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# A Numerical Solution of an Integral Equation Satisfied by the Velocity Distribution around a Body of Revolution in Axial Flow

By D. KERSHAW

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# A Numerical Solution of an Integral Equation Satisfied by the Velocity Distribution around a Body of Revolution in Axial Flow

By D. Kershaw

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*Summary.*

The report contains a description of a method used to find the solution of an integral equation on an automatic computer together with some illustrative examples.

*Introduction.*

Knowledge of the pressure distribution on a body moving in a fluid is required in a variety of applications, such as the prediction of cavitation inception conditions and the study of boundary-layer development. For many purposes the effects of viscosity can be neglected and it is sufficient to consider inviscid incompressible flow. In general this is essentially the problem of solving Laplace's equation with the condition of no flow through the surface of the body, but in the special case of a smooth, simply connected body of revolution in uniform axial flow it can be reduced (Ref. 1) to finding the solution of a linear integral equation. This equation is satisfied by the velocity-distribution function on the body surface from which the pressure distribution can be found by using Bernoulli's equation.

This report describes a numerical solution of the integral equation which is suitable for an automatic digital computer and which has been programmed for the Ferranti 'Pegasus'. Illustrative examples for the particularly difficult case of bodies of revolution with flat heads are given in the Appendix.

The integral equation will not be derived here but it should be remarked that the mathematical model is a system of co-axial vortex rings (replacing the body) at rest in an axial, inviscid, incompressible, uniform flow of unit speed.

*Integral Equation.*

Let the generating curve of a body of revolution be parameterized by the distance along the curve measured from one of the two points where the curve meets the axis of revolution, $y = 0$. Also let the total length of the curve be taken as $L$, then any point on the curve is specified by

with
$$\left.\begin{aligned} x = x(s)\,,\ y = y(s) \\ x(0) = 0\,,\ y(0) = y(L) = 0\,. \end{aligned}\right\} \qquad (1)$$

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If the curve is such that $x' = dx/ds$, $y' = dy/ds$ are continuous and $x'^2 + y'^2 = 1$ the integral equation satisfied by $w(s)$, the speed at points of the body specified by $s$, is (Ref. 1)

$$w(s) = 2x' - \frac{1}{\pi}\int_0^L K(s, \sigma)w(\sigma)d\sigma, \ 0 \leqslant s \leqslant L. \tag{2}$$

The kernel function $K(s, \sigma)$ of this integral equation is

$$\frac{1}{\sqrt{[(x-\xi)^2 + (y+\eta)^2]}}\left\{\frac{[x'y - y'(x-\xi)]}{y}[K(k) - E(k)] - 2\eta\frac{[x'(y-\eta) - y'(x-\xi)]}{(x-\xi)^2 + (y-\eta)^2}E(k)\right\} \tag{3}$$

where

(1) $\quad \xi = x(\sigma), \ \eta = y(\sigma)$

(2) $\quad k^2 = \dfrac{4y\eta}{(x-\xi)^2 + (y+\eta)^2}$

and

(3) $\quad K(k) = \displaystyle\int_0^{\pi/2}\frac{d\varphi}{\sqrt{(1-k^2\sin^2\varphi)}}, \ E(k) = \int_0^{\pi/2}\sqrt{(1-k^2\sin^2\varphi)}d\varphi$

are the complete elliptic integrals of the first and second kinds (Ref. 2 p. 155).

The method described in this report for solving this integral equation for a given body is to approximate to the integral term in (2) using a quadrature formula of the form:

$$\sum_{i=0}^{n} A_i(s)w(\sigma_i), \quad 0 = \sigma_0 < \sigma_1 < \ldots < \sigma_{n-1} < \sigma_n = L. \tag{4}$$

If $s$ is given the values $\sigma_0, \sigma_1, \ldots, \sigma_n$ the integral equation can be replaced by a set of linear simultaneous equations which when solved give $w(s)$ at the quadrature points $\sigma_0, \sigma_1, \ldots, \sigma_n$. The accuracy of the result will evidently depend on the quadrature formula used, which in turn will depend on the form of the integrand. In order to determine an appropriate quadrature formula it will be necessary to investigate the kernel.

*The Kernel.*

It is known that the complete elliptic integral of the first kind $K(k)$ has a logarithmic singularity at $k^2 = 1$, and as

$$1 - k^2 = \frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} \tag{5}$$

the kernel $K(s, \sigma)$ will have a logarithmic singularity when $\sigma = s$. Using the expansions of $K(k)$ and $E(k)$ given in Ref. 2 p. 155 it can be shown that the kernel can be put into the form:

$$K(s, \sigma) = P(s, \sigma)\log|s - \sigma| + Q(s, \sigma) \tag{6}$$

where

$$P(s, \sigma) = \frac{1}{\sqrt{[(x-\xi)^2 + (y-\eta)^2]}}\left\{\frac{[x'y - y'(x-\xi)]}{y}\frac{2}{\pi}E(k_1) - \right.$$
$$\left. - 2\eta\frac{[x'(y-\eta) - y'(x-\xi)]}{(x-\xi)^2 + (y-\eta)^2}\frac{2}{\pi}[K(k_1) - E(k_1)]\right\} \tag{7}$$

$$Q(s, \sigma) = K(s, \sigma) - P(s, \sigma)\log|s - \sigma| \tag{8}$$

and

$$k_1^2 = 1 - k^2 = \frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2}. \tag{9}$$

Analysis of $P(s, \sigma)$ shows that it is a continuous function of $\sigma$ for $0 < s < L$, and that its derivatives with respect to $\sigma$ are continuous except at points of discontinuity of curvature of the generating curve. When $\sigma = s$ it is found that

$$P(s, s) = -\frac{x'}{2y}, \; Q(s, s) = -x''y' + x'y'' + \frac{x'}{2y}(1 - \log 8y). \tag{10}$$

It is seen from the form of $Q(s, s)$ that at points of discontinuity of curvature of the generating curve $Q(s, s)$ will have step discontinuities. For $\sigma \neq s$, $Q(s, \sigma)$ is a continuous function of $\sigma$ for $0 < s < L$ and its derivatives with respect to $\sigma$ will have the same properties as those of $P(s, \sigma)$.

When $s = 0$ or $L$ the kernel has the simple form:

$$\frac{\pi x' \eta^2}{[(x-\xi)^2 + \eta^2]^{3/2}}, \text{ where } x,\, x' \text{ are taken at } s = 0 \text{ or } L \text{ respectively.} \tag{11}$$

Near the end points where $\eta = 0$, $0 < s < L$, $P(s, \sigma)$ and $Q(s, \sigma)$ have the respective forms:

$$\left.\begin{aligned} A(s, \sigma)\eta^2 \log \eta + B(s, \sigma) \\ -A(s, \sigma)\eta^2 \log \eta + C(s, \sigma) \end{aligned}\right\} \tag{12}$$

where $A$, $B$, $C$ are continuous functions of $\sigma$ for $0 < s < L$.

Regarding the form of the solution, little can be said except that it is continuous and that its derivatives will have discontinuities at points of discontinuity of curvature of the body. On physical grounds it is expected that around such points the solution will peak, i.e., that the fluid will move quickly in those regions. For bodies dealt with in this report at points where $y = 0$, $x'$ will be zero so that there $w(s) = 0$, which will be the stagnation points of the flow.

With these remarks in mind, the integral equation will be written:

$$w(s) = 2x' - \frac{1}{\pi}\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}} [P(s, \sigma)\log|s - \sigma| + Q(s, \sigma)]w(\sigma)d\sigma \tag{13}$$

$$0 = s_0 < s_1 < \ldots < s_{n-1} < s_n = L$$

where the points $s_1, s_2, \ldots, s_{n-1}$ are chosen so that $P(s, \sigma)w(\sigma)$ and $Q(s, \sigma)w(\sigma)$ are continuous and have continuous derivatives in $(s_i, s_{i+1})$, $i = 0, 1, \ldots, n-1$. In this case these functions can be approximated by polynomials.

*Quadrature Formula.*

The method of approximating the integral term described here was communicated to the author by G. F. Miller of the National Physical Laboratory.

Consider the numerical quadrature of integrals of the form

$$\int_a^b F(\sigma)m(\sigma - s)d\sigma \tag{14}$$

where $F(\sigma)$ is continuous in $(a, b)$ and $m(t)$, the weight function, may have an integrable singularity at $t = 0$.

If $h = (b-a)/n$ and $\sigma_j = a + jh$, $j = 0, 1, \ldots, n$ (15)

then the integral can be written as the sum of integrals in the form:

$$\sum_{j=0}^{n-1}\int_{\sigma_j}^{\sigma_{j+1}} F(\sigma)m(\sigma - s)d\sigma = h\sum_{j=0}^{n-1}\int_0^1 F(\sigma_j + ph)m(\sigma_j + ph - s)dp. \tag{16}$$

Now Everett's formula (Ref. 3, p. 56) is

$$\left.\begin{aligned} F(\sigma_j+ph) &= (1-p)F(\sigma_j) + pF(\sigma_{j+1}) + E_2\delta^2F(\sigma_j) + F_2\delta^2F(\sigma_{j+1}) + \dots \\ \text{with} \quad E_2 &= -\frac{p(p-1)(p-2)}{3\,!}, \; F_2 = \frac{(p+1)p(p-1)}{3\,!} \end{aligned}\right\} \tag{17}$$

and this series can be used to replace $F(\sigma_j+ph)$ in (16) as long as the tabular values of $F(\sigma)$ required by the central differences do not lie outside the range of integration. If, as in the following, only second differences are taken into account in Everett's formula, only the first and last integrals in (16) require tabular values outside $(a, b)$ and so there, truncated forward and backward difference formulae were used (Ref. 3, p. 54)

$$F(a+ph) = F(a) + p\Delta F(a) + \frac{p(p-1)}{2\,!}\Delta^2F(a) + \dots \tag{18}$$

$$F(b-ph) = F(b) - p\nabla F(b) + \frac{p(p-1)}{2\,!}\nabla^2F(b) - \dots . \tag{19}$$

Substituting the truncated series in (16) will give, on rearrangement, a sum of the form:

$$\sum_{i=0}^{n} B_i(s)F(\sigma_i) \tag{20}$$

where $B_i(s)$ involves integrals of the form:

$$\int_0^1 p^r m(\sigma_j+ph-s)dp, \; r = 0, 1, \dots \tag{21}$$

The integrals

$$\int_{s_i}^{s_{i+1}} w(\sigma)P(s, \sigma)\log|s-\sigma|d\sigma, \quad \int_{s_i}^{s_{i+1}} w(\sigma)Q(s, \sigma)d\sigma \tag{22}$$

can be replaced in this fashion by sums of the form (20) taking:

$$F(\sigma) = w(\sigma)P(s, \sigma) \text{ when } m(t) = \log|t| \tag{23}$$

$$F(\sigma) = w(\sigma)Q(s, \sigma) \text{ when } m(t) = 1 . \tag{24}$$

In (24) the weight function is independent of $s$ and can be evaluated easily, and the quadrature formula will have the simple form:

$$\sum_{i=0}^{n} C_i w(\sigma_i)Q(s, \sigma_i) . \tag{25}$$

With the weight function $m(t) = \log|t|$ it is necessary to compute the integrals

$$J_r = h\int_0^1 p^r \log|\sigma_j + ph - s|dp, \; r = 0, 1, 2, 3, \dots , \tag{26}$$

which can be accomplished by using the recurrence formula

$$\left.\begin{aligned} (r+1)J_r &= (\sigma_{j+1}-s)\log|\sigma_{j+1}-s| + \frac{r}{h}(s-\sigma_j)J_{r-1} - \frac{h}{r+1} \\ J_0 &= (\sigma_{j+1}-s)\log|\sigma_{j+1}-s| + (s-\sigma_j)\log|s-\sigma_j| - h . \end{aligned}\right\} \tag{27}$$

*The Programme.*

The method described above for solving the integral equation was programmed for the Ferranti 'Pegasus' with a drum store of 7,168 words. The computing was carried out mainly in single-length floating-point arithmetic, equivalent to about 9 decimals.

Due to cancellation errors arising in the recurrence (27), it was found necessary to programme the computation of $J_r$ using multiple-length arithmetic. For this reason the interpolation series (17), (18) and (19) were truncated after second differences so that double-length fixed-point arithmetic (about 23 decimals) would be sufficient. The integration formula then requires $J_0$, $J_1$, $J_2$, $J_3$.

To allow maximum storage space the programme was subdivided as follows:

(i) Input of body data and parameters. Computation of body co-ordinates.

(ii) Calculation of quadrature weights.

(iii) Calculation of $P(s, \sigma)$, $Q(s, \sigma)$ and setting up of linear equations.

(iv) Solution of equations and output of results.

The parameters referred to in (i) were integers indicating the number of points to be taken in each interval $(s_i, s_{i+1})$. In order to obtain better definition around the points of the discontinuity small intervals were taken there, and where the curvature of the body was small the intervals $(s_i, s_{i+1})$ were made large.

*Discussion of Results.*

No satisfactory way of evaluating the error involved in this method was found. Results for a sphere using 11 point quadrature are given in the Appendix together with a table of the known flow for comparison, and it is seen that the maximum error involved was $3 \times 10^{-4}$. An increase of the number of points to 21 reduced the maximum error to $4 \times 10^{-5}$.

In order to obtain an estimate of the errors for the flat-nosed bodies considered here, the programme was run twice with different parameters. It was found that for the bodies for which this was done the difference was at most of the order of $5 \times 10^{-3}$ and usually less.

Tables II and III give results for the flat-nosed bodies shown in the diagrams (which show only the generating curve of the body).

In both cases the length of the bodies were taken as 20, but the tables have been shortened to give only the velocity distribution $w(s)$ over the front part.

The generating curve of body II is composed of two perpendicular straight lines joined by a quadrant of a circle to ensure a smooth contour. Body III is generated by the two perpendicular straight lines AB and DE connected by two circular arcs BC and CD.

*Acknowledgement.*

The author is grateful to G. F. Miller of the National Physical Laboratory for his useful advice.

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## APPENDIX

### TABLE I

Body I, Sphere

| $s$ (in degrees) | $W(s)$ | Correct value $= 1 \cdot 5 \sin s$ |
|---|---|---|
| 0 | 0·0 | 0 |
| 18 | 0·4638 | 0·4635 |
| 36 | 0·8816 | 0·8817 |
| 54 | 1·2135 | 1·2135 |
| 72 | 1·4266 | 1·4266 |
| 90 | 1·5000 | 1·5 |
| 108 | 1·4266 | 1·4266 |
| 126 | 1·2135 | 1·2135 |
| 144 | 0·8816 | 0·8817 |
| 162 | 0·4638 | 0·4635 |
| 180 | 0·0 | 0 |

### TABLE II

Body II (*see* Fig. 1)

| $s$ | $x$ | $y$ | $W(s)$ |
|---|---|---|---|
| 0·000 | 0·0 | 0·0 | 0·0 |
| 0·125 | 0·0 | 0·125 | 0·091 |
| 0·250 | 0·0 | 0·250 | 0·187 |
| 0·375 | 0·0 | 0·375 | 0·304 |
| 0·500 | 0·0 | 0·500 | 0·490 |
| 0·696 | 0·038 | 0·691 | 0·922 |
| 0·893 | 0·146 | 0·854 | 1·289 |
| 1·089 | 0·309 | 0·962 | 1·493 |
| 1·285 | 0·500 | 1·0 | 1·353 |
| 1·446 | 0·661 | 1·0 | 1·177 |
| 1·607 | 0·821 | 1·0 | 1·122 |
| 1·767 | 0·982 | 1·0 | 1·093 |
| 1·928 | 1·143 | 1·0 | 1·075 |
| 2·089 | 1·303 | 1·0 | 1·063 |
| 2·249 | 1·464 | 1·0 | 1·055 |
| 2·410 | 1·625 | 1·0 | 1·051 |
| 2·571 | 1·785 | 1·0 | 1·051 |

TABLE III

Body III (*see* Fig. 1)

| $s$ | $x$ | $y$ | $W(s)$ |
|---|---|---|---|
| 0 | 0·0 | 0 | 0 |
| 0·094 | 0·0 | 0·094 | 0·064 |
| 0·188 | 0·0 | 0·188 | 0·129 |
| 0·281 | 0·0 | 0·281 | 0·199 |
| 0·375 | 0·0 | 0·375 | 0·278 |
| 0·469 | 0·0 | 0·469 | 0·373 |
| 0·563 | 0·0 | 0·563 | 0·498 |
| 0·656 | 0·0 | 0·656 | 0·693 |
| 0·750 | 0·0 | 0·750 | 1·406 |
| 0·760 | 0·001 | 0·760 | 1·704 |
| 0·770 | 0·003 | 0·770 | 1·861 |
| 0·781 | 0·007 | 0·780 | 1·987 |
| 0·791 | 0·013 | 0·788 | 2·078 |
| 0·801 | 0·020 | 0·796 | 2·130 |
| 0·811 | 0·028 | 0·802 | 2·138 |
| 0·822 | 0·037 | 0·807 | 2·098 |
| 0·832 | 0·046 | 0·810 | 1·784 |
| 0·979 | 0·189 | 0·846 | 1·209 |
| 1·126 | 0·332 | 0·876 | 1·156 |
| 1·273 | 0·477 | 0·907 | 1·137 |
| 1·420 | 0·622 | 0·932 | 1·131 |
| 1·567 | 0·767 | 0·952 | 1·127 |
| 1·714 | 0·913 | 0·970 | 1·124 |
| 1·861 | 1·059 | 0·983 | 1·120 |
| 2·008 | 1·206 | 0·992 | 1·115 |
| 2·155 | 1·353 | 0·998 | 1·102 |
| 2·302 | 1·500 | 1·000 | 1·083 |
| 2·489 | 1·687 | 1·000 | 1·059 |

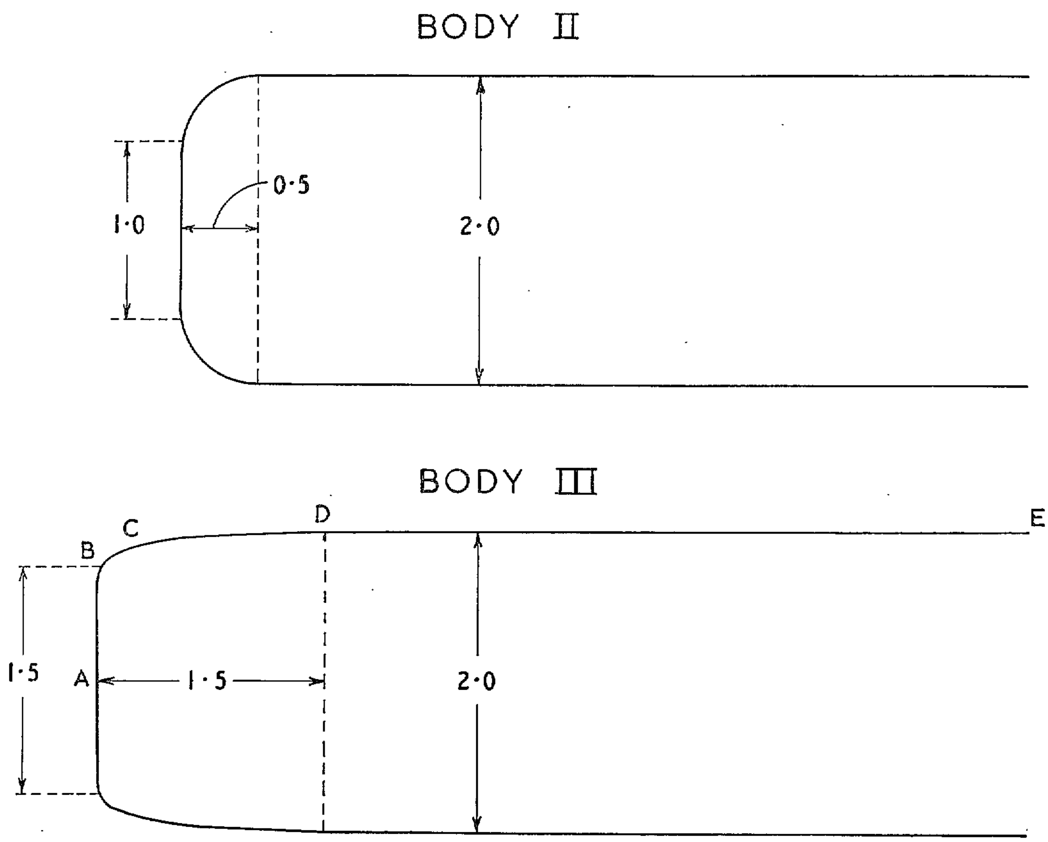


FIG. 1. Generating curves of bodies II and III.



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