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S. NEUMARK, Techn.Sc.D., F.R.Ae.S.

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S. NEUMARK, Techn.Sc.D., F.R.Ae.S.

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Summary.—The concept of complex stiffness in problems of oscillations with viscous or structural (hysteretic) damping is often used in a wrong way, leading to erroneous solutions. It is shown in the Paper that correct expressions for complex stiffness are different in the cases of forced and free oscillations. All fundamental cases for a single degree of freedom are critically re-examined and compared, and fallacious solutions eliminated.

The law of hysteretic damping being only known for a simple harmonic oscillation, all problems involving decaying oscillations, or more than one oscillatory mode, can only be treated tentatively at present, until the general law is found. This requires further experimental work.

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1. *Introduction*.—The first systematic experiments on structural damping were reported as early as 1927 by Kimball and Lovell[1] and 1928 by Becker and Föppl[2]. They both led to the conclusion that this sort of damping, if occurring in oscillatory systems of a single degree of freedom performing simple harmonic oscillations of small amplitudes, might be treated approximately by the usual linear method (as viscous damping), i.e., by assuming that the damping forces or moments were proportional to linear or angular velocity, respectively. There was one important proviso, however, that the coefficients of proportionality were not constants characteristic for the given structure and material, but were themselves functions of the oscillatory frequency, viz., they varied in inverse proportion to the frequency, to a fair degree of approximation. It was found appropriate therefore to modify the typical equation of *forced oscillations*:

$$m\ddot{x} + c\dot{x} + kx = F\cos(\omega t + \alpha) \quad \ldots \quad (1)$$

by the following one :

$$m\ddot{x} + \frac{kg}{\omega}\dot{x} + kx = F\cos(\omega t + \alpha)\,, \quad \ldots \quad (2)$$

i.e., by making the damping coefficient :

$$c = \frac{kg}{\omega}\,, \quad \ldots \quad (3)$$

so that it is assumed inversely proportional to the frequency ω, and proportional to k (stiffness, restoring force constant, spring constant), while g is a dimensionless constant, considered as characteristic for the given structure and material. The inclusion of k in the formula (3) is

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justified not only by the convenience of having a non-dimensional constant (g) but also by the fact that, in this case, the damping is a small additional part of the total reaction of the structure under distortion, of which the major part is the restoring force kx. In this respect, the structural damping differs significantly from the viscous one which (at least in the typical case of a 'dashpot' in the oscillating system) is quite unrelated to the restoring force (*e.g.*, that due to a perfectly elastic spring). The structural damping is often conveniently called 'hysteretic' because the graph of the total reaction $k\{x + (g/\omega)\dot{x}\}$ plotted against the harmonically varying x is a closed loop (ellipse), reminiscent of that of magnetic hysteresis—the area of the loop being independent of frequency.

The above assumption as to the properties of structural damping, deduced from Ref. 2, was at once applied by Küssner[3, 6] in his investigations of wing flutter, where he had to deal with self-excited coupled oscillations of complicated structures of several degrees of freedom, the important problem being merely to determine the critical flight speed V_c at which one of the several oscillatory modes has its effective damping reduced to zero (while other modes are still effectively damped and thus practically non-existent). The oscillation could therefore be treated as simple harmonic, with one definite frequency, so that all damping coefficients could be represented by the formula (3), with various appropriate values of k and g in each elastic element of the system. The difficulty was that the frequency ω was not known in advance being, naturally, one of the quantities to be found, along with V_c, by solving the determinantal (characteristic) equation of the given system of differential equations. It seemed therefore necessary to consider *a characteristic equation in which some coefficients were functions of one of its roots* (the purely imaginary one). Küssner has avoided this difficulty by writing the equations analogous to (2) in *complex form*, viz., by replacing the expression of the total reaction $k\{x + (g/\omega)\dot{x}\}$ by $k\, e^{ig}\, x$, i.e., considering g as a phase advance by which the reaction leads the displacement. All oscillating quantities are then assumed to be complex, and our equation (2) becomes :

$$m\ddot{x} + k\, e^{ig}\, x = F\, e^{i(\omega t + \alpha)}\,. \qquad \ldots \qquad (4)$$

A similar technique was applied by Kassner[9], with a slight alteration, the total reaction being written as $k\,(1 + ig)\,x$, so that Equation (2) became :

$$m\ddot{x} + k\,(1 + ig)\,x = F\, e^{i(\omega t + \alpha)}\,. \qquad \ldots \qquad (5)$$

The difference between (4) and (5) was considered negligible in flutter work, where g was usually quite small so that the differences between real and imaginary parts of the two expressions

$$(e^{ig} = \cos g + i \sin g) \quad \text{and} \quad (1 + ig)$$

were small of 2nd or 3rd order, respectively.

In either way (4) or (5), the frequency ω has disappeared from the damping term. In flutter equations, where there are no external forcing functions (oscillations being self-excited), the frequency is eliminated completely, the penalty being however that *equations with some complex coefficients* must be dealt with. This proved feasible, and the method became general in flutter work as, e.g., in papers by Duncan and Lyon[10], Theodorsen and Garrick[12], Scanlan and Rosenbaum[16], and many others. The structural damping has been found to have an appreciable effect, normally leading to an increase in critical flight speed.

The coefficient of the second term in (4) or (5) has been termed ' complex stiffness ' by several authors[13, 14, 16, 18], or sometimes, less appropriately, ' complex damping ' (*see* Myklestad[17])—it is really ' stiffness and hysteretic damping combined into one complex expression '. The second term itself, e.g., $k\,(1 + ig)\,x$, is aptly called ' complex restoring force '.

The assumption (3), with all its analytical implications and resulting effects on the characteristics of *forced and self-excited coupled oscillations*, was examined theoretically by Kimball[4], Schlippe[5], Pugsley[7], Walker[8] and Coleman[11]. The matter seemed quite clear and free from doubts until, in 1949, Soroka[14] proposed to apply the concept of complex stiffness (in the form of Equation (5)) to studying *free oscillations* of elastic structures. He did not neglect higher powers of g and obtained what he considered a striking result that the frequency would increase with increase of structural damping, instead of decreasing as it does with increase of viscous damping (the stiffness being constant in both cases). He tabulated the rising frequency ratio, up to $g = 1 \cdot 6$, in which case this ratio would have increased by 20 per cent. This treatment was immediately commented on by Pinsker[15] who gave a correct interpretation of Soroka's surprising formulae without, however, challenging the fundamental approach. Apart from this mild criticism, Soroka's theory seems to have been accepted as plausible, and it has been repeated, with no modifications or reservations, in the American textbook on flutter by Scanlan and Rosenbaum (Ref. 16, pp. 86 to 87).

In the meantime, Myklestad[17] proposed to apply the concept of complex stiffness (this time in the form of equation (4)) to both free and forced oscillations, again retaining higher powers of g. As a result, he obtained formulae for magnification factor and phase delay angle in *forced oscillations* differing from the commonly accepted ones (as given by Schlippe[5] and Walker[8]). Myklestad's solution for the case of *free oscillations*, however, seemed plausible although not quite free from doubts ; at least, it gave the frequency of damped free oscillations which decreased with increase of structural damping.

The entire problem of structural (hysteretic) damping was taken up again recently by Bishop[18]. For forced oscillations, he repeated the well-established solution of Schlippe[5] and Walker[8]. For free oscillations, he gave Soroka's solution along with another one (suggested by Collar), the latter agreeing with that proposed by Myklestad.

The present position is thus, that we have two alternative solutions for the free oscillations, and also two for the forced ones. The matter has been further confused by a frequent use by many authors of inaccurate verbal expressions and definitions, resulting from the difficulties in describing some physical phenomena and corresponding algebraic technique in terms of the existing inadequate vocabulary. Thus, e.g., the hysteretic damping is alternatively described as proportional to velocity[4, 8, 11, 13], or to displacement[16, 18], or to restoring force[12], or finally as possessing an amplitude proportional to that of displacement[14]. Also, it is often not clear whether the motion should be described as 'damped' when there exists a force opposing the velocity, or when the amplitude of the oscillation decreases gradually ; for instance, steady forced oscillations would be damped in the former but not in the latter sense. Many difficulties are also encountered because of different notations.

The purpose of the present paper is to clear the existing confusion, to explain the relationships between solutions in real and complex terms, to eliminate clearly erroneous solutions and controversies, to establish and justify a proper (although restricted) definition of complex stiffness, to delimit the regions of the subject matter where basic concepts are well-established or not, and finally to indicate the kind of further experimental work needed for clearing the outstanding questions. To achieve these aims, it has been found indispensable to go through the entire linearised theory of forced and free oscillations, discussing the well-known solutions along with the new and doubtful ones, in unified nomenclature. Particular care has been observed when introducing complex quantities, because most misinterpretations and outright errors had previously been committed in doing this.

It may be mentioned that the many alternative theories, discussed in the present paper, give significantly different numerical results only when the damping coefficient is not very small, i.e., when its second and possibly higher powers cannot be neglected. It is not suggested, therefore, that the choice of method would materially affect, e.g., the numerical results of flutter calculations, where this coefficient has always been quite small heretofore. The matter is interesting, however, and may become important, because high values of structural-damping coefficients are possible and may be useful, so that there have been several proposals to increase them artificially. Also, the neglect of higher powers would leave some questions unanswered ; for instance, the effect of varying structural damping on the frequency of free oscillations would appear to be nil, as its magnitude and its very sign depend critically on second and higher order terms.

Section 2 of the present paper deals with dissipation of energy by damping, not only in simple harmonic but also in 'decaying harmonic' oscillations ; this is essential for understanding the fundamental assumptions of the theory. In Section 3, the simpler problem of steady forced oscillations is considered, with both viscous and hysteretic damping, in real and complex notation, also including Myklestad's theory which is shown to be inadmissible for forced oscillations. The less simple problem of free oscillations is examined in Section 4, again including alternative cases and notations ; Myklestad's theory is shown to be mathematically correct here, and consistent with the orthodox Collar's solution, while Soroka's method is found to be fallacious ; some alternative methods of solution are suggested and discussed, and a need for further experiments pointed out, without which no final decisions can be reached. Conclusions are summarised in Section 5.

The illustrations have been prepared by Miss F. M. Ward.

2. *Dissipation of Energy by Damping*.—The analysis of energy dissipation by damping is important because the results of experimental work on structural damping were given in terms of energy losses and, in such form, have served as a starting point in all subsequent theories. The restoring force is always assumed proportional and opposed to displacement, and the damping force proportional and opposed to velocity, so that the total force acting in the direction of motion is $(-kx - c\dot{x})$. To get the energy loss through damping positive, let us consider the opposite force :

$$f = kx + c\dot{x} \, . \qquad (6)$$

The total energy lost per cycle will be :

$$E = \oint f\,dx = \left|\tfrac{1}{2}kx^2\right|_{t=0}^{T} + c\int \dot{x}\,dx = \tfrac{1}{2}k(x_T^2 - x_0^2) + c\int_0^T \dot{x}^2\,dt \, , \qquad (7)$$

where T is the period and x_0, x_T are the values of x at the beginning and end of the cycle. The first term in (7) is the recoverable energy stored in the spring, while the second one represents the loss through damping.

We are going now to calculate E in two cases, viz., when the oscillation is either simple harmonic, or 'exponentially decaying harmonic'.

2.1. *Dissipation in Simple Harmonic Oscillation*.—The equations of motion are :

$$x = x^* \sin(\omega t + \varphi) \, , \qquad \dot{x} = \omega x^* \cos(\omega t + \varphi) \, , \qquad (8)$$

where φ is the initial phase angle, x^* the amplitude, and the 'circular' frequency is :

$$\omega = 2\pi/T \, . \qquad (9)$$

The resisting force (6) becomes :

$$f = kx + c\omega x^* \cos(\omega t + \varphi) \, . \qquad (10)$$

The first term in (7) is now always 0, because the values of x at $t = 0$ and $t = T$ are always equal to each other, for whatever φ. The second term becomes :

$$E_{d0} = c\,\omega^2\,x^{*2}\int_0^{2\pi/\omega} \cos^2(\omega t + \varphi)\,dt = \pi c\,\omega\,x^{*2} \, , \qquad (11)$$

and is also independent of φ.

This case is illustrated in Fig. 1a, where f, and its respective components kx and $c\dot{x}$, are plotted against x. The curve of f is an oblique ellipse, and its area represents the loss of energy E_{d0}. This ellipse is often referred to as 'hysteresis loop'. The first term in (10) is represented by the doubly covered diameter AB, and its area is obviously nil. The second (damping) term may be represented alone by the (normal-positioned) ellipse in Fig. 1b, of the area equal to that of Fig. 1a.

In the case of *viscous damping*, c is a constant for the given oscillator, independent of the amplitude x^*, frequency ω, or phase angle φ (provided x^* or ω is not large), and (11) shows that the *energy loss is then proportional to the square of amplitude and to the first power of frequency*.

For *structural damping*, the existing experimental results[1,2] have shown that the energy loss per cycle in a simple harmonic motion, while still *proportional to the amplitude squared, does not vary markedly with frequency* and may be assumed not to depend on it at all, with fair approximation, at least through a considerable range of moderate frequencies. This means that $c\omega$ may be assumed as constant and, for reasons already mentioned in the Introduction, conveniently expressed by :

$$c\omega = kg \, , \qquad \qquad (12)$$

where g is the 'dimensionless coefficient of hysteretic damping'. The formula (12) is identical with (3), and (11) now becomes :

$$E_{d0} = \pi \, kg \, x^{*2} \, , \qquad \qquad (13)$$

this leading to the form (2) of the equation of forced oscillations.

2.2. *Dissipation in 'Exponentially Decaying Harmonic Oscillation'*.—In problems of free oscillations with moderate damping, the motion is no longer simple harmonic but, if the linear method still applies, the motion is represented by :

$$x = x^* \, e^{-at} \sin(\omega t + \varphi) \, , \qquad \dot{x} = x^* \, e^{-at} \{\omega \cos(\omega t + \varphi) - a \sin(\omega t + \varphi)\} \, , \qquad (14)$$

where x^* is the 'initial amplitude' and a the 'damping index'. Such a motion will be termed an 'exponentially decaying harmonic oscillation'. It is interesting to find a formula for energy dissipated per cycle in such an oscillation. The total 'resisting force' is still expressed by (6), so it becomes :

$$f = kx + c \, x^* \, e^{-at} \{\omega \cos(\omega t + \varphi) - a \sin(\omega t + \varphi)\} \, . \qquad \qquad (15)$$

The energy loss (7) is now :

$$E = -\tfrac{1}{2} k \, x^{*2} \, (1 - e^{-4\pi a/\omega}) \sin^2\varphi + E_d \, , \qquad \qquad (16)$$

where :

$$E_d = c x^{*2} \int_0^{2\pi/\omega} e^{-2at} \{\omega \cos(\omega t + \varphi) - a \sin(\omega t + \varphi)\}^2 \, dt \, . \qquad \qquad (17)$$

The integral is worked out in Appendix I, and we obtain :

$$E_d = \tfrac{1}{4} c x^{*2} \, (1 - e^{-4\pi a/\omega}) \left(\frac{\omega^2}{a} - \omega \sin 2\varphi + 2a \sin^2\varphi\right) . \qquad \qquad (18)$$

Compared with (11), this formula is very much more complicated, and the energy dissipated, while still proportional to the square of initial amplitude x^{*2} and to the damping coefficient c, is seen to depend also on frequency ω, damping index a, and phase angle φ, in a somewhat involved way. It tends, however, to E_{d0} (as given by (11)) for $a \to 0$, as could be expected. We may write, therefore, conveniently :

$$\frac{E_d}{E_{d0}} = \frac{1 - e^{-4\pi a/\omega}}{4\pi a/\omega} \left(1 - \frac{a}{\omega} \sin 2\varphi + 2 \frac{a^2}{\omega^2} \sin^2\varphi\right) \qquad \qquad (19)$$

or, expanded *for small values of the 'relative damping ratio'* a/ω :

$$\left.\begin{aligned} \frac{E_d}{E_{d0}} = 1 - (2\pi + \sin 2\varphi) \frac{a}{\omega} + \left(\frac{8}{3}\pi^2 + 2\pi \sin 2\varphi + 2 \sin^2\varphi\right) \frac{a^2}{\omega^2} \ldots + \\ + \frac{(4\pi)^{s-2}}{(s-1)!} \left(\frac{16}{s} \frac{\pi^2}{s+1} + \frac{4\pi}{s} \sin 2\varphi + 2 \sin^2\varphi\right) \left(-\frac{a}{\omega}\right)^s \ldots \end{aligned}\right\} ; \qquad (19a)$$

the series converges for any a/ω, but is only convenient for quite small values of this ratio, in view of the large coefficients.

This case is illustrated in Figs. 2a and 2b where f (*see* (15)), or only its second ' damping ' term are plotted, respectively, against x, in one case $\varphi = 0$ only, i.e., when $x = x^* e^{-at} \sin \omega t$, the graphs including one full period. The hatched areas represent the energy lost per cycle and, for this value of φ, they are equal, the first term in (16) being zero. This first term represents the work done by the restoring force kx (obviously negative and interpretable as loss of potential energy stored in the spring), and this work varies with φ, becoming 0 for $\varphi = 0$ or π. It is by no means a dissipated energy, the restoring force being conservative.

As to the truly dissipated energy E_d, given by (18) or (19), it is seen to differ little from E_{d0}, if a/ω is small. The ratio E_d/E_{d0} depends on a/ω and φ only, and is illustrated by Fig. 3, for several values of φ and continuous variation of a/ω. The effect of φ appears not to be large, and the mean values for $0 < \varphi < 2\pi$ are also plotted, giving the mean curve corresponding to :

$$\left.\begin{aligned} \frac{(E_d)_{\text{mean}}}{E_{d0}} &= \frac{(E_d)_{\text{mean}}}{\pi c \omega x^{*2}} = \frac{1 - e^{-4\pi a/\omega}}{4\pi a/\omega}\left(1 + \frac{a^2}{\omega^2}\right) = \\ &= 1 - 2\pi\frac{a}{\omega} + \left(\frac{8}{3}\pi^2 + 1\right)\frac{a^2}{\omega^2} \ldots + \frac{(4\pi)^{s-2}}{(s-1)!}\left(\frac{16}{s}\frac{\pi^2}{s+1} + 1\right)\left(-\frac{a}{\omega}\right)^s \end{aligned}\right\} \quad \ldots \quad (20)$$

In the case of *viscous damping* (c constant), E_d or $(E_d)_{\text{mean}}$ is still mainly proportional to ωx^{*2}, like E_{d0}, being however modified through a/ω, according to Fig. 3. In the case of *structural damping*, the law governing c in exponentially decaying harmonic oscillations is unknown. For small a/ω, the law should differ little from (12), and the energy dissipated should then be mainly proportional to x^{*2}, while being still modified according to Fig. 3, for varying a/ω. The assumption is doubtful for higher values of a/ω, and we shall come back to this question in Section 4.2.4.

3. *Steady Forced Oscillations with Damping (Amplitude of Exciting Force Constant, or Proportional to Frequency Squared).*—Let us consider oscillations of a simple system (mass and spring with viscous or hysteretic damping), excited by an external periodic force varying sinusoidally with time. If we neglect the initial transient stage in which the forced mode is mixed with the natural damped one, then the motion may be considered as steady and simple harmonic, in unison with the exciting force, and the inferences of Section 2 may be applied directly and with no doubts. The differential equation is (4), and its solution may be examined with either c constant for viscous damping, or with c replaced by (3) for hysteretic damping. In the former case the solutions are well known, in the latter somewhat controversial at present. We start by recapitulating briefly the known results for viscous damping, not only for the purpose of having them handy for comparison in unified notation, but also because the case of hysteretic damping then only requires the simple transformation by (3). Real and complex notations, and vectorial representation, are all used in parallel, so as to avoid and explain errors which may creep in (as they have actually done) through inconsistent use of the alternative methods.

3.1. *Viscous Damping.*—3.1.1. *Real notation.*—We consider the differential Equation (*cf.* (1)) :

$$m\ddot{x}_1 + c\dot{x}_1 + kx_1 = F\cos(\omega t + \alpha)\,, \quad \ldots \quad (21)$$

where the suffix ($_1$) has been introduced with a view to later needs, and F, the amplitude of the exciting force, may be assumed as a constant. If the *frequency ω were very small*, then $\dot{x}_1$ and $\ddot{x}_1$ would be also very small compared with x_1, the problem would become nearly static, and the approximate solution would be :

$$x_1 \approx x_{st}\cos(\omega t + \alpha)\,, \quad \ldots \quad (22)$$

where :

$$x_{st} = F/k \quad \ldots \quad (23)$$

is the '*static displacement*' of the mass-spring system under a constant force F. In the general case, for any ω, it is convenient to divide (21) by m, and write it in the form :

$$\ddot{x}_1 + 2\,\lambda\,\omega_n\,\dot{x}_1 + \omega_n^2\,x_1 = \omega_n^2\,x_{st}\cos(\omega t + \alpha)\,, \quad \ldots \quad (24)$$

where, for abbreviation :

$$\omega_n = \sqrt{(k/m)} \qquad \text{(natural undamped frequency)}\,, \quad \ldots \quad (25)$$

$$\lambda = c/c_{cr} = c/2\sqrt{(mk)} = c/2m\omega_n \qquad \text{(relative damping ratio)}\,, \quad (26)$$

$c_{cr} = 2\sqrt{(mk)}$ being the 'critical damping coefficient'.

The required steady solution for x_1 of the Equation (21) or (24) must also be a simple harmonic function of t, similar to (22) but with a modified amplitude and initial phase angle, thus :

$$x_1 = n\,x_{st}\cos(\omega t + \alpha - \eta)\,, \quad \ldots \quad (27)$$

where n (magnification factor) and η (phase delay angle) must be determined. From (27) we have :

$$\dot{x}_1 = -\,n x_{st}\,\omega\sin(\omega t + \alpha - \eta)\,, \qquad \ddot{x}_1 = -\,\omega^2\,x_1 \quad \ldots \quad (28)$$

and, substituting into (24), dividing by ω_n^2, and introducing the symbol :

$$r = \omega/\omega_n \qquad \text{(frequency ratio)}\,, \quad \ldots \quad (29)$$

we obtain :

$$n\{(1 - r^2)\cos(\omega t + \alpha - \eta) - 2\lambda r\sin(\omega t + \alpha - \eta)\} = \cos(\omega t + \alpha)\,, \quad \ldots \quad (30)$$

which equality must be satisfied for any t. Putting, in turn, $\omega t + \alpha = \frac{1}{2}\pi$, and $\omega t + \alpha = \eta$, we get :

$$\left.\begin{aligned} (1 - r^2)\sin\eta - 2\lambda r\cos\eta &= 0 \\ n(1 - r^2) &= \cos\eta \end{aligned}\right\}, \quad \ldots \quad (31)$$

and hence :

$$\tan\eta = \frac{2\lambda r}{1 - r^2}\,, \quad \ldots \quad (32)$$

$$n = \frac{\cos\eta}{1 - r^2} = \frac{1}{\sqrt{\{(1 - r^2)^2 + 4\lambda^2 r^2\}}}\,, \quad \ldots \quad (33)$$

the standard text-book formulae for phase delay and magnification factor. They are illustrated by the familiar 'resonance graphs' reproduced in Figs. 4a and 4b. The curves of n all start from $n = 1$ at $r = 0$, exhibit (if $\lambda < 1/\sqrt{2} = 0{\cdot}7071$) single 'resonance peaks' :

$$n_{\max} = \frac{1}{2\lambda\sqrt{(1 - \lambda^2)}} \qquad \text{at} \qquad r = r_r = \sqrt{(1 - 2\lambda^2)}\,, \quad \ldots \quad (34)$$

or (for $\lambda \geqslant 1/\sqrt{2}$) fall monotonically throughout the r-range ; all curves tend to 0 for $r \to \infty$. The angle η increases with r from 0 to π, all curves intersecting at the value $\frac{1}{2}\pi$ for $r = 1$, as shown in Fig. 4b.

All the above results apply without restriction if F is constant, but some modifications are needed if F varies with ω, the important case in practice being that in which F varies in proportion to ω^2 (this occurs when the exciting force is provided as a component of a centrifugal force of a rotating eccentric mass). In this case, it is convenient to write :

$$F = F_n\frac{\omega^2}{\omega_n^2}\,, \quad \ldots \quad (35)$$

where F_n is the value of F for $\omega = \omega_n$. The 'static displacement', as defined by (23), then also becomes variable :

$$x_{st} = \frac{F_n}{k}\frac{\omega^2}{\omega_n^2}\,, \qquad \text{or} \qquad x_{st} = \bar{x}_{st}\,r^2\,, \quad \ldots \quad (36)$$

but the magnification factor must relate the amplitude of x_1 to some constant, for which purpose $\bar{x}_{st} = F_n/k$ (static displacement at resonance frequency) is particularly suitable. The only resulting modification is that (33) must be replaced by :

$$\bar{n} = nr^2 = \frac{r^2 \cos \eta}{1 - r^2} = \frac{r^2}{\sqrt{\{(1-r^2)^2 + 4\lambda^2 r^2\}}}, \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

and this is illustrated in Fig. 4c. The $\bar{n}$-curves all start from 0, exhibit (if $\lambda < 1/\sqrt{2}$) single resonance peaks :

$$\bar{n}_{\max} = \frac{1}{2\lambda\sqrt{(1-\lambda^2)}} \text{ at } r = \bar{r}_r = \frac{1}{\sqrt{(1-2\lambda^2)}} \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

or (for $\lambda \geqslant 1/\sqrt{2}$) rise monotonically throughout the r-range, all curves tending to 1 for $r \rightarrow \infty$. The angle η behaves exactly as in the previous (F = const.) case, so that the graph 4b is appropriate to either of 4a, 4c.

3.1.2. *Complex notation and vectorial representation.*—Let us now suppose that the same oscillator is subject to the external force $F \sin(\omega t + \alpha) = F \cos(\omega t + \alpha - \frac{1}{2}\pi)$, instead of $F \cos(\omega t + \alpha)$. The displacement, different from x_1, may now be denoted by x_2, and the differential equation written :

$$m\ddot{x}_2 + c\dot{x}_2 + kx_2 = F \sin(\omega t + \alpha), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

and its solution will be obtained directly from the previous section, replacing α, wherever it appears, by $(\alpha - \frac{1}{2}\pi)$. We obtain, instead of (27) :

$$x_2 = n\, x_{st} \sin(\omega t + \alpha - \eta), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

with n, x_{st}, η meaning exactly the same values as before. Formulae (23), (32), (33) and (37) will hold, as none of them depends on α.

Let us now add the equation (39) multiplied by i to (21), introducing simultaneously a new 'complex' variable :

$$x = x_1 + i\, x_2,$$

whereupon we obtain a new 'complex' differential equation for x :

$$m\ddot{x} + c\dot{x} + kx = F\, e^{i(\omega t + \alpha)}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (41)$$

and the corresponding complex solution will be :

$$x = n\, x_{st}\, e^{i(\omega t + \alpha - \eta)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (42)$$

which, resolved into real and imaginary parts, leads back to (27) and (40), respectively. We may also treat (42) as a trial solution of (41) and, on substitution, we obtain immediately (32) and (33).

An important formal modification may now be introduced. From (42) we deduce immediately :

$$\dot{x} = i\omega x, \qquad \ddot{x} = -\omega^2 x \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43)$$

and, using only the first of these relationships, equation (41) can be written :

$$m\ddot{x} + (k + ic\omega)\, x = F\, e^{i(\omega t + \alpha)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44)$$

or, using (25) and (26) :

$$\ddot{x} + (\omega_n^2 + 2i\lambda\omega_n\omega)\, x = \omega_n^2\, x_{st}\, e^{i(\omega t + \alpha)} \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44a)$$

In either form, this is still a linear differential equation of 2nd order, representing exactly the same motion as before. However, the second (damping) term has formally disappeared in (44), while the constant k in the subsequent term has been replaced by $(k + ic\omega)$ which may be called ' complex stiffness ' for the case of viscous damping. An analogous change is seen in (44a). Trivial as this small modification may seem, it is essential for most of the following arguments. The equation (44) may now be solved by the same trial assumption (42) as before. We obtain, using the second of the relationships (43), and dividing by ω_n^2 :

$$n\,(1 - r^2 + 2i\lambda r)\,e^{-i\eta} = 1\,, \qquad \ldots \qquad (45)$$

and (32), (33) follow immediately. The formula (37) for the case of centrifugal excitation also follows.

The usefulness of the entire manipulation becomes apparent when we remove almost the entire sections (3.1.1) and the present one, and only retain equations (44), (45) and (42) and the resulting ones (32), (33) and (37). The advantage consists in :

(i) a convenient way of writing,

(ii) a particular rapid solution ; and

(iii) the use of complex quantities leading to vectorial illustration.

All these benefits become increasingly useful, of course, when dealing with systems of an increasing number of degrees of freedom.

As to the vectorial representation, it is very simple in the given elementary case, and is shown as a (known) graph in Fig. 7a. The open polygon OABC illustrates the left-hand part of (44), and the closing side OC the right-hand part, or forcing function. All vectors must be considered as rotating anti-clockwise at uniform angular speed ω. The velocity vector $\dot{x} = i\omega x$ leads the displacement vector x by 90 deg, and is similarly led by the acceleration vector $\ddot{x}$. All phase angles are shown. In this representation, x_1 and x_2 are the horizontal and vertical component of x, respectively, and the resolutes of all other vectors have a similarly obvious meaning. All vectors may be projected on any other axis, giving analogous motions with arbitrary phase shift relative to x_1 or x_2.

3.2. *Hysteretic Damping.*—The entire theory of Section 3.1 applies here, with the only modification that the damping coefficient c must now be replaced by kg/ω, cf., (3). There is no need to repeat the analysis, and it will suffice to write down the modified main results in the same order, and to discuss them.

3.2.1. *Real notation.*—The differential equation (21) or (24) becomes :

$$m\ddot{x}_1 + \frac{kg}{\omega}\dot{x}_1 + kx_1 = F\cos(\omega t + \alpha) \qquad \ldots \qquad (46)$$

or

$$\ddot{x}_1 + \frac{g}{\omega}\,\omega_n^2\,\dot{x}_1 + \omega_n^2\,x_1 = \omega_n^2\,x_{st}\cos(\omega t + \alpha)\,, \qquad \ldots \qquad (47)$$

the relative damping ratio λ being not used directly in (47). This ratio still has a meaning, however, being related to g in the following way.

$$2\lambda = g\omega_n/\omega = g/r\,. \qquad \ldots \qquad (48)$$

The steady solution still has the form (27) but the formulae (32, 33) become :

$$\tan\eta = \frac{g}{1 - r^2}\,, \qquad \ldots \qquad (49)$$

$$n = \frac{\cos\eta}{1 - r^2} = \frac{1}{\sqrt{\{(1 - r^2)^2 + g^2\}}}\,, \qquad \ldots \qquad (50)$$

as derived and discussed already in Refs. 5, 8, 18, and illustrated in Figs. 5a, 5b. The resonance curves in Fig. 5a, for varying g, start from the various initial values $1/\sqrt{(1+g^2)}$ at $r=0$, exhibit each a single resonance peak :

$$n_{\max} = \frac{1}{g} \qquad \text{at} \qquad r = 1\,, \qquad \qquad (51)$$

and tend to 0 for $r \to \infty$. Angles η (Fig. 5b) start from the various initial values $\tan^{-1} g$ at $r=0$ and increase with r, reaching the common value $\frac{1}{2}\pi$ for $r=1$, and tending to π for $r \to \infty$.

In the case of centrifugal excitation, there is no change in behaviour of η, but n must be replaced by $\bar{n}$, expressed by the formula (37) modified to :

$$\bar{n} = nr^2 = \frac{r^2}{\sqrt{\{(1-r^2)^2+g^2\}}}\,, \qquad \qquad (52)$$

as illustrated in Fig. 5c. All curves of $\bar{n}$ now start from 0, exhibit single resonance peaks :

$$\bar{n}_{\max} = \frac{\sqrt{(1+g^2)}}{g} \qquad \text{at} \qquad r = \bar{r}_r = \sqrt{(1+g^2)}\,, \qquad (53)$$

and tend to 1 for $r \to \infty$.

It may be mentioned that, *for small values of g*, curves of Fig. 5 differ very little from those in Figs. 4 corresponding to $\lambda = \frac{1}{2}g$, and the peaks of both n and $\bar{n}$ are approximately :

$$n_{\max} \approx \bar{n}_{\max} \approx \frac{1}{2\lambda} = \frac{1}{g}\,, \qquad \text{for } r \approx 1\,, \qquad \qquad (54)$$

the second powers of λ or g being neglected. As g increases, the differences become gradually bigger. If g does not exceed 1, then the peak values of n (but not of $\bar{n}$) for hysteretic damping may be made equal to those for viscous damping, if

$$\lambda = \sqrt{\left\{\frac{1-\sqrt{(1-g^2)}}{2}\right\}} \qquad \text{(equivalent to } g = 2\lambda\sqrt{(1-\lambda^2)})\,, \qquad (55)$$

and an example is given in Figs. 8a, b for rather large values $\lambda = 0{\cdot}6$, $g = 0{\cdot}96$. Similarly, if we chose :

$$\lambda = \sqrt{\left\{\frac{1-1/\sqrt{(1+g^2)}}{2}\right\}} \qquad \left(\text{equivalent to } g = \frac{2\lambda\sqrt{(1-\lambda^2)}}{1-2\lambda^2}\right), \qquad (56)$$

then the peaks of $\bar{n}$ (but not of n) become equal for viscous and hysteretic damping, respectively, and an example is given in Fig. 9a, b for $\lambda = 0{\cdot}352$, $g = 0{\cdot}876$. Here, g may have any value, but λ must, of course, not exceed $1/\sqrt{2}$.

3.2.2. *Complex notation.*—The complex notation may be introduced in exactly the same way as in Section 3.1.2, and it will suffice to write here the differential equation, in the forms analogous to (44) and (44a) :

$$m\ddot{x} + k\,(1+ig)\,x = F\,e^{i(\omega t+\alpha)} \qquad \qquad (57)$$

or

$$\ddot{x} + \omega_n^2\,(1+ig)\,x = \omega_n^2\,x_{st}\,e^{i(\omega t+\alpha)}\,. \qquad \qquad (57a)$$

By substituting (42), the solutions (49, 50) follow immediately. The vector diagram of Fig. 7 applies, with the only alteration that the vector $\overline{\mathrm{AB}}$ now denotes $ikgx$ (instead of $ic\omega x$). In the previous case, the ratio $\overline{\mathrm{AB}} : \overline{\mathrm{OA}}$ had the value $c\omega/k$ and hence increased with frequency, but in the present case the ratio is simply g and thus independent of frequency.

3.2.3. *Myklestad's method.*—In Ref. 17, Myklestad introduced the hysteretic damping in a way rather different from that described above. Denoting the (supposedly small) damping force by $k\Delta x$, and taking small differences in the equation of simple harmonic oscillations, written in the complex form :

$$x = x^* e^{i\omega t} , \qquad (58)$$

he obtains :

$$f = k (x + \Delta x) = k x^* e^{i\omega(t+\Delta t)} \qquad (59)$$

and, assuming that :

$$\omega \Delta t = \text{const.} = 2\beta , \qquad (60)$$

he gets :

$$f = k (x + \Delta x) = k e^{2i\beta} x . \qquad (61)$$

The procedure does not seem to be entirely justified, because Δx is not clearly a function of time (and may really be assumed in various ways to suit experimental results), so the process of differencing is doubtful, as is also the assumption (60). But, whatever the method of derivation, the formula (61) agrees with ours :

$$f = k (1 + ig) x , \qquad (62)$$

if we put $g = \sin 2\beta$ or $g = 2\beta$, up to the terms of 1st order in g or β, but not in terms of higher order. In fact, the differencing could be done in an alternative way :

$$\Delta x = i\omega x^* e^{i\omega t} \Delta t = i\omega x \Delta t , \qquad (63)$$

equally justified within 1st order of Δt, and this, combined with the assumption (60), would lead to (62).

The important fact is that Myklestad has used the expression (61) for the complex restoring force in working out formulae for forced oscillations involving higher powers of β (or g), and we may therefore expect differences between his results and those commonly accepted (form (49) and (50)). His equation (61) may be written :

$$\left.\begin{aligned} f &= k (\cos 2\beta + i \sin 2\beta) x = k \{(1 - 2\beta^2 + \ldots) + i (2\beta - \tfrac{4}{3} \beta^3 \ldots)\} x , \\ \text{or} \quad f &= k \{\sqrt{(1 - g^2)} + ig\} x, \qquad \text{if} \qquad g = \sin 2\beta , \end{aligned}\right\} \qquad (64)$$

and it is seen that he admits tacitly of a decrease of the simple static stiffness with an increase of the hysteretic damping coefficient g. The decrease is of the 2nd order in g, thus only admissible if all 2nd order terms are finally neglected, but not otherwise. There is no justification whatsoever at present to assume such a decrease. Let us see, however, what are the results. The equation of forced oscillations, instead of (57) or (57a), becomes :

$$m\ddot{x} + k e^{2i\beta} x = F e^{i(\omega t + \alpha)} \qquad (65)$$

or

$$\ddot{x} + \omega_n^2 e^{2i\beta} x = \omega_n^2 x_{st} e^{i(\omega t + \alpha)} . \qquad (65a)$$

Assuming again the trial solution, analogous to (42) :

$$x = n' x_{st} e^{i(\omega t + \alpha - \eta')} \qquad (66)$$

and substituting it into (65a), we get :

$$n' \left(\omega_n^{\,2}\, e^{2i\beta} - \omega^2\right) e^{-i\eta'} = \omega_n^{\,2}$$

or, separating the real and imaginary parts :

$$\left.\begin{aligned} \cos 2\beta - r^2 &= \cos \eta'/n' \,, \\ \sin 2\beta \qquad &= \sin \eta'/n' \,, \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (67)$$

whence :

$$\tan \eta' = \frac{\sin 2\beta}{\cos 2\beta - r^2}\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (68)$$

$$n' = \frac{\cos \eta'}{\cos 2\beta - r^2} = \frac{1}{\sqrt{\{1 - 2r^2 \cos 2\beta + r^4\}}} = \frac{1}{\sqrt{\{(1 - r^2)^2 + 4r^2 \sin^2\beta\}}}\,, \qquad (69)$$

and similarly, for the case of centrifugal exciting force (proportional to ω^2) :

$$\bar{n}' = \frac{r^2}{\sqrt{\{1 - 2r^2 \cos 2\beta + r^4\}}} = \frac{r^2}{\sqrt{\{(1 - r^2)^2 + 4r^2 \sin^2\beta\}}}\,. \qquad (70)$$

The formulae (68, 69, 70) are analogous to (49, 50, 51), but far from identical—especially for larger values of g, as may be seen from Figs. 6a, b, c which should be compared with 5a, b, c. It should be noticed that Myklestad's formulae have sense only for $g \leqslant 1$, while the standard ones are applicable for any g. A remarkable point is that Myklestad's resonance formula (69) and curves (Fig. 6a) are identical with those (form. (33) and Fig. 4a) for viscous damping and, of course, exhibit the same resonance peaks for $r_r < 1$ (cf., form. (34)) ; the equivalent sets of curves correspond exactly one by one, if we make $\lambda = \sin \beta$ (this has not been done in our plots, where we chose the correspondence $g = \sin 2\beta$). The same remark applies to resonance curves for variable excitation. The formula (68) and graph in Fig. 6b, for the phase delay, differ from both ((32) and Fig. 4b) and ((50) and Fig. 5b). All three sets are hardly distinguishable in the case of low relative damping ($\lambda \approx \frac{1}{2}g \approx \beta$ all small), but the differences increase considerably with relative damping. Two striking cases are illustrated in Figs. 8 and 9, already mentioned before, where Myklestad's curves have also been traced for comparison.

However neat and alluring Myklestad's formulae and graphs may seem, they must be rejected because they are based, as shown, on the tacit assumption that, as g increases, the stiffness decreases in proportion to $\sqrt{(1 - g^2)}$ (*cf.*, form. (64)). There is no reason whatsoever to accept this assumption, which is at variance with the usual way of examining the *effect of increasing damping at constant stiffness*. It is perhaps true that such a variation cannot be simply produced by an experimental apparatus and, from this point of view, the experiments with structural damping must differ considerably from those with viscous damping. In the latter case, the damping may be produced by a dashpot with varying constant, a unit actually separated from the stiffness-producing spring (which itself may practically involve only negligible structural damping), as shown diagrammatically in Fig. 10a. In the former case, the structural damping is naturally and unavoidably supplied by the same spring (or other elastic structure) which produces stiffness ; therefore the pictorial scheme (Fig. 10b) proposed by Bishop[18]—with a seemingly independent damping source h—should rather be replaced by a different scheme (Fig. 10c), where stiffness and damping clearly originate in a common source. It seems unavoidable, in order to change hysteretic damping within wide limits, either to use a number of spring models, so that both stiffness and damping may be chosen at will ; or possibly to have one elastic structure, of more elaborate nature, involving one or more controls permitting a continuous variation of both parameters. Whatever the experimental technique, however, it seems inadmissible to present results, relating to hysteretic damping, for stiffness and damping both varying in an artificial way (*cf.*, 64) to suit arbitrary algebra, while we should rather try to use algebra to suit fundamental concepts.

It may be mentioned that we may come back to the usual (real) form of the differential equation, by using the first of the relationships (43) backwards, so as to eliminate the imaginary part of the complex stiffness term in Myklestad's Equation (65), which then takes the form

$$m\ddot{x} + \frac{kg}{\omega}\dot{x} + kx\sqrt{(1 - g^2)} = F\,e^{i(\omega t + \alpha)}\,, \qquad \qquad (71)$$

where it is clearly seen that the stiffness is assumed to decrease when g increases (*cf.*, equation (46)). It seems clear that such a decrease was not actually intended by Myklestad, and merely resulted from calculating 2nd order effects on the basis of an assumption which was only correct to the 1st order. Myklestad himself hinted that the common way of writing the complex stiffness as $k\,(1 + ig)$ led to peak amplitudes at the natural frequency itself (rather than at the smaller frequency $r_r = \sqrt{\cos 2\beta}$), and claimed that this was at variance with observation. He failed to quote any concrete experimental data, however, and conclusive results could not possibly be obtained for the small values of g he considered, and which have been actually encountered in practice hitherto—not necessarily so in the future. Myklestad also claimed that his way of writing removed some clearly erroneous conclusions in the case of *free oscillations* (presumably referring to Soroka[14]), but this matter is dealt with in Section 4 of this paper.

The theory of Section 3.2.2 is therefore the only one to be recommended, for the time being, for dealing with hysteretic damping in *steady forced oscillations*, and also in other oscillations which are *simple harmonic*, *e.g.*, in established flutter at critical speed. However, even this theory must be understood as based on the existing meagre experimental evidence, and may be modified in the light of future experiments, especially for the case of artificially increased structural damping.

3.3. *Summary of Formulae.*—The table overleaf summarises all relevant formulae for forced oscillations, derived and discussed above, which are usually all that is needed in practice. Myklestad's formulae are included for completeness. Formulae for phase delay angle at resonance peaks are added.

4. *Free Oscillations with Damping.*—As already observed by Bishop[18], the case of free oscillations with hysteretic damping is more difficult than that of forced oscillations, and no satisfactory solution has been given. There are really *two causes* for this. The *first* one is of a mathematical nature, and becomes apparent through the fact that two different solutions have been obtained, and used on the basis of the same assumption as to the damping law, depending on whether real or complex notation was used. This discrepancy has been only partly explained by Bishop. The *second* cause is of more profound nature : the law of hysteretic damping, originating from Kimball[1] and Becker[2], was based on experiments involving simple harmonic oscillations only, while damped free oscillations are never simple harmonic but decaying. It is impossible to guess what the damping law is in such cases, without further experiments. The two solutions mentioned above are both based on the assumption that the law is the same as for simple harmonic oscillations, and it should at least be possible to decide which of these is correct on that assumption. We shall, however, discuss also some alternative plausible ways in which the damping law may be tentatively generalized.

We start again by recapitulating briefly the known theory for the case of viscous damping.

4.1. *Viscous Damping.*—4.1.1. *Real notation.*—The differential equation is now :

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \qquad (72)$$

or, using the notation of Section 3.1.1 (form. (25) and (26)) :

$$\ddot{x} + 2\lambda\omega_n\,\dot{x} + \omega_n{}^2\,x = 0\,. \qquad \qquad (73)$$

			Forced oscillations under simple harmonic force of constant amplitude			
				Forced oscillations under simple harmonic force of amplitude proportional to frequency squared (centrifugal excitation)		References to text and Figures
			Magnification factor n	Phase delay angle η	Magnification factor $\bar{n}$	
Viscous damping		General formulae	$n = \dfrac{1}{\sqrt{\{(1-r^2)^2+4\lambda^2 r^2\}}} = \dfrac{\cos\eta}{1-r^2}$	$\eta = \tan^{-1}\dfrac{2\lambda r}{1-r^2}$	$\bar{n} = \dfrac{r^2}{\sqrt{\{(1-r^2)^2+4\lambda^2 r^2\}}} = \dfrac{r^2\cos\eta}{1-r^2}$	Section 3.1.1 Figs. 4a, b, c
		Resonance peak values	$n_{\max} = \dfrac{1}{2\lambda\sqrt{(1-\lambda^2)}}$	$\eta_r = \dfrac{\pi}{2} - \tan^{-1}\dfrac{\lambda}{\sqrt{(1-2\lambda^2)}}$	$\bar{n}_{\max} = \dfrac{1}{2\lambda\sqrt{(1-\lambda^2)}}$	
			at $r_r = \sqrt{(1-2\lambda^2)}$	$\bar{\eta}_r = \dfrac{\pi}{2} + \tan^{-1}\dfrac{\lambda}{\sqrt{(1-2\lambda^2)}}$	at $\bar{r}_r = \dfrac{1}{\sqrt{(1-2\lambda^2)}}$	
Hysteretic damping	Orthodox treatment	General formulae	$n = \dfrac{1}{\sqrt{\{(1-r^2)^2+g^2\}}} = \dfrac{\cos\eta}{1-r^2}$	$\eta = \tan^{-1}\dfrac{g}{1-r^2}$	$\bar{n} = \dfrac{r^2}{\sqrt{\{(1-r^2)^2+g^2\}}} = \dfrac{r^2\cos\eta}{1-r^2}$	Section 3.2.1 Figs. 5a, b, c
		Resonance peak values	$n_{\max} = \dfrac{1}{g}$	$\eta_r = \dfrac{\pi}{2}$	$\bar{n}_{\max} = \dfrac{\sqrt{(1+g^2)}}{g}$	
			at $r_r = 1$	$\bar{\eta}_r = \dfrac{\pi}{2} + \tan^{-1} g$	at $\bar{r}_r = \sqrt{(1+g^2)}$	
	Myklestad's treatment	General formulae	$n' = \dfrac{1}{\sqrt{\{1-2r^2\cos 2\beta+r^4\}}} = \dfrac{\cos\eta'}{\cos 2\beta - r^2}$	$\eta' = \tan^{-1}\dfrac{\sin 2\beta}{\cos 2\beta - r^2}$	$\bar{n}' = \dfrac{r^2}{\sqrt{\{1-2r^2\cos 2\beta+r^4\}}} = \dfrac{r^2\cos\eta'}{\cos 2\beta - r^2}$	Section 3.2.3 Figs. 6a, b, c
		Resonance peak values	$n'_{\max} = \dfrac{1}{\sin 2\beta}$	$\eta_r' = \dfrac{\pi}{2}$	$\bar{n}'_{\max} = \dfrac{1}{\sin 2\beta}$	
			at $r_r' = \sqrt{\cos 2\beta}$	$\bar{\eta}_r' = \dfrac{\pi}{2} + 2\beta$	at $\bar{r}_r' = \dfrac{1}{\sqrt{\cos 2\beta}}$	

Assuming a trial solution in the form :

$$x_1 = x^* e^{-at} \cos(\omega t + \varphi) , \qquad (74)$$

where x^* and φ are arbitrary constants, we have :

$$\left.\begin{aligned} \dot{x}_1 &= -x^* e^{-at} \{a \cos(\omega t + \varphi) + \omega \sin(\omega t + \varphi)\} , \\ \ddot{x}_1 &= x^* e^{-at} \{(a^2 - \omega^2) \cos(\omega t + \varphi) + 2a\omega \sin(\omega t + \varphi)\} , \end{aligned}\right\} \qquad (75)$$

and, substituting into (73), we see that the following equality must be identically satisfied, for any t :

$$(a^2 - \omega^2 - 2\lambda a \omega_n + \omega_n^2) \cos(\omega t + \varphi) + 2\omega (a - \lambda \omega_n) \sin(\omega t + \varphi) = 0 . \qquad (76)$$

This will be so only if the coefficients of $\cos(\omega t + \varphi)$ and $\sin(\omega t + \varphi)$ are both zero, which gives :

$$a = \lambda \omega_n , \qquad \omega = \omega_n \sqrt{(1 - \lambda^2)} , \qquad (77)$$

hence :

$$\frac{a}{\omega} = \frac{\lambda}{\sqrt{(1 - \lambda^2)}} , \qquad (78)$$

the ratio of amplitudes at the beginning and end of a complete cycle :

$$R = e^{2\pi a/\omega} = e^{2\pi\lambda/\sqrt{(1-\lambda^2)}} , \qquad (79)$$

and the logarithmic decrement :

$$\delta = \ln R = 2\pi \frac{\lambda}{\sqrt{(1 - \lambda^2)}} . \qquad (80)$$

It is seen that (74) is a general solution provided a and ω have the values (77), the condition being, however :

$$\lambda < 1 , \qquad \text{or} \qquad c < 2\sqrt{(mk)} , \qquad (81)$$

i.e., that there is no overdamping. If $\lambda > 1$, the solution (74) is invalid, and the alternative aperiodic solution exists :

$$x_1 = A\, e^{-a't} + B\, e^{-a''t} , \qquad (82)$$

where :

$$a' = \omega_n\{\lambda + \sqrt{(\lambda^2 - 1)}\} , \qquad a'' = \omega_n\{\lambda - \sqrt{(\lambda^2 - 1)}\} \qquad (83)$$

If, finally, $\lambda = 1$, then $a' = a'' = \omega_n$, and the solution is :

$$x_1 = e^{-\omega_n t} (A + Bt) . \qquad (84)$$

The above results are illustrated in Fig. 11, where ω/ω_n, a/ω_n, a/ω, a'/ω_n and a''/ω_n are plotted against λ. The important point to remember is that the critical condition, where the solution ceases to be periodic, occurs when $\lambda = 1$, $\omega = 0$.

4.1.2. *Complex notation and vectorial representation.*—If $\lambda < 1$, the equation (72) or (73) has also a solution :

$$x_2 = x^* e^{-at} \sin(\omega t + \varphi) , \qquad (85)$$

which is obtained from (74) by replacing φ by $(\varphi - \frac{1}{2}\pi)$, and ω and a have the same values as before. Introducing the complex variable $x = x_1 + x_2$, we find that

$$x = x^* e^{-at} \{\cos(\omega t + \varphi) + i \sin(\omega t + \varphi)\} , \qquad (86)$$

or

$$x = x^* e^{-at + i(\omega t + \varphi)} \qquad (87)$$

is also a solution of (73). We may also treat (87) as a trial solution, and then we have :

$$\dot{x} = (i\omega - a)\, x , \qquad \ddot{x} = (i\omega - a)^2\, x , \qquad (88)$$

which substituted into (73), lead immediately to (77).

We may now perform a similar formal transformation as in Section 3.1.2, *viz.*, eliminate $\dot{x}$ from the differential equation. By using only the first of the relationships (88), we write (72) and (73) in the following forms, respectively :

$$m\ddot{x} + \{(k - ac) + ic\omega\} x = 0 \qquad \ldots \qquad (89)$$

and

$$\ddot{x} + \{(\omega_n^2 - 2\lambda a\, \omega_n) + 2i\lambda\, \omega_n\, \omega\} x = 0 . \qquad \ldots \qquad (89a)$$

In either form, this is still a linear differential equation of 2nd order, representing exactly the same motion as before. The damping term has formally disappeared, and the constant k in the subsequent term has been replaced by

$$(k - ac) + ic\omega , \qquad \ldots \qquad (90)$$

which may again be termed *'complex stiffness' for the case of free oscillations with viscous damping.* There are two important differences in comparison with the case of forced oscillations considered in Section 3.1.2. *Firstly,* the complex stiffness (90) now differs from that $(k + ic\omega)$ previously obtained ; it appears that the real part k has now been formally decreased by ac, and this is obviously due to the motion being now not a simple harmonic but a decaying harmonic oscillation. The expression (90) is more general and reduces to the previous form when $a = 0$. *Secondly,* the equations (89) are now peculiar inasmuch as the constant coefficients of x contain both a and ω which are unknown until the equation is solved.* This peculiarity might become very troublesome if it occurred in a more complex system of several degrees of freedom. In the present case, however, we may solve (89a) quite easily, by using the second of relationships (88), whereupon we obtain :

$$(a^2 - \omega^2 + \omega_n^2 - 2\lambda a\omega_n) + 2i\omega\,(\lambda\omega_n - a) = 0 , \qquad \ldots \qquad (91)$$

and hence directly (77).

The advantages of complex notation for free oscillations are the same as for the forced ones and, as known[19], the vectorial representation may still be applied for a combination of oscillatory functions with amplitudes decaying exponentially with time (such as those represented by form (74) or (87)), provided the frequency ω and the damping index a are the same for all functions. The illustration is given in Fig. 14, where the closed triangle OAB represents the three terms of equation (72). All vectors are considered as rotating anticlockwise at uniform angular speed ω, while decreasing exponentially at uniform rate. The velocity vector $\dot{x}$ leads that of displacement x by $(90^\circ + \varepsilon)$, and is similarly lead by the acceleration vector $\ddot{x}$, where the 'damping angle' ε is defined by :

$$\tan \varepsilon = a/\omega . \qquad \ldots \qquad (92)$$

It is obvious that OA = OB, *i.e.*, the triangle is isosceles, and the angle AOB = 2ε. The modified equation (89) is illustrated by the inset diagram in Fig. 14, where the triangle OEB replaces OAB.

4.2. *Hysteretic Damping.*—4.2.1. *Collar's method, real notation.*—The method suggested by Collar and described by Bishop[18] is based on the assumption that the relationship (12) still holds in the case of free oscillations, in spite of the motion being now not simple harmonic. The differential equation (72) or (73) becomes :

$$m\ddot{x} + \frac{kg}{\omega}\dot{x} + kx = 0 \qquad \ldots \qquad (93)$$

or

$$\ddot{x} + \frac{g\omega_n^2}{\omega}\dot{x} + \omega_n^2 x = 0 , \qquad \ldots \qquad (94)$$

so that :

$$\lambda = \frac{g\omega_n}{2\omega} . \qquad \ldots \qquad (95)$$

* It was not so in the case of forced oscillation where the complex stiffness did contain ω (not a which was 0), but ω was known as imposed by the exciting force.

The solution (74) still holds, as also do the conditions (77) which, however, assume the following form :

$$a = \frac{g\omega_n^2}{2\omega}, \qquad \omega = \omega_n \sqrt{\left(1 - \frac{g^2\omega_n^2}{4\omega^2}\right)}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (96)$$

In this case, the coefficient of $\dot{x}$ in (93, 94) depends on ω which is itself unknown, therefore (96) are not final answers ; they may, however, be treated as a system of equations for determining a and ω. The solution is :

$$\frac{\omega}{\omega_n} = \sqrt{\frac{1 + \sqrt{(1 - g^2)}}{2}}, \qquad \frac{a}{\omega_n} = \lambda = \sqrt{\frac{1 - \sqrt{(1 - g^2)}}{2}}, \quad \ldots \quad (97)$$

where the signs of $\sqrt{(1 - g^2)}$ are the only ones consistent with conditions at $g = 0$, and hence :

$$\frac{a}{\omega} = \frac{1 - \sqrt{(1 - g^2)}}{g} = \frac{g}{1 + \sqrt{(1 - g^2)}}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (98)$$

and the logarithmic decrement :

$$\delta = \ln R = 2\pi \frac{a}{\omega} = \frac{2\pi g}{1 + \sqrt{(1 - g^2)}}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (99)$$

All quantities determined by (97 to 99) have a meaning only if $g \leqslant 1$, therefore the oscillatory solution (74) seems to apply for, and only for, such values of g. If $g > 1$, we might expect an aperiodic solution but since the law of structural damping for aperiodic motions is completely unknown at present, it would be futile to attempt to find the corresponding solution.

Formulae (97 and 98) are illustrated in Fig. 12 which shows that the frequency ω decreases and the damping index a increases with increasing g, and so does, of course, the relative damping ratio λ. This behaviour is as might be expected. However, there is one striking difference from the case of viscous damping (Fig. 11). In that case, the limit of validity of the oscillatory solution (74) coincided with the frequency ω falling to zero, λ becoming 1, and a/ω reaching infinity, so that, in the limiting conditions, the solution becomes just aperiodic (84). The complex stability root $(-a + i\omega)$ afterwards splits into two real roots $(-a')$ and $(-a'')$, cf., (82). In the present case, the limit of validity seems to be $\lambda = \omega/\omega_n = a/\omega_n = 1/\sqrt{2} = 0{\cdot}7071$, so that the solution is still fully oscillatory in these limiting conditions. This is a very unusual result. We come back to this question in Section 4.2.4.

4.2.2. *Collar's method in complex notation, leading to Myklestad's formulae.*—The complex notation may be again introduced, in exactly the same way as in Section 4.1.2, and the differential equation will be obtained, on substituting the first of the relationships (88) into (93, 94), in the forms :

$$m\ddot{x} + k\left(1 - \frac{ag}{\omega} + ig\right)x = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (100)$$

or

$$\ddot{x} + \omega_n^2\left(1 - \frac{ag}{\omega} + ig\right)x = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (100a)$$

The damping term has formally disappeared, and the constant k in the subsequent term has been replaced by

$$k\left(1 - \frac{ag}{\omega} + ig\right), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (101)$$

which is now the '*complex stiffness*' *for the case of free oscillations with hysteretic damping*. It must be stressed that, although the real part of this complex stiffness is now less than k, it does not mean that the true stiffness has been modified. This is still k, as seen from equation (93) which is exactly equivalent with (100). The modification of the real constant has taken place merely because the motion is now an exponentially decaying oscillation, and not just simple harmonic, so that the complex ratio $\dot{x} : x$ is now $(-a + i\omega)$, instead of $i\omega$. The complex stiffness (101) is more general, but it becomes again equal to $k(1 + ig)$ when $a = 0$ as, for instance, in the case of steady forced oscillations (Section 3.2).

The equation (100a) can, of course, be solved directly, by using the second of the relationships (88), and then we obtain again the solution (96).

The vector diagram of Fig. 14 applies in the present case, with the only alteration that the vector AB now denotes $kg\dot{x}/\omega$ (instead of $c\dot{x}$). The inset diagram again illustrates the complex restoring term, consisting of components OE and EB.

It may be mentioned that, in equation (100), although the imaginary part in the second term is now known, the real part does depend on a and ω which are unknown until we find the solution. We may, however, use this solution, *e.g.*, form (98), to get rid of them. We find :

$$1 - \frac{ga}{\omega} = \sqrt{(1 - g^2)} \, , \qquad (102)$$

and hence the differential equation may be written in yet another form :

$$m\ddot{x} + k\{\sqrt{(1 - g^2)} + ig\}x = 0 \, , \qquad (103)$$

where the complex stiffness has the form identical with (64), which we encountered when discussing Myklestad's theory. It is seen to be applicable in the present case and, if we put again :

$$g = \sin 2\beta \, ,$$

then the differential equation becomes :

$$m\ddot{x} + k\,e^{2i\beta}\,x = 0 \, , \qquad (104)$$

and the formulae (97 to 99) assume very simple forms :

$$\frac{\omega}{\omega_n} = \cos\beta \, , \qquad \frac{a}{\omega_n} = \lambda = \sin\beta \, , \qquad \frac{a}{\omega} = \tan\beta \, , \qquad \delta = 2\pi\tan\beta \, . \qquad (105)$$

Myklestad's concept of complex stiffness has thus been vindicated for the case of free oscillations with hysteretic damping, of a single degree of freedom. It is not general, however, as it fails, *e.g.*, in the case of forced oscillations. It should, therefore, never be used in any other case although, if g is small, the errors involved may only be small of 2nd order.

Considering the inset diagram in Fig. 14, we observe that, in the present case :

$$\beta = \varepsilon \, . \qquad (106)$$

4.2.3. *Soroka's method.*—Soroka[14] wrote the differential equation of free oscillations with structural damping in the following complex form :

$$m\ddot{x} + k\,(1 + ig)\,x = 0 \, , \qquad (107)$$

which differs from (100) by having a simpler complex stiffness $k(1 + ig)$, just as in the equation (57) which related to steady forced oscillations. Soroka quoted Theodorsen and Garrick[12] as the source of his equation, although these authors applied this sort of complex stiffness only in the problem of critical flutter, i.e., when the oscillation was simple harmonic.

Dividing by m, we may write (107) as follows :

$$\ddot{x} + \omega_n^2\,(1 + ig)\,x = 0 \, . \qquad (107a)$$

It still admits of the solution (87) and, using (88), we obtain :

$$(i\omega - a)^2 + \omega_n^2\,(1 + ig) = 0 \, . \qquad (108)$$

Equating to zero the real and imaginary parts :

$$\omega^2 - a^2 = \omega_n^{\,2}, \qquad 2a\omega = g\,\omega_n^{\,2}, \quad \ldots \quad (109)$$

we find :

$$\frac{\omega}{\omega_n} = \sqrt{\frac{\sqrt{(1+g^2)}+1}{2}}, \qquad \frac{a}{\omega_n} = \sqrt{\frac{\sqrt{(1+g^2)}-1}{2}}, \quad \ldots \quad (110)$$

and hence :

$$\frac{a}{\omega} = \frac{g}{\sqrt{(1+g^2)}+1}, \qquad \delta = \ln R = \frac{2\pi g}{\sqrt{(1+g^2)}+1}. \quad \ldots \quad (111)$$

This is Soroka's solution which, if we put :

$$g = \sinh 2\gamma, \quad \ldots \quad (112)$$

may also be written :

$$\frac{\omega}{\omega_n} = \cosh\gamma, \qquad \lambda = \frac{a}{\omega_n} = \sinh\gamma, \qquad \delta = 2\pi\frac{a}{\omega} = 2\pi\tanh\gamma. \quad \ldots \quad (113)$$

This solution, illustrated in Fig. 13, is paradoxical in the extreme. The formulae have a meaning for any g, up to infinity, so it seems that the oscillatory motion would take place for any amount of damping. The *frequency increases with damping coefficient g,* instead of falling.

The explanation of the error was partly given by Pinsker[15]. Soroka's solution corresponds (unintentionally) to the case when stiffness is not constant but increases itself with g. This can be shown easily by transforming the differential equation (107) back to the real form, by using the first of the relationships (88), whereupon we obtain :

$$m\ddot{x} + \frac{gk}{\omega}\dot{x} + k\left(1 + \frac{ga}{\omega}\right)x = 0, \quad \ldots \quad (114)$$

instead of the correct equation (93), where the stiffness is constant (k). Soroka's stiffness is :

$$k' = k\left(1 + \frac{ga}{\omega}\right), \quad \text{or} \quad k' = k\sqrt{(1+g^2)}, \quad \ldots \quad (115)$$

thus increases indefinitely with g. With such a stiffness, the natural frequency (with no damping) would be expressed by :

$$\omega_n'^2 = \omega_n^2\sqrt{(1+g^2)}, \quad \ldots \quad (116)$$

so that :

$$\omega^2 - \omega_n^{\,2} = \tfrac{1}{2}(\omega_n'^2 - \omega_n^2) \quad \ldots \quad (117)$$

It is seen that the square of frequency has increased by a certain amount owing to the rise of stiffness, and then lost half of this increment due to damping.

The above calculation is illustrated in Fig. 15, where the triangle OAB of Fig. 12 is replaced by OA′B′. The increased stiffness k' is shown, and the resulting damping angle ε' is less than ε.

It is remarkable that Soroka's assumption is completely analogous to that made by Myklestad in the case of forced oscillations. In the latter case, the complex stiffness $k\,e^{2i\beta}$, applicable to free oscillations, was used for the steady forced ones. In the present case, the complex stiffness $k\,(1 + ig)$, applicable only to simple harmonic oscillations, was used for the free, i.e., decaying, oscillations. The procedure cannot be accepted because the concept of stiffness increasing with g in an artificial manner (cf., (115)), to which the theory really applies, has no theoretical or practical meaning.

4.2.4. *An alternative solution based on a plausible assumption on energy dissipation.*—We have seen in Section 4.2.2 that the simple solution based on equation (93) and illustrated in Fig. 12, is somewhat doubtful for large values of g. The reason for that is that the law of hysteretic damping for decaying oscillations is not known, and therefore it was not quite a legitimate generalization to extend the validity of the assumption (12) :

$$c\omega = kg$$

into the field where it has never been proved experimentally. The only reliable answer can be supplied by further experiments whose difficulties should not be underestimated. It may be permissible, however, to try to speculate about the possible alternative generalizations of the law of hysteretic damping.

In the case of sinusoidal oscillations, it was found that the energy dissipated per period through hysteretic damping did not depend on frequency ω and was proportional to the square of amplitude x^* (which was constant throughout the motion). A plausible generalization for decaying oscillations will be that the *mean energy dissipated* per period will not depend on either ω or a, but will be proportional to the square of *some mean amplitude* x_m (provisionally unspecified), because it seems unreasonable to make it proportional to the square of x^* which now denotes the *initial amplitude*. Referring to (20), this will be written :

$$\pi c\omega x^{*2}\left(1+\frac{a^2}{\omega^2}\right)\frac{1-e^{-4\pi a/\omega}}{4\pi a/\omega} = \pi k g x_m^{\ 2}, \qquad (118)$$

and hence :

$$c\omega = kgN, \qquad (119)$$

where :

$$N = \frac{x_m^{\ 2}}{x^{*2}}\frac{4\pi a\omega}{(a^2+\omega^2)\left(1-e^{-4\pi a/\omega}\right)}. \qquad (120)$$

N is a certain function of a/ω, becoming obviously $= 1$ for $a = 0$. The differential equation then takes the form :

$$m\ddot{x} + \frac{kg\,N}{\omega}\dot{x} + kx = 0 \qquad (121)$$

or

$$\ddot{x} + \omega_n^{\ 2}\left(1 - \frac{gNa}{\omega} + igN\right)x = 0. \qquad (121a)$$

Using (88), this leads to :

$$(i\omega - a)^2 + \omega_n^{\ 2}\left(1 - \frac{gNa}{\omega} + igN\right) = 0$$

or, separating the real and imaginary parts :

$$2a\omega = gN\omega_n^{\ 2}, \qquad a^2 - \omega^2 + \omega_n^{\ 2} - \frac{gNa\omega_n^{\ 2}}{\omega} = 0 \qquad (122)$$

Eliminating gN, we obtain

$$\omega^2 + a^2 = \omega_n^{\ 2}, \qquad (123)$$

and hence :

$$\frac{2a\omega}{\omega^2 + a^2} = gN. \qquad (124)$$

Comparing (120) and (124), we obtain finally :

$$g = \frac{x^{*2}}{x_m^{\ 2}}\frac{1-e^{-4\pi a/\omega}}{2\pi}. \qquad (125)$$

The last three equations will solve the problem, once we decide on the definition of x_m. The last word depends on experiment, but we may attempt some trial definition. The simplest and most plausible seems to be that x_m^2 is the *arithmetical mean of the squares of amplitudes at the beginning and end of a cycle, i.e.* :

$$x_m^2 = \tfrac{1}{2}\, x^{*2}\,(1 + e^{-4\pi a/\omega})\,. \qquad \qquad (126)$$

We then obtain from (125) :

$$\pi g = \tanh 2\pi \frac{a}{\omega}\,,$$

or

$$\frac{a}{\omega} = \frac{1}{2\pi} \tanh^{-1} \pi g = \frac{1}{4\pi} \ln \frac{1 + \pi g}{1 - \pi g}\,, \qquad \qquad (127)$$

and from (123) and (124) :

$$\frac{\omega}{\omega_n} = \frac{4\pi}{\sqrt{\left(16\,\pi^2 + \ln^2 \dfrac{1 + \pi g}{1 - \pi g}\right)}}\,, \qquad \frac{a}{\omega_n} = \lambda = \frac{\ln \dfrac{1 + \pi g}{1 - \pi g}}{\sqrt{\left(16\,\pi^2 + \ln^2 \dfrac{1 + \pi g}{1 - \pi g}\right)}}\,, \qquad \qquad (128)$$

and

$$N = \frac{2\pi a \omega}{(\omega^2 + a^2) \tanh 2\pi a/\omega} = \frac{\dfrac{8\pi}{g} \ln \dfrac{1 + \pi g}{1 - \pi g}}{16\,\pi^2 + \ln^2 \dfrac{1 + \pi g}{1 - \pi g}}\,. \qquad \qquad (129)$$

All the above formulae have a meaning only if

$$g < \frac{1}{\pi} = 0{\cdot}3183 \qquad \qquad (130)$$

and, if g tends to this limit, we have $N \to 0$, $\omega \to 0$, $\lambda \to 1$, as it should be. For greater values of g, the motion should be aperiodic. This solution is illustrated in Fig. 16, where ω/ω_n, λ and gN are plotted against g. It is seen that ω decreases slowly with increasing g, to fall very rapidly to 0 near the limiting value, while λ rises first nearly in proportion to g and then shoots up to 1 near the limit. As to gN, it attains its maximum value 1 at g very little less than the limiting value, and then drops very rapidly to 0.

It may be mentioned that, for small values of g, the above solution agrees with that obtained in Section 4.2.1, up to terms of second order in g. Expanding (128), we get :

$$\frac{\omega}{\omega_n} = 1 - \tfrac{1}{8} g^2 \ldots, \qquad \frac{a}{\omega_n} = \lambda = \tfrac{1}{2} g + \mathrm{O}\, g^2 \ldots, \qquad \qquad (131)$$

and identical expansions, to this order, will be obtained from (97). The first of Soroka's formulae (110), however, leads to $\omega/\omega_n = 1 + g^2/8 \ldots$.

It must be clearly stated that the assumption (126) and formulae (128) are not proposed as a proved solution. They are merely given as an example, to show that it is possible to make the frequency fall to 0 at the limiting value of g, on making an assumption at least equally plausible as that made in Section 4.2.1. It is impossible to obtain a reliable solution without further experimental work.

4.3. *Summary of Formulae.*—The following table summarizes all relevant formulae for free oscillations, derived and discussed above. Soroka's formulae are included for completeness.

		ω/ω_n	a/ω_n	δ	References to text and Figures
Viscous damping		$\sqrt{(1-\lambda^2)}$	λ	$2\pi\dfrac{\lambda}{\sqrt{(1-\lambda^2)}}$	Section 4.1.1 Figure 11
Hysteretic damping	After Collar-Bishop, or Myklestad	$\sqrt{\dfrac{1+\sqrt{(1-g^2)}}{2}} = \cos\beta$	$\sqrt{\dfrac{1-\sqrt{(1-g^2)}}{2}} = \sin\beta$	$2\pi\dfrac{g}{1+\sqrt{(1-g^2)}} = 2\pi\tan\beta$	Section 4.2.1 Figure 12
	After Soroka	$\sqrt{\dfrac{\sqrt{(1+g^2)}+1}{2}} = \cosh\gamma$	$\sqrt{\dfrac{\sqrt{(1+g^2)}-1}{2}} = \sinh\gamma$	$2\pi\dfrac{g}{\sqrt{(1+g^2)}+1} = 2\pi\tanh\gamma$	Section 4.2.3 Figure 13
	Solution of Section 4.2.4	$\dfrac{4\pi}{\sqrt{\left(16\pi^2+\ln^2\dfrac{1+\pi g}{1-\pi g}\right)}}$	$\dfrac{\ln\dfrac{1+\pi g}{1-\pi g}}{\sqrt{\left(16\pi^2+\ln^2\dfrac{1+\pi g}{1-\pi g}\right)}}$	$\frac{1}{2}\ln\dfrac{1+\pi g}{1-\pi g}$	Section 4.2.4 Figure 16

5. *Conclusions.*—The main conclusions of the present Paper are as follows :

A. Simple harmonic oscillations, single degree of freedom.

(1) The equation (1) of steady forced oscillations may be transformed to the complex form (44), so that the damping and stiffness terms are combined into a single 'complex-stiffness term', the complex stiffness being

$$k + ic\omega \quad \text{(a)}$$

(2) The expression (a) is convenient in the case of viscous damping when the damping coefficient c is constant (independent of frequency ω). In the case of hysteretic damping, however, when c is inversely proportional to frequency according to (3), the complex stiffness assumes the appropriate form :

$$k(1 + ig)\,, \quad \text{(b)}$$

and the equation of forced oscillations become (57).

(3) The alternative expressions of the complex stiffness

$$k\,e^{ig} \quad \text{or} \quad k\,e^{2i\beta} \quad \text{(c)}$$

are only admissible if g (or β) is very small, and second order effects can be neglected. Using these expressions for larger values of g and deducing effects involving higher orders, as suggested by Myklestad, leads to erroneous results because it implies an artificial assumption that the real stiffness k is replaced by $k\sqrt{(1-g^2)}$, varying with g (*see* (71)). Myklestad's formulae (68 to 70) for the amplitude and phase delay angle are misleading and should not be used.

B. Decaying oscillations, single degree of freedom.

(4) The equation (72) of free oscillations may again be transformed to the complex form (89), but the complex stiffness then becomes different from (a), *viz.* :

$$(k - ac) + ic\omega\,. \quad \text{(d)}$$

(5) The expression (d) applies directly in the case of viscous damping but offers no advantage in practice, as it contains a and ω, both quantities being originally unknown in the typical problem of free oscillations.

(6) The law of hysteretic damping in decaying oscillations is unknown, because all experiments have hitherto been restricted to simple harmonic oscillations. Assuming, however, that the previous law (3) still applies, the complex stiffness becomes :

$$k\left(1 - \frac{ag}{\omega} + ig\right), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad \text{(e)}$$

which differs from (b). On this assumption, the solution of the problem of free oscillations is given by the formulae (97). The solution is plausible for small values of g, but is doubtful for larger values.

(7) Myklestad's expression for complex stiffness in the case of hysteretic damping :

$$k\, e^{2i\beta} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad \text{(f)}$$

leads to the same solution as (e), provided $g = \sin 2\beta$. This expression is therefore admissible in the problem of free oscillations.

(8) Using the complex stiffness in the form (b) in the problem of free oscillations, as proposed by Soroka, leads to erroneous results, because it implies an artificial assumption that the real stiffness k is replaced by $k\sqrt{(1 + g^2)}$, increasing with g. Soroka's formulae (110) are misleading and should not be used.

(9) A tentative alternative solution of the problem of free oscillations with hysteretic damping, based on a plausible assumption on energy dissipation, is given in Section 4.2.4. Its validity depends on experimental confirmation.

C. Damped oscillations with many degrees of freedom.

(10) The complex stiffness in the form (a) or (b), in the cases of viscous or hysteretic damping, respectively, may also be used for systems of many degrees of freedom, provided only an oscillation in a single simple harmonic mode is considered, *i.e.* :

(i) either for steady forced oscillations, where all exciting forces are simple harmonic of the same frequency,

(ii) or for steady self-excited oscillations, *e.g.*, flutter in critical conditions.

In both cases, the form (c) for complex stiffness should be avoided, unless only first order effects of damping are considered.

(11) Whenever the oscillation consists of several modes, whether simple harmonic or decaying, the law of hysteretic damping is unknown. The law applying for a single simple harmonic mode cannot be used, simply because it involves the frequency, and becomes senseless where there are several frequencies. Any attempts to use the concept of complex stiffness in such problems would lead to meaningless solutions.

(12) The general law of structural damping can only be found by new experiments. Such a law should be applicable to any motion, periodic or aperiodic. The difficulties of the experimental technique are very serious, especially as it cannot be anticipated in advance that the law will be linear. The problem will have to be faced if high (artificially augmented) structural damping is to be widely introduced.

D. Remarks on definitions and nomenclature.

(13) To avoid misunderstandings, it is suggested that an oscillation should be termed 'damped' whenever there are damping forces in the system, *i.e.*, dissipative forces opposed

to velocity, irrespective of whether the amplitude decreases or remains constant. The terms 'steady' or 'decaying' oscillation should be used to indicate that the amplitude is constant or decreases as time increases.

(14) In a simple harmonic oscillation with *viscous* damping the amplitude of the damping force ($c\omega x^*$) is proportional to that of velocity (ωx^*), the coefficient c being constant. In the case of *hysteretic* damping, however, c is assumed to be inversely proportional to frequency ω (*see* form. (3)), and hence the amplitude of the damping force (kgx^*) is proportional to that of displacement (x^*), k and g being constant. This has led some authors to describe the hysteretic damping force as 'proportional to displacement but in (counter) phase with velocity'. The expression is wrong, but its use seems plausible when one considers the equation of motion in complex form (*e.g.*, equation (57), where the term $igkx$ represents the damping force). It must be understood that the damping force varies in proportion to the velocity throughout the motion (*see* equation (46)), and only its amplitude is proportional to that of the displacement.

LIST OF SYMBOLS

a	Damping index of a decaying oscillation, *see* (74)
a', a''	Damping indices of an aperiodic motion, *see* (82)
c	Damping coefficient, *see* (1)
c_{cr}	Critical damping coefficient, *see* (26)
E	Energy lost per cycle of an oscillation
E_d	Energy dissipated by damping in a decaying oscillation, per cycle
E_{d0}	Energy dissipated by damping in a simple harmonic oscillation, per cycle
F	Amplitude of applied external force
f	Resisting force, resultant of restoring and damping forces
g	Dimensionless coefficient of hysteretic damping
k	Stiffness (spring constant)
m	Mass
N	Coefficient, *see* (119)
n	Magnification factor for constant amplitude of exciting force
$\bar{n}$	Magnification factor for amplitude of exciting force proportional to frequency squared (centrifugal excitation)
R	Ratio of amplitudes at beginning and end of a cycle
r	Frequency ratio, *see* (29)
t	Time
T	Period
x	Displacement
x^*	Displacement amplitude in simple harmonic oscillation, or initial amplitude in decaying oscillation
$\dot{x}$	Velocity
$\ddot{x}$	Acceleration
α	Initial phase angle of the exciting force
β	Myklestad's angle for hysteretic damping
γ	Auxiliary parameter, *see* (112)
δ	Logarithmic decrement
ε	Damping angle
η	Phase delay angle in forced oscillation
λ	Relative damping ratio
ω	Actual frequency in free or forced oscillation
ω_n	Natural frequency

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APPENDIX (to Section 2.2)

Details of Calculating the Energy Dissipation in an Exponentially Decaying Harmonic Oscillation

To evaluate the integral in (17), we transform :

$$
\begin{aligned}
\left\{\omega \cos(\omega t + \varphi) - a \sin(\omega t + \varphi)\right\}^2 &= \frac{\omega^2 + a^2}{2} + \frac{\omega^2 - a^2}{2} \cos 2(\omega t + \varphi) - a\omega \sin 2(\omega t + \varphi) \\
&= \frac{\omega^2 + a^2}{2} + \left(\frac{\omega^2 - a^2}{2} \cos 2\varphi - a\omega \sin 2\varphi\right) \cos 2\omega t - \\
&\quad - \left(\frac{\omega^2 - a^2}{2} \sin 2\varphi + a\omega \cos 2\varphi\right) \sin 2\omega t \qquad \text{(A.1)}
\end{aligned}
$$

The formula (17) then becomes :

$$
\begin{aligned}
E_d = \tfrac{1}{2} c x^{*2} [(\omega^2 + a^2) I_1 &+ \{(\omega^2 - a^2) \cos 2\varphi - 2a\omega \sin 2\varphi\} I_2 - \\
&- \{(\omega^2 - a^2) \sin 2\varphi + 2a\omega \cos 2\varphi\} I_3] , \qquad \text{(A.2)}
\end{aligned}
$$

where the three integrals are :

$$
I_1 = \int_0^{2\pi/\omega} e^{-2at}\, dt = \frac{1}{2a} (1 - e^{-4\pi a/\omega}) , \qquad \text{(A.3)}
$$

$$
\begin{aligned}
I_2 = \int_0^{2\pi/\omega} e^{-2at} \cos 2\omega t\, dt &= \frac{1}{2(a^2 + \omega^2)} \Big|\, a + (\omega \sin 2\omega t - a \cos 2\omega t)\, e^{-2at} \Big|_0^{2\pi/\omega} \\
&= \frac{a}{2(a^2 + \omega^2)} (1 - e^{-4\pi a/\omega}) , \qquad \text{(A.4)}
\end{aligned}
$$

$$
\begin{aligned}
I_3 = \int_0^{2\pi/\omega} e^{-2at} \sin 2\omega t\, dt &= \frac{1}{2(a^2 + \omega^2)} \Big|\, \omega - (\omega \cos 2\omega t + a \sin 2\omega t)\, e^{-2at} \Big|_0^{2\pi/\omega} \\
&= \frac{\omega}{2(a^2 + \omega^2)} (1 - e^{-4\pi a/\omega}) . \qquad \text{(A.5)}
\end{aligned}
$$

Substituting (A.3, 4, 5) into (A.2), and simplifying, we obtain (18).

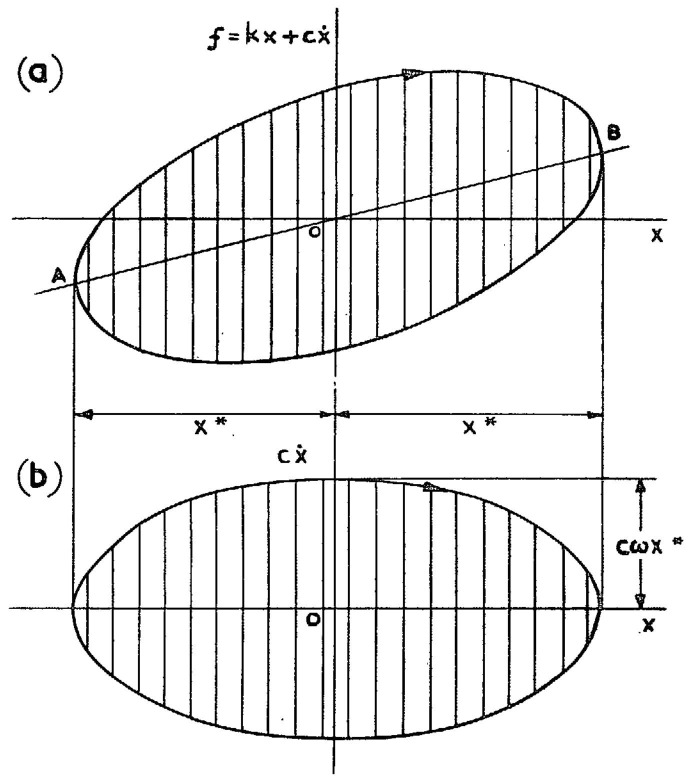


FIG. 1a and b. Energy dissipation in simple harmonic motion, closed (elliptical) hysteresis loops.

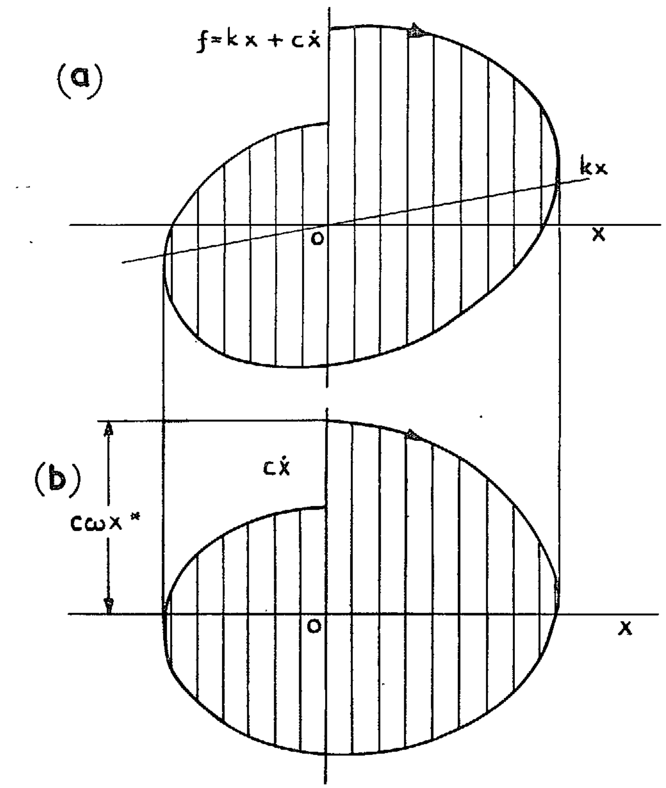


FIG. 2a and b. Energy dissipation in exponentially decaying harmonic oscillations, open hysteresis loops.

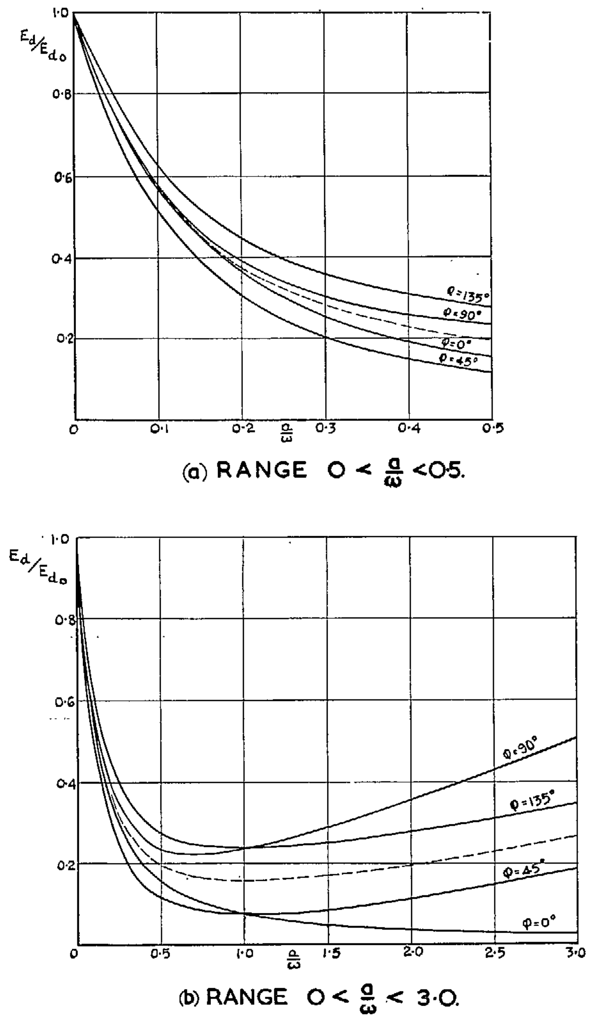


FIG. 3a and b. Variation of energy dissipation E_d/E_{d0} with a/ω.

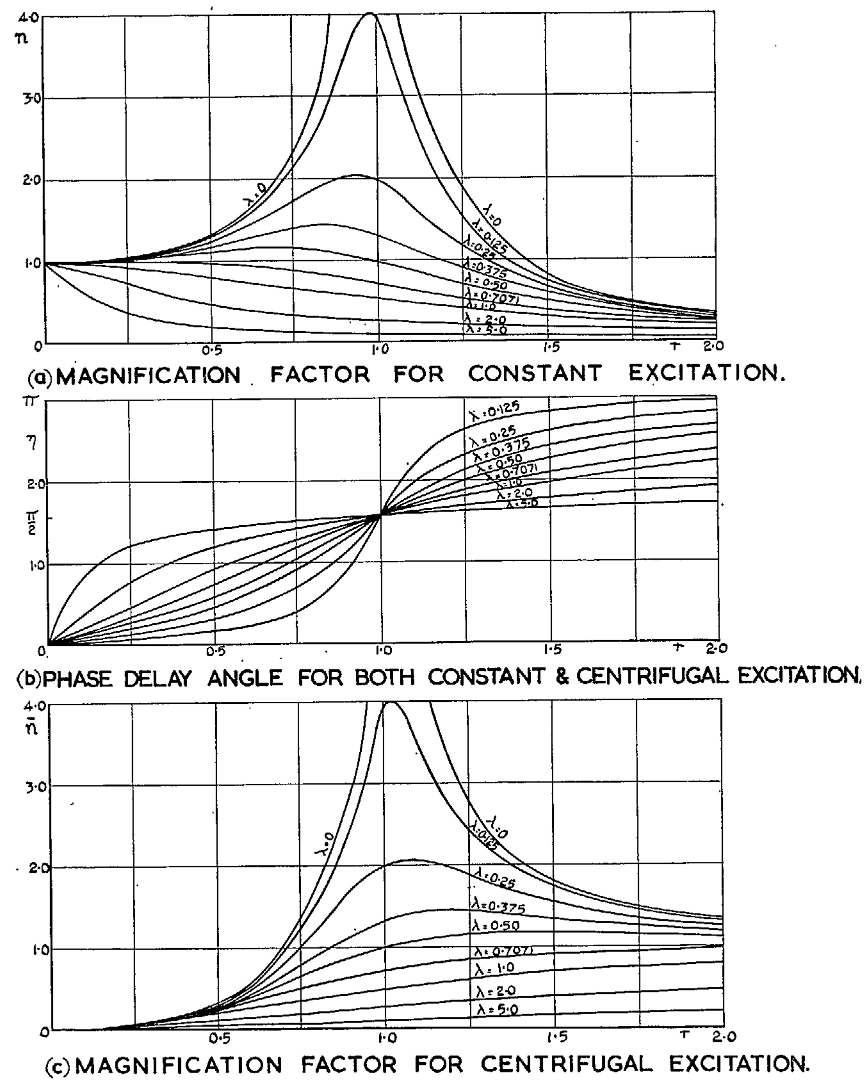
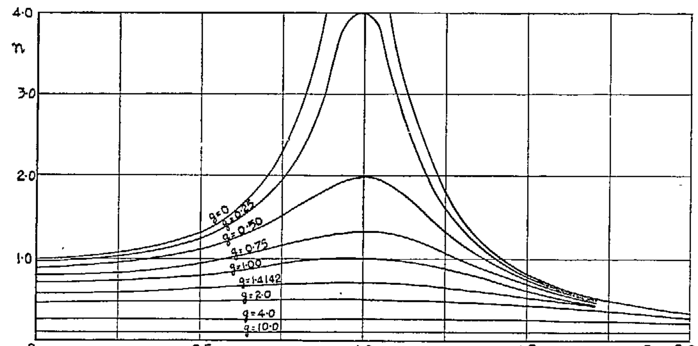
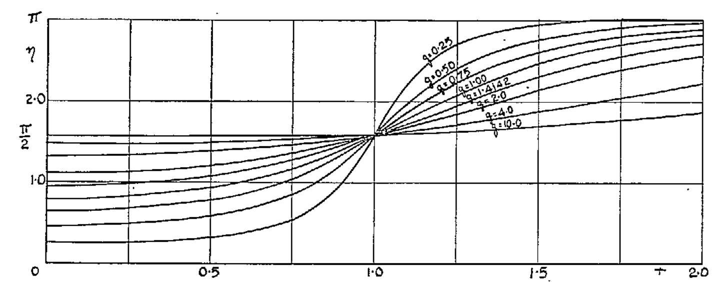


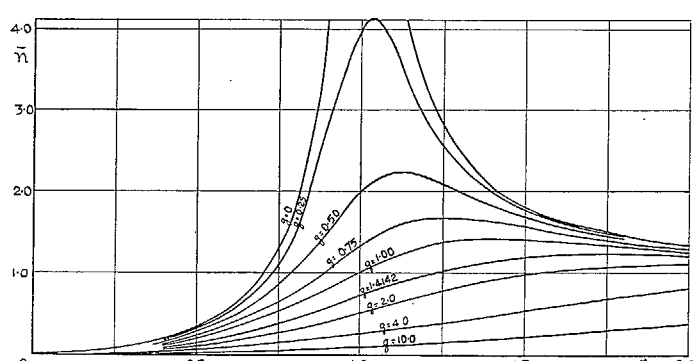
FIG. 4a to c. Phase delay angle and magnification factor for steady forced oscillations with viscous damping.



(a) MAGNIFICATION FACTOR FOR CONSTANT EXCITATION.

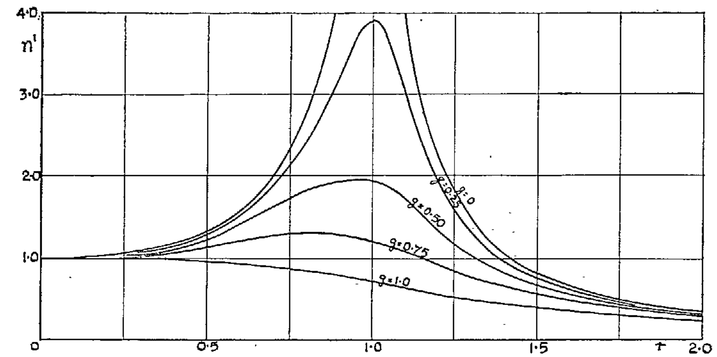


(b) PHASE DELAY ANGLE FOR BOTH CONSTANT & CENTRIFUGAL EXCITATION.

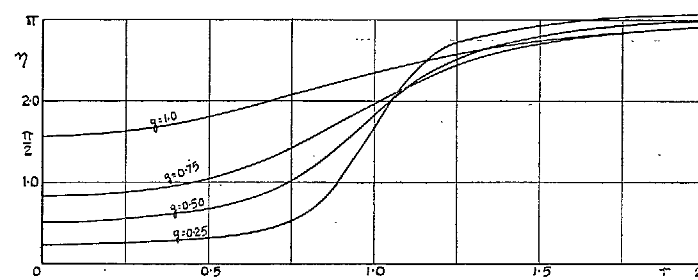


(c) MAGNIFICATION FACTOR FOR CENTRIFUGAL EXCITATION.

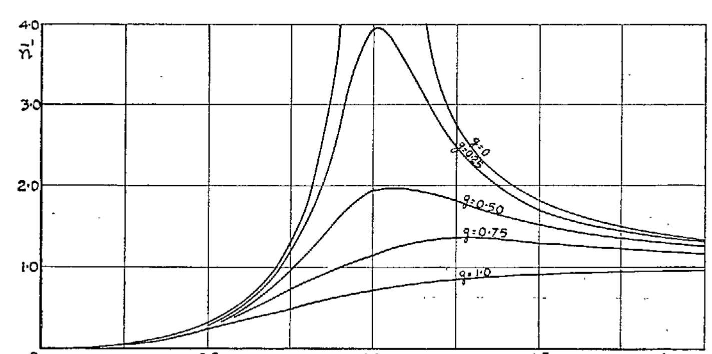
FIG. 5a to c. Phase delay angle and magnification factor for steady forced oscillations with hysteretic damping (standard solution).



(a) MAGNIFICATION FACTOR FOR CONSTANT EXCITATION.



(b) PHASE DELAY ANGLE FOR BOTH CONSTANT & CENTRIFUGAL EXCITATION.



(c) MAGNIFICATION FACTOR FOR CENTRIFUGAL EXCITATION.

FIG. 6a to c. Phase delay angle and magnification factor for steady forced oscillations with hysteretic damping after Myklestad.

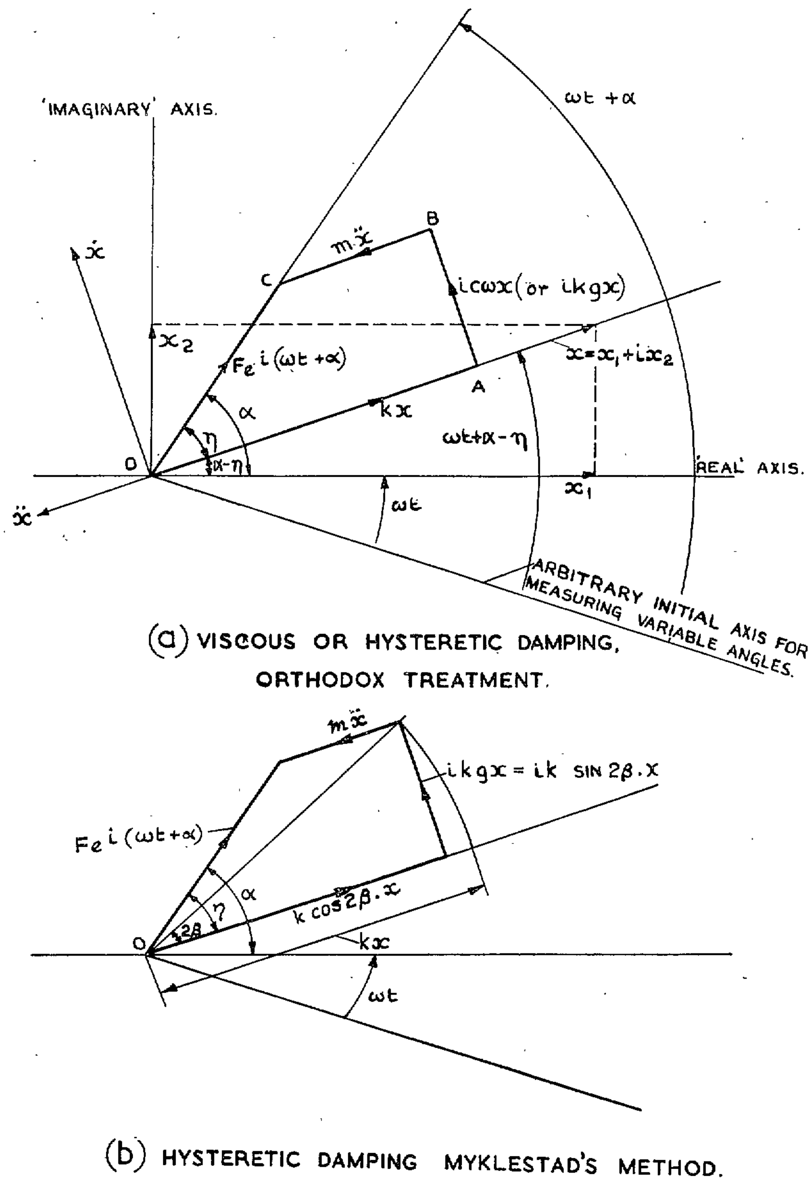


FIG. 7a and b. Vectorial representation of forced oscillations in single degree of freedom, with viscous or hysteretic damping.

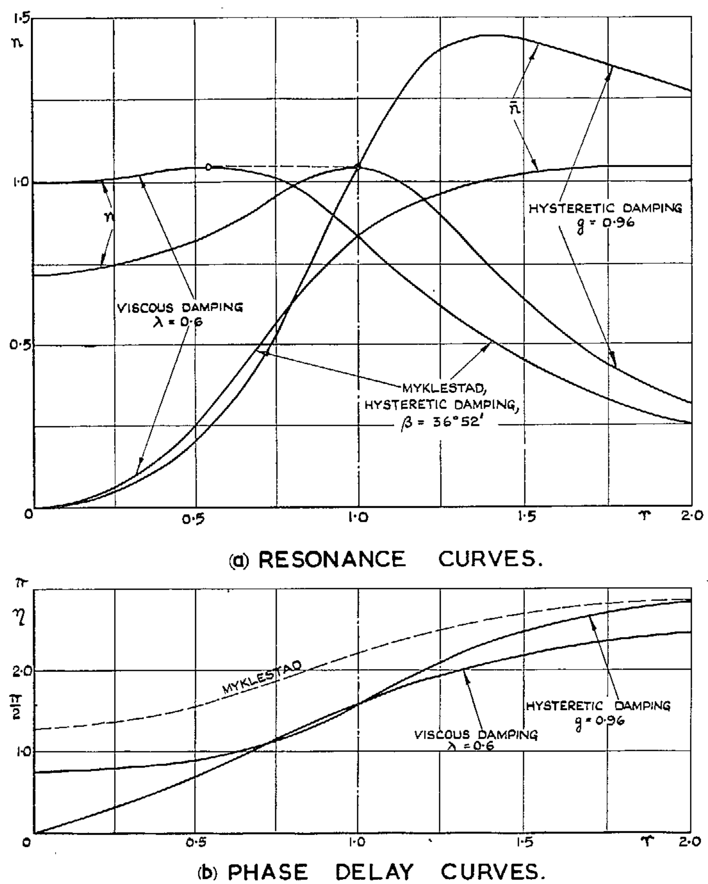


FIG. 8a and b. Example of resonance and phase delay curves at high damping, viscous or orthodox hysteretic or Myklestad's hysteretic, with equal peaks for constant amplitude of the exciting force.

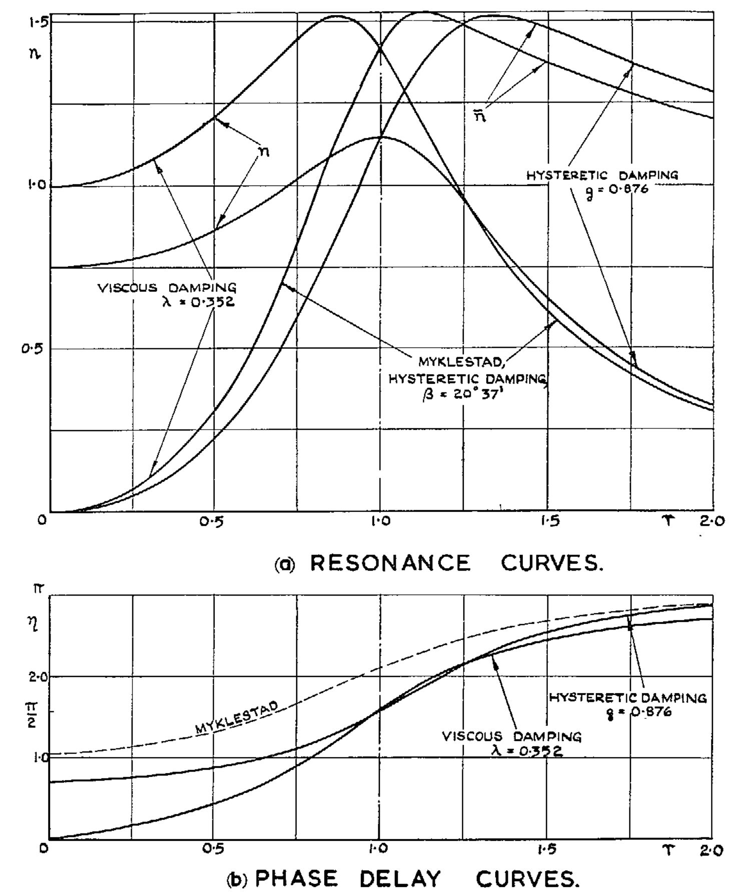


FIG. 9a and b. Example of resonance and phase delay curves at high damping, viscous or orthodox hysteretic or Myklestad's hysteretic, with equal peaks for variable amplitude of the exciting force.

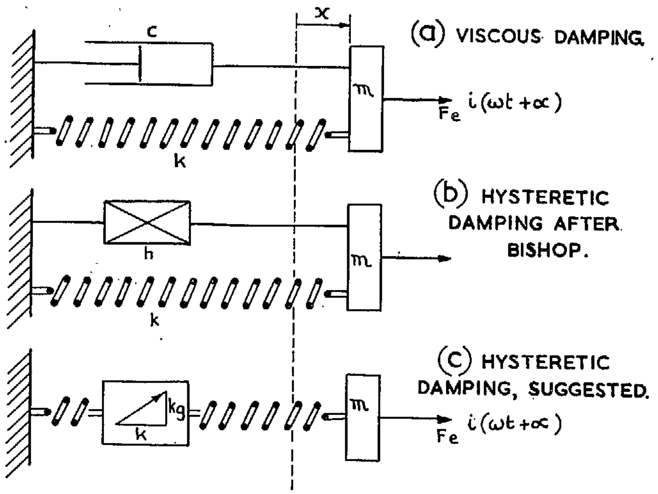


FIG. 10a, b and c. Diagrammatic representation of oscillators with viscous and hysteretic damping.

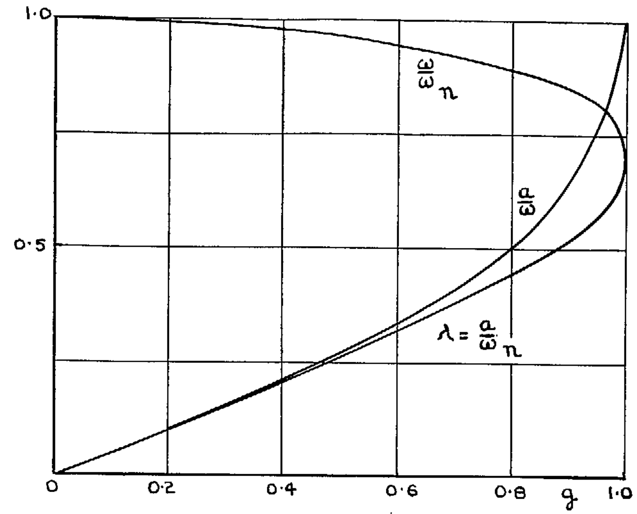


FIG. 12. Characteristics of free oscillation with hysteretic damping after Collar.

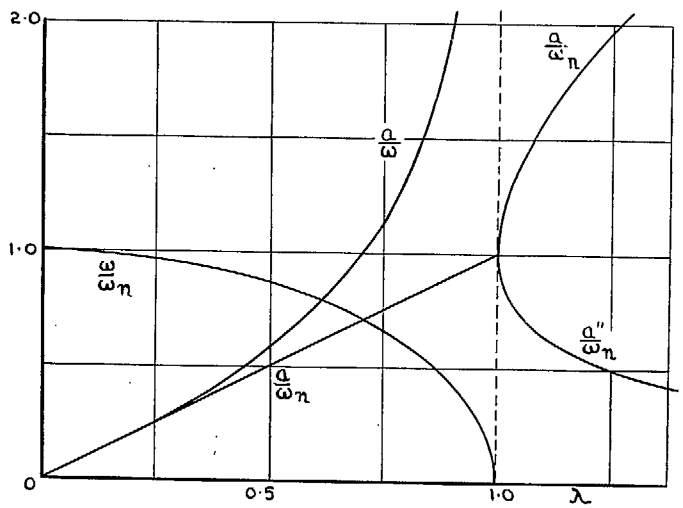


FIG. 11. Characteristics of free oscillation with viscous damping.

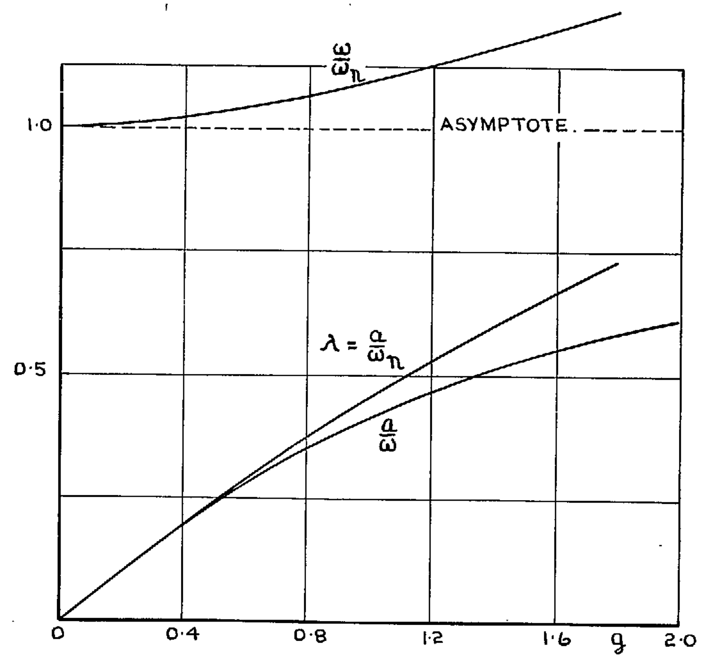


FIG. 13. Characteristics of free oscillation with hysteretic damping after Soroka.

(83916) Wt. 62/1876 K.5 6/62 Hw.

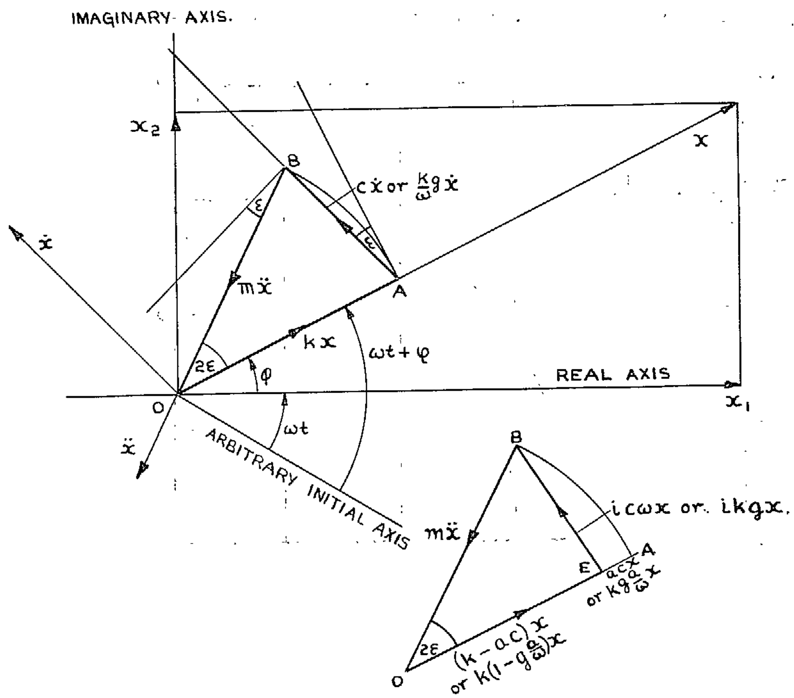


FIG. 14. Vectorial representation of free oscillations in single degree of freedom with viscous or hysteretic damping.

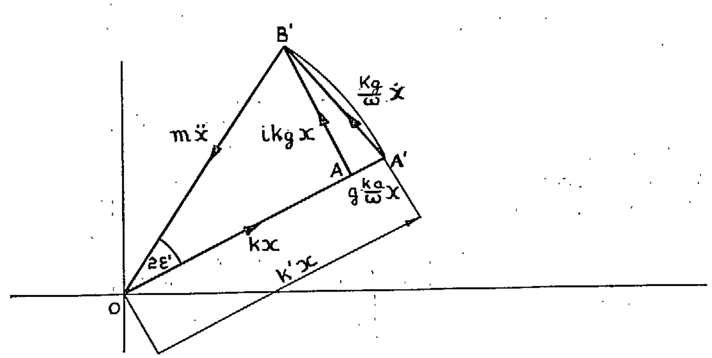


FIG. 15. Vectorial representation of Soroka's theory.

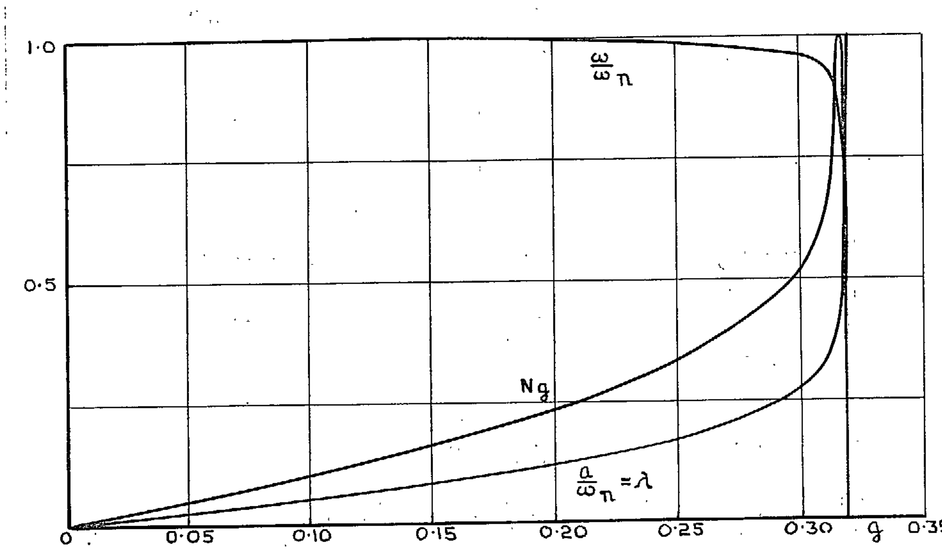


FIG. 16. Characteristics of free oscillation with hysteretic damping, method suggested in Section 4.2.4.

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