.14}$$

* It may be mentioned that the expression in curly brackets in (I.13) is easily shown to be always positive and, as λ' is also positive, we have always $\delta > 0$. Also, the expression in curly brackets is 0 for $\varphi = \varphi_0$ and $\varphi = \varphi_1$, and has its only maximum $2 \sin \frac{1}{2}\theta(1 - \cos \frac{1}{2}\theta)$ at $\varphi = \frac{1}{2}(\varphi_0 + \varphi_1)$. The expression is therefore very small whenever the angle θ is small, and thus is limited to very small values for highly tensioned cables. This explains why δ is so small in such cases.

where

$$\left.\begin{aligned} C &= 1-\cos\theta = \tfrac{1}{2}\theta^2\left(1-\frac{\theta^2}{12}+\frac{\theta^4}{360}\cdots\right), \\ S &= \sin\theta = \theta\left(1-\frac{\theta^2}{6}+\frac{\theta^4}{120}\cdots\right). \end{aligned}\right\} \tag{I.15}$$

Substituting into (I.1) and simplifying, we obtain:

$$\delta = \frac{T_1\theta^4}{12N}\left\{1-\frac{a_1}{2}\,\theta+\left(\frac{3a_2}{20}-\frac{1}{15}\right)\theta^2+\frac{a_1-a_3}{30}\,\theta^3+\left(\frac{a_4}{168}-\frac{17a_2}{1680}+\frac{1}{560}\right)\theta^4\cdots\right\} \tag{I.16}$$

and, dividing (I.16) by (I.8):

$$\frac{\delta}{l} = \frac{\theta^3}{12}\,(1-h_2\theta^2+h_3\theta^3-h_4\theta^4\ldots), \tag{I.17}$$

where:

$$\left.\begin{aligned} h_2 &= \frac{4+a_2}{60} = \frac{1}{30N^2}\,(3w^2-wn\cos\varphi_1+3n^2\sin^2\varphi_1), \\ h_3 &= \frac{a_3-a_1a_2}{120} = \frac{\sin\varphi_1}{60N^3}\,\{6w^3-13w^2n\cos\varphi_1+wn^2(10-3\sin^2\varphi_1)-6n^3\cos\varphi_1\sin^2\varphi_1\}, \\ h_4 &= \frac{1}{5040}\,\{12a_4+21a_1(a_1a_2-a_3)-(9+5a_2+14a_2{}^2)\} = \ldots. \end{aligned}\right\} \tag{I.18}$$

(The expression for h_4 in terms of w, n, φ_1 is prohibitively long, but it is very seldom needed.) The expansion (48) has thus been proved.

To obtain the expansions of the complete longitudinal derivatives (45), we need the following auxiliary series, of which the two last ones are easily derived from (I.8, 11, 12):

$$\left.\begin{aligned} T_1(\sin\varphi_1-\sin\varphi_0) &= T_1\theta\left(\cos\varphi_1+\frac{\sin\varphi_1}{2}\,\theta-\frac{\cos\varphi_1}{6}\,\theta^2-\frac{\sin\varphi_1}{24}\,\theta^3\ldots\right), \\ T_1(\cos\varphi_0-\cos\varphi_1) &= T_1\theta\left(\sin\varphi_1-\frac{\cos\varphi_1}{2}\,\theta-\frac{\sin\varphi_1}{6}\,\theta^2+\frac{\cos\varphi_1}{24}\,\theta^3\ldots\right), \\ z_1-l_1\sin\varphi_0 &= \frac{T_1\theta}{N}\left(\frac{\cos\varphi_1}{2}\,\theta-\frac{a_1\cos\varphi_1-2\sin\varphi_1}{6}\,\theta^2+\right. \\ &\qquad \left.+\frac{a_2\cos\varphi_1-3a_1\sin\varphi_1-3\cos\varphi_1}{24}\,\theta^3\ldots\right), \\ l_1\cos\varphi_0-x_1 &= \frac{T_1\theta}{N}\left(\frac{\sin\varphi_1}{2}\,\theta-\frac{a_1\sin\varphi_1+2\cos\varphi_1}{6}\,\theta^2+\right. \\ &\qquad \left.+\frac{a_2\sin\varphi_1+3a_1\cos\varphi_1-3\sin\varphi_1}{24}\,\theta^3\ldots\right), \end{aligned}\right\} \tag{I.19}$$

and hence, after some more algebraic work, and introducing (I.16):

$$
\left.
\begin{aligned}
X_x &= \frac{12}{\theta^3}\,\frac{N\cos^2\varphi_1 + b_1\theta - c_1\theta^2 - d_1\theta^3\ldots}{1-\dfrac{a_1}{2}\,\theta + \left(\dfrac{3a_2}{20}-\dfrac{1}{15}\right)\theta^2 + \dfrac{a_1-a_3}{30}\,\theta^3\ldots}\,,\\
X_z &= \frac{12}{\theta^3}\,\frac{N\sin\varphi_1\cos\varphi_1 - b_2\theta - c_2\theta^2 + d_2\theta^3\ldots}{1-\dfrac{a_1}{2}\,\theta+\ldots}\,,\\
Z_x &= \frac{12}{\theta^3}\,\frac{N\sin\varphi_1\cos\varphi_1 - b_3\theta - c_3\theta^2 - d_3\theta^3\ldots}{1-\dfrac{a_1}{2}\,\theta+\ldots}\,,\\
Z_z &= \frac{12}{\theta^3}\,\frac{N\sin^2\varphi_1 - b_4\theta - c_4\theta^2 + d_4\theta^3}{1-\dfrac{a_1}{2}\,\theta+\ldots}\,,
\end{aligned}
\right\}
\qquad \text{(I.20)}
$$

where:

$$
\left.
\begin{aligned}
2b_1 &= N\sin 2\varphi_1 - w\sin\varphi_1\cos^2\varphi_1\\
2b_2 = 2b_3 &= N\cos 2\varphi_1 + w\sin^2\varphi_1\cos\varphi_1\\
2b_4 &= N\sin 2\varphi_1 + w\sin^3\varphi_1\\
6c_1 &= \tfrac{1}{2}N(3\cos 2\varphi_1 - 1) + 2w\sin^2\varphi_1\cos\varphi_1 + na_1\sin^3\varphi_1\cos\varphi_1\\
6c_2 &= \tfrac{3}{2}N\sin 2\varphi_1 - 2w\sin\varphi_1\cos^2\varphi_1 + na_1\sin^4\varphi_1\\
6c_3 &= \tfrac{3}{2}N\sin 2\varphi_1 + 2w\sin^3\varphi_1 - (N\cos\varphi_1 + w\sin^2\varphi_1)a_1\cos\varphi_1\\
6c_4 &= -\tfrac{1}{2}N(3\cos 2\varphi_1 + 1) - 2w\sin^2\varphi_1\cos\varphi_1 - (N\cos\varphi_1 + w\sin^2\varphi_1)a_1\sin\varphi_1\\
24d_1 &= 2N\sin 2\varphi_1 - 3w\sin\varphi_1\cos^2\varphi_1 + 3na_1\sin^4\varphi_1 - na_2\sin^3\varphi_1\cos\varphi_1\\
24d_2 &= N(2\cos 2\varphi_1 - 1) + 3w\sin^2\varphi_1\cos\varphi_1 + 3na_1\sin^3\varphi_1\cos\varphi_1 + na_2\sin^4\varphi_1\\
24d_3 &= -N(2\cos 2\varphi_1 + 1) - 3w\sin^2\varphi_1\cos\varphi_1 - 3(N\cos\varphi_1 + w\sin^2\varphi_1)a_1\sin\varphi_1 +\\
&\quad + (N\cos\varphi_1 + w\sin^2\varphi_1)a_2\cos\varphi_1\\
24d_4 &= 2N\sin 2\varphi_1 + 3w\sin^3\varphi_1 - 3(N\cos\varphi_1 + w\sin^2\varphi_1)a_1\cos\varphi_1 -\\
&\quad - (N\cos\varphi_1 + w\sin^2\varphi_1)a_2\sin\varphi_1\,.
\end{aligned}
\right\}
\qquad \text{(I.21)}
$$

Performing the division in (I.20) up to the term in θ, we obtain the formulae (50). Further terms become very complicated, and the formulae (I.20) are more convenient if higher accuracy is required. They will be found particularly useful in Appendix III.

APPENDIX II

Power Expansions for Lateral Cable Derivative

It is required to expand (82), where the denominator is

$$\eta_1 - \eta_0 = \int_{\varphi_0}^{\varphi_1} \sqrt{\frac{n\tau}{n \sin^2\varphi + w \cos \varphi}}\, d\varphi . \tag{II.1}$$

We use again the general expansion (I.1), where:

$$F(\varphi) = \sqrt{\frac{n\tau}{n \sin^2\varphi + w \cos \varphi}} \tag{II.2}$$

and, differentiating twice, and making use of (12), we obtain:

$$\left.\begin{aligned} F'(\varphi) &= F(\varphi) \frac{\sin \varphi(w - n \cos \varphi)}{n \sin^2\varphi + w \cos \varphi}, \\ F''(\varphi) &= F(\varphi) \frac{w^2(1+\sin^2\varphi) - wn \cos \varphi(1+2 \sin^2\varphi) + n^2 \sin^2\varphi(1+\cos^2\varphi)}{(n \sin^2\varphi + w \cos \varphi)^2} . \end{aligned}\right\} \tag{II.3}$$

Substituting in (I.1) and simplifying, we get

$$\eta_1 - \eta_0 = F(\varphi_1) \left\{ 1 - \frac{\sin \varphi_1(w - n \cos \varphi_1)}{2N} \theta + \right.$$

$$\left. + \frac{w^2(1+\sin^2\varphi_1) - wn \cos \varphi_1(1+2 \sin^2\varphi_1) + n^2 \sin^2\varphi_1(1+\cos^2\varphi_1)}{6N^2} \theta^2 \ldots \right\} \tag{II.4}$$

and hence, from (82):

$$Y_y = \frac{N}{\theta} \left\{ 1 + \frac{\sin \varphi_1(w - n \cos \varphi_1)}{2N} \theta - \right.$$

$$\left. - \frac{w^2(2-\sin^2\varphi_1) - 2wn \cos^3\varphi_1 - n^2 \sin^2\varphi_1(1+\sin^2\varphi_1)}{12N^2} \theta^2 \ldots \right\} . \tag{II.5}$$

The expansion (83) has thus been proved, with one additional term.

APPENDIX III

Power Expansions for Characteristics of Longitudinal Oscillatory Modes

It is required to expand (92) and (93), by using (I.20). The transformation is rather lengthy, and only a few major steps will be indicated below:

$$X_x + Z_z = \frac{12}{\theta^3}\,\frac{N - \tfrac{1}{2}w\sin\varphi_1\theta + \dfrac{1}{6}(N + a_1 w\sin\varphi_1)\theta^2 + \dfrac{1}{24}(3w\sin\varphi_1 - 3a_1N - wa_2\sin\varphi_1)\theta^3\ldots}{1 - \dfrac{a_1}{2}\theta + \left(\dfrac{3a_2}{20} - \dfrac{1}{15}\right)\theta^2 + \dfrac{a_1 - a_3}{30}\theta^3\ldots}, \tag{III.1}$$

$$Z_z - X_x = \frac{12}{\theta^3}\,\frac{-N\cos 2\varphi_1 - (N\sin 2\varphi_1 - \tfrac{1}{2}w\sin\varphi_1\cos 2\varphi_1)\theta + \left\{\tfrac{1}{2}N\cos 2\varphi_1 + \dfrac{2w\sin^2\varphi_1\cos\varphi_1}{3} + \dfrac{a_1\sin\varphi_1}{6}(2N\cos\varphi_1 - w\cos 2\varphi_1)\right\}\theta^2\ldots}{1 - \dfrac{a_1}{2}\theta + \ldots}, \quad \ldots \tag{III.2}$$

$$\sqrt{\{(Z_z - X_x)^2 + 4X_zZ_x\}} = \frac{12}{\theta^3}\,\frac{N - \tfrac{1}{2}w\sin\varphi_1\theta + \dfrac{a_1 w\sin\varphi_1}{6}\theta^2 + \dfrac{w\sin\varphi_1 - a_1N - wa_2\sin\varphi_1}{24}\theta^3\ldots}{1 - \dfrac{a_1}{2}\theta + \ldots}. \tag{III.3}$$

Adding (III.1) and (III.3), we obtain:

$$m\omega_1{}^2 = \frac{12N}{\theta^3}\,\frac{1 - \dfrac{w\sin\varphi_1}{2N}\theta + \left(\dfrac{1}{12} + \dfrac{a_1 w\sin\varphi_1}{6N}\right)\theta^2 + \left(\dfrac{w\sin\varphi_1}{12N} - \dfrac{a_1}{12} - \dfrac{a_2 w\sin\varphi_1}{24N}\right)\theta^3\ldots}{1 - \dfrac{a_1}{2}\theta + \left(\dfrac{3a_2}{20} - \dfrac{1}{15}\right)\theta^2 + \dfrac{a_1 - a_3}{30}\theta^3\ldots}. \tag{III.4}$$

Subtracting (III.3) from (III.1) leads to

$$m\omega_2{}^2 = \frac{N}{\theta}\,\frac{1 + \left(\dfrac{w\sin\varphi_1}{2N} - \dfrac{a_1}{2}\right)\theta\ldots}{1 - \dfrac{a_1}{2}\theta\ldots}, \tag{III.5}$$

and it is seen that only two terms of the series have been obtained {as against four terms in (III.4)}. However, we may make use of the relationship {cf. (91) and (47)}:

$$(m\omega_1^2)(m\omega_2^2) = X_x Z_z - X_z Z_x = \frac{T_1 N}{\delta} = \frac{12N^2}{\theta^4 \left\{1 - \frac{a_1}{2}\,\theta + \left(\frac{3a_2}{20} - \frac{1}{15}\right)\theta^2 \ldots\right\}}\,. \tag{III.6}$$

It is seen at a glance that (III.4, 5) satisfy (III.6) up to the terms in θ. We may reverse the procedure using (III.4) and (III.6) to obtain more terms of the expansion for $(m\omega_2^2)$:

$$m\omega_2^2 = \frac{N}{\theta}\,\frac{1}{1 - \frac{w \sin \varphi_1}{2N}\,\theta + \left(\frac{1}{12} + \frac{a_1 w \sin \varphi_1}{6N}\right)\theta^2 + \left(\frac{w \sin \varphi_1}{12N} - \frac{a_1}{12} - \frac{a_2 w \sin \varphi_1}{24N}\right)\theta^3 \ldots} \tag{III.7}$$

and, performing the division in (III.4) and (III.7), we get:

$$\left.\begin{aligned} m\omega_1^2 &= \frac{12N}{\theta^3}\left[1 + \left(\frac{a_1}{2} - \frac{w \sin \varphi_1}{2N}\right)\theta + \left\{\frac{3}{20}(1-a_2) + \frac{a_1^2}{4} - \frac{a_1 w \sin \varphi_1}{12N}\right\}\theta^2 \ldots\right] \\ m\omega_2^2 &= \frac{N}{\theta}\left[1 + \frac{w \sin \varphi_1}{2N}\,\theta - \frac{(n \sin^2\varphi_1 - w \cos \varphi_1)^2 + w^2 \sin^2\varphi_1}{12N^2}\,\theta^2 \ldots\right], \end{aligned}\right\} \tag{III.8}$$

so that (94, 95) have been proved, with additional terms.

Finally, introducing (III.2, 3) into (93), we obtain:

$$\left.\begin{aligned} \left(\frac{B}{A}\right)_1 &= \tan \varphi_1\, \frac{1 - \left(\cot \varphi_1 + \frac{w \sin \varphi_1}{2N}\right)\theta + \left\{\frac{1}{4}(\cot^2\varphi_1 - 1) + \frac{w \cos \varphi_1}{3N} + \frac{a_1}{6}\left(\cot \varphi_1 + \frac{w \sin \varphi_1}{N}\right)\right\}\theta^2 \ldots}{1 + \left\{\frac{1}{2}(\tan \varphi_1 - \cot \varphi_1) - \frac{w \sin \varphi_1}{2N}\right\}\theta - \left\{\frac{1}{2} - \frac{w \cos \varphi_1}{3N} + \frac{a_1 \tan \varphi_1}{6}\left(1 - \frac{w \cos \varphi_1}{N}\right)\right\}\theta^2 \ldots} \\ -\left(\frac{B}{A}\right)_2 &= \cot \varphi_1\, \frac{1 + \left(\tan \varphi_1 - \frac{w \sin \varphi_1}{2N}\right)\theta - \left\{\frac{1}{4}(1 - \tan^2\varphi_1) + \frac{w \sin^2\varphi_1}{3N \cos \varphi_1} + \frac{a_1}{6}\left(\tan \varphi_1 - \frac{w \sin \varphi_1}{N}\right)\right\}\theta^2 \ldots}{1 + \left\{\frac{1}{2}(\tan \varphi_1 - \cot \varphi_1) - \frac{w \sin \varphi_1}{2N}\right\}\theta \ldots} \end{aligned}\right\} \tag{III.9}$$

and this leads directly to (96, 97).

As to the expansions (98, 99), they are simply obtained by dividing (I.12) by (I.11), and *vice versa*.

APPENDIX IV

Longitudinal Oscillatory Modes for a Captive Balloon with Damping but no Disturbance in Pitch

Let us first re-consider the equations of motion with no damping (88). They may be conveniently re-written by eliminating the cable derivatives and introducing instead the frequencies ω_1, ω_2 of the two undamped natural modes, and the slopes of their respective paths $s_1 = (B/A)_1$, $s_2 = -(B/A)_2$. From (92), (93) we get:

$$\left.\begin{aligned} X_x + Z_z &= m(\omega_1^2+\omega_2^2), \qquad & X_x Z_z - X_z Z_x &= m^2\omega_1^2\omega_2^2, \\ \frac{Z_z - X_x}{X_z} &= s_1 - s_2, & \frac{Z_x}{X_z} &= s_1 s_2, \end{aligned}\right\} \tag{IV.1}$$

and hence, solving for the derivatives:

$$\left.\begin{aligned} X_x &= m\frac{s_2\omega_1^2 + s_1\omega_2^2}{s_1+s_2}, \qquad & Z_z &= m\frac{s_1\omega_1^2 + s_2\omega_2^2}{s_1+s_2}, \\ X_z &= m\frac{\omega_1^2 - \omega_2^2}{s_1+s_2}, & Z_x &= m s_1 s_2 \frac{\omega_1^2 - \omega_2^2}{s_1+s_2}. \end{aligned}\right\} \tag{IV.2}$$

Substituting into (88) and dividing by m, we obtain:

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} + \frac{s_2\omega_1^2 + s_1\omega_2^2}{s_1+s_2}\xi + \frac{\omega_1^2-\omega_2^2}{s_1+s_2}\zeta &= 0, \\ \frac{d^2\zeta}{dt^2} + s_1 s_2\frac{\omega_1^2-\omega_2^2}{s_1+s_2}\xi + \frac{s_1\omega_1^2 + s_2\omega_2^2}{s_1+s_2}\zeta &= 0. \end{aligned}\right\} \tag{IV.3}$$

The equations of motion, including aerodynamic damping forces acting on the balloon, may now be written:

$$\left.\begin{aligned} \left(\frac{d^2\xi}{dt^2} + 2jk\frac{d\xi}{dt} + \frac{s_2\omega_1^2 + s_1\omega_2^2}{s_1+s_2}\xi\right) + \frac{\omega_1^2-\omega_2^2}{s_1+s_2}\zeta &= 0, \\ s_1 s_2\frac{\omega_1^2-\omega_2^2}{s_1+s_2}\xi + \left(\frac{d^2\zeta}{dt^2} + 2k\frac{d\zeta}{dt} + \frac{s_1\omega_1^2 + s_2\omega_2^2}{s_1+s_2}\zeta\right) &= 0, \end{aligned}\right\} \tag{IV.4}$$

where $2jk$ and $2k$ are the respective damping coefficients which, of course, are normally not equal (the original damping coefficients have been divided by m, as all other coefficients). In the case of a spherical balloon, these coefficients are easily determined:

$$k = \frac{C_D S_B \rho V}{4m}, \qquad j = 2, \tag{IV.5}$$

the damping in horizontal direction being twice that in the vertical one. For kite balloons we obtain:

$$k = \frac{(a+\frac{1}{2}C_D)S_B\rho V}{2m}, \qquad j = 2\frac{C_D}{C_D+2a}, \tag{IV.6}$$

where a is the lift slope of balloon. The damping in vertical direction may now become much greater than in the horizontal one, and the value of j obviously depends on the balloon shape.

The characteristic quartic equation of the system (IV.4) is:

$$f(\lambda) = \lambda^4 + B'\lambda^3 + C'\lambda^2 + D'\lambda + E' = 0, \tag{IV.7}$$

where

$$\left.\begin{aligned} B' &= 2(j+1)k, \qquad & C' &= \omega_1^2 + \omega_2^2 + 4jk^2, \\ D' &= 2k\frac{(js_1+s_2)\omega_1^2 + (s_1+js_2)\omega_2^2}{s_1+s_2}, & E' &= \omega_1^2\omega_2^2. \end{aligned}\right\} \tag{IV.8}$$

The quartic may be factorised numerically in each particular case. If, however, k may be considered as small (as is often the case), we can factorise (IV.7) approximately, expanding the coefficients of quadratic factors into power series in k, and determining a few terms for each coefficient. Some ordinary algebra leads to the following factorisation:

$$f(\lambda) = \{\lambda^2 + (2kA_1 - 8k^3\alpha_1)\,\lambda + \omega_1^2\,(1 - 4k^2p + 16k^4\chi_1)\} \times$$
$$\times\,\{\lambda^2 + (2kA_2 + 8k^3\alpha_2)\,\lambda + \omega_2^2\,(1 + 4k^2p - 16k^4\chi_2)\} = 0, \qquad \text{(IV.9)}$$

where:

$$\left.\begin{aligned} A_1 &= \frac{s_1 + js_2}{s_1 + s_2}, \quad A_2 = \frac{js_1 + s_2}{s_1 + s_2}, \quad p = \frac{(j-1)^2}{\omega_1^2 - \omega_2^2}\,\frac{s_1 s_2}{(s_1 + s_2)^2}, \\ \alpha_1 &= \alpha_2 = p\,\frac{A_2\omega_1^2 - A_1\omega_2^2}{\omega_1^2 - \omega_2^2}, \quad \chi_1 = \frac{\alpha(A_2 - A_1) - p^2\omega_2^2}{\omega_1^2 - \omega_2^2}, \quad \chi_2 = \frac{\alpha(A_2 - A_1) - p^2\omega_1^2}{\omega_1^2 - \omega_2^2}. \end{aligned}\right\} \qquad \text{(IV.10)}$$

To see how well this approximation works, let us consider the following numerical example (the units of dimensional quantities being unspecified):

$$\omega_1 = 5, \quad \omega_2 = 1, \quad s_1 = 9\cdot 9, \quad s_2 = 0\cdot 1, \quad k = 0\cdot 5, \quad j = 2.$$

The characteristic equation is:

$$f(\lambda) = \lambda^4 + 3\lambda^3 + 28\lambda^2 + 50\cdot 76\lambda + 25 = 0$$

and, factorising it by means of (IV.9, 10) we get:

$$A_1 = 1\cdot 01, \quad A_2 = 1\cdot 99, \quad p = 0\cdot 0004125, \quad \alpha = 0\cdot 0008377,$$
$$\chi_1 = 0\cdot 0000342, \quad \chi_2 = 0\cdot 0000340,$$

thus:

$$f(\lambda) = (\lambda^2 + 1\cdot 0092\lambda + 24\cdot 9905)(\lambda^2 + 1\cdot 9908\lambda + 1\cdot 0004) = 0,$$

and, as it happens, all decimal figures retained here are correct. This example might apply to some spherical balloon. The damping has left the high frequency practically unchanged, and this mode is not strongly damped. The low frequency mode is almost critically damped.

If we now change the values of k and j to:

$$k = 2, \quad j = 0\cdot 1,$$

so as to make the example plausible in some case of kite balloon, the characteristic equation becomes:

$$\lambda^4 + 4\cdot 4\lambda^3 + 27\cdot 6\lambda^2 + 14\cdot 864\lambda + 25 = 0,$$

then

$$A_1 = 0\cdot 991, \quad A_2 = 0\cdot 109, \quad p = 0\cdot 00033414, \quad \alpha = 0\cdot 000024142,$$
$$\chi_1 = -\,0\cdot 000008918, \quad \chi_2 = -\,0\cdot 000001003,$$

and we obtain the factorisation:

$$f(\lambda) = (\lambda^2 + 3\cdot 9625\lambda + 24\cdot 8607)(\lambda^2 + 0\cdot 4375\lambda + 1\cdot 0056) = 0,$$

the correct values of coefficients being 3·9624, 24·8605, 0·4376, 1·0056.

It may be mentioned that, if damping is present, the balloon trajectories in the two modes are no longer straight lines but spirally converging curves (oblong and narrow if the damping is small).

TABLE 1

Numerical Values of Functions $\tau(\varphi)$, $\lambda(\varphi)$, $\sigma(\varphi)$

$\varphi°$	$n/w = 0·1$; $\psi = 5·655°$			$n/w = 0·2$; $\psi = 10·901°$			$n/w = 0·3$; $\psi = 15·482°$			$n/w = 0·4$; $\psi = 19·330°$			$n/w = 0·6$; $\psi = 25·097°$			$n/w = 0·8$; $\psi = 28·997°$		
	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$
0	7·946	0·000	0·000	3·213	0·000	0·000	1·846	0·000	0·000	1·279	0·000	0·000	0·848	0·000	0·000	0·705	0·000	0·000
1	7·948	0·014	0·014	3·214	0·011	0·011	1·846	0·010	0·010	1·279	0·009	0·009	0·848	0·009	0·009	0·705	0·010	0·010
2	7·951	0·027	0·027	3·215	0·022	0·022	1·847	0·019	0·019	1·280	0·018	0·018	0·849	0·018	0·018	0·706	0·020	0·020
3	7·957	0·041	0·041	3·218	0·033	0·033	1·849	0·029	0·029	1·281	0·027	0·027	0·849	0·027	0·027	0·706	0·030	0·029
4	7·966	0·055	0·055	3·221	0·045	0·045	1·851	0·039	0·039	1·282	0·036	0·036	0·850	0·036	0·035	0·707	0·040	0·039
5	7·977	0·069	0·069	3·226	0·056	0·056	1·853	0·048	0·048	1·284	0·045	0·045	0·851	0·045	0·044	0·708	0·050	0·049
6	7·990	0·083	0·083	3·231	0·067	0·067	1·856	0·058	0·058	1·286	0·054	0·054	0·853	0·054	0·053	0·709	0·059	0·059
7	8·006	0·097	0·097	3·237	0·079	0·079	1·860	0·068	0·068	1·288	0·063	0·063	0·854	0·062	0·062	0·711	0·069	0·069
8	8·024	0·111	0·111	3·245	0·090	0·090	1·864	0·078	0·077	1·291	0·072	0·072	0·856	0·071	0·071	0·712	0·079	0·079
9	8·045	0·126	0·125	3·253	0·102	0·101	1·869	0·088	0·087	1·295	0·081	0·080	0·859	0·080	0·080	0·714	0·089	0·088
10	8·069	0·140	0·139	3·263	0·113	0·112	1·874	0·098	0·097	1·298	0·090	0·089	0·861	0·089	0·089	0·716	0·099	0·098
11	8·093	0·154	0·153	3·274	0·125	0·124	1·880	0·108	0·107	1·303	0·099	0·098	0·864	0·098	0·098	0·718	0·109	0·108
12	8·120	0·168	0·167	3·285	0·136	0·135	1·887	0·118	0·116	1·307	0·108	0·107	0·867	0·107	0·106	0·721	0·119	0·117
13	8·150	0·183	0·181	3·298	0·147	0·146	1·894	0·128	0·126	1·312	0·118	0·116	0·870	0·117	0·115	0·723	0·129	0·127
14	8·184	0·197	0·195	3·311	0·159	0·158	1·902	0·138	0·136	1·317	0·127	0·125	0·874	0·126	0·124	0·726	0·139	0·137
15	8·226	0·212	0·210	3·326	0·171	0·169	1·910	0·148	0·146	1·323	0·136	0·134	0·877	0·135	0·133	0·729	0·149	0·146
16	8·261	0·227	0·224	3·340	0·183	0·181	1·919	0·158	0·156	1·329	0·146	0·143	0·881	0·144	0·142	0·733	0·159	0·156
17	8·304	0·242	0·238	3·355	0·195	0·192	1·929	0·169	0·165	1·336	0·155	0·152	0·886	0·153	0·150	0·736	0·169	0·166
18	8·351	0·257	0·253	3·372	0·207	0·204	1·939	0·179	0·175	1·343	0·164	0·161	0·890	0·162	0·159	0·740	0·179	0·175
19	8·400	0·272	0·267	3·391	0·219	0·215	1·951	0·189	0·185	1·351	0·174	0·170	0·895	0·171	0·168	0·744	0·189	0·185
20	8·453	0·288	0·282	3·412	0·232	0·227	1·962	0·199	0·195	1·359	0·183	0·179	0·900	0·181	0·177	0·748	0·199	0·195
21	8·505	0·303	0·296	3·435	0·244	0·239	1·975	0·210	0·205	1·367	0·193	0·188	0·906	0·190	0·186	0·753	0·209	0·204
22	8·561	0·319	0·311	3·459	0·256	0·250	1·988	0·221	0·215	1·376	0·203	0·198	0·912	0·200	0·194	0·757	0·219	0·213
23	8·621	0·334	0·326	3·484	0·269	0·262	2·001	0·232	0·225	1·385	0·213	0·207	0·918	0·209	0·203	0·762	0·229	0·223
24	8·685	0·350	0·341	3·511	0·282	0·274	2·016	0·243	0·235	1·395	0·223	0·216	0·924	0·219	0·212	0·767	0·240	0·232
25	8·759	0·367	0·356	3·539	0·295	0·286	2·031	0·255	0·245	1·406	0·233	0·225	0·931	0·229	0·220	0·773	0·250	0·241
26	8·826	0·384	0·371	3·567	0·308	0·298	2·047	0·266	0·255	1·417	0·243	0·234	0·937	0·238	0·229	0·778	0·260	0·251
27	8·903	0·401	0·386	3·597	0·322	0·310	2·064	0·277	0·265	1·428	0·253	0·243	0·945	0·248	0·238	0·784	0·271	0·260
28	8·983	0·418	0·401	3·628	0·336	0·322	2·081	0·288	0·275	1·440	0·264	0·252	0·952	0·258	0·247	0·790	0·281	0·269
29	9·068	0·436	0·417	3·661	0·350	0·334	2·100	0·299	0·285	1·452	0·274	0·261	0·960	0·267	0·255	0·796	0·291	0·278
30	9·157	0·454	0·432	3·696	0·364	0·346	2·119	0·310	0·295	1·465	0·284	0·270	0·968	0·277	0·264	0·803	0·302	0·288
31	9·247	0·471	0·447	3·731	0·377	0·358	2·139	0·323	0·306	1·479	0·295	0·280	0·977	0·287	0·273	0·809	0·312	0·297
32	9·343	0·489	0·463	3·769	0·392	0·370	2·160	0·336	0·316	1·493	0·306	0·289	0·986	0·298	0·281	0·816	0·323	0·305
33	9·445	0·508	0·479	3·808	0·406	0·383	2·182	0·348	0·327	1·507	0·318	0·298	0·996	0·308	0·290	0·823	0·334	0·314
34	9·551	0·527	0·495	3·850	0·420	0·395	2·205	0·361	0·337	1·523	0·329	0·307	1·005	0·319	0·298	0·831	0·345	0·323
35	9·663	0·546	0·511	3·893	0·436	0·408	2·229	0·374	0·348	1·539	0·340	0·317	1·015	0·329	0·307	0·839	0·356	0·332
36	9·777	0·566	0·527	3·938	0·452	0·420	2·253	0·387	0·358	1·555	0·352	0·326	1·025	0·340	0·315	0·846	0·367	0·341
37	9·899	0·587	0·544	3·984	0·468	0·433	2·279	0·399	0·368	1·572	0·363	0·335	1·035	0·350	0·324	0·854	0·377	0·350
38	10·026	0·609	0·561	4·034	0·484	0·446	2·306	0·412	0·379	1·590	0·374	0·344	1·046	0·361	0·332	0·863	0·388	0·359
39	10·161	0·631	0·577	4·085	0·501	0·459	2·334	0·425	0·389	1·609	0·386	0·354	1·057	0·371	0·341	0·871	0·399	0·367
40	10·302	0·653	0·595	4·139	0·518	0·472	2·364	0·438	0·400	1·628	0·397	0·363	1·068	0·382	0·350	0·880	0·410	0·376
41	10·446	0·674	0·611	4·194	0·534	0·485	2·394	0·451	0·411	1·648	0·410	0·372	1·080	0·393	0·358	0·889	0·422	0·385
42	10·598	0·696	0·629	4·252	0·551	0·498	2·425	0·465	0·421	1·669	0·424	0·382	1·093	0·405	0·366	0·899	0·433	0·393
43	10·759	0·719	0·646	4·313	0·569	0·511	2·458	0·479	0·432	1·690	0·437	0·391	1·106	0·417	0·375	0·909	0·445	0·401
44	10·929	0·743	0·664	4·337	0·587	0·525	2·493	0·494	0·443	1·713	0·451	0·401	1·119	0·429	0·383	0·919	0·457	0·409
45	11·108	0·768	0·682	4·445	0·606	0·538	2·529	0·509	0·454	1·736	0·464	0·410	1·133	0·441	0·391	0·929	0·469	0·418
46	11·289	0·794	0·700	4·513	0·626	0·552	2·565	0·525	0·465	1·760	0·477	0·420	1·147	0·452	0·400	0·939	0·480	0·426
47	11·482	0·822	0·719	4·585	0·646	0·566	2·604	0·541	0·476	1·785	0·491	0·429	1·161	0·464	0·408	0·950	0·492	0·434
48	11·686	0·851	0·737	4·661	0·667	0·580	2·644	0·558	0·487	1·810	0·504	0·439	1·176	0·476	0·416	0·961	0·504	0·443
49	11·904	0·880	0·757	4·741	0·689	0·594	2·686	0·575	0·498	1·837	0·518	0·448	1·192	0·488	0·424	0·972	0·516	0·451
50	12·132	0·911	0·776	4·825	0·711	0·608	2·730	0·593	0·509	1·865	0·531	0·457	1·208	0·500	0·433	0·984	0·527	0·459
51	12·363	0·937	0·795	4·910	0·731	0·622	2·775	0·609	0·520	1·894	0·545	0·467	1·224	0·512	0·441	0·996	0·540	0·467
52	12·610	0·965	0·815	5·000	0·753	0·637	2·822	0·627	0·532	1·924	0·560	0·477	1·241	0·525	0·449	1·008	0·554	0·474
53	12·874	0·996	0·834	5·096	0·776	0·651	2·872	0·645	0·544	1·955	0·575	0·486	1·259	0·538	0·457	1·021	0·567	0·482
54	13·154	1·030	0·855	5·196	0·800	0·666	2·924	0·664	0·555	1·988	0·591	0·496	1·277	0·551	0·465	1·034	0·580	0·489
55	13·451	1·066	0·876	5·302	0·826	0·681	2·978	0·683	0·567	2·022	0·608	0·505	1·296	0·565	0·473	1·047	0·593	0·497
56	13·749	1·104	0·897	5·409	0·853	0·696	3·033	0·704	0·578	2·056	0·625	0·515	1·315	0·579	0·481	1·061	0·606	0·504
57	14·072	1·145	0·919	5·523	0·881	0·711	3·091	0·726	0·590	2·093	0·643	0·525	1·335	0·594	0·489	1·075	0·620	0·512
58	14·418	1·188	0·941	5·645	0·911	0·727	3·153	0·748	0·601	2·131	0·661	0·534	1·355	0·608	0·496	1·089	0·633	0·519
59	14·790	1·234	0·964	5·773	0·943	0·742	3·217	0·772	0·613	2·170	0·680	0·544	1·376	0·624	0·504	1·104	0·646	0·527
60	15·185	1·283	0·987	5·909	0·975	0·758	3·285	0·796	0·624	2·211	0·700	0·553	1·398	0·639	0·512	1·119	0·659	0·534
61	15·576	1·312	1·009	6·046	1·001	0·774	3·353	0·816	0·636	2·253	0·717	0·563	1·421	0·654	0·520	1·135	0·675	0·541
62	16·006	1·348	1·032	6·193	1·029	0·791	3·426	0·839	0·649	2·297	0·736	0·573	1·444	0·669	0·527	1·151	0·691	0·548
63	16·475	1·392	1·056	6·350	1·061	0·807	3·503	0·863	0·661	2·344	0·756	0·582	1·468	0·685	0·534	1·167	0·706	0·554
64	16·982	1·443	1·081	6·518	1·097	0·823	3·585	0·890	0·673	2·393	0·777	0·592	1·493	0·702	0·542	1·184	0·722	0·561
65	17·527	1·501	1·106	6·697	1·135	0·840	3·671	0·918	0·685	2·443	0·800	0·601	1·519	0·719	0·549	1·202	0·737	0·567
66	18·077	1·567	1·132	6·875	1·178	0·857	3·757	0·948	0·697	2·495	0·823	0·611	1·546	0·737	0·556	1·219	0·753	0·574
67	18·683	1·641	1·159	7·069	1·223	0·874	3·850	0·980	0·709	2·549	0·848	0·620	1·573	0·755	0·564	1·238	0·769	0·580
68	19·344	1·721	1·187	7·279	1·272	0·892	3·949	1·014	0·721	2·607	0·874	0·630	1·602	0·774	0·571	1·257	0·784	0·587
69	20·060	1·809	1·215	7·506	1·325	0·909	4·054	1·049	0·733	2·668	0·901	0·640	1·632	0·794	0·578	1·276	0·800	0·593
70	20·832	1·904	1·244	7·750	1·381	0·927	4·165	1·087	0·746	2·731	0·929	0·649	1·663	0·814	0·586	1·296	0·815	0·600
71	21·617	1·954	1·273	8·002	1·429	0·945	4·276	1·119	0·757	2·796	0·955	0·658	1·694	0·833	0·592	1·317	0·835	0·605
72	22·505	2·023	1·303	8·274	1·484	0·964	4·397	1·154	0·769	2·864	0·982	0·667	1·727	0·854	0·598	1·338	0·855	0·610
73	23·495	2·111	1·334	8·563	1·547	0·982	4·527	1·193	0·781	2·937	1·012	0·675	1·761	0·875	0·604	1·360	0·874	0·615
74	24·589	2·218	1·366	8·876	1·618	0·999	4·667	1·236	0·793	3·014	1·044	0·684	1·797	0·897	0·610	1·382	0·894	0·620
75	25·785	2·345	1·398	9·212	1·686	1·018	4·817	1·282	0·805	3·096	1·077	0·693	1·835	0·920	0·616	1·405	0·914	0·625
76	27·084	2·490	1·432	9·574	1·762	1·037	4·970	1·332	0·817	3·181	1·113	0·702	1·873	0·944	0·622	1·429	0·933	0·630
77	28·379	2·654	1·467	9·964	1·842	1·056	5·136	1·385	0·829	3·271	1·151	0·710	1·913	0·970	0·628	1·454	0·953	0·635
78	30·012	2·838	1·502	10·393	1·930	1·075	5·315	1·442	0·841	3·364	1·191	0·719	1·955	0·996	0·634	1·480	0·973	0·640
79	31·868	3·040	1·539	10·854	2·026	1·094	5·506	1·502	0·853	3·465	1·233	0·728	1·999	1·023	0·640	1·506	0·992	0·645
80	33·947	3·262	1·576	11·365	2·128	1·112	5·709	1·566	0·865	3·572	1·277	0·737	2·044	1·051	0·646	1·533	1·012	0·650
81	36·250	3·492	1·615	11·922	2·241	1·131	5·928	1·632	0·876	3·684	1·322	0·744	2·090	1·080	0·650	1·562	1·035	0·654
82	38·891	3·763	1·654	12·537	2·366	1·149	6·165	1·704	0·887	3·804	1·371	0·751	2·141	1·110	0·655	1·591	1·058	0·657
83	41·946	4·063	1·694	13·218	2·502	1·167	6·421	1·781	0·897	3·932	1·422	0·758	2·193	1·141	0·659	1·621	1·083	0·660
84	45·525	4·431	1·735	13·977	2·656	1·184	6·699	1·866	0·907	4·068	1·478	0·764	2·248	1·175	0·663	1·653	1·108	0·663
85	49·771	4·841	1·776	14·828	2·824	1·201	7·001	1·957	0·915	4·215	1·536	0·770	2·305	1·209	0·666	1·685	1·134	0·666
86	54·890	5·371	1·816	15·789	3·020	1·216	7·333	2·057	0·923	4·371	1·599	0·775	2·366	1·245	0·669	1·719	1·162	0·668
87	61·183	5·959	1·857	16·882	3·254	1·229	7·696	2·165	0·930	4·540	1·666	0·779	2·429	1·283	0·671	1·754	1·190	0·670
88	69·103	6·794	1·889	18·138	3·490	1·240	8·098	2·287	0·935	4·723	1·740	0·782	2·496	1·324	0·673	1·791	1·219	0·671
89	79·376	7·877	1·912	19·594	3·786	1·247	8·543	2·422	0·938	4·920	1·820	0·784	2·566	1·366	0·674	1·829	1·250	0·672
90	93·229	9·208	1·927	21·304	4·123	1·251	9·040	2·571	0·939	5·135	1·905	0·785	2·641	1·411	0·674	1·869	1·282	0·672

TABLE 1 (*continued*)

Numerical Values of Functions $\tau(\varphi)$, $\lambda(\varphi)$, $\sigma(\varphi)$

$\varphi°$	$n/w = 1{\cdot}0$; $\psi = 31{\cdot}717°$			$n/w = 1{\cdot}2$; $\psi = 33{\cdot}690°$			$n/w = 1{\cdot}4$; $\psi = 35{\cdot}173°$			$n/w = 1{\cdot}6$; $\psi = 36{\cdot}323°$			$n/w = 2{\cdot}0$; $\psi = 37{\cdot}982°$			$n/w = 2{\cdot}5$; $\psi = 39{\cdot}345°$		
	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$	$\tau(\varphi)$	$\lambda(\varphi)$	$\sigma(\varphi)$
0	0·650	0·000	0·000	0·629	0·000	0·000	0·624	0·000	0·000	0·626	0·000	0·000	0·639	0·000	0·000	0·661	0·000	0·000
1	0·650	0·011	0·011	0·629	0·013	0·013	0·624	0·015	0·015	0·626	0·017	0·017	0·639	0·022	0·022	0·661	0·028	0·028
2	0·651	0·023	0·023	0·630	0·026	0·026	0·624	0·030	0·030	0·626	0·035	0·035	0·639	0·044	0·044	0·661	0·057	0·057
3	0·651	0·034	0·034	0·630	0·039	0·039	0·625	0·046	0·045	0·627	0·052	0·052	0·640	0·066	0·066	0·662	0·085	0·085
4	0·652	0·045	0·045	0·631	0·053	0·052	0·625	0·061	0·060	0·627	0·069	0·069	0·640	0·088	0·088	0·662	0·114	0·113
5	0·653	0·057	0·056	0·632	0·066	0·065	0·626	0·076	0·076	0·628	0·087	0·086	0·641	0·110	0·110	0·663	0·142	0·141
6	0·654	0·068	0·068	0·633	0·079	0·079	0·627	0·091	0·091	0·629	0·104	0·104	0·642	0·132	0·132	0·664	0·170	0·170
7	0·655	0·079	0·079	0·634	0·092	0·092	0·628	0·106	0·106	0·630	0·122	0·121	0·643	0·155	0·154	0·665	0·199	0·198
8	0·656	0·091	0·090	0·635	0·105	0·105	0·629	0·121	0·121	0·631	0·139	0·138	0·644	0·177	0·176	0·666	0·227	0·226
9	0·658	0·102	0·102	0·637	0·118	0·118	0·631	0·137	0·136	0·633	0·156	0·156	0·646	0·199	0·198	0·668	0·256	0·255
10	0·660	0·113	0·113	0·639	0·132	0·131	0·633	0·152	0·151	0·635	0·174	0·173	0·648	0·221	0·220	0·670	0·284	0·283
11	0·663	0·125	0·124	0·641	0·145	0·143	0·635	0·167	0·165	0·637	0·191	0·189	0·651	0·242	0·241	0·673	0·310	0·309
12	0·665	0·136	0·135	0·644	0·158	0·156	0·638	0·182	0·180	0·640	0·207	0·205	0·653	0·263	0·262	0·675	0·337	0·335
13	0·667	0·148	0·146	0·646	0·171	0·169	0·640	0·196	0·194	0·642	0·224	0·221	0·655	0·284	0·283	0·677	0·363	0·360
14	0·669	0·159	0·157	0·648	0·184	0·181	0·642	0·211	0·208	0·645	0·241	0·238	0·658	0·305	0·303	0·680	0·389	0·385
15	0·672	0·170	0·168	0·651	0·197	0·194	0·645	0·226	0·223	0·647	0·258	0·254	0·660	0·325	0·323	0·682	0·415	0·410
16	0·676	0·182	0·179	0·654	0·210	0·206	0·648	0·241	0·237	0·650	0·275	0·270	0·663	0·346	0·343	0·685	0·441	0·435
17	0·679	0·193	0·190	0·657	0·223	0·219	0·651	0·256	0·251	0·653	0·291	0·286	0·666	0·367	0·363	0·688	0·468	0·461
18	0·683	0·204	0·201	0·660	0·236	0·231	0·654	0·271	0·266	0·656	0·308	0·303	0·669	0·388	0·383	0·691	0·494	0·486
19	0·686	0·216	0·212	0·664	0·249	0·244	0·657	0·286	0·280	0·659	0·325	0·319	0·672	0·409	0·403	0·694	0·520	0·511
20	0·690	0·227	0·223	0·667	0·262	0·257	0·661	0·300	0·294	0·662	0·342	0·335	0·675	0·430	0·422	0·697	0·546	0·536
21	0·694	0·239	0·233	0·671	0·275	0·268	0·665	0·315	0·307	0·666	0·358	0·350	0·679	0·449	0·440	0·701	0·569	0·560
22	0·698	0·250	0·243	0·675	0·288	0·280	0·669	0·329	0·320	0·670	0·374	0·364	0·683	0·468	0·458	0·705	0·592	0·583
23	0·703	0·261	0·254	0·679	0·300	0·291	0·672	0·344	0·333	0·674	0·390	0·378	0·687	0·488	0·476	0·708	0·616	0·606
24	0·707	0·273	0·264	0·683	0·313	0·303	0·676	0·358	0·346	0·678	0·406	0·393	0·690	0·507	0·494	0·712	0·639	0·628
25	0·711	0·284	0·274	0·687	0·326	0·315	0·680	0·372	0·359	0·682	0·422	0·407	0·694	0·526	0·511	0·716	0·662	0·649
26	0·717	0·296	0·285	0·692	0·339	0·326	0·685	0·387	0·373	0·686	0·438	0·422	0·698	0·545	0·528	0·720	0·685	0·669
27	0·722	0·307	0·295	0·697	0·352	0·338	0·690	0·401	0·386	0·691	0·454	0·436	0·703	0·564	0·545	0·724	0·708	0·688
28	0·727	0·319	0·305	0·702	0·365	0·350	0·694	0·415	0·399	0·695	0·470	0·451	0·707	0·583	0·562	0·728	0·731	0·707
29	0·733	0·330	0·316	0·707	0·377	0·361	0·699	0·430	0·412	0·700	0·486	0·466	0·711	0·603	0·579	0·732	0·754	0·726
30	0·738	0·342	0·326	0·712	0·390	0·373	0·704	0·444	0·425	0·704	0·502	0·480	0·716	0·622	0·595	0·736	0·777	0·745
31	0·744	0·353	0·336	0·718	0·403	0·383	0·709	0·458	0·436	0·709	0·517	0·493	0·721	0·639	0·611	0·741	0·797	0·764
32	0·750	0·365	0·345	0·724	0·416	0·394	0·715	0·472	0·447	0·715	0·532	0·505	0·725	0·657	0·626	0·745	0·818	0·782
33	0·757	0·377	0·355	0·729	0·429	0·404	0·720	0·486	0·459	0·720	0·547	0·517	0·730	0·674	0·641	0·750	0·838	0·799
34	0·763	0·388	0·364	0·735	0·441	0·415	0·725	0·500	0·470	0·725	0·562	0·529	0·735	0·692	0·656	0·754	0·858	0·815
35	0·769	0·400	0·374	0·741	0·454	0·425	0·731	0·514	0·482	0·730	0·577	0·542	0·740	0·709	0·670	0·759	0·878	0·831
36	0·776	0·412	0·384	0·747	0·467	0·436	0·737	0·528	0·493	0·736	0·593	0·555	0·745	0·727	0·684	0·763	0·898	0·847
37	0·783	0·424	0·393	0·754	0·480	0·446	0·743	0·542	0·505	0·741	0·608	0·568	0·750	0·744	0·698	0·768	0·919	0·863
38	0·791	0·435	0·403	0·760	0·493	0·457	0·749	0·556	0·516	0·747	0·623	0·580	0·755	0·762	0·712	0·773	0·939	0·879
39	0·798	0·447	0·412	0·767	0·505	0·467	0·755	0·570	0·527	0·753	0·638	0·592	0·761	0·779	0·726	0·778	0·959	0·895
40	0·805	0·459	0·422	0·773	0·518	0·477	0·761	0·584	0·539	0·758	0·653	0·604	0·766	0·797	0·739	0·782	0·979	0·910
41	0·813	0·471	0·431	0·780	0·531	0·487	0·768	0·598	0·549	0·765	0·668	0·615	0·771	0·813	0·752	0·787	0·997	0·923
42	0·821	0·483	0·439	0·788	0·544	0·496	0·774	0·611	0·558	0·771	0·682	0·625	0·777	0·829	0·764	0·792	1·015	0·936
43	0·829	0·495	0·448	0·795	0·557	0·505	0·781	0·625	0·568	0·777	0·697	0·637	0·782	0·846	0·776	0·797	1·033	0·949
44	0·838	0·508	0·456	0·802	0·570	0·514	0·788	0·639	0·578	0·783	0·712	0·647	0·788	0·862	0·788	0·802	1·051	0·962
45	0·846	0·520	0·465	0·809	0·583	0·523	0·794	0·653	0·588	0·790	0·726	0·657	0·794	0·878	0·800	0·807	1·069	0·975
46	0·855	0·532	0·474	0·817	0·596	0·533	0·802	0·667	0·597	0·796	0·741	0·667	0·800	0·894	0·811	0·813	1·087	0·988
47	0·864	0·544	0·482	0·826	0·609	0·542	0·809	0·680	0·607	0·803	0·756	0·677	0·805	0·911	0·822	0·818	1·105	1·001
48	0·873	0·557	0·491	0·834	0·622	0·551	0·816	0·694	0·617	0·810	0·770	0·687	0·811	0·927	0·833	0·823	1·124	1·014
49	0·882	0·569	0·500	0·842	0·635	0·560	0·823	0·708	0·627	0·816	0·785	0·697	0·817	0·943	0·844	0·828	1·142	1·026
50	0·892	0·581	0·508	0·850	0·648	0·569	0·831	0·722	0·636	0·823	0·800	0·707	0·823	0·959	0·854	0·833	1·160	1·038
51	0·902	0·594	0·516	0·858	0·662	0·577	0·839	0·736	0·644	0·830	0·814	0·716	0·829	0·975	0·863	0·839	1·176	1·048
52	0·912	0·606	0·523	0·867	0·675	0·585	0·846	0·750	0·653	0·837	0·829	0·725	0·836	0·991	0·872	0·844	1·193	1·058
53	0·922	0·619	0·531	0·876	0·689	0·593	0·854	0·764	0·661	0·845	0·843	0·734	0·842	1·006	0·881	0·849	1·210	1·067
54	0·932	0·632	0·539	0·885	0·702	0·600	0·862	0·778	0·669	0·852	0·858	0·743	0·848	1·022	0·890	0·855	1·226	1·077
55	0·943	0·645	0·546	0·894	0·716	0·608	0·870	0·792	0·677	0·859	0·873	0·752	0·854	1·037	0·899	0·860	1·243	1·087
56	0·954	0·659	0·554	0·904	0·730	0·616	0·879	0·806	0·685	0·867	0·887	0·761	0·861	1·053	0·908	0·866	1·260	1·096
57	0·966	0·672	0·561	0·913	0·743	0·624	0·887	0·820	0·693	0·875	0·902	0·769	0·867	1·069	0·917	0·871	1·277	1·106
58	0·977	0·686	0·569	0·923	0·757	0·631	0·896	0·835	0·701	0·882	0·916	0·777	0·874	1·084	0·926	0·877	1·293	1·116
59	0·989	0·700	0·576	0·933	0·770	0·639	0·904	0·849	0·709	0·890	0·931	0·784	0·880	1·100	0·935	0·882	1·310	1·125
60	1·000	0·714	0·584	0·942	0·784	0·647	0·913	0·863	0·717	0·898	0·946	0·791	0·887	1·115	0·943	0·888	1·327	1·134
61	1·013	0·727	0·590	0·953	0·799	0·653	0·922	0·878	0·723	0·906	0·961	0·798	0·894	1·131	0·950	0·894	1·343	1·141
62	1·026	0·741	0·596	0·964	0·814	0·659	0·932	0·893	0·730	0·914	0·976	0·805	0·901	1·147	0·957	0·899	1·359	1·148
63	1·038	0·755	0·603	0·974	0·829	0·666	0·941	0·908	0·736	0·922	0·991	0·812	0·908	1·162	0·964	0·905	1·375	1·155
64	1·051	0·770	0·609	0·985	0·843	0·672	0·950	0·922	0·742	0·931	1·006	0·819	0·914	1·178	0·971	0·911	1·391	1·162
65	1·064	0·785	0·615	0·996	0·858	0·678	0·959	0·937	0·748	0·939	1·021	0·826	0·921	1·193	0·978	0·917	1·408	1·168
66	1·078	0·800	0·621	1·008	0·873	0·684	0·969	0·952	0·755	0·948	1·036	0·832	0·929	1·209	0·985	0·923	1·424	1·175
67	1·093	0·815	0·627	1·019	0·888	0·691	0·979	0·967	0·761	0·957	1·051	0·838	0·936	1·225	0·992	0·928	1·440	1·182
68	1·107	0·831	0·634	1·031	0·903	0·697	0·989	0·982	0·767	0·966	1·067	0·844	0·943	1·240	0·998	0·934	1·456	1·189
69	1·121	0·847	0·640	1·043	0·918	0·703	1·000	0·997	0·774	0·974	1·082	0·850	0·950	1·256	1·004	0·940	1·472	1·196
70	1·136	0·863	0·646	1·055	0·933	0·709	1·010	1·012	0·780	0·983	1·097	0·855	0·958	1·271	1·009	0·946	1·488	1·202
71	1·152	0·879	0·650	1·067	0·948	0·714	1·020	1·029	0·784	0·993	1·113	0·859	0·965	1·288	1·013	0·953	1·505	1·207
72	1·168	0·896	0·655	1·080	0·964	0·718	1·031	1·045	0·788	1·002	1·130	0·863	0·973	1·304	1·017	0·959	1·521	1·212
73	1·184	0·913	0·660	1·093	0·980	0·723	1·042	1·062	0·793	1·012	1·146	0·867	0·980	1·321	1·021	0·965	1·537	1·216
74	1·200	0·930	0·664	1·106	0·997	0·727	1·053	1·079	0·797	1·021	1·163	0·872	0·988	1·337	1·026	0·971	1·553	1·220
75	1·217	0·947	0·669	1·120	1·014	0·731	1·065	1·095	0·801	1·031	1·179	0·876	0·996	1·353	1·030	0·978	1·570	1·224
76	1·235	0·966	0·673	1·133	1·031	0·736	1·076	1·112	0·806	1·041	1·196	0·880	1·004	1·370	1·034	0·984	1·586	1·229
77	1·253	0·984	0·678	1·147	1·049	0·740	1·088	1·129	0·810	1·051	1·212	0·884	1·012	1·386	1·038	0·990	1·603	1·233
78	1·272	1·003	0·682	1·162	1·067	0·744	1·100	1·146	0·814	1·062	1·229	0·889	1·020	1·403	1·042	0·997	1·620	1·237
79	1·290	1·023	0·687	1·177	1·085	0·749	1·112	1·162	0·818	1·072	1·245	0·893	1·028	1·419	1·047	1·004	1·637	1·241
80	1·309	1·042	0·692	1·192	1·103	0·753	1·125	1·179	0·823	1·083	1·262	0·897	1·037	1·435	1·051	1·010	1·654	1·245
81	1·329	1·063	0·695	1·207	1·123	0·756	1·137	1·197	0·826	1·094	1·279	0·900	1·045	1·452	1·054	1·017	1·671	1·248
82	1·349	1·083	0·698	1·223	1·142	0·759	1·150	1·215	0·828	1·105	1·297	0·903	1·054	1·470	1·056	1·024	1·688	1·250
83	1·370	1·105	0·701	1·240	1·162	0·762	1·164	1·234	0·831	1·116	1·315	0·905	1·063	1·487	1·059	1·031	1·705	1·252
84	1·392	1·127	0·703	1·256	1·181	0·764	1·177	1·253	0·833	1·128	1·334	0·907	1·071	1·505	1·061	1·038	1·723	1·254
85	1·414	1·149	0·705	1·273	1·202	0·766	1·191	1·273	0·835	1·139	1·353	0·909	1·080	1·523	1·063	1·045	1·740	1·256
86	1·437	1·172	0·707	1·291	1·223	0·768	1·205	1·293	0·837	1·151	1·372	0·910	1·090	1·542	1·064	1·052	1·758	1·257
87	1·461	1·196	0·709	1·309	1·245	0·769	1·220	1·313	0·838	1·163	1·391	0·911	1·099	1·560	1·065	1·059	1·776	1·258
88	1·486	1·221	0·710	1·327	1·267	0·770	1·235	1·334	0·839	1·176	1·411	0·912	1·108	1·579	1·066	1·066	1·794	1·259
89	1·512	1·247	0·711	1·346	1·290	0·771	1·250	1·355	0·839	1·189	1·432	0·913	1·118	1·598	1·066	1·074	1·812	1·260
90	1·538	1·273	0·711	1·366	1·313	0·771	1·265	1·377	0·839	1·201	1·453	0·913	1·128	1·618	1·066	1·081	1·831	1·260

(87440)

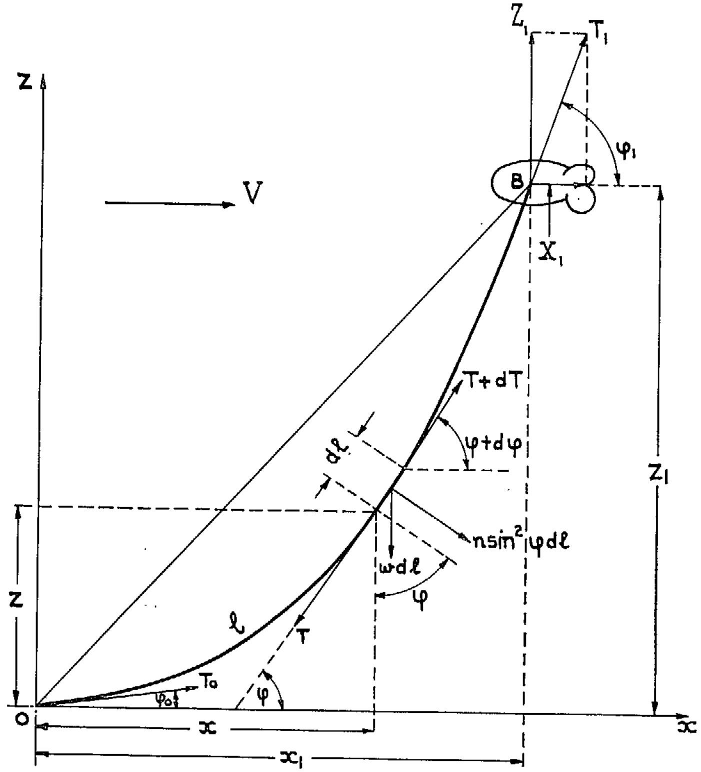


FIG. 1. Typical two-dimensional cable configuration, and forces acting on cable.

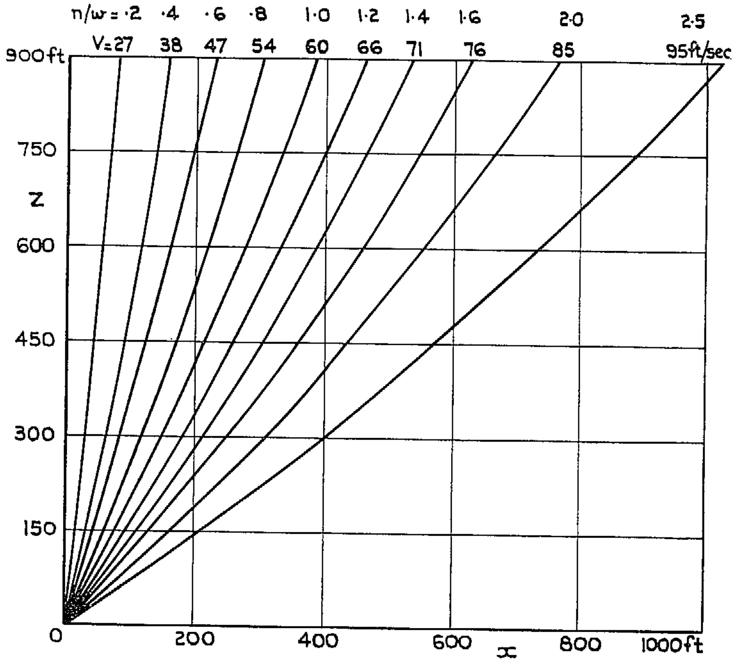
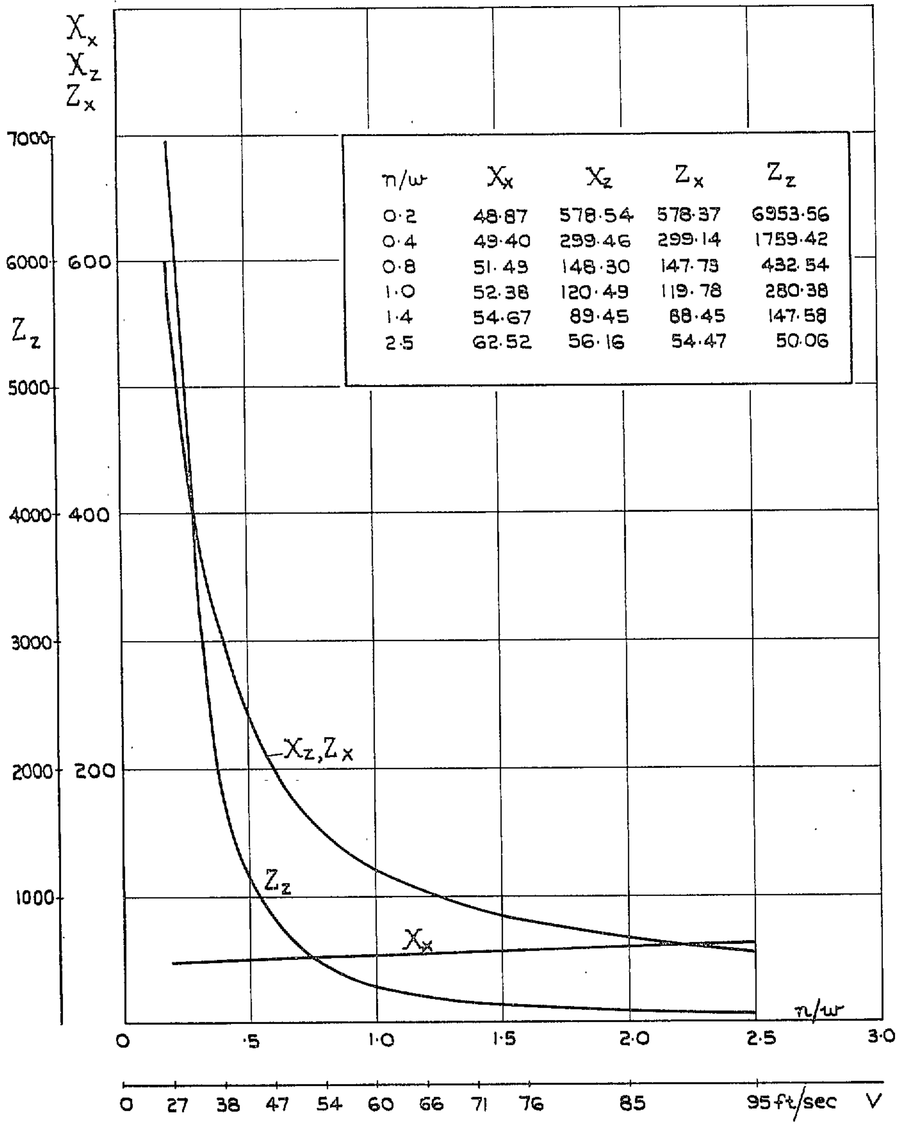


FIG. 2. Example of cable configurations for varying winds, with constant balloon height.



n/w	X_x	X_z	Z_x	Z_z
0·2	48·87	578·54	578·37	6953·56
0·4	49·40	299·46	299·14	1759·42
0·8	51·49	148·30	147·73	432·54
1·0	52·38	120·49	119·78	280·38
1·4	54·67	89·45	88·45	147·58
2·5	62·52	56·16	54·47	50·06

NOTE: DIFFERENT SCALE FOR Z_z

FIG. 3. Variation of longitudinal cable derivatives for configurations of Fig. 2.

(87440) Wt. 64/1857 K 5 11/63 Hw.

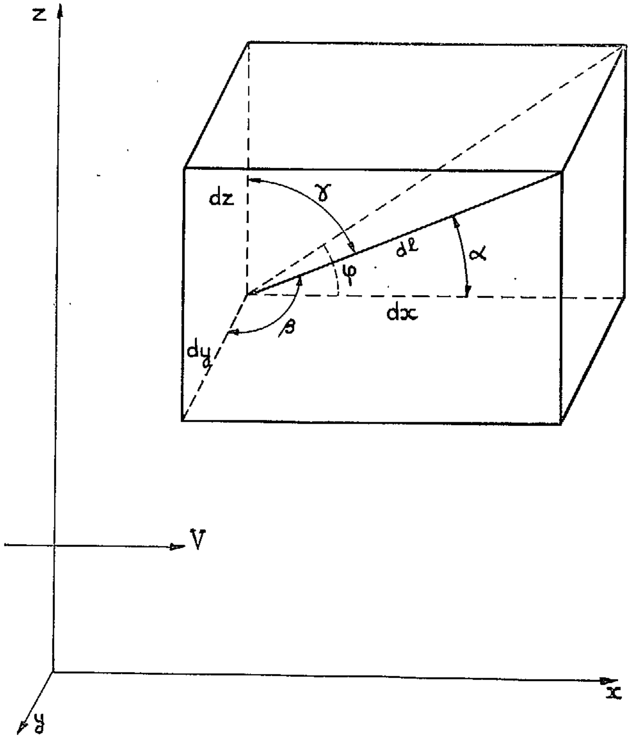


FIG. 4. Element of a cable in three dimensions in the case of a lateral displacement of a balloon.

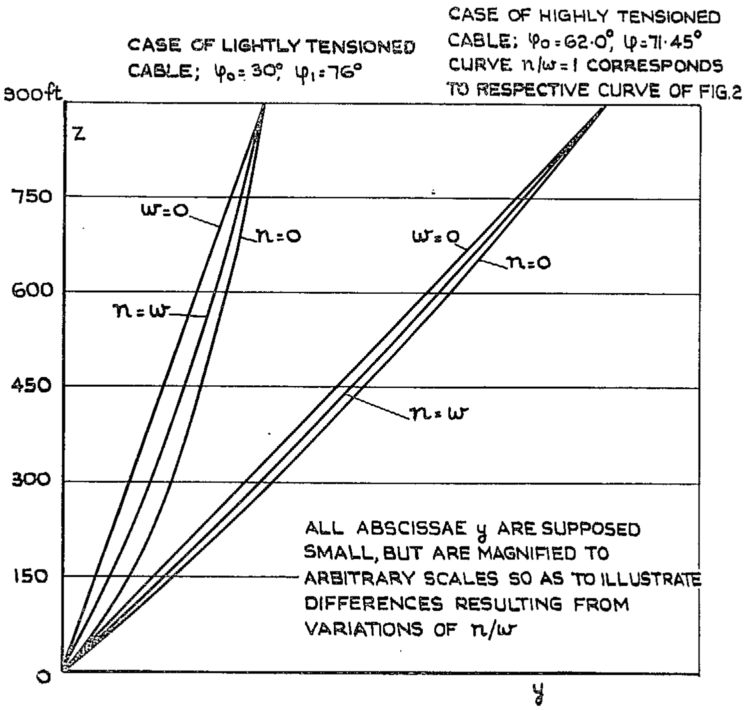


FIG. 5. Example of laterally distorted cable curves, cases of lightly or highly tensioned cables.

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