R. & M. No. 3249

MINISTRY OF AVIATION

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

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LONDON: HER MAJESTY'S STATIONERY OFFICE
1962
PRICE 15s. 0d. NET

Properties of a Two-Parameter Family of Thin Conically Cambered Delta Wings by Slender-Body Theory

By J. C. Cooke

Communicated by the Deputy Controller Aircraft (Research and Development), Ministry of Aviation

*Reports and Memoranda No. 3249**
July, 1960

Summary. Slender theory with exact boundary conditions is used to calculate the flow past a thin conically cambered delta wing. The (spanwise) camber-line consists of a straight central part with two drooping pieces at the sides, the points of junction being termed the shoulders. The shape is not given at the start but is the result of a series of conformal transformations. Attention is concentrated on flow which is attached at the leading edges ; incidence and lift and drag coefficients are worked out for this condition. It is found that, for a given lift coefficient, moving the shoulder outboard reduces the droop and droop angle required and also reduces the error introduced by neglecting second order terms. For low lift coefficients the lowest drag is obtained with shoulder position well outboard, but for high lift coefficients least drag is obtained when the shoulder position is along the centre-line and the camber-line reduces to a circular arc.

1. *Introduction.* The equations of motion for an inviscid fluid flowing past a body are usually linearised, that is to say the squares and products of the derivatives of the perturbation velocity potential are ignored. This is the first step in a simplification procedure, and may be called 'linearised potential theory' without any qualifying adjectives. A further simplification may be introduced by applying the boundary condition (the vanishing of the normal component of the velocity on the surface of the body) *not* on the body surface but on some simple surface near to the real one. In other words we may say that the vanishing normal velocity component at a point A on the surface is to apply not at A but at a suitably near point A′. There are of course many ways of doing this. Even when the second surface has been decided upon, the choice of the point A′ on it to correspond to A is still open.

A further simplification is 'slender' theory in which the potential equation is further simplified beyond the original linearisation, so as to become effectively two-dimensional. In this simplification, if Cartesian axes, x, y and z are taken, with x in the direction of the main stream, the velocity potential φ is supposed to satisfy the equation

$$\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0,$$

at each station x = constant.

* Previously issued as R.A.E. Tech. Note No. Aero. 2698—A.R.C. ~~22,327~~. 22,237

Here again there are two forms depending on the manner in which the boundary conditions are applied. If they are applied properly on the surface of the body the theory is called 'slender-body' theory. If the boundary conditions are applied on a plane in the sense explained above the theory is called 'slender-wing' theory. The name arises since it was first applied for wings; in this case the boundary condition was taken on a plane which did not depart very far from the wing surface. In slender-wing theory, second-order terms are ignored in calculating pressure and forces.

Smith[1] gave an account of the flow past a delta wing without thickness, cambered so as to form part of a circular cone, by slender-body theory. He showed that the higher order terms (which are neglected in slender-wing theory) could be large. Spanwise camber, applied in practical designs, differs noticeably from a circular arc. It was therefore thought worthwhile to study a more general type of camber (which may have a more practical application) by slender-body theory.

The characteristic appearance of this type of spanwise camber is shown in Fig. 1. The camber-line is straight near the centre-line of the wing, but droops down at the edges. Such camber-lines have been dealt with in Refs. 2 and 3, applying slender-wing theory.

In order to make satisfactory study by slender-body theory it is necessary to be able to find a conformal transformation which transforms the camber-line into a circle. For a general shape this cannot be done in closed form, and here we start with a circle and make a series of transformations which give a camber-line of the right general type; its actual shape comes out of the transformation used.

These transformations are shown in Fig. 3 and are due to Maskell. A two-parameter family is produced and any member may be specified by the position of the 'shoulder' B (*see* Fig. 1) and (a) the amount of droop or (b) the angle ϵ at A, which we shall call the 'droop angle'. The family includes the circular arc camber-line of Smith as a special case. ($c = 0$.) Some examples are shown in Figs. 4 and 5. This paper only deals with conically cambered thin wings for which there is a certain incidence at which the velocity is finite at the leading edges and the flow is likely to be attached, at least near the edges. This incidence is most closely studied here because it gives a theoretical flow which is most likely to apply for real fluids.

The theory can be extended to non-conical cases at the cost of additional complication.

2. *The Transformations.* We use a co-ordinate system as illustrated in Fig. 2; in a cross-flow plane and transformations of it we use the complex co-ordinates

$$Z_n = y_n + iz_n .$$

We start with the circle $|Z_0| = a \sec \delta$ in the Z_0-plane (*see* Fig. 3).

The transformation

$$Z_1 = Z_0 + ia \tan \delta \tag{1}$$

changes the origin, and then

$$Z_2 = Z_1 + \frac{a^2}{Z_1} \tag{2}$$

transforms the circle into part of a circular arc.

After another change of origin

$$Z_3 = Z_2 - 2ia \tan \delta , \tag{3}$$

we use the transformation

$$Z_4^2 = 4c^2 + Z_3^2 . \tag{4}$$

It will be noticed that the part of the imaginary axis $-2ic \leqslant Z_3 \leqslant 2ic$ is still on the imaginary axis in the Z_0 and Z_1 planes, but becomes part of the real axis

$$-2c \leqslant Z_4 \leqslant 2c$$

in the Z_4-plane.

The transformations leave the point at infinity unchanged.

By this means we have succeeded in transforming a circle with two straight pieces on the imaginary axis into a straight piece on the real axis, together with two 'droops' whose shape can be calculated. It is found that although these droops are tangential to the straight part, the curvature is not continuous at the join. In fact the droops join up to the straight part in half of a cusp, that is, its radius of curvature is zero at the join. This may be a disadvantage. It can be avoided by replacing the transformation (4) by

$$Z_4^4 = Z_3^4 + r'(iZ_3^3R + Z_3^2R^2) + k^4R^4, \tag{5}$$

where r' and k are real numbers, and R is the radius of the circle in the Z_3 plane. Shapes obtained by this transformation, with $r' = 1$ and for various values of k, are shown in Fig. 6. However, the analysis is made more complicated by this transformation and it will not be pursued further here. It should be noted that Smith's[1] case corresponds to $c = 0$, or in other words, to stopping at transformations (2) or (3).

If a point P on the circle is given by

$$Z_0 = a \sec \delta \, e^{i\lambda}, \qquad Z_1 = re^{i\theta},$$

then it can easily be shown that

$$r^2 - a^2 = 2ra \tan \delta \sin \theta, \tag{6}$$

$$r = a \tan \delta \sin \theta + a \sqrt{(1+\tan^2\delta \sin^2\theta)}, \tag{7}$$

and that

$$y_3 = 2a \cos \theta \sqrt{(1+\tan^2\delta \sin^2\theta)},$$

$$z_3 = -2a \tan \delta \cos^2\theta.$$

If we write

$$\sin \phi = \sin \delta \cos \theta \tag{8}$$

we have

$$y_3 = R \sin 2\phi,$$

$$z_3 = -R(1-\cos 2\phi),$$

where

$$R = 2a \operatorname{cosec} 2\delta.$$

Hence

$$Z_3 = 2R \sin \phi \, e^{-i\phi}, \tag{9}$$

and ϕ is the angle shown in Fig. 3, and R is the radius of the circle in the Z_3-plane.

3. *Properties of the Family of Cross-Section Shapes.* If the semi-span is s, and the 'shoulder' (the end of the straight part) is distance ns from the centre, and the amount of droop is Hs (*see* Fig. 1) there are simple relations between the various parameters.

One end of the section is the point A where $\theta = 0$, $\phi = \delta$, and so by Equation (9)

$$Z_3 = 2a \sec \delta \, e^{-i\delta}.$$

Hence at this point by Equation (4)

$$(y_4+iz_4)^2 = 4c^2 + 4a^2 \sec^2 \delta \, e^{-2i\delta}.$$

On separating real and imaginary parts we obtain

$$s^2(1-H^2) = 4c^2 + 4a^2 \sec^2\delta \cos 2\delta = 4c^2 + 4a^2(1-\tan^2\delta), \tag{10}$$

$$s^2 H = 2a^2 \sec^2\delta \sin 2\delta = 4a^2 \tan \delta. \tag{11}$$

The end of the straight part is at the point B where $Z_3 = 0$ and so $y_4 = 2c$ at this point. Hence

$$ns = 2c. \tag{12}$$

From Equations (10), (11) and (12) we find

$$\frac{1 - H^2 - n^2}{H} = \frac{1 - \tan^2\delta}{\tan \delta},$$

or

$$\cot 2\delta = \frac{1 - n^2 - H^2}{2H}.$$

Given n and H this determines δ, and then c/a may be found from the equation

$$\frac{c^2}{a^2} = \frac{n^2 \tan \delta}{H}.$$

In Tables 1 and 2 we give values of δ and $\bar{c} = c/a$ for various values of n and H.

TABLE 1

Values of δ

H \ n	0	0·2	0·4	0·6	0·7	0·8	0·9
0·1	0·0997	0·1037	0·1182	0·1537	0·1903	0·2596	0·4190
0·2	0·1974	0·2051	0·2318	0·2940	0·3526	0·4480	0·6060
0·3	0·2915	0·3019	0·3374	0·4144	0·4800	0·5740	0·7028
0·4	0·3805	0·3927	0·4332	0·5152	0·5792	0·6629	

TABLE 2

Values of $\bar{c} = c/a$

H \ n	0	0·2	0·4	0·6	0·7	0·8	0·9
0·1	0	0·2040	0·4356	0·7468	0·9716	1·6999	3·6076
0·2	0	0·2040	0·4344	0·7382	0·9496	1·5378	2·8065
0·3	0	0·2038	0·4327	0·7265	0·9221	1·3796	1·5123
0·4	0	0·2035	0·4303	0·7139	0·8951	1·1175	

It will be noticed that, for the lower values of n, $\bar{c}$ does not vary very much for fixed n.

4. *Preliminary Relations.* In order to simplify the account that follows, we collect here for reference certain formulae which will be required later.

We note first that if ds_0 is an element of the arc of the circle, then

$$ds_0 = a \sec \delta d\lambda, \tag{13}$$

and also

$$ds_0 = \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{1/2} d\theta$$

$$= \frac{r}{\cos \phi} d\theta, \tag{14}$$

as may be shown by direct use of Equations (7) and (8).

From Equations (4) and (9) we have

$$2Z_4 \left(\frac{dy_4}{d\phi} + i\frac{dz_4}{d\phi}\right) = 8R^2 \sin \phi \, e^{-3i\phi}$$

for points on the droops.

Hence, on the droops

$$\frac{dy_4}{d\phi} + i\frac{dz_4}{d\phi} = \frac{4R^2 \sin \phi}{Z_4 e^{3i\phi}}.$$

If we let

$$Z_4 e^{3i\phi} = e + if,$$

we find that, on the droops

$$\frac{dz_4}{dy_4} = -\frac{f}{e}. \tag{15}$$

If ds_4 is the element of arc on the droops we note that

$$\frac{ds_4}{d\phi} = \left|\frac{dy_4}{d\phi} + i\frac{dz_4}{d\phi}\right| = \frac{4R^2 \sin \phi}{|Z_4|}. \tag{16}$$

We note also that, on the droops

$$\left|\frac{dZ_2}{dZ_1}\right| = \left|1 - \frac{a^2}{Z_1{}^2}\right| = \left|1 - \frac{a^2}{r^2 e^{2i\theta}}\right| = \frac{|r^2 - a^2 e^{-2i\theta}|}{r^2}$$

$$= \frac{\sqrt{(r^4 - 2a^2 r^2 \cos 2\theta + a^4)}}{r^2}$$

$$= \frac{\sqrt{\{(r^2 - a^2)^2 + 4r^2 a^2 \sin^2\theta\}}}{r^2}$$

$$= \frac{2a \sin \theta}{r \cos \delta}, \tag{17}$$

using Equation (6).

In many cases we shall consider δ so small that we may ignore δ^4 compared with unity.

We then have

$$|Z_4|^2 = 4a^2 \left\{\bar{c}^4 + \frac{\cos^4\theta}{\cos^4\delta} + \frac{2\bar{c}^2\cos^2\theta}{\cos^2\delta}(1-2\sin^2\delta\cos^2\theta)\right\}^{1/2}$$

$$= 4a^2(\bar{c}^2+\cos^2\theta)\left\{1 + \frac{\delta^2(\cos^4\theta+\bar{c}^2\cos^2\theta-2\bar{c}^2\cos^4\theta)}{(\bar{c}^2+\cos^2\theta)^2}\right\}. \tag{18}$$

If $\mathscr{I}$ denotes 'the imaginary part of' we note that

$$\mathscr{I}(Z_4^2e^{3i\phi}) = \mathscr{I}\{e^{3i\phi}(4c^2+4R^2\sin^2\phi\, e^{-2i\phi})\}$$

$$= 4(c^2\sin 3\phi + R^2\sin^3\phi) \tag{19}$$

$$= 4a^2\left[\bar{c}^2(3\sin\phi - 4\sin^3\phi) + \frac{\sin^3\phi}{\sin^2\delta\cos^2\delta}\right]$$

$$= 4a^2\sin\delta\,[3\bar{c}^2\cos\theta + \cos^3\theta + \delta^2\cos^3\theta(1-4\bar{c}^2)]. \tag{20}$$

5. *The Normal Velocity on the Surface.* We take a conical surface as seen in Fig. 2. The axes are as shown with the origin O at the apex, Oxy is the plane through the flat part, and it is from this plane that the angle of incidence α will be measured. We differ here from Smith, who measured α from the plane through O and the leading edges. These edges are the lines $|y| = s = Kx$, $z = -Hs$. The wing being supposed slender, K is small.

We write

$$z_4 = z, \qquad y_4 = y,$$

and since the surface is a cone through the origin O its equation must be homogeneous in x, y, z and may be written

$$\frac{z}{Kx} = f\left(\frac{y}{Kx}\right).$$

We have

$$\frac{\partial z}{\partial x} = Kf\left(\frac{y}{Kx}\right) - \frac{y}{x}f'\left(\frac{y}{Kx}\right)$$

$$= K\frac{z_4}{Kx} - \frac{Ky_4}{Kx}\frac{dz_4}{dy_4}$$

$$= \frac{K}{se}(ez_4+fy_4),$$

using Equation (15), and putting $Kx = s$.

Hence

$$\frac{\partial z}{\partial x} = \frac{K}{se}\mathscr{I}\{(e+if)(y_4+iz_4)\}$$

$$= \frac{K}{se}\mathscr{I}(Z_4^2e^{3i\phi}).$$

We have also by Equation (15)

$$\frac{\partial z}{\partial y} = \frac{dz_4}{dy_4} = -\frac{f}{e}.$$

We split the free-stream velocity U into two components, $U \cos \alpha$ parallel to the plane $z = 0$ and $U \sin \alpha$ perpendicular to it. The velocity normal to the contour in the cross-flow plane due to the component $U \cos \alpha$ is on the 'droops' (*see* Ref. 4)

$$\frac{\partial \varphi}{\partial n} = v_{n\,4} = \frac{U \cos \alpha \dfrac{\partial z}{\partial x}}{\left\{1 + \left(\dfrac{\partial z}{\partial y}\right)^2\right\}^{1/2}}$$

$$= \frac{U \cos \alpha \dfrac{K}{se} \mathscr{I}(Z_4{}^2 e^{3i\phi})}{\sqrt{\{1 + (f/e)^2\}}}$$

$$= \frac{KU \cos \alpha \, \mathscr{I}(Z_4{}^2 e^{3i\phi})}{s\,|Z_4|}. \tag{21}$$

$v_{n\,4}$ is zero on the flat part of the section.

In order to find the velocity potential we must find the component $v_{n\,0}$ in the Z_0 plane which is related to $v_{n\,4}$ by the mapping ratio $|dZ_4/dZ_0|$.

Hence

$$v_{n\,0} = v_{n\,4} \left|\frac{dZ_4}{dZ_0}\right|$$

$$= v_{n\,4} \left|\frac{dZ_4}{dZ_3}\right| \left|\frac{dZ_3}{dZ_2}\right| \left|\frac{dZ_2}{dZ_1}\right| \left|\frac{dZ_1}{dZ_0}\right|$$

$$= \frac{KU \cos \alpha \, \mathscr{I}(Z_4{}^2 e^{3i\phi})}{s\,|Z_4|} \left|\frac{Z_3}{Z_4}\right| \frac{2a \sin \theta}{r \cos \delta}$$

$$= \frac{16KUa^2 \cos \alpha}{s \cos^2 \delta} \frac{(c^2 \sin 3\phi + R^2 \sin^3 \phi) \sin \theta \cos \theta}{|Z_4|^2 r}, \tag{22}$$

using Equations (4), (9), (17), (19) and (21).

The signs must be checked here. We adopt the convention that v_n represents the component of the velocity along the outward drawn normal. This being so v_n is positive on the upper surface and negative on the lower surface of the droops.

In going round the surface the signs of $\sin \theta$, $\cos \theta$, $\sin \phi$, $\cos \phi$ and $v_{n\,0}$ are as shown in Table 3. In the Table are shown also the signs of B' (*see* Equation (27) below).

TABLE 3

Signs of $\sin \theta$, $\cos \theta$, $\sin \phi$, $\cos \phi$, $v_{n\,0}$ *and* B'

Points in Fig. 3	$\sin \theta$	$\cos \theta$	$\sin \phi$	$\cos \phi$	$v_{n\,0}$	B'
A to B	+	+	+	+	+	+
D to E	+	−	−	+	+	+
E to F	−	−	−	+	−	−
H to A	−	+	+	+	−	−

This verifies that $v_{n\,0}$ takes the correct signs according to the conventions adopted. (We do not consider values of ϕ greater than 45 deg.)

6. *The Velocity Potential.* According to Weber[4] the complex velocity potential in the Z_0-plane required to produce a normal velocity $v_{n\,0}$ at the surface is found by means of a source of strength $2\pi a \sec \delta\, \overline{v}_{n\,0}$ at the centre of the circle together with a source distribution on the circumference of strength $2(v_{n\,0} - \overline{v}_{n\,0})$ per unit length, where $\overline{v}_{n\,0}$ is the mean velocity on the circumference, given by

$$\overline{v}_{n\,0} = \frac{1}{2\pi} \int_0^{2\pi} v_{n\,0}\, d\lambda .$$

Using the value of $d\lambda$ given by (13) and (14) we verify that $\overline{v}_{n\,0}$ vanishes, as expected for a thin wing. Hence the complex velocity potential is

$$\frac{1}{2\pi} \int 2v_{n\,0}' \log (Z_0 - Z_0')ds_0' , \qquad (= w_2, \text{ say}), \tag{23}$$

where a prime attached to any expression such as $v_{n\,0}'$ means that a prime must be attached to all variables in the formula for $v_{n\,0}'$. Z_0' is a point on the contour, but Z_0 is any point in the complex plane.

In order to obtain the complete formula for the velocity potential we must consider the other component $U \sin \alpha$ in the cross-flow plane, which must produce a flow with zero normal velocity on the surface. The complex velocity potential due to this is

$$- iU \sin \alpha \left(Z_0 - \frac{a^2 \sec^2 \delta}{Z_0} \right), \qquad (= w_1, \text{ say}). \tag{24}$$

Hence the velocity potential φ is given by

$$\varphi = Re(w_1 + w_2) + Ux \cos \alpha ,$$

the last term arising from the velocity component $U \cos \alpha$ in the x-direction. $v_{n\,0}'$ is given by Equation (22) with primes attached to all the variables.

From Equations (14), (22) and (23) we may write

$$w_2 = C \int_0^{2\pi} B' \log (Z_0 - Z_0')d\theta' , \tag{25}$$

where

$$C = \frac{16KUa^2 \cos \alpha}{\pi s \cos^2 \delta} , \tag{26}$$

$$B' = \frac{(c^2 \sin 3\phi' + R^2 \sin^3 \phi') \sin \theta' \cos \theta'}{\cos \phi' \, |Z_4'|^2} . \tag{27}$$

Denoting the real part of w_2 by φ_2 we have

$$\varphi_2 = C \int_0^{2\pi} B' \log |Z_0 - Z_0'| d\theta' .$$

If for the moment we denote the radius of the circle by d and we wish to find the value of φ_2 on the circle, we have

$$\varphi_2 = C \int_{-\pi/2}^{\pi/2} B' \log |y_0 + iz_0 - y_0' - iz_0'| \, |y_0 + iz_0 + y_0' - iz_0'| d\theta' .$$

from the symmetry about the z_0-axis. Hence, on the circle

$$\varphi_2 = \tfrac{1}{2}C\int_{-\pi/2}^{\pi/2} B' \log\,[(y_0-y_0')^2+(z_0-z_0')^2]\,[(y_0+y_0')^2+(z_0-z_0')^2]\,d\theta'$$

$$= \tfrac{1}{2}C\int_{-\pi/2}^{\pi/2} B' \log\,[2d^2-2y_0y_0'-2z_0z_0']\,[2d^2+2y_0y_0'-2z_0z_0']\,d\theta'$$

$$= \tfrac{1}{2}C\int_{-\pi/2}^{\pi/2} B' \log 4\{(d^2-z_0z_0')^2-y_0^2y_0'^2\}\,d\theta'.$$

On multiplying out and putting $y_0^2 = d^2 - z_0^2$, etc, we have

$$\varphi_2 = C\int_{-\pi/2}^{\pi/2} B' \log 2d\,|z_0-z_0'|\,d\theta'$$

on the surface of the section.

Since $z_0 - z_0' = z_1 - z_1'$ we have on the surface, taking note of the signs of B',

$$\varphi_2 = C\int_0^{\pi/2} B' \log\left|\frac{z_1-z_{1u}'}{z_1-z_{1l}'}\right| d\theta', \tag{28}$$

where z_{1u}' is the value of z_1' at a point specified by θ', and z_{1l}' is its value at the corresponding point on the lower surface specified by $-\theta'$.

7. *The Condition for Attached Flow at the Edges.* As Smith[1] points out the flow will be attached if the cross-flow velocity is non-singular at the leading edges, that is, if dw/dZ_4 is non-singular at the point $\theta = 0$, where $w = w_1 + w_2$.

We proceed then to find dw/dZ_4 at this point.

From Equations (24) and (25)

$$\frac{dw}{dZ_0} = C\int_0^{2\pi} \frac{B'd\theta'}{Z_0-Z_0'} - iU\sin\alpha\left(1+\frac{a^2\sec^2\delta}{Z_0^2}\right).$$

At the point A (*see* Fig. 3)

$$Z_0 = ae^{-i\delta}\sec\delta,\quad Z_0-Z_0' = Z_1-Z_1' = a-Z_1' = a-r'e^{i\theta'},$$

and so

$$\frac{dw}{dZ_0} = C\int_0^{\pi/2} B'\left\{\frac{1}{a-r'e^{i\theta'}}-\frac{1}{a-\mathbf{r}'e^{-i\theta'}}+\frac{1}{a-r'e^{i(\pi-\theta)}}\right.$$

$$\left.-\frac{1}{a-\mathbf{r}'e^{-i(\pi-\theta')}}\right\}d\theta' - iU\sin\alpha\,(1+e^{2i\delta}),$$

where $\mathbf{r}'$ is the value of r' when θ' is replaced by $-\theta'$. (We may refer here to the signs given in Table 3.)

Now by combining together the first and third, and the second and fourth terms, we find that

$$\frac{1}{a-r'e^{i\theta'}}-\frac{1}{a-\mathbf{r}'e^{-i\theta'}}+\frac{1}{a-r'e^{i(\pi-\theta)}}-\frac{1}{a-\mathbf{r}'e^{-i(\pi-\theta)}} = \frac{2ie^{i\delta}\cos\phi'}{a\sin\theta'},$$

using Equations (6), (7) and (8), in particular noting that

$$\left.\begin{aligned} r'-\mathbf{r}' &= 2a\tan\delta\sin\theta',\\ r'+\mathbf{r}' &= 2a\sec\delta\cos\phi',\\ r'\,\mathbf{r}' &= a^2. \end{aligned}\right\} \tag{29}$$

Hence we find

$$\frac{dw}{dZ_0} = 2U \cos \delta \, ie^{i\delta} \cos \alpha \left[\frac{16Ka}{s \cos^3 \delta} I_1 - \tan \alpha\right],$$

at $\theta = 0$, where

$$I_1 = \frac{1}{\pi} \int_0^{\pi/2} \frac{(c^2 \sin 3\phi' + R^2 \sin^3\phi') \cos \theta' d\theta'}{|Z_4'|^2}. \tag{30}$$

Now

$$\left|\frac{dw}{dZ_4}\right| = \left|\frac{dw}{dZ_0}\right| \left|\frac{dZ_0}{dZ_1}\right| \left|\frac{dZ_1}{dZ_2}\right| \left|\frac{dZ_2}{dZ_3}\right| \left|\frac{dZ_3}{dZ_4}\right|,$$

and $|dZ_1/dZ_2|$ is infinite at $\theta = 0$ as Equation (17) shows, whilst the other derivatives of the transformations are finite; hence for finite $|dw/dZ_4|$ we must have

$$\frac{dw}{dZ_0} = 0$$

at $\theta = 0$, and so the condition for attached flow becomes

$$\frac{\tan \alpha}{K} = \frac{16a}{s \cos^3\delta} I_1. \tag{31}$$

Since K is assumed small we must have α small and $\tan \alpha/K$ may be replaced by α/K.

It seems only possible to evaluate I_1 numerically in general, but if we suppose that δ is so small that δ^4 may be neglected compared with unity we find in Appendix II that

$$I_1 = \frac{\sin \delta}{16} J,$$

where

$$J = 1 + 4\bar{c}^2 - \frac{4\bar{c}^2}{h} - \delta^2 \left(6\bar{c}^2 - 8\bar{c}^4 - \frac{2\bar{c}^2}{h} + 8\frac{\bar{c}^4}{h} - \frac{4}{h^3} + \frac{3}{h^5}\right),$$

$$h^2 = 1 + \frac{1}{\bar{c}^2}, \qquad \bar{c} = c/a.$$

Hence we have

$$\frac{\tan \alpha}{K} = \frac{a \tan \delta}{s \cos^2\delta} J. \tag{32}$$

In Smith's case, where $\bar{c} = 0$, $s = 2a$, $Z_4 = Z_3$ we can show that

$$I_1 = \frac{\sin \delta}{16},$$

and putting this value in Equation (31) we find

$$\frac{\tan \alpha}{K} = \frac{\sin \delta}{2 \cos^3\delta} = \tfrac{1}{2} \tan \delta(1+\tan^2\delta). \tag{33}$$

Smith's[1] definition of α which we may call α_s differs from the present definition, which we call α_c, the relation between them being

$$\frac{\tan \alpha_s}{K} = \frac{\tan \alpha_c}{K} + \tan \delta. \tag{34}$$

When allowance is made for this, Equation (33) agrees with Smith's result.

8. *The Lift Coefficient.* Following Smith[1] we may write for the lift L the equation

$$L = -\rho U R\{\textstyle\int_C w\, dZ_4\}$$

taken round a contour which surrounds the wake. Since w is analytic outside the wake the contour may be replaced by a large circle. If we perform the integration in the Z_1-plane we have

$$L = -\rho U R\left\{\int w \frac{dZ_4}{dZ_3}\frac{dZ_3}{dZ_2}\frac{dZ_2}{dZ_1}\, dZ_1\right\}.$$

We expand the integrand in powers of $1/Z_1$ in the form

$$a_1 Z_1 + b_1 + a_{-1} Z_1^{-1} + \dots,$$

and we have

$$L = 2\pi\rho U \mathscr{I}(a_{-1}).$$

We have

$$\frac{dZ_4}{dZ_3} = \frac{Z_3}{Z_4} = 1 - \frac{2c^2}{Z_3^2} + \dots = 1 - \frac{2c^2}{Z_1^2} + \dots,$$

$$\frac{dZ_2}{dZ_1} = 1 - \frac{a^2}{Z_1^2}, \qquad \frac{dZ_3}{dZ_2} = 1,$$

$$w = -iU\sin\alpha\left(Z_1 - ia\tan\delta - \frac{a^2\sec^2\delta}{Z_1} + \dots\right) + C\int_0^{2\pi} B'\log(Z_1 - Z_1')\, d\theta'.$$

The last integral may be written

$$C\int_0^{\pi} B'\log\frac{Z_1 - Z_{1u}'}{Z_1 - Z_{1l}'}\, d\theta',$$

where Z_{1u}' is a point specified by θ' on the upper surface, and Z_{1l}' is the corresponding point specified by $-\theta'$ on the lower surface.

Now, expanding in powers of $1/Z_1$,

$$\log\frac{Z_1 - Z_{1u}'}{Z_1 - Z_{1l}'} = -\frac{Z_{1u} + Z_{1l}'}{Z_1} + \dots$$

$$= -\frac{r'e^{i\theta} + \mathbf{r}'e^{-i\theta'}}{Z_1} + \dots$$

$$= \frac{(\mathbf{r}' - r')\cos\theta' - i(\mathbf{r}' + r')\sin\theta'}{Z_1} + \dots .$$

Hence

$$a_{-1} = iU\sin\alpha[a^2\sec^2\delta + 2c^2 + a^2] +$$

$$+ C\int_0^{\pi} B'\{(\mathbf{r}' - r')\cos\theta' - i(\mathbf{r}' + r')\sin\theta'\}\, d\theta',$$

and so

$$L = 2\pi\rho U^2\sin\alpha[a^2(1 + \sec^2\delta) + 2c^2] -$$

$$- C\rho U\int_0^{\pi} B' 2\pi\sin\theta'(r' + \mathbf{r}')\, d\theta'.$$

Hence, using Equation (29) and putting in the values of A and B', Equations (26) and (27), we have

$$L = 2\pi\rho U^2 \sin\alpha\{a^2(2+\tan^2\delta) + 2c^2\} -$$

$$- 128\rho U^2 \frac{a^3}{s}\frac{K\cos\alpha}{\cos^3\delta}\int_0^{\pi/2}\frac{c^2\sin 3\phi' + R^2\sin^3\phi'}{|Z_4'|^2}\sin^2\theta'\cos\theta' d\theta',$$

and, noting that the projected area of the wing is s^2/K,

$$\frac{C_L}{\pi K^2} = 4\left(\frac{a}{s}\right)^2(2+\tan^2\delta+2\bar{c}^2)\frac{\sin\alpha}{K} - 256\left(\frac{a}{s}\right)^3\frac{\cos\alpha}{\cos^3\delta}(I_1-I_2),$$

where I_1 is defined in Equation (30), and

$$I_2 = \frac{1}{\pi}\int_0^{\pi/2}\frac{(c^2\sin 3\phi' + R^2\sin^3\phi')\cos^3\theta' d\theta'}{|Z_4'|^2}. \tag{35}$$

$I_1 - I_2$ is evaluated in Appendix II on the assumption of small δ as before, and the final result is

$$\frac{C_L}{\pi K^2} = 4\left(\frac{a}{s}\right)^2(2+2\bar{c}^2+\tan^2\delta)\frac{\sin\alpha}{K} -$$

$$- 4\left(\frac{a}{s}\right)^3\frac{\tan\delta}{\cos^2\delta}(M-\delta^2 N)\cos\alpha, \tag{36}$$

where

$$\left.\begin{aligned} M &= 1 + 8\bar{c}^2 + 16\bar{c}^4 - 16\frac{\bar{c}^2}{h} - 16\frac{\bar{c}^4}{h}, \\ N &= 2\left(5\bar{c}^2 + 8\bar{c}^4 - 40\bar{c}^6 - 14\frac{\bar{c}^2}{h} + 12\frac{\bar{c}^4}{h} + 40\frac{\bar{c}^6}{h} + \frac{2}{h^3}\right). \end{aligned}\right\} \tag{37}$$

We may show that in Smith's case

$$I_1 - I_2 = \frac{1}{64}\sin\delta,$$

and hence

$$\frac{C_L}{\pi K^2} = 2\left\{\frac{\sin\alpha}{K}\left(1+\frac{1}{2}\tan^2\delta\right) - \frac{1}{4}\cos\alpha\tan\delta(1+\tan^2\delta)\right\}.$$

This agrees with Smith's result when α is small and allowance is made for its different definition in the two cases. (Equation (34).)

If α is small we may write

$$\frac{C_L}{\pi K^2} = R'\left(\frac{\alpha}{K}\right) - S', \tag{38}$$

where

$$\left.\begin{aligned} R' &= 4\left(\frac{a}{s}\right)^2(2+2\bar{c}^2+\tan^2\delta), \\ S' &= 4\left(\frac{a}{s}\right)^3\frac{\tan\delta}{\cos^2\delta}(M-\delta^2 N). \end{aligned}\right\} \tag{39}$$

9. *The Drag Coefficient.* In order to find the drag we shall find the drag at zero incidence and then use a result of Ward.

At zero incidence the drag D is given by[1, 5]

$$D = -\tfrac{1}{2}\rho \int \varphi \frac{\partial\varphi}{\partial n}\, ds_4 = -\tfrac{1}{2}\rho \int (\varphi_u - \varphi_l) \left|\frac{\partial\varphi}{\partial n}\right| ds_4,$$

the first integral being taken right round the trailing edge and the second over the upper half.

We note that $\partial\varphi/\partial n = 0$ on the straight parts and so we need only evaluate the integral over the droops.

We take the value of φ_2 from Equation (28), noting that φ_1 is zero at zero incidence, $\partial\varphi/\partial n$ from Equations (21) and (19), and note that by Equations (8) and (16), (taking s_4 to increase as θ increases),

$$ds_4 = \frac{4R^2 \sin^2\delta \sin\theta \cos\theta}{|Z_4| \cos\phi}\, d\theta\,.$$

We also put $\alpha = 0$. Hence

$$D = -\tfrac{1}{2}\rho \int_0^{\pi} \frac{4KU(c^2 \sin 3\phi + R^2 \sin^3\phi)}{s\,|Z_4|} \cdot \frac{4R^2 \sin^2\delta \sin\theta\cos\theta}{|Z_4| \cos\phi}\, d\theta \times$$

$$\times\, C \int_0^{\pi/2} B' \log \left|\frac{z_{1u} - z_{1u}'}{z_{1u} - z_{1l}'}\right| \left|\frac{z_{1l} - z_{1l}'}{z_{1l} - z_{1u}'}\right| d\theta'$$

$$= -\frac{128K^2U^2a^4\rho}{\pi s^2 \cos^4\delta} \int_0^{\pi} B d\theta \int_0^{\pi/2} B' \log \left|\frac{z_{1u} - z_{1u}'}{z_{1u} - z_{1l}'}\right| \left|\frac{z_{1l} - z_{1l}'}{z_{1l} - z_{1u}'}\right| d\theta'\,. \tag{40}$$

The integral may be simplified by the substitution

$$\sin\psi = \frac{\tan\phi}{\tan\delta}.$$

We show in Appendix III that the log term then becomes

$$2 \log \left|\frac{\cos\psi - \cos\psi'}{\cos\psi + \cos\psi'}\right|.$$

We show also that if δ^4 is neglected compared with unity

$$B d\theta = -\frac{1}{4} \tan\delta \cos^2\delta\, g(\psi) d\psi,$$

where

$$g(\psi) = \frac{(3\bar{c}^2 + \sin^2\psi)\sin^2\psi\cos\psi}{\bar{c}^2 + \sin^2\psi} +$$

$$+ \frac{\delta^2 \sin^2\psi \cos\psi}{\bar{c}^2 + \sin^2\psi}\Bigg[6\bar{c}^2 - 10\bar{c}^2 \sin^2\psi + 4\sin^2\psi -$$

$$- 3\sin^4\psi - \frac{\sin^2\psi(3\bar{c}^2 + \sin^2\psi)}{(\bar{c}^2 + \sin^2\psi)^2} \times$$

$$\times (2\bar{c}^2 - 3\bar{c}^2 \sin^2\psi + 2\sin^2\psi - \sin^4\psi)\Bigg]. \tag{41}$$

Hence

$$D = -\frac{16K^2\rho U^2 a^4 \sin^2\delta}{\pi s^2 \cos^2\delta}\int_0^{\pi} g(\psi)d\psi \int_0^{\pi/2} g(\psi')\log\left|\frac{\cos\psi - \cos\psi'}{\cos\psi + \cos\psi'}\right| d\psi'$$

$$= -\frac{16K^2\rho U^2 a^4 \tan^2\delta}{\pi s^2}\int_0^{\pi} g(\psi)d\psi \int_0^{\pi} g(\psi')\log|\cos\psi - \cos\psi'|\,d\psi',$$

since $g(\pi-\psi') = -g(\psi')$.

Hence the drag coefficient for $\alpha = 0$, denoted by $C_D(0)$, is given by

$$\frac{C_D(0)}{\pi K^3} = -32\tan^2\delta\left(\frac{a}{s}\right)^4 I, \tag{42}$$

where

$$I = \frac{1}{\pi^2}\int_0^{\pi} g(\psi)d\psi \int_0^{\pi} g(\psi')\log|\cos\psi - \cos\psi'|\,d\psi'. \tag{43}$$

A programme for the numerical evaluation of this integral by the automatic computer DEUCE at the Royal Aircraft Establishment has been devised by J. A. Beasley (using an approximate formula for integrals of the form

$$\int_{-1}^{1}\int_{-1}^{1} f(\eta)f(\eta')\log|\eta - \eta'|\,d\eta d\eta'$$

derived in Ref. 6).

This gives the drag coefficient at $\alpha = 0$. To find the drag at any other incidence we use Ward's theorem for slender bodies. If α_z is the value of α at zero lift, according to Ward[5] we have

$$C_D = C_{D0} + \frac{1}{2}\left(\frac{dC_L}{d\alpha}\right)_{\alpha=\alpha_z}(\alpha-\alpha_z)^2,$$

where C_{D0} is the drag at zero lift.

Hence, putting $\alpha = 0$, we have

$$C_D(0) = C_{D0} + \frac{1}{2}\left(\frac{dC_L}{d\alpha}\right)_{\alpha=\alpha_z}\alpha_z^2,$$

and so by subtraction

$$C_D = C_D(0) + \frac{1}{2}\pi K R'(\alpha^2 - 2\alpha\alpha_z),$$

using Equation (38). Using this equation again we have

$$\frac{C_D}{\pi K^3} = \frac{C_D(0)}{\pi K^3} + \frac{1}{2}R'\left(\frac{\alpha}{K}\right)^2 - S'\left(\frac{\alpha}{K}\right). \tag{44}$$

10. *The First Order Approximation.* The theory outlined above is exact within the limitations of slender-body theory. In slender-wing theory the procedure is equivalent to neglecting second- and higher-order terms in some parameter depending on the slope of the wing surface. Thus in the special case $c = 0$ Smith[1] neglected $\tan^2\delta (= \beta^2)$ compared with unity. We shall do the same here.

In order to obtain the first-order solution from Equations (32), (36) and (42) it is necessary to replace $\cos\delta$ by 1 and to drop the δ^2 terms in J, R', S' and $g(\psi)$. Some of the results of the first-order theory are shown in Figs. 12 to 17.

Examination of Equations (32), (36), (39), (41), (42) and (43) shows that the first-order solution has an error of order β^3 and not of order β^2. Therefore the first-order solution is in fact a second-order solution. This explains why in Figs. 12 to 17 the exact and first-order solutions are so close even when β is only moderately small.

The procedure applied in this Note to derive the first-order solution differs somewhat from the 'usual' method of slender-wing theory, as applied for example in Refs. 2 and 3. There the boundary condition of slender-body theory, namely

$$\frac{\partial\varphi(x, y, z)}{\partial n} = \frac{U \cos\alpha \dfrac{\partial z(x, y)}{\partial x}}{\sqrt{\left\{1+\left(\dfrac{\partial z}{\partial y}\right)^2\right\}}}, \tag{45}$$

is replaced by the condition

$$\frac{\partial\varphi(x, y, z=0)}{\partial z} = U \cos\alpha \frac{\partial z(x, y)}{\partial x}. \tag{46}$$

This is equivalent, firstly, to approximating the velocity component normal to the surface, $\partial\varphi/\partial n$, by the velocity component normal to a chordal plane, $\partial\varphi(x, y, z)/\partial z$. In other words squares of the spanwise surface slope are ignored compared with unity. Secondly the velocity component $\partial\varphi(x, y, z)/\partial z$ at the surface is approximated by the normal velocity in the median plane $\partial\varphi(x, y, z = 0)/\partial z$. (*See* Fig. 19.) We do not know whether this involves in all cases an error of the second order rather than the first order compared with unity although in the case of the circular arc ($n = 0$) the procedure does yield such an error in the calculated values of angle of incidence and lift and drag coefficients for attached flow, as we shall now show.

To investigate the difference between the 'usual' slender-wing theory and the exact and the first-order slender-body theories we have approximated the circular cone

$$\frac{z(x, y)}{Kx} = -\frac{1-\beta^2}{2\beta} + \frac{1+\beta^2}{2\beta}\left\{1 - \left[\frac{2y\beta}{Kx(1+\beta^2)}\right]^2\right\}^{1/2}$$

by

$$\frac{z(x, y)}{Kx} = \beta\left(1 - \frac{y^2}{K^2x^2}\right) + \beta^3\left(\frac{y^2}{K^2x^2} - \frac{y^4}{K^4x^4}\right) + O(\beta^5). \tag{47}$$

To the cone given by Equation (47) we have applied the 'usual' slender-wing theory as described above, determined the potential function and derived from this the incidence for attached flow and the lift and drag coefficients. The results are given in Table 4.

TABLE 4

Values for Different Approximation Methods. $n = 0$

	Exact	First order	'Usual method'
α/K	$\frac{1}{2}\beta(1+\beta^2)$	$\frac{1}{2}\beta$	$\frac{1}{2}\beta\left(1+\frac{5}{12}\beta^2\right) + O(\beta^5)$
$C_L/\pi K^2$	$\frac{1}{2}\beta(1+2\beta^2+\beta^4)$	$\frac{1}{2}\beta$	$\frac{1}{2}\beta\left(1+2\beta^2\right) + O(\beta^5)$
κ	$\frac{4}{3}\left(1-\frac{3}{4}\beta^2\right) + O(\beta^4)$	$\frac{4}{3}$	$\frac{4}{3}\left(1-\frac{1}{2}\beta^2\right) + O(\beta^4)$

We notice that in this case the 'usual' slender-wing theory is correct to the second order and not merely to the first order. To keep, in a slender-wing calculation, terms of order β^3 is of course permissible, but although it here leads to closer agreement with the exact results than ignoring the β^3 terms, this need not be true in general.

11. *Results.* As can be seen in Table 2, $\bar{c}$ does not vary very much for fixed n. In order to simplify the calculations $\bar{c}$ was considered fixed, the values chosen being $\bar{c} = 0$, $0 \cdot 433$, $0 \cdot 730$ and $1 \cdot 071$. These give nominal values of n equal to 0, 0·4, 0·6, and 0·75 respectively. A few calculations were also done for $\bar{c} = 0 \cdot 2$. At the value $\bar{c} = 1 \cdot 071$, n varies more than at lower values of $\bar{c}$; in fact the values of n for $H = 0 \cdot 1$, $0 \cdot 2$ and $0 \cdot 3$ are 0·74, 0·75 and 0·76 respectively.

The results in Figs. 7 and 8 show that the farther outboard the shoulder is taken the less the amount of droop and droop angle required to secure a given lift coefficient with attached flow.

In the family considered by Brebner[2] drag can be reduced by moving the shoulder position outboard. Fig. 11 shows that this not necessarily true for the camber-lines used here, though it is true outboard of the humps in Fig. 11. Much depends on the lift coefficient desired. If this is low then increasing the value of n beyond about 0·6 gives great improvement in the value of κ. If, however, the value of $C_L/\pi K^2$ is to be greater than 0·5 then the best value of n is zero, that is to say the circular arc camber gives the least value of κ. Indeed the value is less than unity, whilst even $n = 1$ only gives κ equal to unity. For a value of $C_L/\pi K^2$ equal to 0·3, which may be a typical value in practice, values of n greater than 0·7 give the least drag.

The shape of the curves in Fig. 11 is somewhat surprising, as each one rises to a maximum for some value of n. This general trend was checked by performing independent calculations for $C_L/\pi K^2 = 0$, which may be done exactly.

Figs. 12 to 17 show that the error made by applying first-order theory when designing for a given lift coefficient is smaller the farther out the shoulder position. For a given lift coefficient it can be shown that moving the shoulder outboard reduces the value of $\beta(= \tan \delta)$ and, since we expect better agreement with smaller values of β, this improvement is to be expected.

To fix ideas let us suppose we are designing for attached flow at a C_L of 0·1 for a wing of 72 deg sweep. In such a case we have $C_L/\pi K^2 = 0 \cdot 3$. This value would also apply for a wing with a C_L of 0·05 and a sweep of 77 deg. We will consider the cases $n = 0$, $0 \cdot 6$ and $0 \cdot 75$. For these, some results are given in Table 5.

TABLE 5

Three Wings with $C_L/\pi K^2 = 0 \cdot 3$, $K = \tan 18$ *deg*, $C_L = 0 \cdot 1$

	$n = 0$ (Circular arc)	$n = 0 \cdot 6$	$n = 0 \cdot 75$
Droop/semi-span	0·44	0·18	0·12
Angle of droop ϵ	47 deg.	38 deg.	35 deg.
Angle of incidence α_E	4·8 deg.	4·1 deg.	3·6 deg.
$C_{D\,E}$	0·0028	0·0030	0·0028
κ_E	1·15	1·23	1·16
$\Delta\alpha/\alpha_E$	0·16	0·04	0·02
$\Delta C_L/C_{L\,E}$	0·09	0·02	0·01
$\Delta C_D/C_{D\,E}$	0·36	0·10	0·06
$\Delta\kappa/\kappa_E$	−0·16	−0·03	−0·02

In the last four items Δ is to be interpreted in the sense 'exact minus first order'. The table shows that the errors are greatly reduced by moving the shoulder position outboard, and are very small when $n = 0 \cdot 75$.

These results apply on the assumption that the flow is attached at the leading edges and remains attached over the surface.

12. *Conclusions.* It is possible by a series of simple conformal transformations, starting with a circle, to generate a camber-line likely to be used in practice. It is thus possible, using slender-body theory, to apply the boundary conditions on the surface exactly, instead of using some method of approximation which is usually found necessary. The camber-line is straight in the middle and droops at the sides, outboard of a 'shoulder' position.

It is found that moving the shoulder position outboard reduces the amount of droop and the droop angle required for attached flow at a given lift coefficient. It also reduces the error in first-order theory.

The effect on drag of moving the shoulder position is not so simple. Outboard movement first causes a rise in drag but a maximum is reached and the drag goes down again. For high lift coefficients the value of the lift-dependent drag factor κ may be less than unity when the shoulder position is near the centre-line but it increases to a maximum and then goes down, approaching the limit unity. For low lift coefficients κ starts much higher, but still rises at first, finally tending to the limit unity as before.

LIST OF SYMBOLS

a	OA in Z_0-plane, Fig. 3
A	Aspect ratio
B'	Defined by Equation (27)
C	Defined by Equation (26)
c	CB in Z_4-plane, Fig. 3
$\bar{c}$	c/a
C_L	Lift coefficient
C_D	Drag coefficient
D	Drag
e, f	Defined by $Z_4 e^{3i\phi} = e + if$
G	Defined in Equations (48)
h	Given by $h^2 = 1 + (1/\bar{c}^2)$
H	Defined in Fig. 1
I_1	Defined by Equation (30)
I_2	Defined by Equation (35)
$I(m, n)$	Integrals occurring in Appendices I and II
J	Defined just before Equation (32)
k	Parameter in Equation (5)
K	= Cot (angle of sweep)
L	Lift
M, N	Defined in Equation (37)
n	Defined in Fig. 1
R	Radius of circle in Z_3-plane in Fig. 3 ($= a/\sin\delta\cos\delta$)
r	OP in Z_1-plane, Fig. 3
$\mathbf{r}$	OP′ in Z_1-plane, Fig. 3
r'	Parameter in Equation (5)
R'	Defined in Equation (39)
S'	Defined in Equation (39)

s		Semi-span
U		Velocity at infinity
v_n		Velocity component normal to surface in cross-flow plane
w		Complex velocity potential
x, y, z		Cartesian co-ordinates
Z_n	=	$y_n + iz_n$
α		Angle of incidence
α_z		Angle of incidence at zero lift
β	=	Tan δ
δ		Half the angle AO′B in Z_2-plane, Fig. 3
Δ		Difference; exact minus first order
ϵ		Droop angle as defined in Fig. 1
θ		Angle POA in Z_1-plane, Fig. 3
κ		Lift-dependent drag factor $= \pi A C_D / C_L^2$
λ		Angle PO′y_0 in Z_0-plane, Fig. 3
ρ		Density
ϕ		Half angle POB in Z_2-plane, Fig. 3
φ		Velocity potential
$\dfrac{\partial\varphi}{\partial n}$		Normal component of velocity in cross-flow plane
ψ		Defined by $\sin \psi = \tan \phi / \tan \delta$

Suffices

u	Refers to the upper surface
l	Refers to the lower surface
E	Refers to the exact theory
F	Refers to first-order theory

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APPENDIX I

Values of Certain Integrals

$$I(0, 1) = \int_0^{\pi/2} \frac{d\theta}{c^2 + \cos^2\theta} = \int_0^{\infty} \frac{dt}{c^2(1+t^2)+1}$$

$$= \frac{1}{c^2} \int_0^{\infty} \frac{dt}{t' + h^2} = \frac{1}{c^2 h} \int_0^{\pi/2} d\phi = \frac{\pi}{2c^2 h},$$

putting $t = \tan\theta$, $h^2 = 1 + c^{-2}$, $t = h \tan\phi$.

$$I(2, 1) = \int_0^{\pi/2} \frac{\cos^2\theta d\theta}{c^2 + \cos^2\theta} = \int_0^{\pi/2} d\theta - c^2 I(0, 1) = \frac{\pi}{2}\left(1 - \frac{1}{h}\right).$$

$$I(4, 1) = \int_0^{\pi/2} \frac{\cos^4\theta d\theta}{c^2 + \cos^2\theta} = \int_0^{\pi/2} \cos^2\theta d\theta - c^2 I(2, 1) = \frac{\pi}{4}\left[1 - 2c^2 + \frac{2c^2}{h}\right].$$

$$I(6, 1) = \int_0^{\pi/2} \frac{\cos^6\theta d\theta}{c^2 + \cos^2\theta} = \int_0^{\pi/2} \cos^4\theta d\theta - c^2 I(4, 1)$$

$$= \pi\left(\frac{3}{16} - \frac{c^2}{4} + \frac{c^4}{2} - \frac{c^4}{2h}\right).$$

$$I(2, 3) = \int_0^{\pi/2} \frac{\cos^2\theta d\theta}{(c^2 + \cos^2\theta)^3} = \frac{1}{c^6} \int_0^{\infty} \frac{1 + t^2}{(h^2 + t^2)^3} dt$$

$$= \frac{1}{h^5 c^6} \int_0^{\pi/2} (\cos^4\phi + h^2 \sin^2\phi \cos^2\phi) d\phi$$

$$= \frac{\pi}{16 h^5 c^6}(3 + h^2).$$

$$I(4, 3) = \int_0^{\pi/2} \frac{\cos^4\theta d\theta}{(c^2 + \cos^2\theta)^3} = \int_0^{\infty} \frac{dt}{c^6(t^2 + h^2)^3}$$

$$= \frac{1}{h^5 c^6} \int_0^{\pi/2} \cos^4\phi d\phi = \frac{3\pi}{16 h^5 c^6}.$$

$$I(6, 3) = I(2, 1) - 2c^2 I(4, 3) - c^4 I(2, 3)$$

$$= \pi\left(\frac{1}{2} - \frac{15}{16h} + \frac{5}{8h^3} - \frac{3}{16h^5}\right).$$

$$I(8, 3) = I(4, 1) - 2c^2 I(6, 3) - c^4 I(4, 3)$$

$$= \pi\left(\frac{1}{4} - \frac{3c^2}{2} + \frac{3c^2}{2h} + \frac{11}{16h^3} - \frac{3}{16h^5}\right).$$

$$I(10, 3) = I(6, 1) - 2c^2 I(8, 3) - c^4 I(6, 3)$$

$$= \pi\left(\frac{3}{16} - \frac{3c^2}{4} + 3c^4 - \frac{3c^2}{4h} - 3\frac{c^4}{h} + \frac{3}{4h^3} - \frac{3}{16h^5}\right).$$

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APPENDIX II

The Values of I_1 and I_2

$$I_1 = \frac{1}{\pi}\int_0^{\pi/2} \frac{(c^2 \sin 3\phi' + R^2 \sin^3 \phi') \cos \theta' d\theta'}{|Z_4'|^2}.$$

By Equations (18), (19) and (20), keeping terms in δ^2 in the integrand, we have, dropping the primes,

$$I_1 = \frac{\sin\delta}{4\pi}\int_0^{\pi/2} \frac{3\bar{c}^2\cos^2\theta + \cos^4\theta + \delta^2(1-4\bar{c}^2)\cos^4\theta}{\bar{c}^2+\cos^2\theta} \times$$

$$\times \left\{1 - \frac{\delta^2(\cos^4\theta + \bar{c}^2\cos^2\theta - 2\bar{c}^2\cos^4\theta)}{(\bar{c}^2+\cos^2\theta)^2}\right\} d\theta$$

$$= \frac{\sin\delta}{4\pi}\int_0^{\pi/2} \left\{\frac{3\bar{c}^2\cos^2\theta + \cos^4\theta + \delta^2(1-4\bar{c}^2)\cos^4\theta}{\bar{c}^2+\cos^2\theta} - \right.$$

$$\left. - \delta^2 \frac{3\bar{c}^4\cos^4\theta + (4\bar{c}^2 - 6\bar{c}^4)\cos^6\theta + (1-2\bar{c}^2)\cos^8\theta}{(\bar{c}^2+\cos^2\theta)^3}\right\} d\theta$$

$$= \frac{\sin\delta}{4\pi}\left[3\bar{c}^2 I(2, 1) + I(4, 1) + \delta^2\left\{(1-4\bar{c}^2)I(4, 1) - 3\bar{c}^4 I(4, 3) - \right.\right.$$

$$\left.\left. - (4\bar{c}^2 - 6\bar{c}^4)I(6, 3) - (1-2\bar{c}^2)I(8, 3)\right\}\right]$$

$$= \frac{\sin\delta}{16}\left[1 + 4\bar{c}^2 - \frac{4\bar{c}^2}{h} - \delta^2\left(6\bar{c}^2 - 8\bar{c}^4 - 2\frac{\bar{c}^2}{h} + 8\frac{\bar{c}^4}{h} - \frac{4}{h^3} + \frac{3}{h^5}\right)\right],$$

using the results of Appendix I.

I_2 is the same as I_1 except that the integrand is multiplied throughout by $\cos^2\theta$.

Hence we find

$$I_2 = \frac{\sin\delta}{4}\left[\frac{3}{16} + \frac{\bar{c}^2}{2} - \bar{c}^4 + \frac{\bar{c}^4}{h} - \right.$$

$$\left. - \delta^2\left(\frac{7}{8}\bar{c}^2 - 3\bar{c}^4 + 5\bar{c}^6 + \frac{5}{4}\frac{\bar{c}^2}{h} + \frac{1}{2}\frac{\bar{c}^4}{h} - \frac{5\bar{c}^6}{h} - \frac{5}{4h^3} + \frac{3}{4h^5}\right)\right].$$

On subtraction we have

$$I_1 - I_2 = \frac{\sin\delta}{64}\left[1 + 8\bar{c}^2 + 16\bar{c}^4 - 16\frac{\bar{c}^2}{h} - 16\frac{\bar{c}^4}{h} - \right.$$

$$\left. - 2\delta^2\left(5\bar{c}^2 + 8\bar{c}^4 - 40\bar{c}^6 - 14\frac{\bar{c}^2}{h} + 12\frac{\bar{c}^4}{h} + 40\frac{\bar{c}^6}{h} + \frac{2}{h^3}\right)\right].$$

APPENDIX III

The Transformation to ψ

Given

$$\sin\psi = \frac{\tan\phi}{\tan\delta}, \qquad \sin\phi = \sin\delta\cos\theta,$$

we have

$$\left.\begin{aligned} \cos\theta &= \frac{\sin\psi}{G}, & \sin\theta &= \frac{\cos\psi\cos\delta}{G}, \\ \cos\phi &= \frac{\cos\delta}{G}, & \sin\phi &= \frac{\sin\psi\sin\delta}{G}, \\ & G^2 = 1-\cos^2\psi\sin^2\delta, \end{aligned}\right\} \tag{48}$$

as may be verified.

We also have

$$r = a\tan\delta\sin\theta + a\sqrt{(1+\tan^2\delta\sin^2\theta)}$$

$$= \frac{a}{G}(1+\sin\delta\cos\psi),$$

$$\mathbf{r} = \frac{a}{G}(1-\sin\delta\cos\psi),$$

$$r\mathbf{r} = a^2.$$

We also have

$$d\theta = -\frac{\cos^3\phi\cos\psi}{\cos\delta\sin\theta}\,d\psi.$$

In Equation (40) the value of the log term is

$$\log\left|\frac{r\sin\theta - r'\sin\theta'}{r\sin\theta + \mathbf{r}'\sin\theta'}\right|\left|\frac{-\mathbf{r}\sin\theta + \mathbf{r}'\sin\theta'}{-\mathbf{r}\sin\theta - r'\sin\theta'}\right|$$

$$= \log\left|\frac{a^2\sin^2\theta + a^2\sin^2\theta' - (r\mathbf{r}'+\mathbf{r}r')\sin\theta\sin\theta'}{a^2\sin^2\theta + a^2\sin^2\theta' + (rr'+\mathbf{r}\mathbf{r}')\sin\theta\sin\theta'}\right|,$$

since $r\mathbf{r} = a^2$.

Hence the value of the term is

$$\log\left|\frac{\dfrac{\cos^2\psi}{G^2}+\dfrac{\cos^2\psi'}{G'^2}-\dfrac{2-2\sin^2\delta\cos\psi\cos\psi'}{G^2G'^2}\cos\psi\cos\psi'}{\dfrac{\cos^2\psi}{G^2}+\dfrac{\cos^2\psi'}{G'^2}+\dfrac{2+2\sin^2\delta\cos\psi\cos\psi'}{G^2G'^2}\cos\psi\cos\psi'}\right|$$

$$= \log\frac{(\cos\psi-\cos\psi')^2}{(\cos\psi+\cos\psi')^2}$$

$$= 2\log\left|\frac{\cos\psi-\cos\psi'}{\cos\psi+\cos\psi'}\right|.$$

We may show also that

$$Bd\theta = -\frac{(c^2 \sin 3\phi + R^2 \sin^3 \phi) \cos^2 \phi \cos \theta \cos \psi \, d\psi}{\cos \delta \, |Z_4|^2}$$

$$= -\frac{a^2 \tan \delta \cos^2 \delta \sin^2 \psi \cos \psi \{3\bar{c}^2 + \sin^2 \psi + \delta^2(-3\bar{c}^2 - \bar{c}^2 \sin^2 \psi + \sin^2 \psi)\}}{G^6 \, |Z_4|^2} d\psi .$$

We have

$$\frac{1}{G^6} = 1 + 3\delta^2 \cos^2 \psi ,$$

and by Equations (18) and (48) we have

$$\frac{1}{|Z_4|^2} = \frac{1}{4a^2(c^2 + \sin^2 \psi)} \left\{ 1 - \frac{\delta^2 \sin^2 \psi \cos^2 \psi}{c^2 + \sin^2 \psi} \right\} \left\{ \frac{1 - \delta^2(\sin^4 \psi + c^2 \sin^2 \psi - 2c^2 \sin^4 \psi)}{(c^2 + \sin^2 \psi)^2} \right\} ,$$

as far as the second order in δ.

Hence

$$Bd\theta = -\frac{1}{4} \tan \delta \cos^2 \delta \, g(\psi) d\psi ,$$

where

$$g(\psi) = \frac{(3\bar{c}^2 + \sin^2 \psi) \sin^2 \psi \cos \psi}{\bar{c}^2 + \sin^2 \psi} +$$

$$+ \frac{\delta^2 \sin^2 \psi \cos \psi}{\bar{c}^2 + \sin^2 \psi} \Big[6\bar{c}^2 - 10\bar{c}^2 \sin^2 \psi + 4 \sin^2 \psi - 3 \sin^4 \psi -$$

$$- \frac{\sin^2 \psi (3\bar{c}^2 + \sin^2 \psi)}{(\bar{c}^2 + \sin^2 \psi)^2} (2\bar{c}^2 - 3\bar{c}^2 \sin^2 \psi + 2 \sin^2 \psi - \sin^4 \psi) \Big] .$$

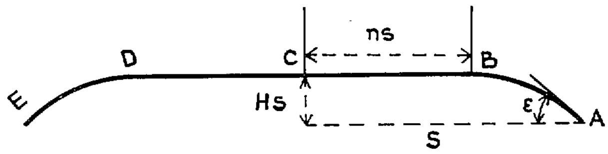

FIG. 1. The Parameters.

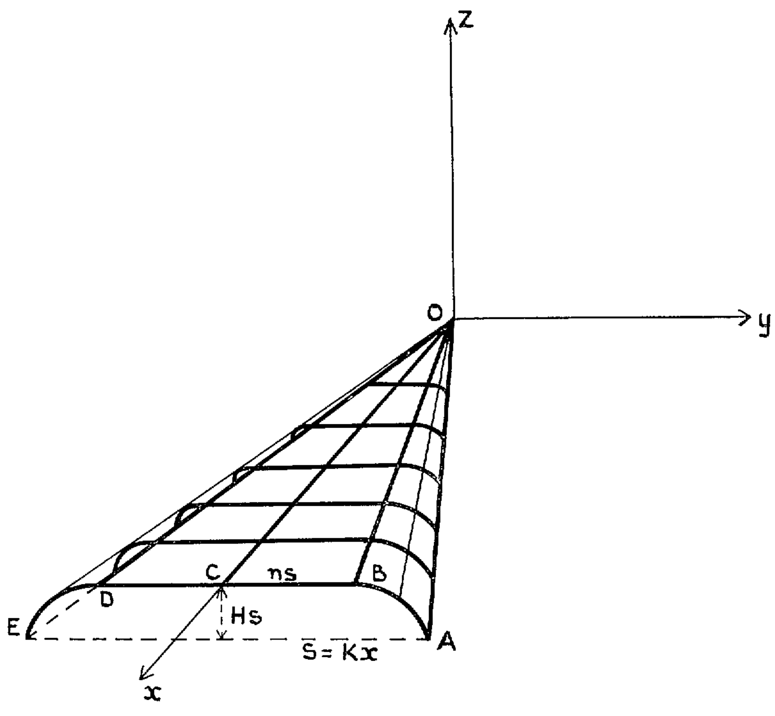

FIG. 2. The co-ordinate system.

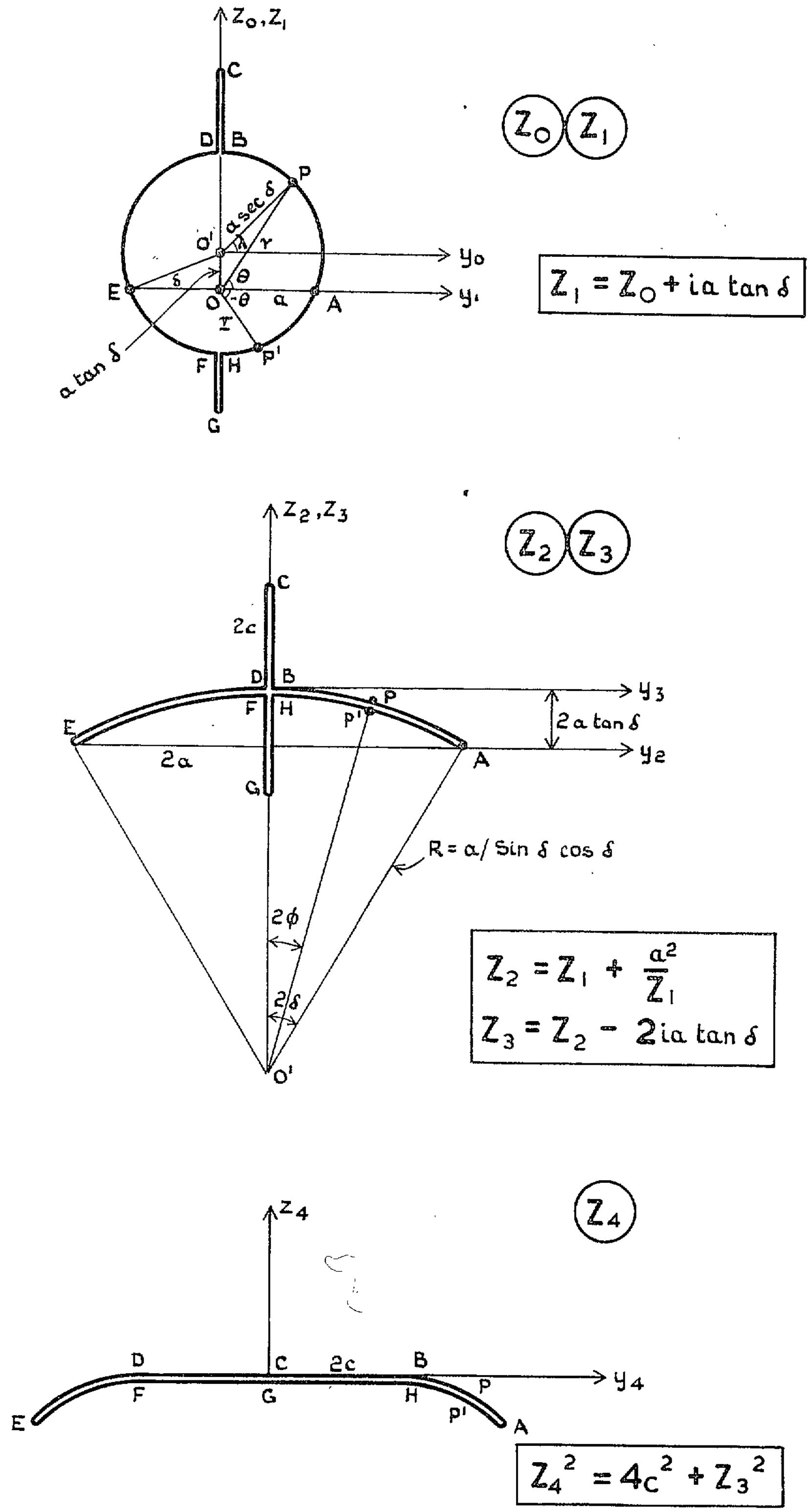

FIG. 3. The conformal transformations.

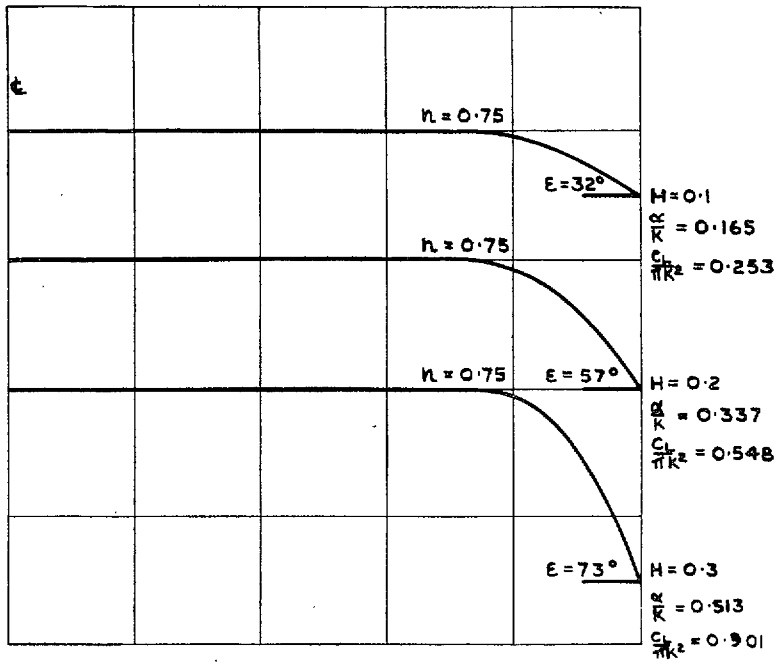

FIG. 4. Sections with shoulder at 0·75*s*.

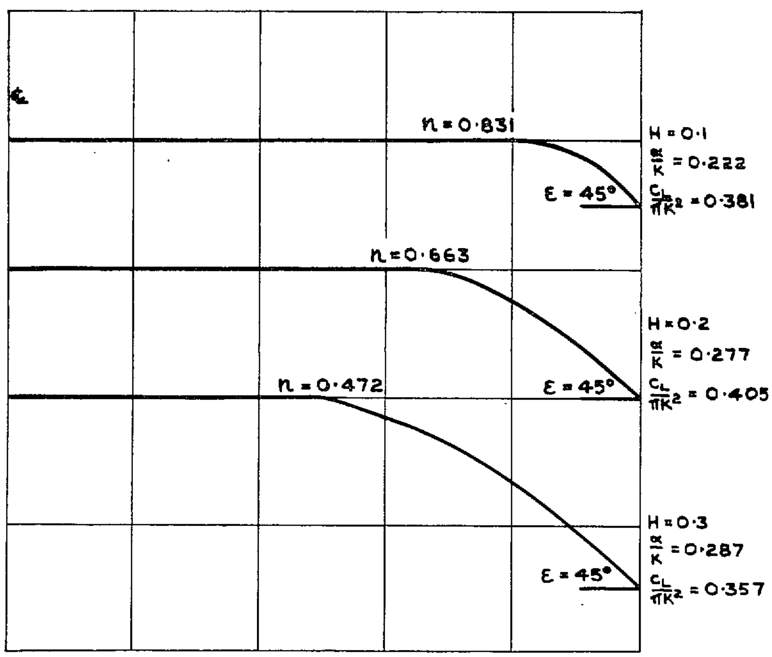

FIG. 5. Sections with droop angle of 45 deg.

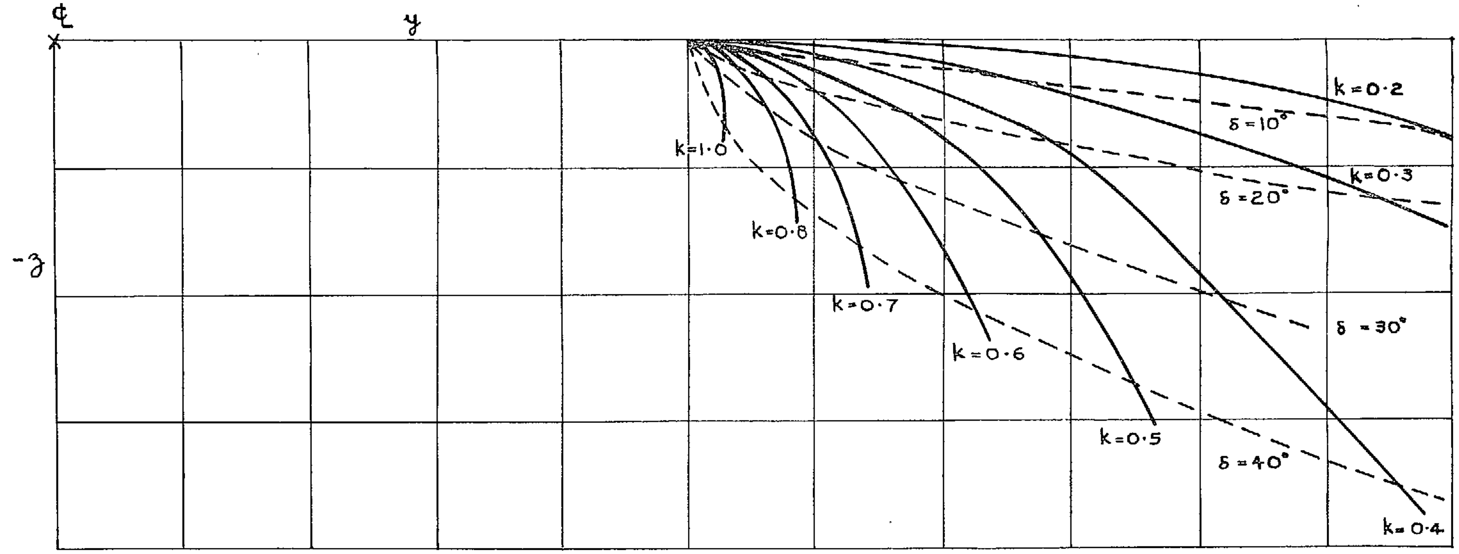

FIG. 6. Camber-lines generated by the transformation $Z_4^4 = Z_3^4 + iZ_3^3 R + Z_3^2 R^2 + k^4 R^4$ for various values of k.

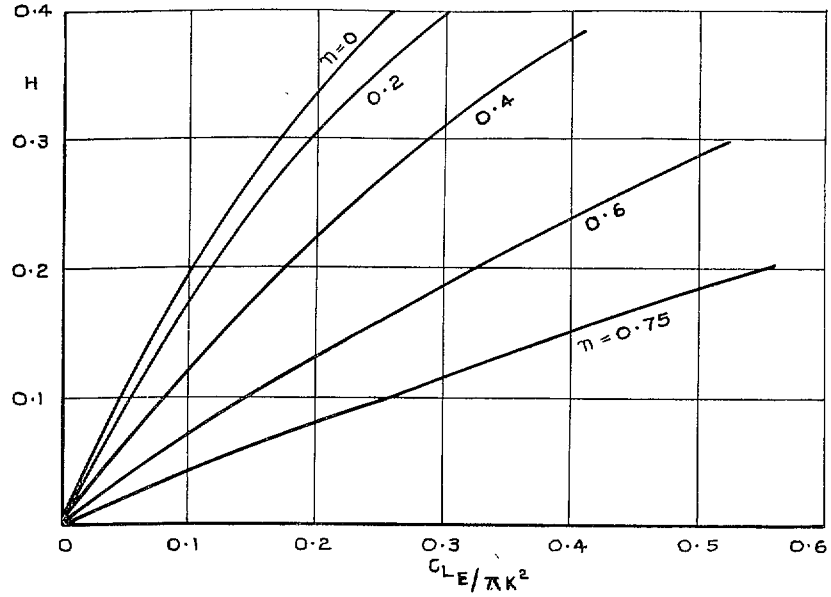

FIG. 7. Droop required for given C_L.

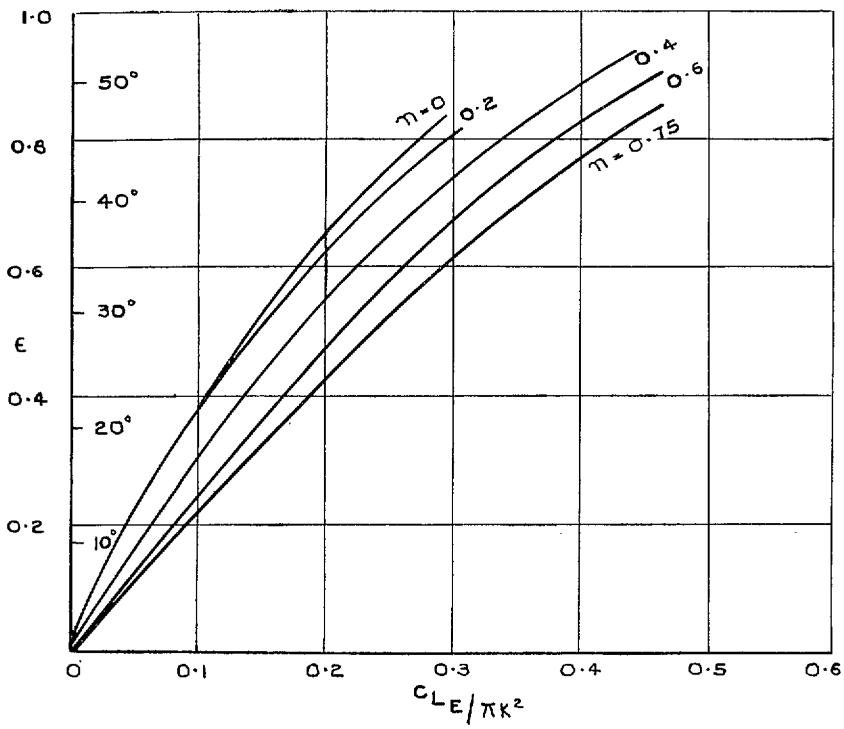

FIG. 8. Droop angle required for given C_L.

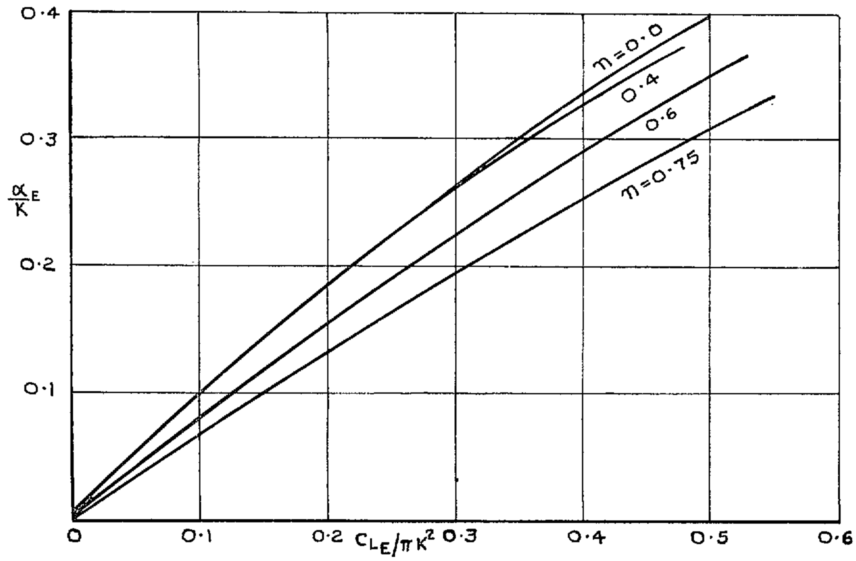

FIG. 9. Incidence required for given C_L.

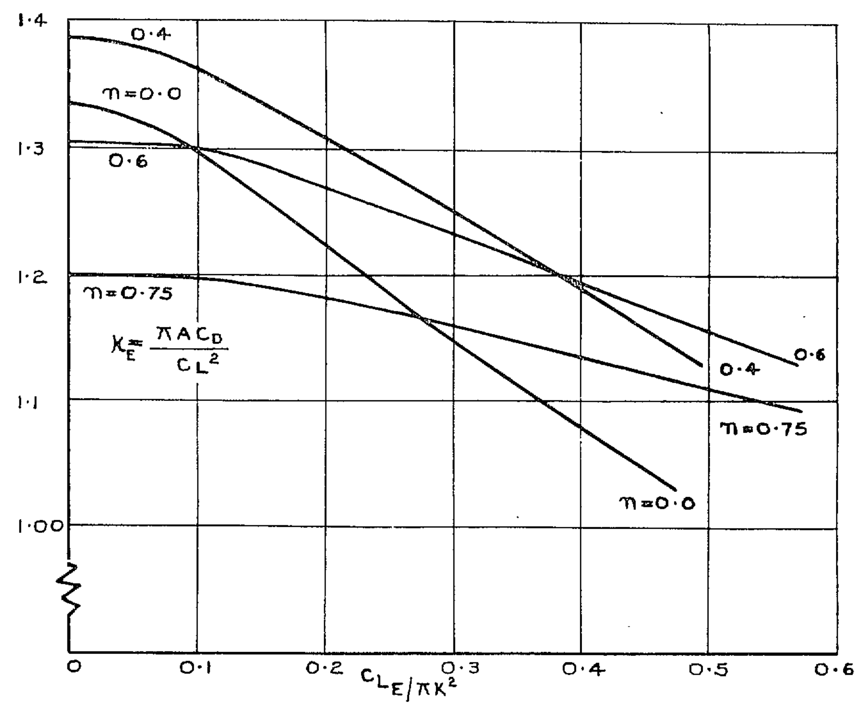

FIG. 10. Lift-dependent drag factor for given C_L.

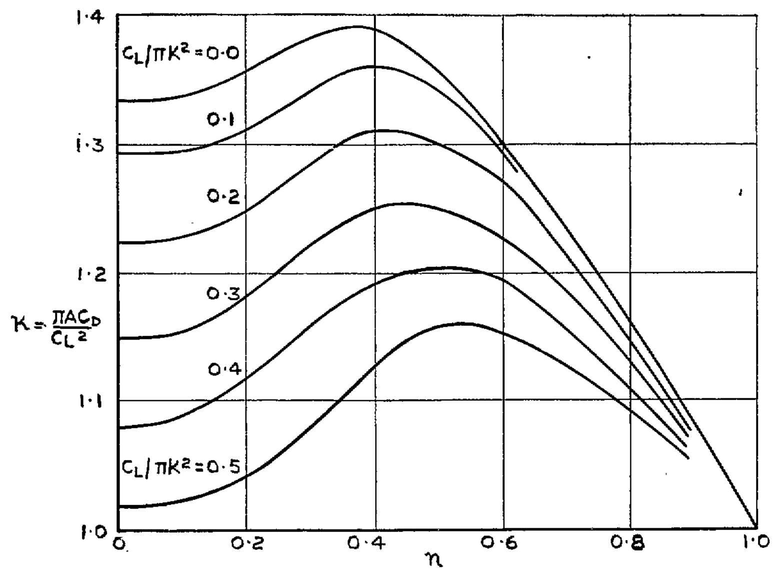

FIG. 11. Lift-dependent drag factor versus shoulder position.

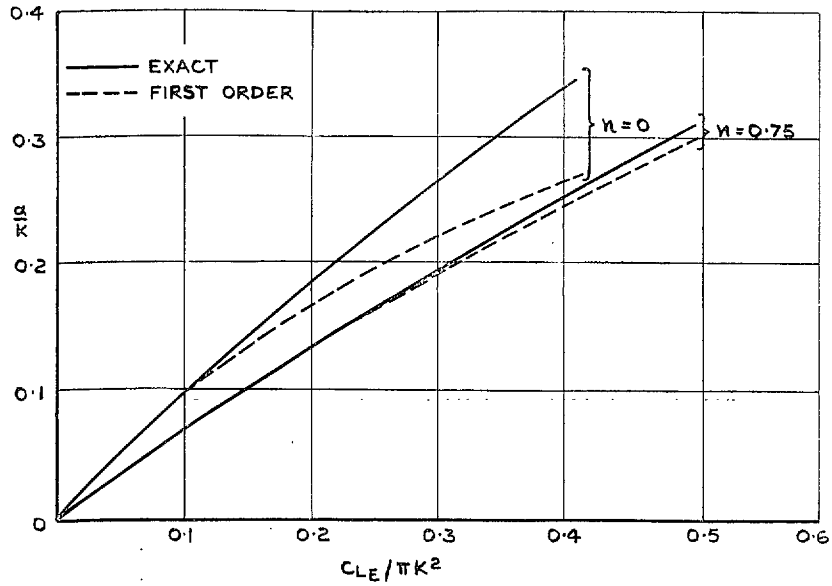

Fig. 12. α/K versus lift coefficient.

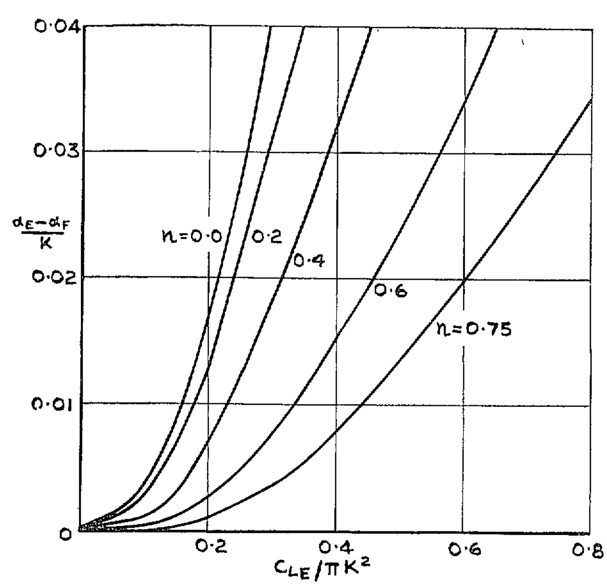

Fig. 13. $\triangle\ (\alpha/K)$ versus lift coefficient.

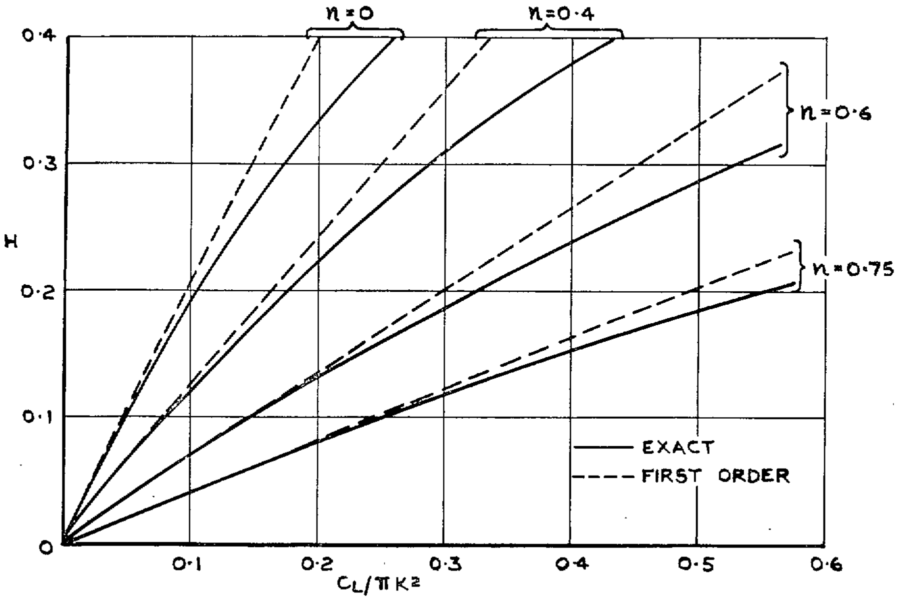

FIG. 14. Droop versus lift coefficient.

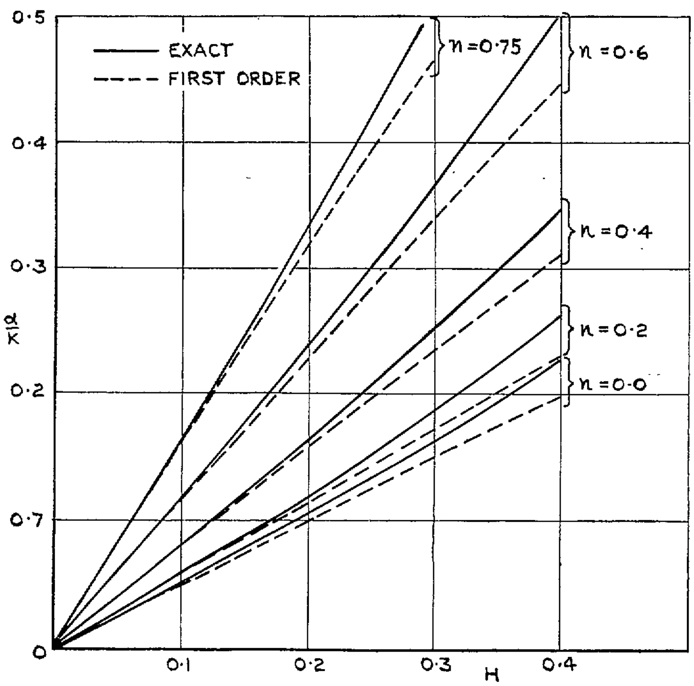

FIG. 15. α/K versus droop.

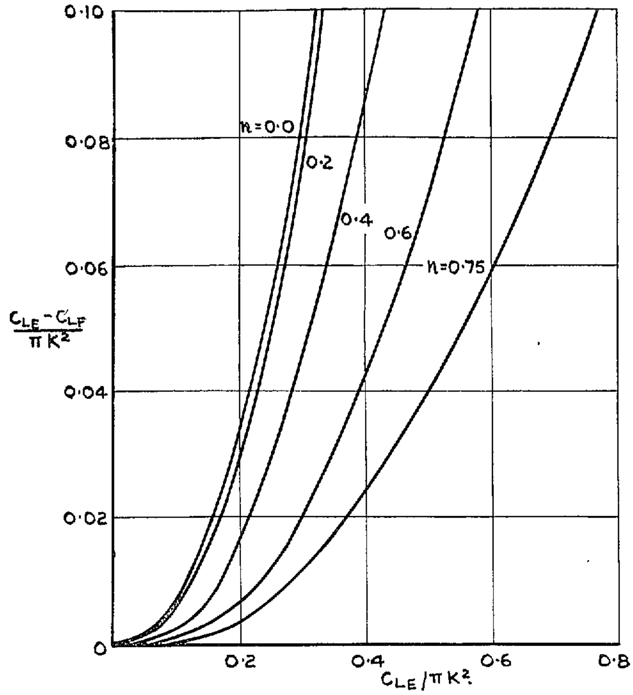

FIG. 16. $\Delta\,(C_L/\pi K^2)$ versus lift coefficient.

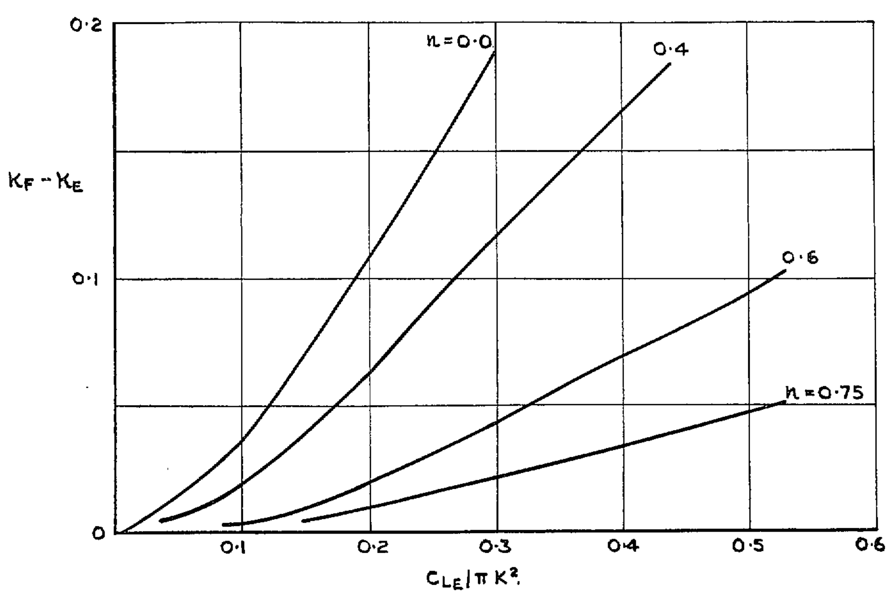

FIG. 17. $\Delta\kappa$ versus lift coefficient.

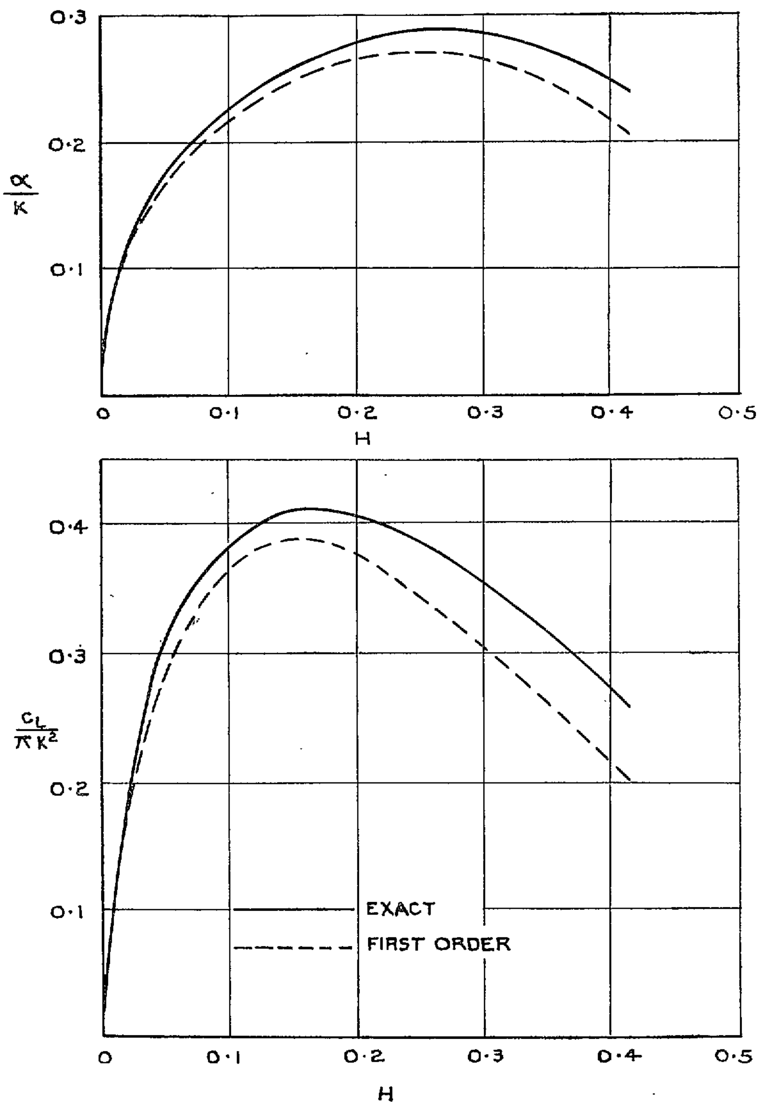

FIG. 18. Incidence and lift coefficient for series of Fig. 5 with droop angle equal to 45 deg.

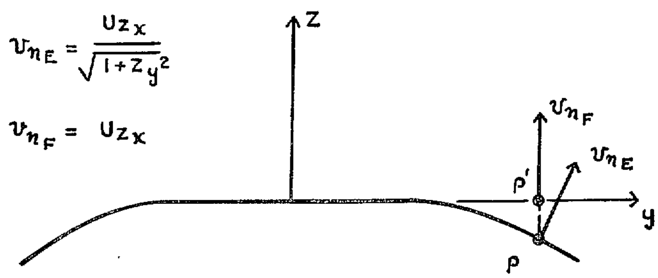

FIG. 19. 'Usual' slender-wing method of applying boundary conditions.

(83192) Wt. 67/1876 K.5 3/62 Hw.

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