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The Effect of Solar Radiation Pressure on the Attitude Control of an Artificial Earth Satellite

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The Effect of Solar Radiation Pressure on the Attitude Control of an Artificial Earth Satellite

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Summary.

This report presents an account of the demand made by solar radiation pressure on the attitude-control system of an earth satellite whose external configuration is in the shape of a rectangular prism, the surfaces being assumed to be perfectly reflecting. Expressions determining the amount of angular impulse that must be supplied by an attitude-control system in the course of a year in order to provide perfect stabilisation for a space-stabilised satellite, and an earth-pointing satellite in a non-precessing orbit, are developed. Examples are given for particular cases and further examples include a comparison of the radiation-pressure torque with the torque set up by the earth's gravitational field, and the attitude deviations arising as a result of radiation pressure on an earth-pointing satellite employing gravity-gradient stabilisation alone.

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1. *Introduction and General Discussion.*

Radiation falling on a surface may be partly absorbed, partly reflected and, unless the body be very thick or opaque, partly transmitted. Assuming no transmission, any surface may be specified by its reflectivity or absorptivity, the former being the fraction of the radiation incident on the surface which is reflected and the latter the fraction which is absorbed. A surface having an absorptivity of unity for all wavelengths is called an 'ideal' black surface. In general, the absorptivity varies greatly with wavelength and, to a lesser extent, with the temperature of the absorber.

The amount of the sun's radiant energy in the vicinity of the earth is generally expressed in terms of the solar constant, this being the amount of the sun's radiation received on unit area in unit time, the receiving area being perpendicular to the sun's rays and at a distance from the sun equal to the mean radius of the earth's orbit. The solar constant S' may be expressed as $1 \cdot 94$ cal cm^{-2} min^{-1} or $1 \cdot 3 \times 10^6$ erg cm^{-2} sec^{-1}. If the radiation is completely absorbed by the body then $1 \cdot 3 \times 10^6$ ergs will be absorbed by 1 cm^2 every second. In accordance with Einstein's energy-mass relationship ($E = mc^2$ where c is the velocity of the radiation) the mass associated with a unit erg of radiation is $1/c^2$ so that the momentum imparted to unit area per second is cS'/c^2. Thus, taking $c = 3 \times 10^{10}$ cm sec^{-1}, the radiation pressure due to radiation at normal incidence on a completely absorbing body in the vicinity of the earth is

$$P = \frac{S'}{c} = \frac{1 \cdot 3 \times 10^6}{3 \times 10^{10}} = 4 \cdot 3 \times 10^{-5} \text{ erg cm}^{-3},$$

i.e. $$P = 4 \cdot 3 \times 10^{-5} \text{ dynes/cm}^2$$

since erg cm^{-3} $\equiv$ dyne cm^{-2}.

In the F.P.S. system of units, $P = 2 \cdot 9 \times 10^{-6}$ pdl/ft^2.

Radiation pressure will be capable of producing a torque about the centre of mass of an orbiting earth satellite. The magnitude of the radiation-pressure torque will be influenced by the nature of the satellite's surface, i.e. whether the surface reflects or absorbs most of the radiation. Bombardment by cosmic dust (micrometeorites) will inevitably produce a gradual erosion of the outer metallic skin of the satellite, similar to the erosion of a metal by the impact of high-speed molecules.

This report presents an analysis of the effect of the sun's radiation pressure on the attitude control of a satellite in the shape of a rectangular prism and whose surfaces completely reflect the radiation incident on them; both space-stabilised and earth-pointing satellites are considered. Although the theory is carried out specifically for the rectangular-prism configuration, the method can be generalised to allow corresponding theoretical deductions to be made for many of the more characteristic satellite configurations, for example cylinders, spheres and cones. Whenever curved surfaces are involved in the calculations the effective areas presented to the sun must be used in any theoretical analysis, and it is perhaps best to carry out the analysis for a given configuration rather than attempt a more general treatment to cover a range of configurations. Alternatively, the results for the rectangular prism may be used as an approximation for the sphere or cylinder if, as may well be the case, very accurate estimates are not required. Simple calculations will show the order of the error that may be expected by employing such an approximation.

A detailed analysis of the more general problem of absorption and reflection at the satellite's surface proves to be more complex than the analysis for a completely reflecting surface. In the latter case only the normal component of the force due to radiation pressure on a given surface need be considered, whereas in the case of a 'black body' surface the force on the surface will be in the direction of the incident radiation, and has therefore a tangential component as well as a normal component of force. A further complication may arise as a result of re-radiation from a surface at a given temperature; if the surface acts as a diffuse emitter, i.e. emits radiation uniformly in all directions, there will be a reaction on the surface along the direction normal to the surface. However, such considerations have not been included in the present work, which is restricted to the case of complete reflection.

As defined in Ref. 1, 'earth-pointing' satellite generally refers to a satellite whose principal axis about which the moment of inertia is least points continually towards the centre of the earth; the earth's gravitational field provides a stabilising effect for attitude motion about the other two principal axes. However, for relatively large orbital radii, when the stabilising effect may be very small, it may still be desired to have one axis continually pointing at the earth—and such a satellite would still be an earth-pointing satellite even though a built-in attitude-control system would probably be required to achieve this.

2. *Assumptions.*

(1) The sun's radiation is the only source of attitude perturbation acting on the satellite.

(2) The magnitude of the solar radiation pressure in the vicinity of the earth is constant for all positions of the earth along its orbit.

(3) The satellite's surface is homogeneous and has a reflectivity of unity.

(4) Translational motion of the satellite relative to the earth is neglected, so that the intensity of the radiation in the vicinity of the satellite is taken to be constant and equal to that amount defined by the solar constant.

(5) Eclipses of the satellite as it passes through the earth's shadow are neglected so that the satellite is regarded as being continuously under the influence of the radiation. The expressions developed in the text for the amount of angular impulse will therefore represent maximum values. The actual values will clearly depend on the amount of time spent by the satellite in the earth's shadow; for example, if the satellite spends half of its time in the shadow, the angular impulse will be approximately half the corresponding maximum value. If the control system is designed to provide this maximum angular impulse it will be capable of providing compensation at all times, irrespective of the changing shadowing effect due, for example, to orbital regression.

(6) Attitude control will in general be provided about the principal axes of inertia of the satellite, but the theory presented in this report applies to any systems of orthogonal right-handed axes (origin at the centre of mass of the satellite) assumed to be rigidly embedded in the vehicle.

It will be assumed that the chosen axes lie along directions parallel to the faces of the satellite so that each pair of axes lies in a plane parallel to two surfaces of the configuration (Fig. 1).

Assumptions (4) and (5) applied to the case of the space-stabilised satellite reduce the problem to an investigation into the effect of solar radiation pressure on the attitude control of a space-stabilised body following the same orbit round the sun as that of the earth. In the case of the earth-pointing satellite (in a non-precessing orbit) the problem reduces to a similar investigation for a body spinning about an axis whose direction remains fixed in space.

3. *Radiation Pressure on a Perfectly Reflecting Rectangular Prism.*

By assumption (6) it follows that the normal passing through the centre of pressure of a given surface is along a direction parallel to one of the satellite axes. As the satellite is carried round the sun by the earth the direction of the sun will change relative to the surface geometry of the satellite, the only exception being a satellite which continually points the same surface towards the sun.

Let XYZ be an orthogonal right-handed system of axes (origin at the centre of mass of the satellite) having a fixed orientation in space, and let $\mathbf{n}$ be the unit vector specifying the direction of the sun's radiation relative to these axes. Let $[\cos\gamma_X, \cos\gamma_Y, \cos\gamma_Z]$ be the direction cosines such that

$$\mathbf{n} = \cos\gamma_X\,\mathbf{i} + \cos\gamma_Y\,\mathbf{j} + \cos\gamma_Z\,\mathbf{k} \tag{1}$$

where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are unit vectors along X, Y, Z respectively.

Let X_S, Y_S, Z_S be a similar system of axes representing satellite axes, with $(\mathbf{i}_S, \mathbf{j}_S, \mathbf{k}_S)$ unit vectors along X_S, Y_S, Z_S. Suppose, for the present, that satellite axes are coincident with X, Y, Z and let $S_{X_SY_S}$, $S_{X_SZ_S}$ and $S_{Y_SZ_S}$ be the surface areas parallel to the planes containing the X_S, Y_S; X_S, Z_S; and Y_S, Z_S axes.

Consider radiation incident on face 1 of area $S_{X_S Z_S}$—Figs. 2 and 3. The incident radiation is at an angle γ_Y with the surface normal and so, assuming complete reflection at the surface, the force exerted on this surface is

$$\mathbf{F} = 2PS_{X_S Z_S} \cos^2\gamma_Y \mathbf{j}_S \tag{2}$$

where P is the radiation pressure on a 'black body' surface placed at right angles to the direction of the radiation. The normal component given by equation (2) is proportional to $\cos^2\gamma_Y$ because of the obliquity of the radiation on the surface as explained in Appendix II.

The torque about the centre of mass due to this force component is

$$\mathbf{L}_1 = \mathbf{r}_1 \times \mathbf{F} \tag{3}$$

where $\mathbf{r}_1 = (x_1\mathbf{i}_S + y_1\mathbf{j}_S + z_1\mathbf{k}_S)$ is the position vector of the centre of pressure of face 1 relative to the centre of mass,

i.e.
$$\mathbf{L}_1 = 2PS_{X_S Z_S} \cos^2\gamma_Y (x_1\mathbf{k}_S - z_1\mathbf{i}_S) . \tag{4}$$

Since the force is normal to the surface, the torque acting about the centre of mass, due to radiation incident on the face opposite to face 1 must be exactly the same as if the radiation was supposed incident on the inner side of face 1. Hence, if the force given by equation (2) is written as

$$\mathbf{F} = 2PS_{X_S Z_S}|\cos\gamma_Y|\cos\gamma_Y\, \mathbf{j}_S \tag{5}$$

then this equation represents the magnitude and the direction of the force for all γ_Y. If $|\cos\gamma_Y|\cos\gamma_Y$ is written instead of $\cos^2\gamma_Y$ in equation (4) then this equation represents the torque for all γ_Y, since the torque-arm components entering into the equation are the same whether the radiation is incident on face 1 or the opposite face. Thus,

$$\mathbf{L}_1 = 2PS_{X_S Z_S}|\cos\gamma_Y|\cos\gamma_Y\, (x_1\mathbf{k}_S - z_1\mathbf{i}_S) . \tag{6}$$

A similar argument applies for radiation incident on the other two pairs of faces giving the torque components

$$\mathbf{L}_2 = 2PS_{Y_S Z_S}|\cos\gamma_X|\cos\gamma_X\, (z_2\mathbf{j}_S - y_2\mathbf{k}_S) \tag{7}$$

due to radiation incident on face 2 or the opposite face, and

$$\mathbf{L}_3 = 2PS_{X_S Y_S}|\cos\gamma_Z|\cos\gamma_Z\, (y_3\mathbf{i}_S - x_3\mathbf{j}_S) \tag{8}$$

due to radiation incident on face 3 or the opposite face.

The torque-arm components in the latter two equations are the components defined by the position vectors $\mathbf{r}_2 = (x_2\mathbf{i}_S + y_2\mathbf{j}_S + z_2\mathbf{k}_S)$, $\mathbf{r}_3 = (x_3\mathbf{i}_S + y_3\mathbf{j}_S + z_3\mathbf{k}_S)$ representing the positions of the centres of pressure of faces 2 and 3 relative to the centre of mass, Fig. 2. The directions of the satellite axes relative to three faces of the configuration can always be chosen to make the torque-arm components entering equations (6), (7) and (8) positive quantities, and all torque-arm components will be regarded as positive throughout the remainder of this report.

The resultant torque about the centre of mass may be written

$$\begin{aligned}\mathbf{\Gamma} = 2P[&(y_3S_{X_S Y_S}|\cos\gamma_Z|\cos\gamma_Z - z_1S_{X_S Z_S}|\cos\gamma_Y|\cos\gamma_Y)\mathbf{i}_S + \\ &+ (z_2S_{Y_S Z_S}|\cos\gamma_X|\cos\gamma_X - x_3S_{X_S Y_S}|\cos\gamma_Z|\cos\gamma_Z)\mathbf{j}_S + \\ &+ (x_1S_{X_S Z_S}|\cos\gamma_Y|\cos\gamma_Y - y_2S_{Y_S Z_S}|\cos\gamma_X|\cos\gamma_X)\mathbf{k}_S] .\end{aligned} \tag{9}$$

If any changes in the directions of the satellite axes, and any shifts in the position of the centre of mass relative to the fixed surface geometry of the satellite are neglected, the radiation-pressure torque will vary only as the direction cosines vary. For a space-stabilised satellite the change in the direction cosines will be due to the satellite's motion about the sun; in the case of an earth-pointing satellite, i.e. one which points the same axis continually towards the centre of the earth, the torque components will also vary as a result of the changing orientation of the satellite as it revolves around the earth.

4. *The Space-Stabilised Satellite.*

Certain theoretical investigations which follow require integrations along the path described by the earth in its journey round the sun, and to perform such integrations it is convenient to express the radiation-pressure torque in terms of the angle θ measured in the plane of the ecliptic, θ being zero when the earth is at perigee. Let X', Y', Z' define a system of space-fixed axes with Z' perpendicular to the plane of the ecliptic and X' and Y' along the major and minor axes respectively of the orbit followed by the earth in its path round the sun, the axes having origin at the centre of mass of the satellite (Fig. 4). Let $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ be unit vectors along X', Y', Z' and have direction cosines $[(l_1, m_1, n_1); (l_2, m_2, n_2); (l_3, m_3, n_3)]$ relative to axes X, Y, Z. Then, since satellite axes are coincident with X, Y, Z,

$$\left.\begin{aligned} \mathbf{i}' &= l_1\mathbf{i}_S + m_1\mathbf{j}_S + n_1\mathbf{k}_S \\ \mathbf{j}' &= l_2\mathbf{i}_S + m_2\mathbf{j}_S + n_2\mathbf{k}_S \\ \mathbf{k}' &= l_3\mathbf{i}_S + m_3\mathbf{j}_S + n_3\mathbf{k}_S\,. \end{aligned}\right\} \tag{10}$$

Relative to the two sets of axes,

$$\mathbf{n} = \cos\gamma_X\ \mathbf{i}_S + \cos\gamma_Y\ \mathbf{j}_S + \cos\gamma_Z\ \mathbf{k}_S \tag{11}$$

$$\mathbf{n} = \cos\theta\mathbf{i}' + \sin\theta\mathbf{j}' \tag{12}$$

since $\mathbf{n}$ moves in the plane of the ecliptic and θ defines the angular position in the ellipse as indicated in Fig. 4. Combining (10) and (12) we have

$$\mathbf{n} = (l_1\mathbf{i}_S + m_1\mathbf{j}_S + n_1\mathbf{k}_S)\cos\theta + (l_2\mathbf{i}_S + m_2\mathbf{j}_S + n_2\mathbf{k}_S)\sin\theta\,,$$

i.e.

$$\mathbf{n} = (l_1\cos\theta + l_2\sin\theta)\mathbf{i}_S + (m_1\cos\theta + m_2\sin\theta)\mathbf{j}_S + (n_1\cos\theta + n_2\sin\theta)\mathbf{k}_S\,. \tag{13}$$

Equating coefficients of $\mathbf{i}_S$, $\mathbf{j}_S$, $\mathbf{k}_S$ in equations (11) and (13) gives the relations

$$\left.\begin{aligned} \cos\gamma_X &= l_1\cos\theta + l_2\sin\theta \\ \cos\gamma_Y &= m_1\cos\theta + m_2\sin\theta \\ \cos\gamma_Z &= n_1\cos\theta + n_2\sin\theta\,. \end{aligned}\right\} \tag{14}$$

The torque due to radiation pressure may therefore be written, using (9) and (14),

$$\begin{aligned} \mathbf{\Gamma} = {} & 2P[y_3S_{X_SY_S}|n_1\cos\theta + n_2\sin\theta|(n_1\cos\theta + n_2\sin\theta) - \\ & - z_1S_{X_SZ_S}|m_1\cos\theta + m_2\sin\theta|(m_1\cos\theta + m_2\sin\theta)]\mathbf{i}_S + \\ & + 2P[z_2S_{Y_SZ_S}|l_1\cos\theta + l_2\sin\theta|(l_1\cos\theta + l_2\sin\theta) - \\ & - x_3S_{X_SY_S}|n_1\cos\theta + n_2\sin\theta|(n_1\cos\theta + n_2\sin\theta)]\mathbf{j}_S + \\ & + 2P[x_1S_{X_SZ_S}|m_1\cos\theta + m_2\sin\theta|(m_1\cos\theta + m_2\sin\theta) - \\ & - y_2S_{Y_SZ_S}|l_1\cos\theta + l_2\sin\theta|(l_1\cos\theta + l_2\sin\theta)]\mathbf{k}_S\,. \end{aligned} \tag{15}$$

4.1. *Amount of Angular Impulse per Year.*

The amount of angular impulse required from the attitude control in order that the satellite shall remain completely space-stabilised throughout one year may be determined by summing the angular impulses required to maintain each of the satellite axes in the desired direction throughout the year; satellite axes are chosen since the attitude-control torques are assumed to be provided about satellite axes. The total amount of angular impulse per year is therefore

$$J_T = \int_{\theta=0}^{2\pi} |\Gamma_1| \left(\frac{dt}{d\theta}\right) d\theta + \int_{\theta=0}^{2\pi} |\Gamma_2| \left(\frac{dt}{d\theta}\right) d\theta + \int_{\theta=0}^{2\pi} |\Gamma_3| \left(\frac{dt}{d\theta}\right) d\theta \tag{16}$$

where $(\Gamma_1, \Gamma_2, \Gamma_3)$ are the components of $\mathbf{\Gamma}$ along $(\mathbf{i}_S, \mathbf{j}_S, \mathbf{k}_S)$.

Taking each component in turn,

$$\Gamma_1(\theta) = 2P[y_3 S_{X_S Y_S} |n_1 \cos\theta + n_2 \sin\theta| (n_1 \cos\theta + n_2 \sin\theta) +$$

$$+ z_1 S_{X_S Z_S} |m_1 \cos\theta + m_2 \sin\theta| (m_1 \cos\theta + m_2 \sin\theta)] .$$

Let

$$f_n(\theta) = n_1 \cos\theta + n_2 \sin\theta$$

and

$$f_m(\theta) = m_1 \cos\theta + m_2 \sin\theta ,$$

and write

$$k_1^2 = y_3 S_{X_S Y_S} , \qquad k_2^2 = z_1 S_{X_S Z_S} ,$$

then the expression for the torque component becomes

$$\Gamma_1(\theta) = 2P[k_1^2 |f_n(\theta)| f_n(\theta) - k_2^2 |f_m(\theta)| f_m(\theta)] .$$

If

$$f_n(\theta), f_m(\theta) > 0: \ \Gamma_1(\theta) = 2P[k_1^2 f_n^2(\theta) - k_2^2 f_m^2(\theta)]$$

$$f_n(\theta), f_m(\theta) < 0: \ \Gamma_1(\theta) = 2P[k_2^2 f_m^2(\theta) - k_1^2 f_n^2(\theta)]$$

$$f_n(\theta) > 0, f_m(\theta) < 0: \ \Gamma_1(\theta) = 2P[k_1^2 f_n^2(\theta) + k_2^2 f_m^2(\theta)] > 0$$

$$f_n(\theta) < 0, f_m(\theta) > 0: \ \Gamma_1(\theta) = 2P[-k_1^2 f_n^2(\theta) - k_2^2 f_m^2(\theta)] < 0 .$$

Consequently, $\Gamma_1(\theta)$ is zero only when $k_1 f_n(\theta) - k_2 f_m(\theta) = 0$ giving

$$\theta = \tan^{-1}\left(\frac{k_1 n_1 - k_2 m_1}{k_2 m_2 - k_1 n_2}\right) + N'\pi \tag{17}$$

where N' is an integer or zero.

Hence, as θ increases over a range of 2π, $\Gamma_1(\theta)$ will be zero when $\theta = \theta_1'$ and $\theta = \theta_1' + \pi$, where θ_1' is the value given by equation (17) with $N' = 0$. Therefore

$$\int_{\theta=0}^{2\pi} |\Gamma_1(\theta)| \left(\frac{dt}{d\theta}\right) d\theta = \left| \int_{\theta=\theta_1'}^{\theta_1'+\pi} \Gamma_1(\theta) \left(\frac{dt}{d\theta}\right) d\theta - \int_{\theta=\theta_1'+\pi}^{\theta_1'+2\pi} \Gamma_1(\theta) \left(\frac{dt}{d\theta}\right) d\theta \right| . \tag{18}$$

Similarly, $\Gamma_2(\theta)$ is zero when $k_3 f_l(\theta) - k_4 f_n(\theta) = 0$ where $f_l(\theta) = l_1 \cos\theta + l_2 \sin\theta$, $k_3^2 = z_2 S_{Y_S Z_S}$ and $k_4^2 = x_3 S_{X_S Y_S}$, giving $\Gamma_2(\theta)$ zero when

$$\theta = \tan^{-1}\left(\frac{k_3 l_1 - k_4 n_1}{k_4 n_2 - k_3 l_2}\right) + N'\pi . \tag{19}$$

Again, $\Gamma_3(\theta)$ is zero when $k_5 f_m(\theta) - k_6 f_l(\theta) = 0$ where $k_5^2 = x_1 S_{X_S Z_S}$ and $k_6^2 = y_2 S_{Y_S Z_S}$, giving $\Gamma_3(\theta)$ zero when

$$\theta = \tan^{-1}\left(\frac{k_5 m_1 - k_6 l_1}{k_6 l_2 - k_5 m_2}\right) + N'\pi. \tag{20}$$

The expressions corresponding to equation (18) are therefore

$$\int_{\theta=0}^{2\pi} |\Gamma_2(\theta)| \left(\frac{dt}{d\theta}\right) d\theta = \left| \int_{\theta=\theta_2'}^{\theta_2'+\pi} \Gamma_2(\theta) \left(\frac{dt}{d\theta}\right) d\theta - \int_{\theta=\theta_2'+\pi}^{\theta_2'+2\pi} \Gamma_2(\theta) \left(\frac{dt}{d\theta}\right) d\theta \right| \tag{21}$$

and

$$\int_{\theta=0}^{2\pi} |\Gamma_3(\theta)| \left(\frac{dt}{d\theta}\right) d\theta = \left| \int_{\theta=\theta_3'}^{\theta_3'+\pi} \Gamma_3(\theta) \left(\frac{dt}{d\theta}\right) d\theta - \int_{\theta=\theta_3'+\pi}^{\theta_3'+2\pi} \Gamma_3(\theta) \left(\frac{d\theta}{dt}\right) d\theta \right| \tag{22}$$

where θ_2' and θ_3' are given by equations (19) and 20) respectively, with $N' = 0$. Returning to equation (18),

let

$$I = \int_{\theta=\theta_1'}^{\theta_1'+\pi} \Gamma_1(\theta) \left(\frac{dt}{d\theta}\right) d\theta.$$

Now

$$\left(\frac{dt}{d\theta}\right) = \frac{l_E^2}{h_E(1+e_E \cos\theta)^2},$$

where l_E is the semi-latus rectum and e_E the eccentricity of the earth's orbit round the sun, h_E being the angular momentum per unit mass of the earth. However, since $e_E \ll 1$, ($e_E = 0{\cdot}0167$) a sufficiently good approximation for the present analysis will be to assume $e_E = 0$, i.e. a circular orbit.

Thus,

$$I = \frac{l_E^2}{h_E}\int_{\theta=\theta_1'}^{\theta_1'+\pi} \Gamma_1(\theta)\, d\theta = \frac{2\rho l_E^2}{h_E}\int_{\theta=\theta_1'}^{\theta_1'+\pi} [k_1^2|f_n(\theta)|f_n(\theta) - k_2^2|f_m(\theta)|f_m(\theta)]\, d\theta.$$

Consider

$$I_1 = \int_{\theta=\theta_1'}^{\theta_1'+\pi} |f_n(\theta)| f_n(\theta)\, d\theta$$

$$= \int_{\theta=\theta_1'}^{\theta_1'+\pi} |n_1 \cos\theta + n_2 \sin\theta| (n_1 \cos\theta + n_2 \sin\theta)\, d\theta.$$

Put $n_1 = n\cos\chi$ and $n_2 = n\sin\chi$, $n \geqslant 0$; giving $(n_1\cos\theta + n_2\sin\theta) = n\cos\phi_n$, where $\phi_n = (\theta - \chi)$.

Hence

$$I_1 = n^2 \int_{\phi_1'}^{\phi_1'+\pi} |\cos\phi_n| \cos\phi_n\, d\phi_n$$

where

$$\phi_1' = (\theta_1' - \chi),$$

i.e.

$$I_1 = n^2 \int_{\phi_1'}^{\phi_1'+\pi} (\text{sign} \cos\phi_n) \cos^2\phi_n\, d\phi_n. \tag{23}$$

If ϕ_n is considered to lie in the range $-\pi/2 \leqslant \phi_n \leqslant 3\pi/2$ then

$$I_1 = I_1' = n^2 \left(\int_{\phi_1'}^{\pi/2} (\text{sign} \cos \phi_1') \cos^2\phi_n \, d\phi_n - \int_{\pi/2}^{\phi_1'+\pi} (\text{sign} \cos \phi_1') \cos^2\phi_n \, d\phi_n \right)$$

if

$$-\pi/2 < \phi_1' < \pi/2, \text{ since } (\text{sign} \cos \phi_n) \text{ changes when } \phi_n = \pi/2,$$

i.e.

$$I_1' = n^2 (\text{sign} \cos \phi_1') \left(\int_{\phi_1'}^{\pi/2} \cos^2\phi_n \, d\phi_n - \int_{\pi/2}^{\phi_1'+\pi} \cos^2\phi_n \, d\phi_n \right),$$

giving

$$I_1' = -n^2 (\text{sign} \cos \phi_1') \left(\phi_1' + \frac{\sin 2\phi_1'}{2} \right).$$

If $\pi/2 < \phi_1' < 3\pi/2$, the integral is

$$I_1 = I_1'' = n^2 \left(\int_{\phi_1'}^{3\pi/2} (\text{sign} \cos \phi_1') \cos^2\phi_n \, d\phi_n - \int_{-\pi/2}^{\phi_1'-\pi} (\text{sign} \cos \phi_1') \cos^2\phi_n \, d\phi_n \right)$$

since $(\text{sign} \cos \phi_n)$ changes when $\phi_n = 3\pi/2$.

If we write $\phi_1' = \phi_1'' + \pi$, then since $\pi/2 < \phi_1' < 3\pi/2$, ϕ_1'' must lie in the range $-\pi/2 < \phi_1'' < \pi/2$. The integral becomes

$$I_1'' = n^2 \left(\int_{\phi_1''+\pi}^{3\pi/2} \text{sign} \cos (\phi_1''+\pi) \cos^2\phi_n \, d\phi_n - \int_{-\pi/2}^{\phi_1''} \text{sign} \cos (\phi_1''+\pi) \cos^2\phi_n \, d\phi_n \right).$$

This may be written

$$I_1'' = -n^2 (\text{sign} \cos \phi_1'') \left(\int_{\phi_1''}^{\pi/2} \cos^2\phi_n \, d\phi_n - \int_{\pi/2}^{\phi_1''+\pi} \cos^2\phi_n \, d\phi_n \right).$$

Hence I_1'' is simply equal to $-I_1'$ with ϕ_1' replaced by ϕ_1''. ϕ_1'' [defined in the open interval $(-\pi/2, \pi/2)$] is, of course, simply the value of ϕ_n in this interval corresponding to $\Gamma_1(\theta) = 0$ so that when ϕ_1' lies in the range $\pi/2 < \phi_1' < 3\pi/2$ the integral may be taken between limits $(\phi_1' - \pi)$ and ϕ_1' and the sign of the integral reversed. A general expression for the integral may therefore be written

$$I_1 = -n^2 (\text{sign} \cos \phi_1') \left(\phi_1 + \frac{\sin 2\phi_1}{2} \right) \tag{24}$$

where ϕ_1 is defined in the range $-\pi/2 < \phi_1 < \pi/2$, ϕ_1' in the range $-\pi/2 < \phi_1' < 3\pi/2$, excluding $\phi_1' = \pi/2$, and $\phi_1 = \phi_1'$ if ϕ_1' falls in the range $-\pi/2 < \phi_1' < \pi/2$ or $\phi_1 = \phi_1' - \pi$ if ϕ_1' falls in the range $\pi/2 < \phi_1' < 3\pi/2$.

If $\phi_1' = -\pi/2$, equation (23) gives

$$I_1 = n^2 \int_{-\pi/2}^{\pi/2} (\text{sign} \cos \phi_n) \cos^2\phi_n \, d\phi_n, \tag{25}$$

i.e. $I_1 = n^2\pi/2$ since $\cos \phi_n$ remains positive throughout the integration. Similarly, if $\phi_1' = \pi/2$, $I_1 = -n^2\pi/2$ since $\cos \phi_n$ is now always negative.

Treating $I_2 = \int_{\theta=\theta_1'}^{\theta_1'+\pi} |f_m(\theta)| f_m(\theta)\, d\theta$ in a similar manner, with $m_1 = m \cos\psi$ and $m_2 = m \sin\psi$, $m \geqslant 0$ and $\phi_m = (\theta - \psi)$, $\phi_2' = (\theta_1' - \psi)$ with ϕ_2' restricted to the range $-\pi/2 < \phi_2' < 3\pi/2$ and excluding $\phi_2' = \pi/2$, then

$$I_2 = -m^2(\text{sign}\cos\phi_2')\left(\phi_2 + \frac{\sin 2\phi_2}{2}\right) \tag{26}$$

with ϕ_2 and ϕ_2' defined in a similar manner to ϕ_1 and ϕ_1' in equation (24). When applying these results it is probably better to evaluate the integrals separately and then determine $I = I_1 - I_2$, rather than combine the two integrals and then perform a numerical substitution.

Since the integral of the $\Gamma_1(\theta)$ function over a complete cycle is zero, then

$$\int_{\theta=\theta_1'}^{\theta_1'+\pi} \Gamma_1(\theta)\, d\theta = -\int_{\theta_1'+\pi}^{\theta_1'+2\pi} \Gamma_1(\theta)\, d\theta .$$

Hence

$$\int_{\theta=0}^{2\pi} |\Gamma_1(\theta)| \left(\frac{dt}{d\theta}\right) d\theta = \frac{2l_E^2}{h_E} \left| \int_{\theta=\theta_1'}^{\theta_1'+\pi} \Gamma_1(\theta)\, d\theta \right| ,$$

i.e.

$$J_1 = \frac{4Pl_E^2}{h_E} |(k_1^2 I_1 - k_2^2 I_2)| \tag{27}$$

where the integrals I_1 and I_2 are given by equations (24) and (26) and $J_1 = \int_{\theta=0}^{2\pi} |\Gamma_1(\theta)| \left(\frac{dt}{d\theta}\right) d\theta$.

The analysis for J_2 and J_3 follows exactly the same line of argument as that presented for J_1 and the results will merely be stated.

$$J_2 = \frac{4Pl_E^2}{h_E} |(k_3^2 I_3 - k_4^2 I_4)| \tag{28}$$

where $I_3 = -l^2(\text{sign}\cos\phi_3')\left(\phi_3 + \frac{\sin 2\phi_3}{2}\right)$

with the relations $l_1 = l\cos\xi$ and $l_2 = l\sin\xi$, $l \geqslant 0$, $\phi_3' = (\theta_2' - \xi)$

and

$$I_4 = -n^2(\text{sign}\cos\phi_4')\left(\phi_4 + \frac{\sin 2\phi_4}{2}\right), \qquad \text{with } \phi_4' = (\theta_2' - \chi).$$

$$J_3 = \frac{4Pl_E^2}{h_E} |(k_5^2 I_5 - k_6^2 I_6)| \tag{29}$$

where

$$I_5 = -m^2(\text{sign}\cos\phi_5')\left(\phi_5 + \frac{\sin 2\phi_5}{2}\right), \qquad \text{with } \phi_5' = (\theta_3' - \psi)$$

and

$$I_6 = -l^2(\text{sign}\cos\phi_6')\left(\phi_6 + \frac{\sin 2\phi_6}{2}\right), \qquad \text{with } \phi_6' = (\theta_3' - \xi).$$

The angles ϕ_3', ϕ_4', ϕ_5', ϕ_6' are defined in the same range as ϕ_1' and ϕ_2' and the integrals should be treated exactly as I_1 and I_2 in the detailed theory.

Substitution of J_1, J_2, J_3 into equation (16) allows a determination of the total amount of angular impulse per year which would be necessary to keep the satellite completely space-stabilised when under the continuous influence of the sun's radiation. However, as the satellite may spend almost half of its time in the earth's shadow, equation (16) will represent an upper estimate of the quantity under discussion.

4.2. *Numerical Example on Amount of Angular Impulse.*

Suppose the centre of mass of the satellite to lie along the Z_S axis, and suppose that the orientation of the satellite is such that $l_1 = m_2 = n_3 = 1$ (all other direction cosines being zero) i.e. satellite axes are coincident with X', Y' Z', making Z_S perpendicular to the plane of the ecliptic. There is no torque component about the Z_S axis since $x_1 = y_2 = 0$. From equation (17) the torque about the X_S axis is zero when $\theta_1' = \tan^{-1}(-m_1/m_2) = 0$, giving $\phi_2' = -\pi/2$ since $\psi = \pi/2$ from the relations $m_1 = m \cos\psi$ and $m_2 = m \cos\psi$, m_1 being zero and m_2 unity.

Since

$$\phi_2' = -\pi/2,\ I_2 = m^2\,\pi/2 = \pi/2 \text{ since } m = 1 .$$

Hence,

$$J_1 = \left| -\frac{2Pl_E^2}{h_E}\,\pi k_2^2 \right| .$$

From equation (19) the torque about the Y_S axis is zero when $\theta_2' = \tan^{-1}(-l_1/l_2) = -\infty$, giving $\phi_3' = -\pi/2$ since $\xi = 0$ from the relations $l_1 = l\cos\xi$ and $l_2 = l\sin\xi$, l_1 being unity and l_2 zero.

Since

$$\phi_3' = -\pi/2,\ I_3 = l^2\pi/2 = \pi/2 \text{ since } l = 1 .$$

Hence

$$J_2 = \left| \frac{2Pl_E^2}{h_E}\,\pi k_3^2 \right| .$$

Therefore

$$J_T = \frac{2Pl_E^2}{h_E}\,\pi(k_2^2 + k_3^2) = \frac{2Pl_E^2}{h_E}\,\pi z(S_{X_S Z_S} + S_{Y_S Z_S}) .$$

If T_E is the period of the earth's orbit then $T_E = 2\pi l_E^2/h_E$ (since we are neglecting e_E in comparison to unity).

Therefore

$$J_T = P T_E z(S_{X_S Z_S} + S_{Y_S Z_S}) . \tag{30}$$

Taking $P = 2{\cdot}9 \times 10^{-6}$ pdl ft^{-2}, $T_E = 365$ days, $z = 3$ ft, $S_{X_S Z_S} = S_{Y_S Z_S} = 30$ ft^2, equation (30) gives $J_T \simeq 1{\cdot}6 \times 10^4$ lb ft^2 sec^{-1}.

4.3. *Development of a Nett Amount of Angular Impulse.*

If the earth moved round the sun in an exactly circular orbit the nett angular impulse about a given satellite axis during one complete orbit round the sun would be zero. However, because of the eccentricity of the orbit there may be a nett angular impulse per orbit, determined by the expression

$$J_R = \int_{\theta=0}^{2\pi} \Gamma_1(\theta)\left(\frac{dt}{d\theta}\right) d\theta + \int_{\theta=0}^{2\pi} \Gamma_2(\theta)\left(\frac{dt}{d\theta}\right) d\theta + \int_{\theta=0}^{2\pi} \Gamma_3(\theta)\left(\frac{dt}{d\theta}\right) d\theta \tag{31}$$

where $\Gamma(\theta)$ is given by equation (15) and $\dfrac{dt}{d\theta} = \dfrac{l_E^2}{h_E(1 + l_E\cos\theta)^2}$.

Since the eccentricity is small, any nett amount of angular impulse about a given satellite axis per orbit will be small compared with the build up between successive positions along the orbit at which the torque about the axis in question is zero. Shadowing of the satellite by the earth will also cause unequal amounts of angular impulse to be developed, but the amount of angular impulse produced in this way is not easily calculated. However, in the case of a reaction-jet attitude control system any such nett build up is irrelevant since it is simply the result of ejecting gas which provides the compensating torque. Other systems involving, perhaps reaction-wheels should be designed, to cater for any tendency to develop any unidirectional store of angular momentum.

5. *The Earth-Pointing Satellite in a Non-Precessing Orbit.*

Suppose the Z_S axis of the satellite to point continuously towards the centre of the earth as the satellite orbits the earth along an elliptic orbit of eccentricity e'. The problem now is to specify the radiation-pressure torque components along axes rotating with an angular velocity about the space-fixed Y_S axis of the satellite (the Y_S axis being chosen normal to the orbital plane containing the motion of the satellite about the earth). Let Y be coincident with Y_S so that the satellite orbit is contained in the XZ plane, Fig. 5; furthermore let satellite axes be coincident with X, Y, Z when the satellite is at its perigee position. The unit vector specifying the direction of the sun's radiation relative to the satellite has components $(n_{X_S}, n_{Y_S}, n_{Z_S})$ along satellite axes, where

$$\left.\begin{aligned} n_{X_S} &= \cos\gamma_X \cos\theta_2 - \cos\gamma_Z \sin\theta_2 \\ n_{Y_S} &= \cos\gamma_Y \\ n_{Z_S} &= \cos\gamma_X \sin\theta_2 + \cos\gamma_Z \cos\theta_2 \end{aligned}\right\} \tag{32}$$

and θ_2 is the true anomaly of the satellite in its orbit, i.e. the angle between the current radius vector and the radius vector at perigee; $(n_{X_S}, n_{Y_S}, n_{Z_S})$ are in fact the direction cosines of the unit vector $\mathbf{n}$ relative to the rotating satellite axes. Following the reasoning leading to equation (9) it may be shown that the radiation-pressure torque for a given θ_2 will be

$$\begin{aligned} \mathbf{\Gamma} = 2P[&\{k_1^2|\cos\gamma_X \sin\theta_2 + \cos\gamma_Z \cos\theta_2|(\cos\gamma_X \sin\theta_2 + \cos\gamma_Z \cos\theta_2) - \\ &- k_2^2|\cos\gamma_Y|(\cos\gamma_Y)\}\mathbf{i}_S + \\ &+ \{k_3^2|\cos\gamma_X \cos\theta_2 - \cos\gamma_Z \sin\theta_2|(\cos\gamma_X \cos\theta_2 - \cos\gamma_Z \sin\theta_2) - \\ &- k_4^2|\cos\gamma_X \sin\theta_2 + \cos\gamma_Z \cos\theta_2|(\cos\gamma_X \sin\theta_2 + \cos\gamma_Z \cos\theta_2)\mathbf{j}_S + \\ &+ \{k_5^2|\cos\gamma_Y|(\cos\gamma_Y) - \\ &- k_6^2|\cos\gamma_X \cos\theta_2 - \cos\gamma_Z \sin\theta_2|(\cos\gamma_X \cos\theta_2 - \cos\gamma_Z \sin\theta_2)\}\mathbf{k}_S]\,. \end{aligned} \tag{33}$$

Using the relations given by expression (14) in the torque equation gives

$$\begin{aligned} \mathbf{\Gamma} = 2P[\![&[k_1^2|l_1\cos\theta + l_2\sin\theta)\sin\theta_2 + (n_1\cos\theta + n_2\sin\theta)\cos\theta_2| \times \\ &\times \{(l_1\cos\theta + l_2\sin\theta)\sin\theta_2 + (n_1\cos\theta + n_2\sin\theta)\cos\theta_2\} - \\ &- k_2^2|m_1\cos\theta + m_2\sin\theta|(m_1\cos\theta + m_2\sin\theta)]\mathbf{i}_S + \\ &+ [k_3^2|(l_1\cos\theta + l_2\sin\theta)\cos\theta_2 - (n_1\cos\theta + n_2\sin\theta)\sin\theta_2| \times \\ &\times \{(l_1\cos\theta + l_2\sin\theta)\cos\theta_2 - (n_1\cos\theta + n_2\sin\theta)\sin\theta_2\} - \\ &- k_4^2|(l_1\cos\theta + l_2\sin\theta)\sin\theta_2 + (n_1\cos\theta + n_2\sin\theta)\cos\theta_2| \times \\ &\times \{(l_1\cos\theta + l_2\sin\theta)\sin\theta_2 + (n_1\cos\theta + n_2\sin\theta)\cos\theta_2\}]\mathbf{j}_S + \\ &+ [k_5^2|m_1\cos\theta + m_2\sin\theta|(m_1\cos\theta + m_2\sin\theta) - \\ &- k_6^2|(l_1\cos\theta + l_2\sin\theta)\cos\theta_2 - (n_1\cos\theta + n_2\sin\theta)\sin\theta_2| \times \\ &\times \{(l_1\cos\theta + l_2\sin\theta)\cos\theta_2 - (n_1\cos\theta + n_2\sin\theta)\sin\theta_2\}\mathbf{k}_S]\!]\,. \end{aligned} \tag{34}$$

The determination of the points at which the components of equation (34) become zero proves to be a long and tedious operation and an alternative method of analysis is employed in the theory which follows.

5.1. *Amount of Angular Impulse per Satellite Orbit.*

For the purpose of simplifying the problem it will be assumed that the satellite completes one orbit round the earth for a fixed position of the earth relative to the sun, i.e. the orbital period of the satellite is small compared with the orbital period of the earth. This assumption allows equation (34) to be treated as a function of θ_2 alone.

Let

$$A = (l_1 \cos\theta + l_2 \sin\theta), \qquad B = (n_1 \cos\theta + n_2 \sin\theta), \qquad C = (m_1 \cos\theta + m_2 \sin\theta).$$

Taking each component in turn

$$\Gamma_1(\theta_2) = 2P[k_1^2|f_{AB}(\theta_2)|f_{AB}(\theta_2) - k_2^2|C|C],$$

where

$$f_{AB}(\theta_2) = (A \sin\theta_2 + B\cos\theta_2).$$

If

$$f_{AB}(\theta_2) > 0,\ C > 0\colon \Gamma_1(\theta_2) = 2P[k_1^2 f_{AB}^2(\theta_2) - k_2^2C^2]$$
$$f_{AB}(\theta_2) < 0,\ C < 0\colon \Gamma_1(\theta_2) = 2P[-k_1^2 f_{AB}^2(\theta_2) + k_2^2C^2]$$
$$f_{AB}(\theta_2) > 0,\ C < 0\colon \Gamma_1(\theta_2) = 2P[k_1^2 f_{AB}^2(\theta_2) + k_2^2C^2] > 0$$
$$f_{AB}(\theta_2) < 0,\ C > 0\colon \Gamma_1(\theta_2) = 2P[-k_1^2 f_{AB}^2(\theta_2) - k_2^2C^2] < 0.$$

Consequently, $\Gamma_1(\theta_2)$ is zero only when $k_1 f_{AB}(\theta_2) - k_2 C = 0$,

i.e.

$$A\sin\theta_2 + B\cos\theta_2 = \frac{k_2 C}{k_1}.$$

Put $A = r\cos\eta$ and $B = r\sin\eta$, $r \geqslant 0$, then $\sin(\theta_2+\eta) = k_2C/rk_1 = E$, say.

Let Δ be the principal value of $\sin^{-1}E$, then for a variation of θ_2 over an angle 2π

$$\theta_2 + \eta = \Delta$$

or

$$\theta_2 + \eta = \pi - \Delta,$$

i.e.

$$\theta_2 = (\Delta - \eta) \text{ or } \theta_2 = (\pi - \Delta - \eta)$$

where $-\pi/2 \leqslant \Delta \leqslant \pi/2$.

Let $\theta_{2(1)}' = (\Delta - \eta)$ and $\theta_{2(1)}'' = (\pi - \Delta - \eta)$ be the successive values of θ_2 at which $\Gamma_1(\theta_2)$ is zero. The magnitude of the angular impulse about the X_S axis of the satellite per satellite orbit is

$$\int_{\theta_2=0}^{2\pi} |\Gamma_1(\theta_2)| \left(\frac{dt}{d\theta_2}\right) d\theta_2 = \left| \int_{\theta_2=\theta_{2(1)}'}^{\theta_{2(1)}''} \Gamma_1(\theta_2)\left(\frac{dt}{d\theta_2}\right) d\theta_2 - \int_{\theta_{2(1)}''}^{\theta_{2(1)}'+2\pi} \Gamma_1(\theta_2)\left(\frac{dt}{d\theta_2}\right) d\theta_2 \right|. \tag{35}$$

Similarly, $\Gamma_2(\theta_2)$ is zero when

$$k_3(A\cos\theta_2 - B\sin\theta_2) = k_4(A\sin\theta_2 + B\cos\theta_2)$$

giving

$$\theta_2 = \tan^{-1}\left(\frac{k_3A - k_4B}{k_4A + k_3B}\right) + N'\pi \tag{36}$$

where N' is an integer or zero, and $\Gamma_3(\theta_2)$ is zero when

$$k_6(A\cos\theta_2 - B\sin\theta_2) = k_5C.$$

Put $A = r\sin\beta$ and $B = r\cos\beta$, $r \geqslant 0$, then $\sin(\beta-\theta_2) = k_5C/k_6r = D$, say. Let δ be the principal value of $\sin^{-1}D$, then for a variation of θ_2 over an angle 2π,

$$\beta - \theta_2 = \delta$$

or

$$\beta - \theta_2 = \pi - \delta,$$

i.e.

$$\theta_2 = \beta - \delta \qquad \text{or} \qquad \theta_2 = \beta + \delta - \pi$$

where $-\pi/2 \leqslant \delta \leqslant \pi/2$.

Let $\theta_{2(2)}'$, $\theta_{2(2)}''$ and $\theta_{2(3)}'$, $\theta_{2(3)}''$ be the successive values of θ_2 at which $\Gamma_2(\theta_2)$ and $\Gamma_3(\theta_2)$ become zero respectively. The magnitude of the angular impulses about the Y_S and Z_S axes of the satellite per satellite orbit is then given by

$$\int_{\theta_2=0}^{2\pi} |\Gamma_2(\theta_2)| \left(\frac{dt}{d\theta_2}\right) d\theta_2 = \left| \int_{\theta_2=\theta_{2(2)}'}^{\theta_{2(2)}''} \Gamma_2(\theta_2) \left(\frac{dt}{d\theta_2}\right) d\theta_2 - \int_{\theta_{2(2)}''}^{\theta_{2(2)}'+2\pi} \Gamma_2(\theta_2) \left(\frac{dt}{d\theta_2}\right) d\theta_2 \right| \tag{37}$$

for the Y_S axis, and

$$\int_{\theta_2=0}^{2\pi} |\Gamma_3(\theta_2)| \left(\frac{dt}{d\theta_2}\right) d\theta_2 = \left| \int_{\theta_2=\theta_{2(3)}'}^{\theta_{(2)3}''} \Gamma_3(\theta_2) \left(\frac{dt}{d\theta_2}\right) d\theta_2 - \int_{\theta_2=\theta_{2(3)}''}^{\theta_{2(3)}'+2\pi} \Gamma_3(\theta_2) \left(\frac{dt}{d\theta_2}\right) d\theta_2 \right| \tag{38}$$

for the Z_S axis.

The total amount of angular impulse per satellite orbit that must be supplied about satellite axes is simply the sum of the expressions given by equations (35), (37) and (38). If J_T' denotes this total angular impulse then

$$J_T' = \sum_{r=1}^{3} \int_{\theta_2=0}^{2\pi} |\Gamma_r(\theta_2)| \left(\frac{dt}{d\theta_2}\right) d\theta_2 . \tag{39}$$

Consider first the integral of the torque about the X_S axis as given by equation (35), written out at length we wish to evaluate the quantity

$$J_1' = \left| \frac{2Pl'^2}{h'} \left[\int_{\theta_{2(1)}'}^{\theta_{2(1)}''} \frac{k_1^2|A\sin\theta_2 + B\cos\theta_2|(A\sin\theta_2 + B\cos\theta_2) - k_2^2|C|C}{(1+e'\cos\theta_2)^2} d\theta_2 - \right.\right.$$
$$\left.\left. - \int_{\theta_{2(1)}''}^{\theta_{2(1)}'+2\pi} \frac{k_1^2|A\sin\theta_2 + B\cos\theta_2|(A\sin\theta_2 + B\cos\theta_2) - k_2^2|C|C}{(1+e'\cos\theta_2)^2} d\theta_2 \right] \right| \tag{40}$$

since $(dt/d\theta_2) = l'^2/h'(1+e'\cos\theta_2)^2$, where l' is the semi-latus rectum and h' the angular momentum per unit mass of the satellite in its orbit round the earth. The expression will be split into two parts $I_{2(1)}$ and $I_{2(2)}$ where

$$I_{2(1)} = \frac{2Pl'^2}{h'} \left[\int_{\theta_{2(1)}'}^{\theta_{2(1)}''} \frac{k_1^2|A\sin\theta_2 + B\cos\theta_2|(A\sin\theta_2 + B\cos\theta_2)}{(1+e'\cos\theta_2)^2} d\theta_2 - \right.$$
$$\left. - \int_{\theta_{2(1)}''}^{\theta_{2(1)}'+2\pi} \frac{k_1^2|A\sin\theta_2 + B\cos\theta_2|(A\sin\theta_2 + B\cos\theta_2)}{(1+e'\cos\theta_2)^2} d\theta_2 \right] \tag{41}$$

and

$$I_{2(2)} = \frac{2Pl'^2}{h'} \left[\int_{\theta_{2(1)}'}^{\theta_{2(1)}''} \frac{k_2^2|C|C}{(1+e'\cos\theta_2)^2} d\theta_2 - \int_{\theta_{2(1)}''}^{\theta_{2(1)}'+2\pi} \frac{k_2^2|C|C}{(1+e'\cos\theta_2)^2} d\theta_2 \right]. \tag{42}$$

Taking the integrals in the expression for $I_{2(1)}$, put $A = r \sin \beta$ and $B = r \cos \beta$, $r \geqslant 0$, giving $(B \cos \theta_2 + A \sin \theta_2) = r \cos \phi_{2(1)}$, where $\phi_{2(1)} = (\theta_2 - \beta)$. Hence the integrals involved are of the form

$$\int \frac{|\cos \phi_{2(1)}| \cos \phi_{2(1)}}{[1 + e' \cos (\phi_{2(1)} + \beta)]^2} \, d\phi_{2(1)} = \int (\text{sign} \cos \phi_{2(1)}) \frac{\cos^2\phi_{2(1)}}{[1 + e' \cos (\phi_{2(1)} + \beta)]^2} \, d\phi_{2(1)}.$$

Expanding

$$\cos^2\phi_{2(1)} \text{ as } \cos^2[(\phi_{2(1)} + \beta) - \beta] \equiv$$

$$\cos^2(\phi_{2(1)} + \beta) \cos^2\beta - 2 \sin \beta \cos \beta \sin (\phi_{2(1)} + \beta) \cos (\phi_{2(1)} + \beta) + \sin^2 (\phi_{2(1)} + \beta) \sin^2\beta$$

enables the integral to be expressed as

$$\int (\text{sign} \cos \phi_{2(1)}) \times$$

$$\times \frac{[\cos^2\beta \cos^2(\phi_{2(1)}+\beta) - 2 \sin \beta \cos \beta \sin (\phi_{2(1)}+\beta) \cos (\phi_{2(1)}+\beta) + \sin^2(\phi_{2(1)}+\beta) \sin^2\beta]}{[1 + e' \cos (\phi_{2(1)}+\beta)]^2} \, d\phi_{2(1)}$$

and the integrals $\displaystyle\int \frac{\cos^2\nu \, d\nu}{(1 + e' \cos \nu)^2}$, $\displaystyle\int \frac{\sin^2\nu \, d\nu}{(1 + e' \cos \nu)^2}$ and $\displaystyle\int \frac{\sin \nu \cos \nu \, d\nu}{(1 + e' \cos \nu)^2}$,

where $\nu = (\phi_{2(1)}+\beta)$, are given in Appendix I. Since the integration is over a complete cycle of 2π radians $\phi_{2(1)}$ may be chosen to lie within the range

$$-\frac{\pi}{2} \leqslant \phi_{2(1)} \leqslant +\frac{3\pi}{2}.$$

Let

$$\phi_{2(1)}' = (\theta_{2(1)}' - \beta)$$

and

$$\phi_{2(1)}'' = (\theta_{2(1)}'' - \beta).$$

The situations which can arise are

(1) $\phi_{2(1)}'$ in the lower half plane, $\phi_{2(1)}''$ in the upper half plane.

(2) $\phi_{2(1)}'$ in the upper half plane, $\phi_{2(1)}''$ in the lower half plane.

(3) $\phi_{2(1)}'$ and $\phi_{2(1)}''$ in the upper half plane $\left]\begin{array}{l} \phi_{2(1)}' > \phi_{2(1)}'' \\ \phi_{2(1)}' < \phi_{2(1)}''. \end{array}\right.$

(4) $\phi_{2(1)}'$ and $\phi_{2(1)}''$ in the lower half plane $\left]\begin{array}{l} \phi_{2(1)}' > \phi_{2(1)}'' \\ \phi_{2(1)}' < \phi_{2(1)}''. \end{array}\right.$

Since (sign cos $\phi_{2(1)}$) changes when $\phi_{2(1)} = \pm \pi/2$, the possibilities are

(a) $$I_{2(1)} = \frac{2Pl'^2}{h'} k_1^2 r^2 \left[\left(\int_{\phi_{2(1)}'}^{\pi/2} - \int_{\pi/2}^{\phi_{2(1)}''}\right) - \left(-\int_{\phi_{2(1)}''}^{3\pi/2} + \int_{-\pi/2}^{\phi_{2(1)}'}\right)\right] \tag{43}$$

where the integrand of each integral is $\cos^2\phi_{2(1)}/[1 + e' \cos (\phi_{2(1)}+\beta)]^2$, $\phi_{2(1)}'$ is in the lower half plane and $\phi_{2(1)}''$ in the upper half plane and (sign cos $\phi_{2(1)}'$) is therefore positive. If $\phi_{2(1)}'$ is in the upper half plane and $\phi_{2(1)}''$ in the lower half plane, equation (43) holds if $\phi_{2(1)}'$ and $\phi_{2(1)}''$ are interchanged and the signs of the integrals reversed.

(b) $$I_{2(1)} = \frac{(-1)2Pl'^2}{h'} k_1^2 r^2 \left[\left(\int_{\phi_{2(1)}'}^{\phi_{2(1)}''}\right) - \left(\int_{\phi_{2(1)}''}^{3\pi/2} - \int_{-\pi/2}^{\pi/2} + \int_{\pi/2}^{\phi_{2(1)}'}\right)\right] \tag{44}$$

if $\phi_{2(1)}'$ and $\phi_{2(1)}''$ are in the upper half plane and $\phi_{2(1)}' < \phi_{2(1)}''$, sign $\cos \phi_{2'1)}'$ being negative. If $\phi_{2(1)}' > \phi_{2(1)}''$ equation (44) holds if $\phi_{2(1)}'$ and $\phi_{2(1)}''$ are interchanged and the signs of the integrals reversed.

(c) $$I_{2(1)} = \frac{2Pl'^2}{h'} k_1^2 r^2 \left[\left(\int_{\phi_{2(1)}'}^{\phi_{2(1)}''} \right) - \left(\int_{\phi_{2(1)}''}^{\pi/2} - \int_{\pi/2}^{3\pi/2} + \int_{-\pi/2}^{\phi_{2(1)}'} \right) \right] \tag{45}$$

if $\phi_{2(1)}'$ and $\phi_{2(1)}''$ are in the lower half plane and $\phi_{2(1)}' < \phi_{2(1)}''$, sign $\cos \phi_{2(1)}'$ being positive. If $\phi_{2(1)}' < \phi_{2(1)}''$ equation (45) holds if $\phi_{2(1)}'$ and $\phi_{2(1)}''$ are interchanged and the signs of the integrals reversed.

Let

$$F_1(\phi_{2(1)}) = \int \frac{\cos^2\phi_{2(1)} d\phi_{2(1)}}{[1 + e' \cos(\phi_{2(1)} + \beta)]^2},$$

so that the solutions may be expressed in the form

(a) $$I_{2(1)} = W_1 \left[F_1\left(\frac{\pi}{2}\right) + \frac{F_1\left(\frac{3\pi}{2}\right) + F_1\left(-\frac{\pi}{2}\right)}{2} - F_1(\phi_{2(1)}') - F_1(\phi_{2(1)}'') \right] \tag{46}$$

(b) $$I_{2(1)} = W_1 \left[F_1(\phi_{2(1)}'') - F_1(\phi_{2(1)}') + F_1\left(\frac{\pi}{2}\right) - \frac{F_1\left(\frac{3\pi}{2}\right) + F_1\left(-\frac{\pi}{2}\right)}{2} \right] \tag{47}$$

(c) $$I_{2(1)} = W_1 \left[F_1(\phi_{2(1)}'') - F_1(\phi_{2(1)}') - F_1\left(\frac{\pi}{2}\right) + \frac{F_1\left(\frac{3\pi}{2}\right) + F_1\left(-\frac{\pi}{2}\right)}{2} \right] \tag{48}$$

where $W_1 = 4Pl'^2 k_1^2 r^2/h'$. When $\phi_{2(1)}'$ or $\phi_{2(1)}'' = \pm \pi/2$ the solution may be obtained by the same process as in Section 4.1.

The expression for $I_{2(2)}$ is of a simpler form since C is a constant in the integration, i.e.

$$I_{2(2)} = \frac{2Pl'^2}{h'} k_2^2 C^2 (\text{sign } C) \left[\int_{\theta_{2(1)}'}^{\theta_{2(1)}''} - \int_{\theta_{2(1)}''}^{\theta_{2(1)}'+2\pi} \right] \tag{49}$$

where the integrand of each integral is $(1 + e' \cos \theta_2)^{-2}$.

Let

$$F_2(\theta_2) = \int \frac{d\theta_2}{(1 + e' \cos \theta_2)^2}, \text{ giving the solution}$$

$$I_{2(2)} = W_2 (\text{sign } C) \left[F_2(\theta_{2(1)}'') - \frac{F_2(\theta_{2(1)}') + F_2(\theta_{2(1)}' + 2\pi)}{2} \right] \tag{50}$$

where

$$W_2 = \frac{4Pl'^2}{h'} k_2^2 C^2.$$

Hence,

$$J_1' = \int_{\theta_2=0}^{2\pi} |\Gamma_1(\theta_2)| \left(\frac{dt}{d\theta_2}\right) d\theta_2 = |I_{2(1)} - I_{2(2)}| \tag{51}$$

where $I_{2(1)}$ is given by the appropriate expression {equations (46), (47) or (48)} and $I_{2(2)}$ is determined by equation (50).

Let $F_3(\phi_{2(3)}) = \int \frac{\cos^2\phi_{2(3)}}{[1 + e' \cos(\phi_{2(3)} - \epsilon)]^2} \, d\phi_{2(3)}$ so that equation (55) may be written

$$I_{2(3)} = W_3 \left[F_3\left(\frac{\pi}{2}\right) + \frac{F_3\left(\frac{3\pi}{2}\right) + F_3\left(-\frac{\pi}{2}\right)}{2} - F_3(\phi_{2(3)}') - F_3(\phi_{2(3)}'') \right] \tag{56}$$

where

$$W_3 = \frac{4Pl'^2}{h'} k_3^2 r^2 .$$

The expression for $I_{2(4)}$ is identical with that for $I_{2(1)}$ if k_4^2 is substituted for k_1^2 and $\theta_{2(1)}'$ and $\theta_{2(1)}''$ replaced by $\theta_{2(2)}'$ and $\theta_{2(2)}''$ respectively. However, since $\theta_{2(2)}'$ and $\theta_{2(2)}''$ are always π radians apart the only case to be considered is that corresponding to equation (43),

i.e.

$$I_{2(4)} = \frac{2Pl'^2}{h'} k_4^2 r^2 \left[\left(\int_{\phi_{2(4)}'}^{\pi/2} - \int_{\pi/2}^{\phi_{2(4)}''} \right) - \left(- \int_{\phi_{2(4)}''}^{3\pi/2} + \int_{-\pi/2}^{\phi_{2(4)}'} \right) \right] \tag{57}$$

if $\phi_{2(4)}'$ is in the lower half plane. If $\phi_{2(4)}''$ is in the lower half plane equation (57) applies if $\phi_{2(4)}'$ and $\phi_{2(4)}''$ are interchanged and the signs of the integrals reversed. Here, $\phi_{2(4)}' = (\theta_{2(2)}' - \beta)$ and $\phi_{2(4)}'' = (\theta_{2(2)}'' - \beta)$.

Hence

$$I_{2(4)} = W_4 \left[F_1\left(\frac{\pi}{2}\right) + \frac{F_1\left(\frac{3\pi}{2}\right) + F_1\left(-\frac{\pi}{2}\right)}{2} - F_1(\phi_{2(4)}') - F_1(\phi_{2(4)}'') \right] \tag{58}$$

where

$$F_1(\phi) = \int \frac{\cos^2\phi}{[1 + e' \cos(\phi + \beta)]^2} \, d\phi \quad \text{and} \quad W_4 = \frac{4Pl'^2}{h'} k_4^2 r^2 .$$

Hence,

$$J_2' = \int_{\theta_2=0}^{2\pi} |\Gamma_2(\theta_2)| \left(\frac{dt}{d\theta_2}\right) d\theta_2 = |I_{2(3)} - I_{2(4)}| \tag{59}$$

where $I_{2(3)}$ and $I_{2(4)}$ are given by equations (56) and (58). When $\phi_{2(3)}''$, $\phi_{2(3)}'$ or $\phi_{2(4)}''$, $\phi_{2(4)}' = \pm \pi/2$ the solution may be obtained by the process used in Section 4.1.

Finally, in order to develop an expression for equation (38) we must evaluate the quantity

$$J_3' = \left| \frac{2Pl'^2}{h'} \left[\int_{\theta_{2(3)}'}^{\theta_{2(3)}''} \frac{k_5^2 |C| C - k_6^2 |A \cos\theta_2 - B \sin\theta_2| (A \cos\theta_2 - B \sin\theta_2)}{(1 + e' \cos\theta_2)^2} \, d\theta_2 - \right.\right.$$

$$\left.\left. - \int_{\theta_{2(3)}''}^{\theta_{2(3)}'+2\pi} \frac{k_5^2 |C| C - k_6^2 |A \cos\theta_2 - B \sin\theta_2| (A \cos\theta_2 - B \sin\theta_2)}{(1 + e' \cos\theta_2)^2} \, d\theta_2 \right] \right| \tag{60}$$

Let

$$I_{2(5)} = \frac{2Pl'^2}{h'} \left[k_5^2 \int_{\theta_{2(3)}'}^{\theta_{2(3)}''} \frac{|C| C}{(1 + e' \cos\theta_2)^2} \, d\theta_2 - k_5^2 \int_{\theta_{2(3)}''}^{\theta_{2(3)}'+2\pi} \frac{|C| C}{(1 + e' \cos\theta_2)^2} \, d\theta_2 \right] \tag{61}$$

and

$$I_{2(6)} = \frac{2Pl'^2}{h'} \left[k_6^2 \int_{\theta_{2(3)}'}^{\theta_{2(3)}''} \frac{|A \cos\theta_2 - B \sin\theta_2| (A \cos\theta_2 - B \sin\theta_2)}{(1 + e' \cos\theta_2)^2} \, d\theta_2 - \right.$$

$$\left. - k_6^2 \int_{\theta_{2(3)}''}^{\theta_{2(3)}'+2\pi} \frac{|A \cos\theta_2 - B \sin\theta_2| (A \cos\theta_2 - B \sin\theta_2)}{(1 + e' \cos\theta_2)^2} \, d\theta_2 \right] . \tag{62}$$

The expression for $I_{2(5)}$ is identical with that for $I_{2(2)}$ if k_5^2 is substituted for k_2^2 and $\theta_{2(1)}'$ and $\theta_{2(1)}''$ replaced by $\theta_{2(3)}'$ and $\theta_{2(3)}''$ respectively,

i.e.

$$I_{2(5)} = \frac{2Pl'^2}{h'} k_5^2 C^2 (\text{sign } C) \left[\int_{\theta_{2(3)}'}^{\theta_{2(3)}''} - \int_{\theta_{2(3)}''}^{\theta_{2(3)}'+2\pi} \right] \tag{63}$$

where the integrand of each integral is $(1 + e' \cos \theta_2)^{-2}$,

i.e.

$$I_{2(5)} = W_5 (\text{sign } C) \left[F_2(\theta_{2(3)}'') - \frac{F_2(\theta_{2(3)}') + F_2(\theta_{2(3)}' + 2\pi)}{2} \right] \tag{64}$$

where

$$W_5 = \frac{4Pl'^2}{h'} k_5^2 C^2 .$$

The expression for $I_{2(6)}$ is identical with that for $I_{2(3)}$ if k_6^2 is substituted for k_3^2 and $\theta_{2(2)}'$ and $\theta_{2(2)}''$ replaced by $\theta_{2(3)}'$ and $\theta_{2(3)}''$ respectively. However, the solution must be expressed as one of three alternatives as in the case of $I_{2(1)}$.

(a) If $\phi_{2(6)}'$ is in the lower half plane and $\phi_{2(6)}''$ in the upper half plane then

$$I_{2(6)} = W_6 \left[F_3\left(\frac{\pi}{2}\right) + \frac{F_3\left(\frac{3\pi}{2}\right) + F_3\left(-\frac{\pi}{2}\right)}{2} - F_3(\phi_{2(6)}') - F(\phi_{2(6)}'') \right]. \tag{65}$$

If $\phi_{2(6)}'$ is in the upper half plane and $\phi_{2(6)}''$ in the lower half plane equation (65) applies with $\phi_{2(6)}'$ and $\phi_{2(6)}''$ interchanged and the sign reversed.

(b) If both $\phi_{2(6)}'$ and $\phi_{2(6)}''$ are in the upper half plane then

$$I_{2(6)} = W_6 \left[F_3(\phi_{2(6)}'') - F_3(\phi_{2(6)}') + F_3\left(\frac{\pi}{2}\right) - \frac{F_3\left(\frac{3\pi}{2}\right) + F_3\left(-\frac{\pi}{2}\right)}{2} \right] \tag{66}$$

if $\phi_{2(6)}' < \phi_{2(6)}''$, and if $\phi_{2(6)}' > \phi_{2(6)}''$ the equation applies with $\phi_{2(6)}'$ and $\phi_{2(6)}''$ interchanged and the sign reversed.

(c) If both $\phi_{2(6)}'$ and $\phi_{2(6)}''$ are in the lower half plane then

$$I_{2(6)} = W_6 \left[F_3(\phi_{2(6)}'') - F_3(\phi_{2(6)}') - F_3\left(\frac{\pi}{2}\right) + \frac{F_3\left(\frac{3\pi}{2}\right) + F_3\left(-\frac{\pi}{2}\right)}{2} \right] \tag{67}$$

when $\phi_{2(6)}' < \phi_{2(6)}''$, and if $\phi_{2(6)}' > \phi_{2(6)}''$ the equation applies with $\phi_{2(6)}'$ and $\phi_{2(6)}''$ interchanged and the sign reversed.

In these equations

$$W_6 = \frac{4Pl'^2}{h'} k_6^2 \gamma^2 \quad \text{and} \quad F_3(\phi) = \int \frac{\cos^2\phi \, d\phi}{[1 + e' \cos(\phi - \epsilon)]^2} .$$

Hence

$$J_3' = \int_{\theta_2=0}^{2\pi} |\Gamma_3(\theta_2)| \left(\frac{dt}{d\theta_2}\right) d\theta_2 = |I_{2(5)} - I_{2(6)}| \tag{68}$$

where $I_{2(5)}$ is given by equation (64) and $I_{2(6)}$ by the appropriate equation (65) to (67).

The total amount of angular impulse per satellite orbit can therefore be expressed as

$$J_T' = \sum_{r=1}^{3} \int_{\theta_2=0}^{2\pi} |\Gamma_r(\theta_2)| \left(\frac{dt}{d\theta_2}\right) d\theta_2 = |I_{2(1)} - I_{2(2)}| + |I_{2(3)} - I_{2(4)}| + |I_{2(5)} - I_{2(6)}| . \tag{69}$$

The quantities contained in the modulus signs of equation (69) are straightforward to evaluate for a given case and the author feels that the result is best expressed by this equation, rather than attempt to combine the individual constituents into one expression.

The total amount of angular impulse J_T' is a function of the angle θ and will be a maximum for some angular position $\theta = \theta_M$ of the satellite in the plane of the ecliptic. If J_M' is the maximum value of J_T' as determined from equation (69), then if the satellite completes N orbits during one year, an upper estimate for the magnitude of angular impulse per year is simply

$$J_N' = NJ_M' . \tag{70}$$

Alternatively one may take the mean value of J_T', denoted by $\bar{J}_T'$, for a change in θ of 2π and use the expression

$$J_N'' = N\bar{J}_T' \tag{71}$$

as the total amount of angular impulse over a period of a year, as obtained by taking the mean value of the total angular impulse per satellite orbit.

An example will serve to illustrate the procedure in a particular case.

Suppose $l_1 = m_2 = n_3 = 1$, (all other direction cosines being zero) so that X, Y, Z and X', Y', Z' are coincident sets of axes. The satellite now orbits in a plane normal to the plane of the ecliptic, the normal to the orbital plane being parallel with the direction of the minor axis of the earth's orbit. Let $z_1 = z_2 = z$, (all other torque arms being zero) so that the centre of mass of the satellite is along the Z_S axis at a distance z from the geometrical centre of the configuration. The surface areas involved in the problem are $S_{X_S Z_S}$ and $S_{Y_S Z_S}$; assume these to be equal areas—denoted by S.

We have

$$A = \cos\theta , \qquad B = 0 , \qquad C = \sin\theta$$

$$k_2^2 = k_3^2 = zS .$$

The formula given in the general theory, for determining the points at which $\Gamma_1(\theta_2) = 0$ cannot be satisfied for any choice of θ_2, i.e. the torque is unidirectional about the X_S axis.

Also

$$I_{2(1)} = 0 \qquad \text{since} \qquad k_1^2 = 0$$

and

$$I_{2(2)} = W_2(\text{sign}\sin\theta)\left[F_2(\theta_{2(1)}'') - \frac{F_2(\theta_{2(1)}'') + F_2(\theta_{2(1)}'' + 2\pi)}{2}\right]$$

since

$$\theta_{2(1)}' = \theta_{2(1)}'' .$$

If we assume that the satellite orbit is circular then $F_2(\theta_2) = \theta_2$ and $I_{2(2)}$ reduces to give

$$I_{2(2)} = W_2(\text{sign}\sin\theta)(-\pi)$$

therefore

$$J_1' = \left| -I_{2(2)} \right| = \left| \frac{4Pl'^2}{h'} zS\pi \sin^2\theta \right| ,$$

i.e.

$$J_1' = 2PT'zS\sin^2\theta$$

since all the components of the expression are positive quantities, and $T' = 2\pi l'^2/h'$ is the orbital period of the satellite about the earth.

The conditions for $\Gamma_2(\theta_2) = 0$ give $\theta_{2(2)}' = \pi/2$, $\theta_{2(2)}'' = 3\pi/2$. We have $\phi_{2(3)} = \pi/2$ and $\phi_{2(3)}'' = 3\pi/2$, and this is therefore the special case when the integration limits are coincident with the limiting points. Consequently we may write

$$I_{2(3)} = \frac{2Pl'^2}{h'} k_3^2 r^2 \left[-\int_{\pi/2}^{3\pi/2} \cos^2\phi_{2(3)} d\phi_{2(3)} - \int_{-\pi/2}^{\pi/2} \cos^2\phi_{2(3)} d\phi_{2(3)} \right] \tag{72}$$

since sign $\cos \phi_2$ is negative over the range $\pi/2$ to $3\pi/2$ and positive over the range $-\pi/2$ to $\pi/2$, i.e.

$$I_{2(3)} = -\frac{2Pl'^2}{h'} k_3^2 r^2 \pi$$

$$= -PT'zS\cos^2\theta.$$

Now $I_{2(4)} = 0$ since $k_4^2 = 0$, therefore

$$J_2' = PT'zS\cos^2\theta. \tag{73}$$

Since there is no torque about the Z_s axis we have $J_3' = 0$.
Hence

$$J_T' = PT'zS(2\sin^2\theta + \cos^2\theta). \tag{74}$$

The total amount of angular impulse J_T' is a maximum when $\theta = \pi/2$ or $3\pi/2$ and thus

$$J_M' = 2PT'zS \tag{75}$$

and

$$J_N' = 2PzST_E \tag{76}$$

where $T_E = NT'$ is the period of the earth in its orbit round the sun.

The mean value of J_T' is

$$\bar{J}_T' = \frac{PT'zS}{2\pi} \int_{\theta=0}^{2\pi} (2\sin^2\theta + \cos^2\theta) d\theta,$$

i.e.

$$\bar{J}_T' = \frac{3}{2} PT'zS. \tag{77}$$

6. *Attitude Deviations of an Earth-Pointing Gravity-Gradient-Stabilised Satellite due to Radiation Pressure.*

Consider the case where the satellite is moving along an orbit whose plane is perpendicular to the plane of the ecliptic and where the Y_S rotational axis is pointing towards the sun. Let the centre of mass lie along the 'long-axis' (i.e. the axis of minimum moment of inertia) of the vehicle, the 'long-axis' being assumed to pass through the centre of symmetry as indicated by Fig. 6. In the case of a gravity-gradient-stabilised satellite the 'long-axis' is the earth pointing axis. Let z be the distance between the centre of symmetry and the centre of mass. The force on the satellite due to radiation pressure is $2PS$ (assuming complete reflection) where S is the surface area presented to the sun. Also the torque about the centre of mass is $2PSz$, acting along the X_S axis, and is constant for all positions of the satellite in its orbit.

Ref. 1 shows that the attitude motion about the X_S axis for an earth-pointing satellite (in a circular orbit) relying solely on the gravitational restoring torque for its stabilisation, is described by the equation

$$I_{X_S}\ddot{\theta}_1 + 4(I_{Y_S} - I_{Z_S})\omega_0^2\theta_1 + (I_{X_S} + I_{Z_S} - I_{Y_S})\dot{\theta}_3 = L_{X_S}', \text{ (for small angles)}$$

where I_{X_S}, I_{Y_S} and I_{Z_S} are the principal moments of inertia about the X_S, Y_S and Z_S axes, θ_1 and θ_3 are the angular deviations about the X_S and Z_S axes and ω_0 is the angular velocity of the satellite in its orbit; L_{X_S}' is the component of any external disturbing torque about X_S. In the present instance $L_{X_S}' = 2PSz$, and since this is a constant only the steady-state solution is of interest, resulting in an attitude deviation given by

$$\theta_1 = \frac{2PSz}{4(I_{Y_S} - I_{Z_S})\omega_0^2}.$$

Referring to Fig. 6 if $a = d$ and the dimensions of the configuration are such that $2l = 8$ ft, $2b = 2$ ft and $2a = 3$ ft, then $z = 3$ ft, $S = (2l+2b)2a = 30$ sq. ft., and if the mass of the satellite is taken to be 500 lb then $(I_{Y_S} - I_{Z_S}) \sim 5300$ lb. ft². Taking ω_0 as 10^{-3} rad. sec^{-1}, the attitude deviation is of the order of $1\frac{1}{2}$ degrees.

7. *A Comparison of the Torques due to Radiation Pressure and the Earth's Gravitational Field for a Space-Stabilised Satellite of Given Orientation.*

Suppose $l_1 = m_2 = n_3 = 1$, (see Section 4.2). As in Section 6 let the centre of mass be a distance z from the centre of symmetry so that the radiation-pressure torque given by equation (15) reduces to

$$\mathbf{\Gamma} = 2PSz[-|\sin\theta|\sin\theta\,\mathbf{i}_S + |\cos\theta|\cos\theta\,\mathbf{j}_S]$$

if

$$S_{X_SZ_S} = S_{Y_SZ_S} = S.$$

Therefore

$$|\mathbf{\Gamma}| = 2PSz\sqrt{(\sin^4\theta + \cos^4\theta)}$$

and attains a maximum value given by

$$|\mathbf{\Gamma}|_{\max} = 2PSz.$$

From Section 3.2 of Ref. 1 it is shown that with the X_S and Z_S axes in the plane of the satellite's orbit (making the orbital plane at right angles to the plane of the ecliptic in the present case) the maximum value of the gravitational torque is given by

$$|\mathbf{\Gamma}_G|_{\max} = \frac{3}{2}\omega_0^2|I_{Z_S} - I_{X_S}|.$$

Hence, for this *particular satellite* having *the orientation stated above*, $|\mathbf{\Gamma}|_{\max} = |\mathbf{\Gamma}_G|_{\max}$ when $\omega_0^2 = 4PSz/3|I_{Z_S} - I_{X_S}|$. For a circular orbit $\omega_0^2 = GM_E/R^3$ where M_E is the mass of the earth, G the gravitational constant and R the distance between the satellite and the centre of the earth, i.e.

$$R^3 = \frac{3GM_E|I_{Z_S} - I_{X_S}|}{4PSz}$$

giving $R \sim 10{,}000$ miles if we chose $S = 30$ sq. ft, $z = 3$ ft and $|I_{Z_S} - I_{X_S}| = 5300$ lb. ft² as in Section 6.

8. *Concluding Remarks.*

The equations developed in the text may be used to estimate the annual amount of angular impulse required from a satellite attitude-control system to counter the solar radiation-pressure torque. Two situations are considered, the space-stabilised satellite and an earth-stabilised satellite in a non-precessing orbit. The analysis assumes a specific geometrical configuration, namely that of a rectangular prism, and also complete reflection at all surfaces. The necessary amount of angular impulse may be provided by any suitable control system, for example by reaction jets or accelerating flywheels.

The expressions for the space-stabilised satellite indicate that as the satellite completes one orbit round the sun the nett angular impulse about any one of the satellite axes is zero (the effect of the eccentricity of the earth's orbit being neglected). If this were strictly true, the radiation-pressure torque could be most effectively compensated by the use of accelerating flywheels, one such wheel along each of the satellite axes. The angular momentum attained by a given flywheel during one half of the orbit would be nullified during the second half of the orbit, resulting in no nett increase or decrease of the flywheel's angular velocity. In general, however, the situation is not so simple as this since the satellite spends a certain proportion of its time passing through the earth's shadow. Owing to orbital regression and the fact that the satellite's orbit about the earth may be an ellipse, the unidirectional change of angular momentum about a given axis during one half of the earth's orbit will not in general be compensated exactly during the remainder of the orbit. Thus, an accelerating flywheel would gain or loose angular momentum during the course of one year. Such a nett change of flywheel angular momentum may be very small in comparison to the build up of angular momentum between successive positions at which the torque component concerned is zero, and therefore be of little significance. However, if it were desired to use small flywheels for control over a period of several years the shadowing effect of the earth should be considered, especially if relatively large torque arms were present. This may very well be the case when the principal axes of inertia of the satellite have been made as nearly equal as possible in order to minimise the torque arising from the earth's gravitational-field gradient, especially if air-drag torques are small and meteoroid hazards not serious.

The relative importance of the perturbation torques due to radiation pressure and the earth's gravitational-field gradient may be inspected by comparing the angular impulse given by equation (16) of Section 4.1 with the corresponding quantity determined by considering the angular impulse produced by the gravity-gradient torque over a period of a year.

The equation for the radiation-pressure torque about the centre of mass of the rectangular prism configuration indicates, as one might expect, that the torque is always zero if the centre of mass coincides with the centre of symmetry of the external shell. This may also be the case when all the surfaces are completely absorbing, and if so, the situation arising when the different surfaces have varying absorption characteristics will result in a nett torque acting on the vehicle even when the centre of mass coincides with the centre of symmetry.

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c	Velocity of light *in vacuo*
e'	Eccentricity of satellite orbit round the earth
e_E	Eccentricity of earth's orbit round the sun
h'	Angular momentum per unit mass of satellite in orbit round the earth
h_E	Angular momentum per unit mass of earth in its orbit round the sun
$(I_{X_S}, I_{Y_S}, I_{Z_S})$	Principal moments of inertia of satellite
J_T	Total amount of angular impulse per year imparted to space-stabilised satellite to maintain perfect stabilisation against radiation-pressure torques
J_r	Amount of angular impulse per year about individual axes of space-stabilised satellite (r taking values 1, 2, 3)
J_R	Nett angular impulse imparted to space-stabilised satellite during one year by radiation-pressure torques
J_T'	Total amount of angular impulse per satellite orbit that has to be supplied about satellite axes for perfect stabilisation of earth-pointing satellite against radiation-pressure torques
J_r'	Amount of angular impulse per satellite orbit about individual axes of earth-pointing satellite (r taking values 1, 2, 3)
J_M'	Maximum value of J_T'
J_N'	Upper estimate for amount of angular impulse per year for earth-pointing satellite
$\bar{J}_T'$	Mean value of J_T'
J_N''	Estimate for amount of angular impulse per year for earth-pointing satellite based on the mean value of J_T'
k_r^2	Positive constants (r taking values 1 to 6)
l'	Semi-latus rectum of satellite orbit round the earth
l_E	Semi-latus rectum of the earth's orbit round the sun
$\mathbf{L}_r$	Torque components about centre of mass of satellite due to radiation incident on faces 1, 2, 3 (or opposite faces) *see* Fig. 2 (r = 1, 2, 3)
$\mathbf{n}$	Unit vector specifying direction of the sun's radiation relative to satellite axes
$(n_{X_S}, n_{Y_S}, n_{Z_S})$	Components of **n** along satellite axes (earth-pointing satellite)
N	Number of orbits round the earth made by the satellite during one year
P	Pressure of sun's radiation on a 'black-body' surface placed at right angles to the direction of the radiation at the mean distance of the earth from the sun

$\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$	Position vectors of centres of pressure of faces 1, 2, 3 relative to centre of mass of satellite—*see* Fig. 2
S'	Solar constant
$S_{X_S Z_S}, S_{Y_S Z_S}, S_{X_S Y_S}$	Areas of faces of rectangular prism
T'	Period of satellite in its orbit round the earth
T_E	Period of earth's orbit
x_r, y_r, z_r	Components of $\mathbf{r}_r$ along satellite axes (r taking values 1, 2, 3)
$\mathbf{\Gamma}$	Torque about centre of mass of satellite due to radiation pressure (complete reflection)
$\Gamma_1, \Gamma_2, \Gamma_3$	Components of $\mathbf{\Gamma}$ along satellite axes
θ	Angle defining motion of the satellite in the plane of the ecliptic
θ_2	Angle defining the angular motion of earth-pointing satellite in its orbit

Axes

X', Y', Z'	System of space-fixed axes, origin at the centre of mass of the satellite and having Z' normal to the plane of the ecliptic and X' and Y' along the major and minor axes respectively of the earth's orbit round the sun
$(\mathbf{i}', \mathbf{j}', \mathbf{k}')$	Unit vectors along (X', Y', Z')
X, Y, Z	Set of space-fixed axes, origin at the centre of mass of the satellite
$(\mathbf{i}, \mathbf{j}, \mathbf{k})$	Unit vectors along (X, Y, Z)
X_S, Y_S, Z_S	Satellite axes (origin at the centre of mass)
$(\mathbf{i}_S, \mathbf{j}_S, \mathbf{k}_S)$	Unit vectors along (X_S, Y_S, Z_S)
$[(l_1, m_1, n_1); (l_2, m_2, n_2); (l_3, m_3, n_3)]$	Direction cosines of X', Y', Z' axes relative to XYZ
$\cos\gamma_X, \cos\gamma_Y, \cos\gamma_Z$	Direction cosines of $\mathbf{n}$ relative to axes X, Y, Z

REFERENCE

No.	*Author*	*Title, etc.*
1	N. E. Ives	Principles of attitude control of artificial satellites. A.R.C. R. & M. 3276. November, 1959.

APPENDIX I

Integrals Occurring in the Text

$$\int \frac{d\nu}{(1+e\cos\nu)^2} = \left[-\frac{e}{(1-e^2)}\frac{\sin\nu}{(1+e\cos\nu)} + \frac{2}{(1-e^2)^{3/2}}\tan^{-1}\left(\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\nu}{2}\right)\right]$$

$$\int \frac{\cos^2\nu\, d\nu}{(1+e\cos\nu)^2} = \frac{1}{e^2}\left[\nu - \frac{e}{(1-e^2)}\frac{\sin\nu}{(1+e\cos\nu)} + \frac{2(1-2e^2)}{(1-e^2)^{3/2}}\tan^{-1}\left(\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\nu}{2}\right)\right]$$

$$\int \frac{\sin^2\nu\, d\nu}{(1+e\cos\nu)^2} = \frac{1}{e^2}\left[\frac{e\sin\nu}{(1+e\cos\nu)} + \frac{2}{\sqrt{(1-e^2)}}\tan^{-1}\left(\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\nu}{2}\right) - \nu\right]$$

$$\int \frac{\sin\nu\cos\nu\, d\nu}{(1+e\cos\nu)^2} = \frac{1}{e^2}\left[\frac{e\cos\nu}{(1+e\cos\nu)} - \log_e(1+e\cos\nu)\right].$$

These integrals are taken between the limits specified in the appropriate sections of the text.

APPENDIX II

Radiation Incident Obliquely on a Flat Surface

Consider radiation incident at an angle γ with the normal to a surface of area S and having an absorptivity of unity, Fig. 7. The projection of the area S in a plane normal to the direction of the incident radiation is $S\cos\gamma$, and the force on such an area is $PS\cos\gamma$. Since the same amount of radiation is incident on the surface S, the force on this surface must also be $PS\cos\gamma$, acting through the centre of pressure in the direction of the incident radiation. This force may be resolved into two components:

$PS\cos^2\gamma$ normal to the surface S,

and

$PS\cos\gamma\sin\gamma$ tangential to the surface.

If the radiation suffers complete reflection at the surface the resultant tangential component of force is zero and the normal component is $2PS\cos^2\gamma$.

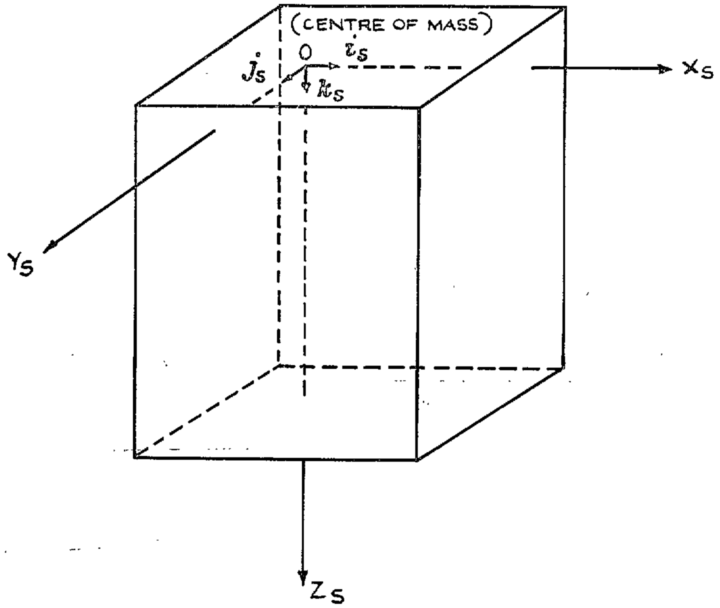

FIG. 1. Orientation of satellite axes relative to the outer geometrical shell.

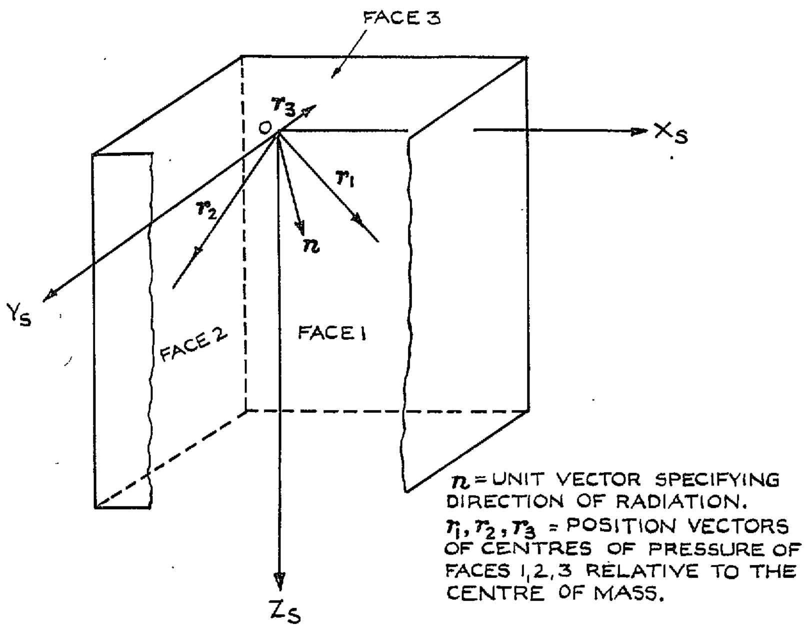

FIG. 2. 'Cut-away' view of satellite.

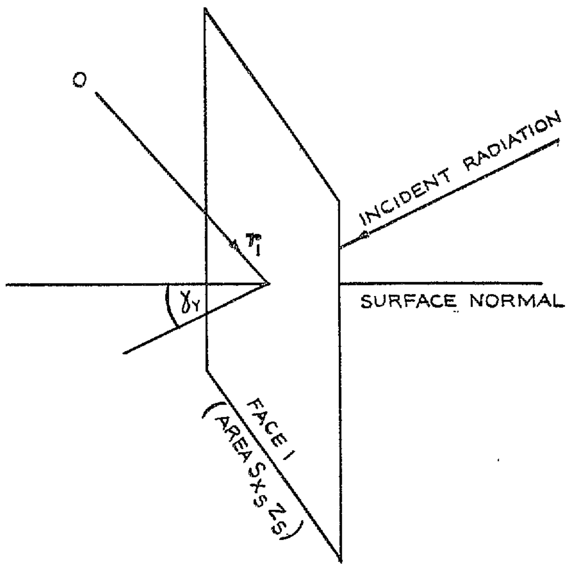

FIG. 3. Radiation incident on face 1.

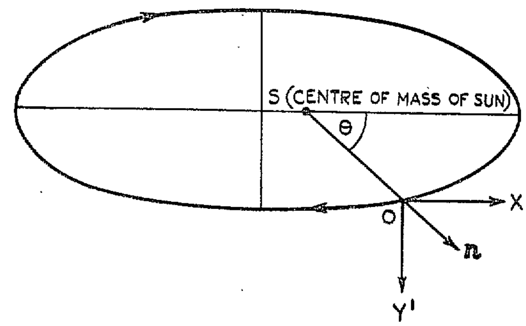

FIG. 4. Motion in the plane of the ecliptic.

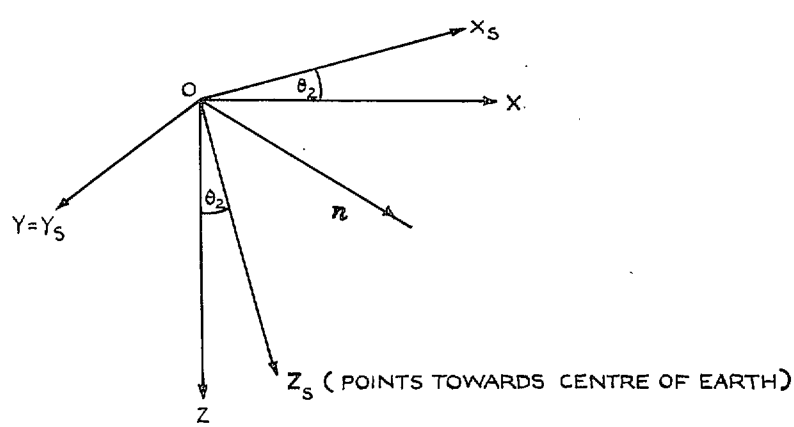

FIG. 5. Rotation of satellite axes for an earth-pointing satellite.

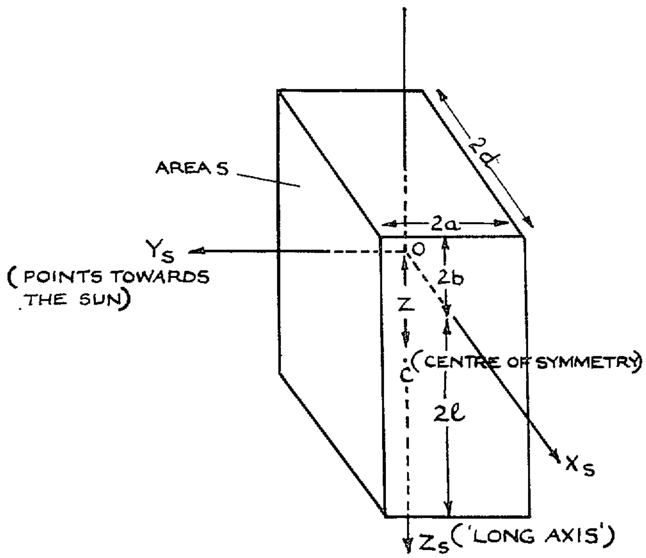

FIG. 6. Earth-pointing satellite employing gravity-gradient stabilisation.

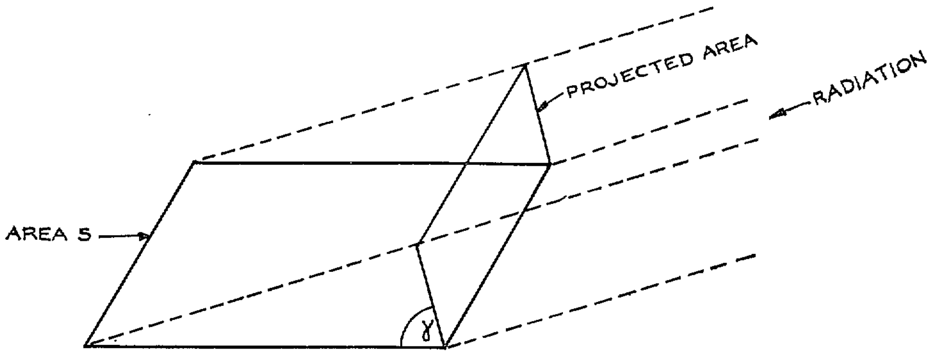

FIG. 7. Radiation incident obliquely on a flat surface.

(87419) Wt. 64/1857 K.5 8/63 Hw.

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