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Thermal Stresses near the Roots of Rectangular Wings

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Summary. A continuous solution is derived for the stress distribution in a simple wing surface represented by a uniformly reinforced strip bounded by equal constant area edge members, when the sections in the airstream experience a uniform temperature rise. The section shielded by the fuselage is assumed to remain at a constant temperature. This analysis, which takes into account the bending, shear and direct stiffness of the edge members, is used to evaluate the shear stress distribution in a specific strip used as an example. The stress distribution in this same specimen strip is also calculated by the Argyris matrix force method assuming a finite spanwise temperature gradient at the edges of the fuselage, both for a constant chordwise temperature and for a parabolic chordwise temperature variation in the airstream.

1. *Introduction.* When a supersonic aircraft with a continuous wing structure passing through its fuselage accelerates rapidly, temperature differences arise between the section of the wing shielded by the fuselage and the remainder of the wing in the airstream. A self-equilibrating system of stresses is thus produced in the wing so that the sections inside and outside the fuselage remain compatible. In this Report the stresses set up in this way in the surface of a rectangular wing are considered. Restraints imposed on the thermal expansion of the surface by the fuselage-wing connections and by temperature variations through the webs are neglected. The surface is represented by a flat isotropic strip reinforced by closely spaced stringers and ribs, and by equal constant area edge members.

Expressions are derived for the thermal stresses in the strip when the sections in the airstream experience a uniform temperature rise while the section shielded by the fuselage remains at a constant temperature. An upper limit is thus obtained for the stresses produced by a uniform temperature rise in the airstream. In this analysis the bending, direct and shear flexibilities of the edge members are taken fully into account.

The Argyris matrix force method[2, 3] is used to calculate the stress distribution in the strip due to a parabolic variation of temperature in the chordwise direction on the sections in the airstream. This analysis, which is repeated for a uniform temperature in the chordwise direction, is readily

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applicable to any chordwise temperature variation. The temperature of the section within the fuselage is again taken as constant but the transition from the conditions within the fuselage to those in the airstream is here assumed to take place over a finite distance.

The analysis of thermal stresses in rectangular wings in general is discussed in Appendix III.

2. *Assumptions.* The following assumptions are made throughout this Report:

(*a*) Stress-strain relations are linear.

(*b*) Buckling does not take place.

(*c*) Rivet flexibility is negligible.

(*d*) The temperature is constant through the surface analysed.

(*e*) The stiffening effect of the stringers and ribs may be represented by an effective increase in the sheet thickness in the appropriate directions.

(*f*) The wing-fuselage connections do not restrict the thermal expansion of the wing surface.

The following assumption is made in the 'continuous' analysis of Section 4 only:

(*g*) The strip is infinitely long.

The following assumptions are made in the matrix analysis of Section 5 only:

(*h*) The strip may be represented as an orthogonal grid of direct stress carrying members separating panels each of which carries a constant shear flow only.

(*j*) The cross-sectional area of these direct stress carrying members is the sum of the cross-sectional area of any reinforcing member along the grid lines together with half the cross-sectional area of the adjacent shear carrying fields.

(*k*) The influence of Poisson's ratio on the interaction of the direct stresses in the orthogonal flanges is negligible.

(*l*) The bending and shear stiffnesses of the edge members are negligible.

(*m*) The temperature varies linearly between nodal points of the grid.

The detailed assumptions made with regard to the temperature distribution at the edges of the fuselage will be found in Section 5.

Assumptions (*h*) to (*m*) are standard assumptions made in the application of the Argyris matrix force method to wing structures. It is seen from the results of Section 4 that the gradients of shear stress in the strip are sufficiently low for the limitations implicit in assumption (*h*) to be unimportant in the present application. Assumptions (*j*) and (*k*) have previously been justified in the context of the diffusion of end load from edge members into rectangular panels by comparison both with more elaborate theoretical analyses and with strain gauge results[5].

3. *The Simple Strip used as an Example.* All calculations in this Report are performed on the simple specimen strip shown in Fig. 1. While the symmetry of the strip simplifies the analyses, it is not essential to them.

The stresses calculated in this strip are expressed in dimensionless form as fractions of $E\alpha \times$ (maximum temperature difference). To give an indication of the order of these stresses, the following approximate values of $E\alpha T$ are quoted for a temperature difference of 200 deg C.

Steel, $E\alpha T = 68{,}400$ lb/in.^{2}

Light alloy, $E\alpha T = 46{,}000$ lb/in.^{2}.

4. *Temperature Variation in a Spanwise Direction only.* In this section analytical expressions are obtained for the stress distribution in the simple strip described above when the parts protruding from the fuselage are subjected to a uniform temperature rise, leaving the remainder at a constant temperature. The method used here was suggested to the author by a paper by Horvay[1] in which a similar method is used to analyse an isotropic strip without edge members.

Expressions for the stress distribution in an infinitely long strip subjected to a uniform load over half its length are derived in Appendix I. These expressions are quoted below. The strip lies parallel to the x axis and is subjected to a load intensity S from $x = -\infty$ to $x = 0$.

$$\sigma_x = \frac{S}{\rho_1 \pi t} \int_0^\infty \frac{(1+Q_1\theta^2)(U_2 p_1^2 \cosh p_1\theta Y - U_1 p_2^2 \cosh p_2\theta Y)}{R_1U_2 - R_2U_1} \frac{\sin \theta X}{\theta} d\theta \tag{1}$$

$$\sigma_y = \frac{S}{\rho_2 t}\left[\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{(1+Q_1\theta^2)(U_2 \cosh p_1\theta Y - U_1 \cosh p_2\theta Y)}{R_1U_2 - R_2U_1} \frac{\sin \theta X}{\theta} d\theta\right] \tag{2}$$

$$\tau_{xy} = -\frac{S}{\pi t}\int_0^\infty \frac{(1+Q_1\theta^2)(U_2 p_1 \sinh p_1\theta Y - U_1 p_2 \sinh p_2\theta Y)}{R_1U_2 - R_2U_1} \frac{\cos \theta X}{\theta} d\theta. \tag{3}$$

In order to find the stresses in the strip under the prescribed temperature distribution, two stress systems of the kind quoted above are superposed in such a way that the section of the strip between $x = -\infty$ and $x = 0$ corresponds to the portions of the wing in the airstream. Putting $S = \rho_2 E\alpha Tt$ and subtracting the local value of $E\alpha T$ from σ_y at every point in the strip, the following expressions for the stresses in the strip are obtained. The axes of reference are at the centre of the strip as shown in Fig. 1.

$$\sigma_x = \frac{\rho_2}{\rho_1}\frac{E\alpha T}{\pi}\int_0^\infty \frac{(1+Q_1\theta^2)(U_2 p_1^2 \cosh p_1\theta Y - U_1 p_2^2 \cosh p_2\theta Y)W_1}{R_1U_2 - R_2U_1}\frac{d\theta}{\theta} \tag{4}$$

$$\tau_{xy} = -\frac{\rho_2 E\alpha T}{\pi}\int_0^\infty \frac{(1+Q_1\theta^2)(U_2 p_1 \sinh p_1\theta Y - U_1 p_2 \sinh p_2\theta Y)W_2}{R_1U_2 - R_2U_1}\frac{d\theta}{\theta} \tag{5}$$

where

$$W_1 = \sin \theta(X+X') - \sin \theta(X-X')$$

$$W_2 = \cos \theta(X+X') - \cos \theta(X-X').$$

Within the fuselage

$$\sigma_y = E\alpha T\left[1 - \frac{1}{\pi}\int_0^\infty \frac{(1+Q_1\theta^2)(U_2 \cosh p_1\theta Y - U_1 \cosh p_2\theta Y)W_1}{R_1U_2 - R_2U_1}\frac{d\theta}{\theta}\right]. \tag{6}$$

In the airstream

$$\sigma_y = -\frac{E\alpha T}{\pi}\int_0^\infty \frac{(1+Q_1\theta^2)(U_2 \cosh p_1\theta Y - U_1 \cosh p_2\theta Y)W_1}{R_1U_2 - R_2U_1}\frac{d\theta}{\theta}. \tag{7}$$

The evaluation of these integrals is laborious and has been restricted to finding the shear stress distribution along the temperature discontinuity and adjacent to the edge members when the infinite strip is heated over half its length. The calculated shear stresses are not significantly different from those produced near the edges of the fuselage under the prescribed temperature distribution, as the stresses die out rapidly along the strip. The shear stresses are plotted in Fig. 5 together with the corresponding values obtained by Horvay for an isotropic strip. It will be seen that for an edge member of the size considered here the peak shear stress is graphically indistinguishable from the value for an isotropic strip ($E\alpha T/\pi$). The peak shear stress is however dependent on the width of the strip and would tend to $1 \cdot 131 E\alpha T/\pi$ if the width of this strip were increased to infinity.

The greatest values of σ_x and σ_y achieved in an unreinforced isotropic strip are both equal to $E\alpha T/2$. It is seen by symmetry that the peak value of σ_y is unchanged by the reinforcements introduced in the present example. The peak value of σ_x is reduced however by the presence of the edge members and the region of high shear stress is increased.

5. *Temperature Variation in both Spanwise and Chordwise Directions.* In this section an analysis, by the Argyris matrix force method, of the stresses in the specimen strip is summarised when the wing is subjected to typical temperature distributions incorporating a rapid rise in temperature in the spanwise direction at the section at which the wing protrudes from the fuselage. The strip is idealised into an orthogonal grid of direct stress carrying members separating fields carrying shear stresses only, making the assumptions specified in Section 2. The grid lines are more closely spaced in those parts of the strip where large gradients of stress are likely to occur. The assembly of direct and shear stress carrying elements which constitute the idealised structure are analysed in matrix notation using self-equilibrating stress systems as unknowns. The matrix analysis of redundant structures including the effect of thermal stresses was developed in a series of papers by Argyris[3] where the structural idealisation used here was also introduced. The relevant sections of the matrix analysis are summarised in Appendix II.

The grid pattern used here is shown in Fig. 3 and a typical self-equilibrating stress system is shown in Fig. 4.

The temperature of the flange in the idealised structure corresponding to the edge of the fuselage is assumed to be at the mean of the local temperature of the strip in the airstream and the temperature within the fuselage. Hence the temperature effectively changes from the conditions within the fuselage to those in the airstream over a distance of 3 in. in the spanwise direction (*i.e.*, 6 per cent of the chord).

The following temperature distributions are considered; in each case the temperature of the strip within the fuselage is assumed to be constant.

(*a*) Parabolic chordwise variation of temperature over the sections of the strip in the airstream.

As the temperature is assumed to vary linearly along each direct stress carrying element, the parabolic chordwise distribution is here approximately represented by defining the temperature at the nodes of the grid according to the expression

$$T = \frac{T_1}{2}(Y^2+1)$$

where T_1 is the temperature at the spanwise edges.

(*b*) Uniform temperature rise over the sections of the strip in the airstream.

The spanwise members are assumed to experience a constant temperature right up to the edges of the fuselage. This approximation will not materially affect the stresses under temperature distribution (*a*) and will have no effect at all under temperature distribution (*b*).

The stresses corresponding to these temperature variations are shown in Figs. 6 to 11. Comparison of the shear stresses produced by an abrupt change in temperature at the edges of the fuselage (which are virtually the same as those shown in Fig. 5) with those produced by the temperature distribution (b) considered here (Fig. 11) shows that a considerable alleviation of the shear stresses results from assuming a high finite temperature gradient at the edges of the fuselage in place of an abrupt temperature change.

6. *Acknowledgement.* The author would like to thank Mrs. C. A. Mason for her assistance in the computational work incorporated in this Report.

7. *Conclusions.* The thermal stresses set up in the surfaces of a rectangular wing structure continuous through a fuselage have been considered when those portions of the wing in the airstream are subjected to a rise in temperature. Firstly a continuous solution was derived for the stress distribution in the surface of a hypothetical wing structure represented by a uniformly reinforced isotropic strip with equal constant area edge members, when those parts of the strip in the airstream experience a uniform temperature rise. The section of the strip shielded by the fuselage was assumed to remain at a constant temperature. The bending, direct and shear flexibilities of the edge members were taken fully into account. The shear stresses calculated for a specific example suggest that the peak shear stress in a reinforced strip with edge members of reasonable practical proportions is unlikely to differ greatly from that in an isotropic strip; the region of high shear stress is however larger. A matrix method using a finite grid size was employed to analyse the stress distribution produced by a parabolic chordwise temperature variation over the portions of the strip in the airstream. A finite spanwise temperature gradient was assumed at the edges of the fuselage. It was shown by repeating the latter analysis with a constant chordwise temperature that the stresses corresponding to the assumed spanwise temperature gradient at the edges of the fuselage were considerably less than those due to the abrupt change in temperature assumed in the first analysis. The analysis of thermal stresses in rectangular wings in general was discussed in an Appendix.

NOTATION

x, y		Orthogonal axes, x being in the spanwise direction
a		Semi-chord
X, Y		x/a, y/a
aX'		Half-width of fuselage
v		Deflection parallel to y axis
t		Thickness of shear carrying sheet
ρ_1, ρ_2		$\frac{\text{Effective thickness of sheet parallel to } x, y \text{ axis}}{\text{Thickness of shear carrying sheet}}$
σ_x, σ_y		Direct stresses
τ_{xy}		Shear stress
ν		Poisson's ratio
ϕ		Stress function
κ	=	$2\rho_2\rho_1(1+\nu) - (\rho_2+\rho_1)\nu$
S		Load on strip from $x = -\infty$ to $x = 0$
P		Local load on strip
λ		Defined by Equation (10)
θ	=	λa
f		Function of y
p_1, p_2		Defined by Equation (12)
A_1, A_2		Arbitrary constants
α		Coefficient of expansion
E		Young's modulus
T		Temperature difference
G		Shear modulus
B		Cross-sectional area of edge member
I		Second moment of area of edge member for bending in xy plane
A'		Effective cross-sectional area of edge member for shearing in xy plane
Q_1	=	$2(1+\nu)\dfrac{I}{A'a^2}$
Q_2	=	$\dfrac{I}{\rho_1 a^3 t}$

$$Q_3 = \frac{B}{\rho_1 ta}$$

$$q_1 = \frac{\rho_1/\rho_2 + p_1{}^2\nu}{p_1}$$

$$q_2 = \frac{\rho_1/\rho_2 + p_2{}^2\nu}{p_2}$$

$$R_1 = (1+Q_1\theta^2)\cosh p_1\theta + Q_2\theta^3 q_1 \sinh p_1\theta$$

$$R_2 = (1+Q_1\theta^2)\cosh p_2\theta + Q_2\theta^3 q_2 \sinh p_2\theta$$

$$U_1 = p_1 \sinh p_1\theta + Q_3\theta\left(p_1{}^2 + \frac{\rho_1}{p_2}\nu\right)\cosh p_1\theta$$

$$U_2 = p_2 \sinh p_2\theta + Q_3\theta\left(p_2{}^2 + \frac{\rho_1}{\rho_2}\nu\right)\cosh p_2\theta$$

$$W_1 = \sin\theta(X+X') - \sin\theta(X-X')$$

$$W_2 = \cos\theta(X+X') - \cos\theta(X-X')$$

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APPENDIX I

The Stress Distribution in an Infinite Strip Bounded by Edge Members with Bending, Direct and Shear Stiffness when the Edges are Subjected to a Uniform Load over Half their Length

Consider an infinite uniform strip of width $2a$, symmetrical about its centre-line. The strip, which is bounded by edge members of constant finite bending, direct and shear stiffnesses is subjected to a uniform load over half its length as shown in Fig. 2. The effective thicknesses of the strip in the longitudinal and transverse directions are $\rho_1 t$ and $\rho_2 t$ respectively. To satisfy the equilibrium and compatibility conditions within the strip a stress function ϕ must be found such that

$$\left.\begin{aligned} \rho_1 \sigma_x &= \frac{\partial^2 \phi}{\partial y^2} \\ \rho_2 \sigma_y &= \frac{\partial^2 \phi}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} \end{aligned}\right\} \tag{8}$$

and

$$\rho_2 \frac{\partial^4 \phi}{\partial y^4} + \kappa \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \rho_1 \frac{\partial^4 \phi}{\partial x^4} = 0 \tag{9}$$

where

$$\kappa = 2\rho_2\rho_1(1+\nu) - (\rho_2+\rho_1)\nu .$$

The solution to this equation for the boundary conditions considered here may be written in the form

$$\phi = \frac{Sx^2}{4t} - \frac{S}{\pi t}\int_0^\infty f(y)\,\frac{\sin \lambda x}{\lambda^3}\,d\lambda . \tag{10}$$

Substituting for ϕ in Equation (9) and differentiating with respect to λ gives

$$\rho_2 \frac{\partial^4 f}{\partial y^4} - \kappa\lambda^2 \frac{\partial^2 f}{\partial y^2} + \rho_1 \lambda^4 f = 0 . \tag{11}$$

As the strip is symmetrical about the x axis, the solution to this equation is

$$f = A_1 \cosh p_1 \lambda y + A_2 \cosh p_2 \lambda y$$

where p_1 and p_2 are given by

$$p^2 = \frac{\kappa \pm \sqrt{(\kappa^2 - 4\rho_1\rho_2)}}{2\rho_2} . \tag{12}$$

Hence

$$\phi = \frac{Sx^2}{4t} - \frac{Sa^2}{\pi t}\int_0^\infty \left[A_1 \cosh p_1\lambda y + A_2 \cosh p_2 \lambda y\right] \frac{\sin \lambda x}{\lambda^3}\,d\lambda . \tag{13}$$

To satisfy the equilibrium boundary conditions at $y = \pm a$

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho_1}\frac{\partial^2 \phi}{\partial y^2} - \frac{\nu}{\rho_2}\frac{\partial^2 \phi}{\partial x^2}\right) = -\frac{t}{B}\frac{\partial^2 \phi}{\partial x \partial y} . \tag{14}$$

Substituting Equation (13) in Equation (14) gives

$$A_2 = -\frac{U_1}{U_2} A_1 \tag{15}$$

where

$$U_1 = p_1 \sinh p_1\theta + Q_3\theta \left(p_1^2 + \frac{\rho_1}{\rho_2}\nu\right) \cosh p_1\theta$$

$$U_2 = p_2 \sinh p_2\theta + Q_3\theta \left(p_2^2 + \frac{\rho_1}{\rho_2}\nu\right) \cosh p_2\theta$$

$$\theta = \lambda a$$

$$Q_3 = \frac{B}{\rho_1 ta}.$$

Finally the compatibility boundary condition at $y = \pm a$ must be considered. From Equation (13) for the stress function, the deflection v at $y = \pm a$ can be deduced to be

$$\frac{Sa}{Et}\left[\frac{1}{2\rho_2} + \frac{1}{\rho_1\pi}\int_0^\infty (q_1A_1 \sinh p_1\theta + q_2A_2 \sinh p_2\theta)\,\frac{\sin\theta X}{\theta^2}\,d\theta\right] \tag{16}$$

where

$$X = \frac{x}{a}$$

$$q_1 = \frac{\rho_1/\rho_2 + p_1^2\nu}{p_1}$$

$$q_2 = \frac{\rho_1/\rho_2 + p_2^2\nu}{p_2}$$

The vertical deflection of the edge member can be separated into three parts.

(1) The deflection v_0 when $x = 0$

$$v_0 = \frac{1}{2\rho_2}\frac{Sa}{Et} \tag{17}$$

(2) The bending deformation given by

$$EI\frac{\partial^4 v_1}{\partial x^4} = P - t\frac{\partial^2\phi}{\partial x^2} \tag{18}$$

where P is the local load per unit length applied to the edge of the strip, the loading considered here being

$$P = S,\ x < 0$$

$$P = 0,\ x > 0.$$

P can be expressed as a Fourier integral

$$P = \frac{S}{2} - \frac{S}{\pi}\int_0^\infty \frac{\sin\lambda x}{\lambda}\,d\lambda \tag{19}$$

(3) The shear deformation given by

$$-GA'\frac{\partial^2 v_2}{\partial x^2} = P - t\frac{\partial^2\phi}{\partial x^2}. \tag{20}$$

From Equations (17), (18), (19) and (20), the total vertical deflection of the edge member is

$$\frac{1}{E\rho_2}\frac{Sa}{2t} - \frac{Sa^2}{\pi}\int_0^\infty \left\{\frac{1}{GA'} + \frac{a^2}{\theta^2 EI}\right\}\{1 + A_1 \cosh p_1\theta + A_2 \cosh p_2\theta\}\frac{\sin\theta X}{\theta^3}\,d\theta. \tag{21}$$

Equating expressions (16) and (21) and differentiating with respect to θ

$$Q_2\theta^3(q_1A_1 \sinh p_1\theta + q_2A_2 \sinh p_2\theta) = -(1+Q_1\theta^2)(1+A_1 \cosh p_1\theta + A_2 \cosh p_2\theta) \tag{22}$$

where

$$Q_1 = \frac{EI}{GA'a^2} = 2(1+\nu)\frac{I}{A'a^2}$$

$$Q_2 = \frac{I}{\rho_1 a^3 t}.$$

Combining Equations (15) and (22), the following expressions for the arbitrary constants are obtained

$$\left.\begin{aligned} A_1 &= \frac{U_2(1+Q_1\theta^2)}{R_2U_1 - R_1U_2} \\ A_2 &= \frac{U_1(1+Q_1\theta^2)}{R_1U_2 - R_2U_1} \end{aligned}\right\} \tag{23}$$

where

$$R_1 = (1+Q_1\theta^2)\cosh p_1\theta + Q_2\theta^3 q_1 \sinh p_1\theta$$

and

$$R_2 = (1+Q_1\theta^2)\cosh p_2\theta + Q_2\theta^3 q_2 \sinh p_2\theta.$$

Substituting in Equation (10) and putting $X = x/a$, $Y = y/a$ gives

$$\phi = \frac{Sa^2}{t}\left[\frac{X^2}{4} + \frac{1}{\pi}\int_0^\infty \frac{(1+Q_1\theta^2)(U_2 \cosh p_1\theta Y - U_1 \cosh p_2\theta Y)}{R_1U_2 - R_2U_1}\frac{\sin\theta X}{\theta^3}\,d\theta\right]. \tag{24}$$

Hence the stresses are given by

$$\sigma_x = \frac{S}{\rho_1\pi t}\int_0^\infty \frac{(1+Q_1\theta^2)(U_2p_1^2 \cosh p_1\theta Y - U_1p_2^2 \cosh p_2\theta Y)}{R_1U_2 - R_2U_1}\frac{\sin\theta X}{\theta}\,d\theta \tag{25}$$

$$\sigma_y = \frac{S}{2\rho_2 t} - \frac{S}{\rho_2\pi t}\int_0^\infty \frac{(1+Q_1\theta^2)(U_2 \cosh p_1\theta Y - U_1 \cosh p_2\theta Y)}{R_1U_2 - R_2U_1}\frac{\sin\theta X}{\theta}\,d\theta \tag{26}$$

$$\tau_{xy} = -\frac{S}{\pi t}\int_0^\infty \frac{(1+Q_1\theta^2)(U_2p_1 \sinh p_1\theta Y - U_1p_2 \sinh p_2\theta Y)}{R_1U_2 - R_2U_1}\frac{\cos\theta X}{\theta}\,d\theta. \tag{27}$$

If the strip bounded by the edge members were isotropic (i.e., $p_1 = p_2 = 1$), the stress function would be given by

$$\phi = \frac{Sa^2}{t}\left[\frac{X^2}{4} + \frac{1}{\pi}\int_0^\infty \frac{(1+Q_1\theta^2)(U_2' \cosh\theta Y - U_1'\theta Y \sinh\theta Y)}{R_1'U_2' - R_2'U_1'}\frac{\sin\theta X}{\theta^3}\,d\theta\right]$$

where

$$R_1' = (1+Q_1\theta^2)\cosh\theta + Q_2(1+\nu)\theta^3 \sinh\theta$$

$$R_2' = (1+Q_1\theta^2)\theta \sinh\theta + Q_2\theta^3[(1+\nu)\theta\cosh\theta - (1-\nu)\sinh\theta]$$

$$U_1' = \sinh\theta + Q_3(1+\nu)\theta\cosh\theta$$

$$U_2' = [1+Q^3(1+\nu)\theta^2]\sinh\theta + \theta(1+2Q_3)\cosh\theta.$$

APPENDIX II

The Matrix Analysis of the Thermal Stresses in the Strip

Algebraic symbols in bold type are used in this Appendix to denote matrices.

The matrix analysis given below is summarised from Ref. 1.

As has already been described in Section 5 of this Report, the strip is idealised into an orthogonal grid of direct stress carrying members separating fields in each of which the shear stress is assumed constant. Hence the load in the direct stress carrying members varies linearly from node to node. The strip can therefore be thought of as an assembly of direct stress carrying members in each of which the load varies linearly, and shear fields in each of which the shear flow is constant.

Let $\mathbf{X}$ be the column matrix of the n redundancies in the idealised structure. In the present application the redundancies are selected as self-equilibrating stress systems of the kind shown in Fig. 4. As the structure and the temperature distribution are both doubly symmetrical, only one quadrant need be analysed.

Let $\mathbf{b}_1$ be a load transformation matrix such that the product $\mathbf{b}_1\mathbf{X}$ gives the column matrix of the loads in the elements of the idealised structure due to the redundancies. The $\mathbf{b}_1$ matrix has n columns and has two rows corresponding to each direct stress carrying element and one row corresponding to each shear carrying field.

Let $\mathbf{f}$ be a matrix consisting of a diagonal assembly of the flexibility matrices of the unassembled elements of the structure. As the cross-sectional area of the direct stress carrying members is assumed constant, the flexibility matrix of each of these is of the form

$$\frac{l}{6EA}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

where

l	Length of element
A	Cross-sectional area of element.

The flexibility of each shear field is a scalar of magnitude

$$\frac{\Phi}{Gt}$$

where Φ is the area of the field.

If $\mathbf{H}$ is the column matrix of the extension of the members of the structure due to their unrestrained thermal expansion and if $\mathbf{V}$ is the column matrix of the total extensions of the elements, then

$$\mathbf{V} = \mathbf{f}\mathbf{b}_1\mathbf{X} + \mathbf{H}.$$

By the principle of virtual forces, the compatibility conditions at the (generalised) points of application of the redundancies $\mathbf{X}$ may be expressed as

$$\mathbf{b}_1'\mathbf{V} = \mathbf{b}_1'[\mathbf{f}\mathbf{b}_1\mathbf{X} + \mathbf{H}] = 0.$$

Hence

$$\mathbf{X} = -(\mathbf{b}_1'\mathbf{f}\mathbf{b}_1)^{-1}\mathbf{b}_1'\mathbf{H}$$

and the column matrix $\mathbf{S}$ of the loads in the elements of the structure is given by

$$\mathbf{S} = \mathbf{b}_1\mathbf{X} = -\mathbf{b}_1(\mathbf{b}_1'\mathbf{f}\mathbf{b}_1)^{-1}\mathbf{b}_1'\mathbf{H}.$$

The direct stresses are obtained by dividing the relevant loads by the cross-sectional areas of the members in the idealised structure.

APPENDIX III

Review of the Analysis of Thermal Stresses in Rectangular Wings

The surfaces of thin-skinned wing structures are usually too complicated for a 'continuous' stress analysis to be practicable. When however all the members in the surface conform to an orthogonal grid, the matrix method given in this Report can always be used to obtain an approximate solution. Furthermore, provided the same grid size is used, a major portion of the computations involved in the thermal stress analysis is also incorporated in the analysis of the stresses due to external loads by the same method. Provided therefore the expected temperature gradients are not too severe, this method requires very little extra computation.

In the analysis of thin solid rectangular wings, complications due to the presence of reinforcing members do not arise and, while the matrix method used in this Report is still applicable if it is assumed that the wing thickness does not vary significantly within any one cell of a finite grid, an alternative 'continuous' approximate method due to Mendelson and Hirschberg[4] can also be used if the temperature variation is purely in the plane of the wing. In this method the governing equation for the stress function is satisfied at a finite number of chordwise stations by a polynominal approximation along the chord. If only a few temperature distributions are considered and if the thickness variation of the wing has a simple mathematical form, this method may be preferable.

No analytical solution has yet been obtained for a temperature distribution including both an abrupt change in temperature in the spanwise direction at the edges of the fuselage and a chordwise temperature variation. It is however possible to modify the matrix analysis used in this Report for such a temperature distribution. If the portions of the wing inside and outside the fuselage are considered as independent grids with no connection between them, each may be subjected to the relevant temperature distribution over its entire area. The relative deflections of the nodes of the two grids along the edges of the fuselage can then be found by the method given here as can the redundant forces necessary at these nodes to make the different sections of the wing compatible.

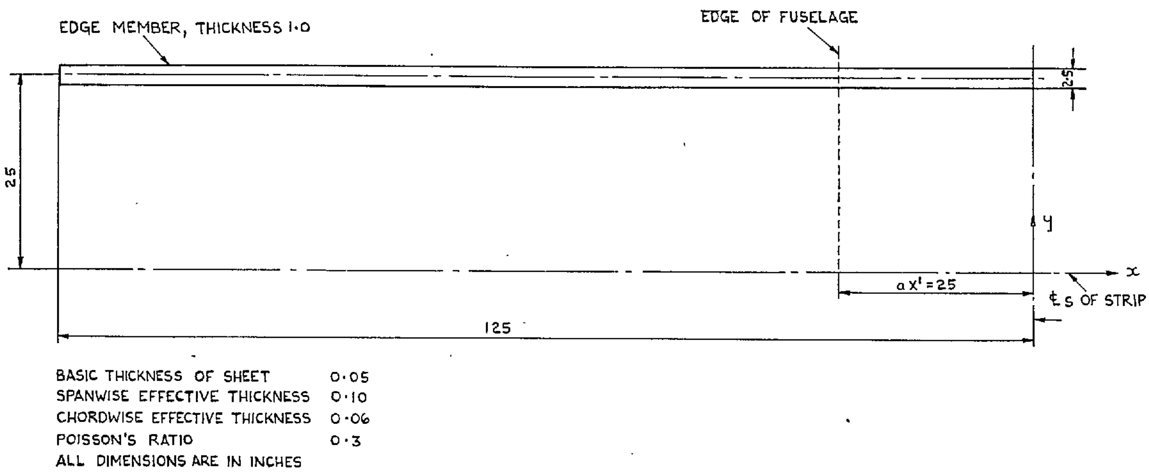

FIG. 1. Simple strip analysed.

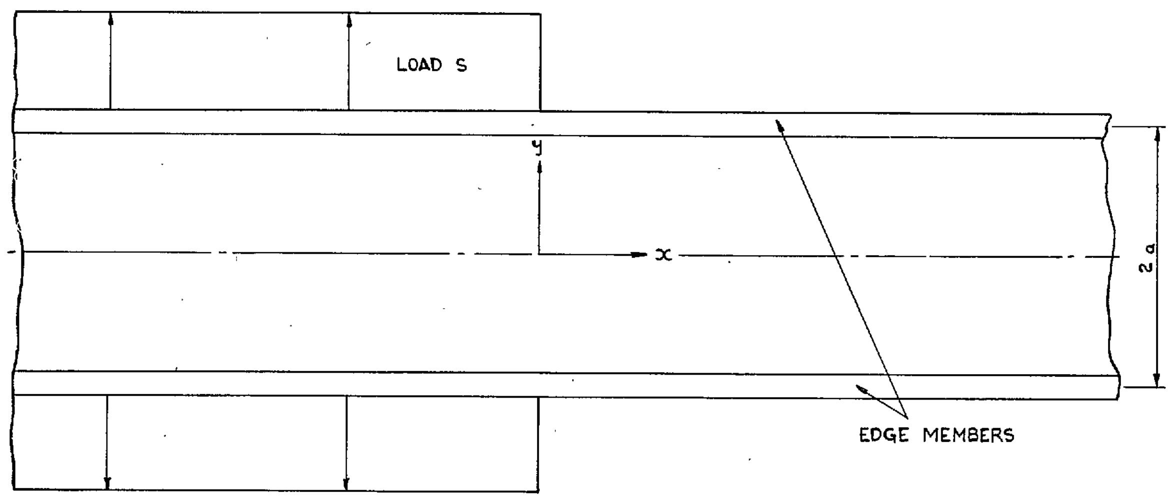

FIG. 2. Infinite strip loaded over half its length.

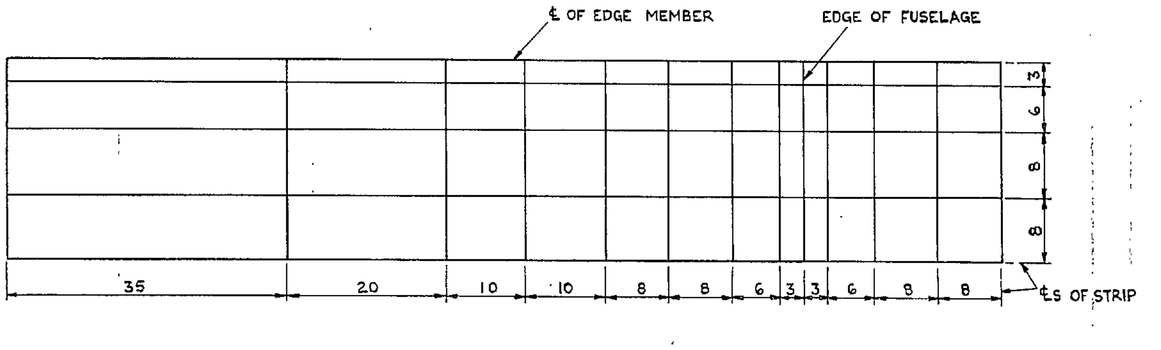

FIG. 3. Grid pattern used in matrix analysis—25 unknowns.

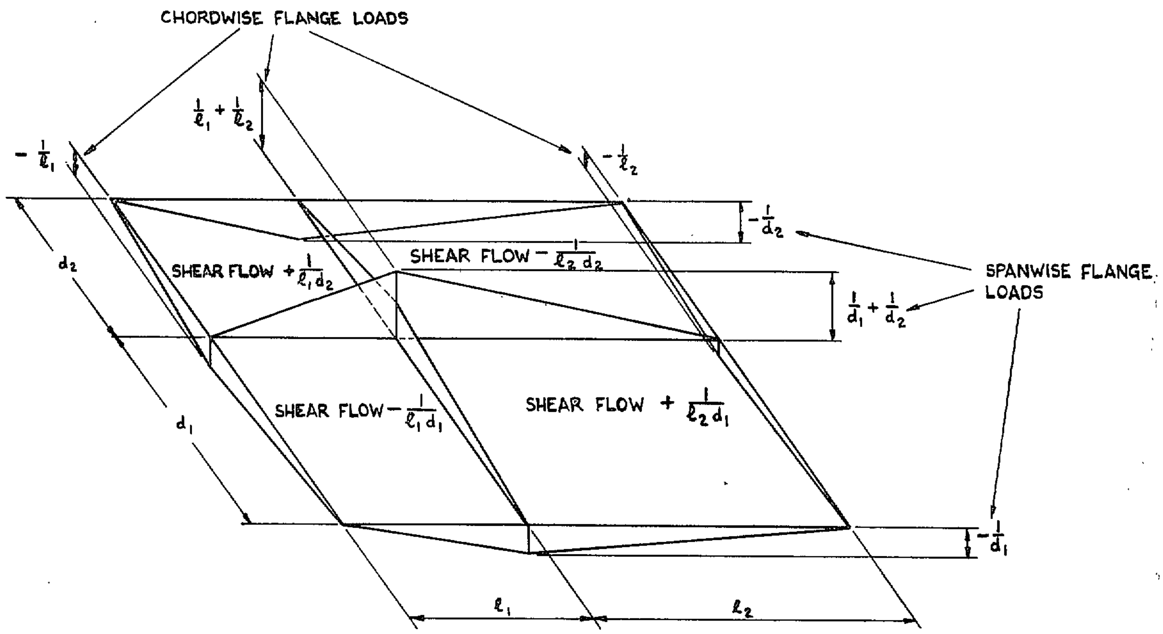

FIG. 4. Unit value of typical self-equilibrating stress system.

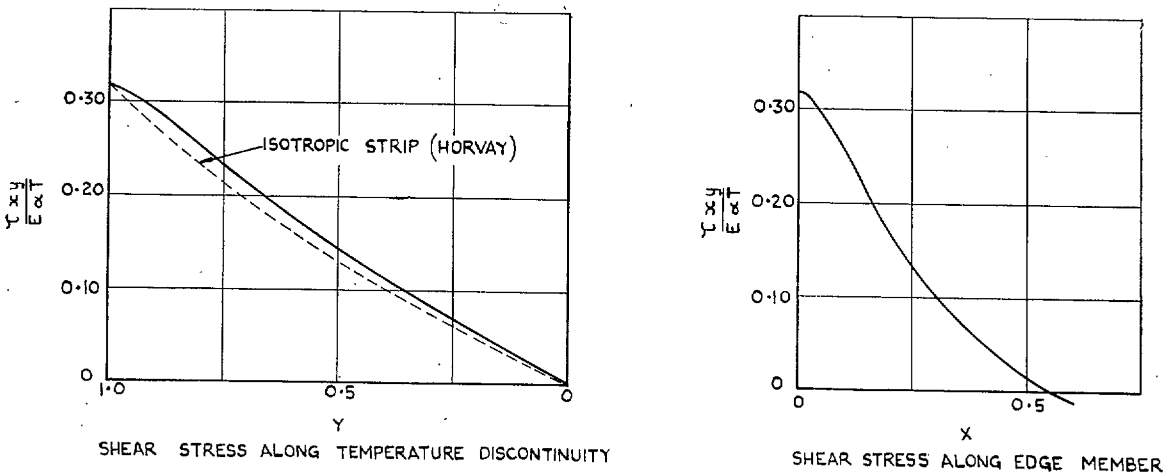

FIG. 5. Shear stresses due to a uniform temperature rise over half the length of an infinite strip with edge members.

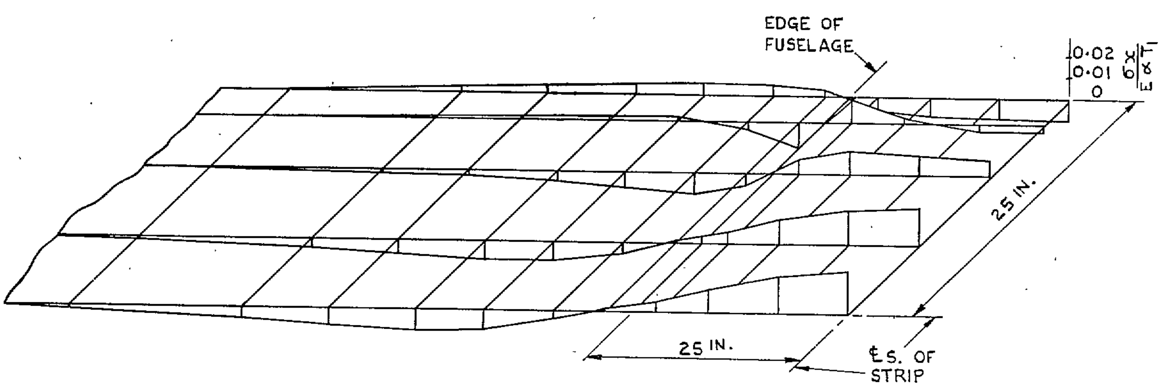

FIG. 6. Spanwise stress in strip—parabolic chordwise temperature distribution outside fuselage.

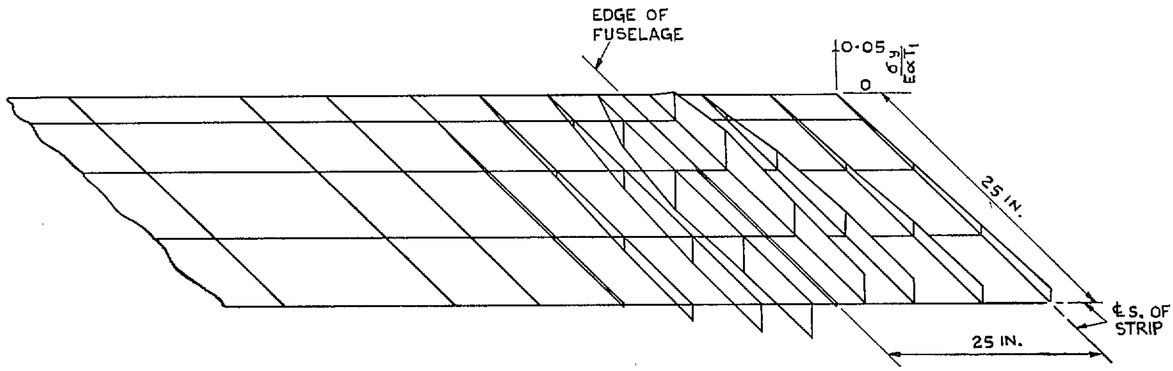

FIG. 7. Chordwise stress in strip—parabolic chordwise temperature outside fuselage.

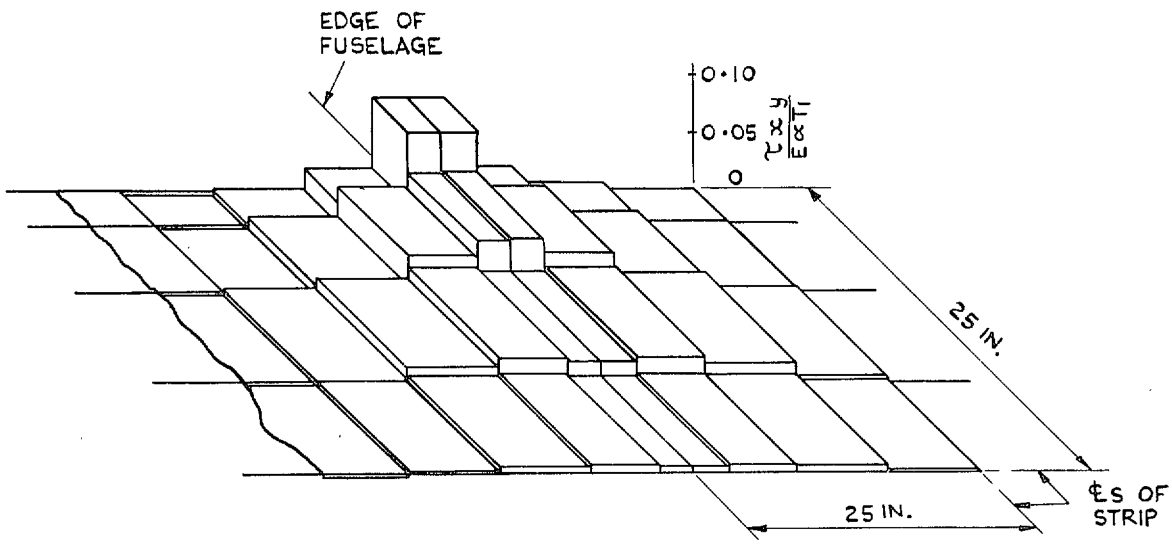

FIG. 8. Shear stress in strip—parabolic chordwise temperature distribution outside fuselage.

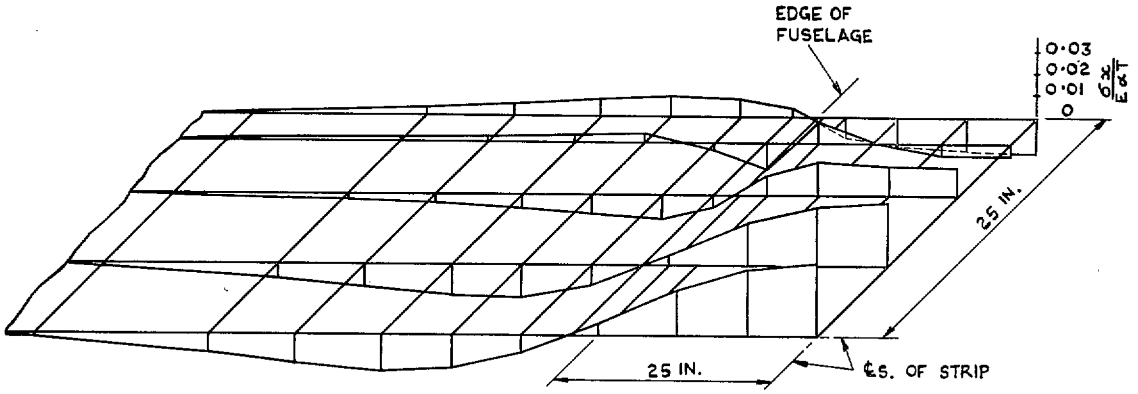

FIG. 9. Spanwise stress in strip—constant chordwise temperature.

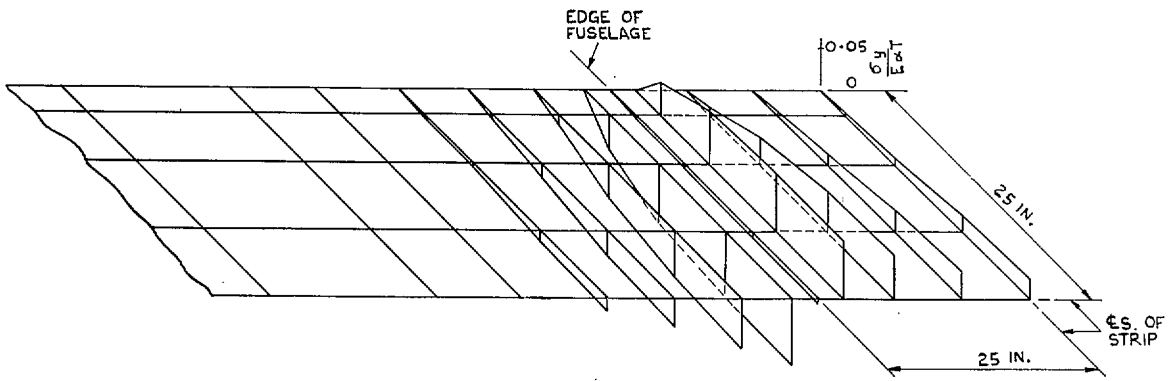

FIG. 10. Chordwise stress in strip—constant chordwise temperature.

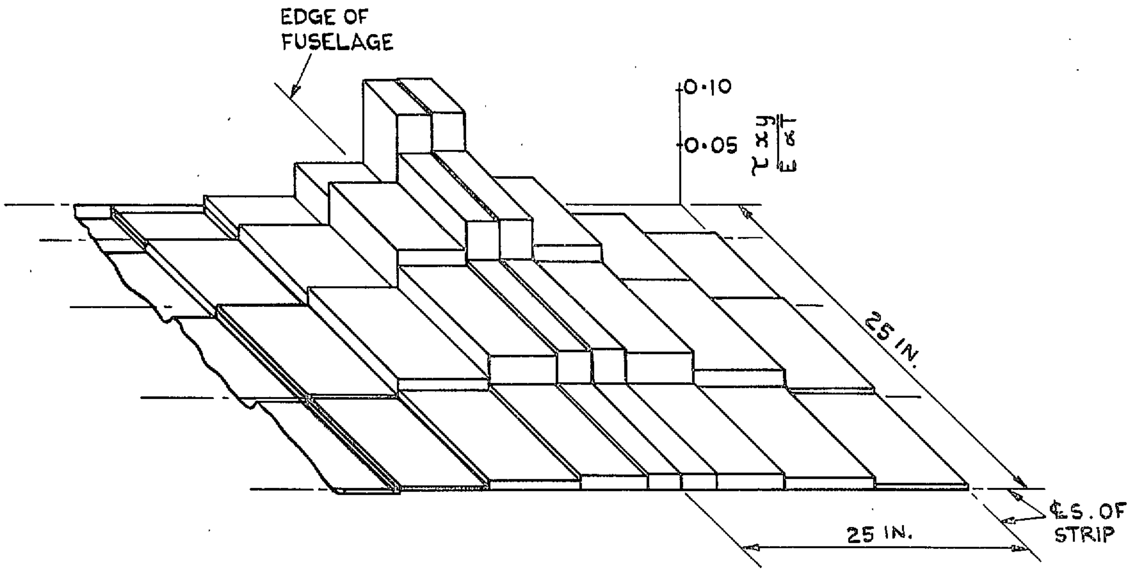

FIG. 11. Shear stress in strip—constant chordwise temperature.

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