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Numerical Methods for Calculating the Zero-Lift Wave Drag and the Lift-Dependent Wave Drag of Slender Wings

By J. WEBER

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Numerical Methods for Calculating the Zero-Lift Wave Drag and the Lift-Dependent Wave Drag of Slender Wings

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Summary. Numerical methods are given for calculating the double integral

$$\int_a^b \int_a^b F(x)F(x') \log |x - x'| \, dx \, dx'$$

for the three cases:

(i) $F(x)$ is given numerically,
(ii) $F(x)$ is the first derivative of a numerically given function,
(iii) $F(x)$ is the second derivative $S''(x)$ of a numerically given function $S(x)$.

For the third case the method of Eminton[3] is extended to functions $S(x)$ for which the first derivative at $x = b$ is not zero. For the other cases the functions are approximated by finite Fourier series which have given values at certain fixed points.

1. *Introduction.* The calculation of the wave drag due to volume as well as that due to lift requires the evaluation of double integrals of the form

$$\int_a^b \int_a^b F(x)F(x') \log |x - x'| \, dx \, dx'.$$

Since in many practical cases the function $F(x)$ is not given in analytical form the integration cannot be performed explicitly but numerical methods must be applied.

There occur three different cases:

(i) The function $F(x)$ is given numerically.
(ii) $F(x)$ is the first derivative $L'(x)$ of a numerically given function $L(x)$.
(iii) $F(x)$ is the second derivative $S''(x)$ of a numerically given function $S(x)$.

There exist several numerical methods to deal with case (i) (*see* for example Refs. 1 and 2). The application of these methods to the third case requires the determination of the second derivative of the given function. Due to the inevitable inaccuracy of the second derivative when determined by numerical or graphical methods, this procedure is often not appropriate.

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In the third case, it seems more advisable to apply the technique of Eminton[3]. The given function $S(x)$ is approximated by one which has the given values $S(x_i)$ at certain fixed positions x_i and which is chosen so as to make the double integral a minimum. To apply this method $S(x_i)$ need be known less accurately and at a considerably smaller number of positions x_i than for the direct application of the numerical techniques developed for case (i). Eminton has treated only cases for which the first derivatives of $S(x)$ at the ends of the range of integration $x = a$ and $x = b$ are zero. In this report, we extend the method to cases for which $S'(b) \neq 0$; an extension to $S'(a) \neq 0$ is not needed for slender configurations.

In cases (i) and (ii) a procedure similar to that in case (iii) is not possible. In case (ii) we approximate $L(x)$ by a finite Fourier series which has the given values $L(x_\mu)$ at fixed points x_μ and express the double integral as a double sum of the products $L(x_\mu)\, L(x_\nu)$ multiplied by fixed coefficients $f_{\mu\nu}$.

Though in case (i) the method of Ref. 1 is directly applicable, we derive another formula by means of a Fourier analysis of $F(x)$ since this seems to be more appropriate in certain cases.

2. *The Numerical Calculation of the Zero-Lift Wave Drag According to Slender-Body Theory.*

2.1. *The Drag Formula.* For slender bodies with a pointed apex, the wave drag due to volume is given by the relation (*see* for example Ref. 4):

$$\frac{D}{q} = -\frac{1}{2\pi}\int_0^1\int_0^1 S''(x)S''(x')\log|x-x'|\,dx\,dx' + \frac{S'(1)}{\pi}\int_0^1 S''(x)\log(1-x)\,dx + \frac{[S'(1)]^2}{2\pi}[k - \log\beta s]\,. \tag{1}$$

The x-axis is taken in the direction of the free stream and the body length is taken as unity. $S(x)$ is the cross sectional area in the plane $x = \text{const}$; $S'(x)$ and $S''(x)$ are the first and second derivatives of $S(x)$ with respect to x.

k depends only on the geometry near the trailing edge (*see* for example Ref. 4). For wings with sharp unswept trailing edge:

$$k = \log 2 - \frac{\int_{-1}^{+1}\int_{-1}^{+1}\epsilon(\eta)\epsilon(\eta')\log|\eta-\eta'|\,d\eta\,d\eta'}{\left[\int_{-1}^{+1}\epsilon(\eta)\,d\eta\right]^2} \tag{2}$$

where

$$\epsilon(y) = \left[\frac{\partial z(x, y)}{\partial x}\right]_{x=1}, \tag{3}$$

$$\eta = \frac{y}{s}\,. \tag{4}$$

x, y, z is a rectangular co-ordinate system, with z normal to the wing plane. s is the semi-span of the wing at the trailing edge.

2.2. *A Numerical Method for Determining the Double Integral*

$$-\frac{1}{2\pi}\int_0^1\int_0^1 S''(x)S''(x')\log|x-x'|\,dx\,dx'.$$

As mentioned in the introduction Eminton[3] has derived a method for calculating the double integral in Equation (1) for area distributions for which the first streamwise derivative is zero at the two ends: $S'(o) = S'(1) = 0$. For wings with unswept trailing edge $S'(1) \neq 0$, except for wings with cusped trailing edge. It is therefore desirable to extend Eminton's method to area distributions with $S'(1) \neq 0$. It is not necessary to consider the case $S'(o) \neq 0$, since the assumption $S'(o) = 0$ is a requirement of slender-body theory which permits only bodies with pointed apex to be treated.

We introduce the co-ordinate ϑ by

$$\cos \vartheta = 1 - 2x. \tag{5}$$

The first derivative of $S(x)$ can be written in the form:

$$S'(x) = \frac{S'(1)}{\pi} \vartheta + \sum_{n=1}^{\infty} a_n \sin n\vartheta. \tag{6}$$

Integrating this relation, we find that the area distribution is given by:—

$$\begin{aligned} S(x) &= S(o) + \frac{S'(1)}{2\pi} (\sin \vartheta - \vartheta \cos \vartheta) + \frac{a_1}{4} (\vartheta - \sin \vartheta \cos \vartheta) \\ &\quad + \frac{1}{4} \sum_{n=2}^{\infty} a_n \left[\frac{\sin (n-1)\vartheta}{n-1} - \frac{\sin (n+1)\vartheta}{n+1} \right]. \end{aligned}$$

$S(x)$ has for $x = 1$, i.e., $\vartheta = \pi$, the value $S(1)$, if

$$a_1 = \frac{4}{\pi} \left[S(1) - S(o) - \frac{S'(1)}{2} \right]. \tag{7}$$

Therefore:

$$\begin{aligned} S(x) &= S(o) + \frac{S(1) - S(o)}{\pi} (\vartheta - \sin \vartheta \cos \vartheta) \\ &\quad - \frac{S'(1)}{2\pi} (1 + \cos \vartheta)(\vartheta - \sin \vartheta) \\ &\quad + \frac{1}{4} \sum_{n=2}^{\infty} a_n \left[\frac{\sin (n-1)\vartheta}{n-1} - \frac{\sin (n+1)\vartheta}{n+1} \right]. \end{aligned} \tag{8}$$

For the double integral the following relation is obtained:

$$\begin{aligned} I_1 &= -\frac{1}{2\pi} \int_0^1 \int_0^1 S''(x) S''(x') \log |x - x'| \, dx \, dx' \\ &= \frac{1}{2\pi} [S'(1)]^2 \log 2 \\ &\quad - \frac{1}{2\pi} \int_0^\pi \int_0^\pi \left[\frac{S'(1)}{\pi} + \sum_{n=1}^{\infty} n a_n \cos n\vartheta \right] \left[\frac{S'(1)}{\pi} + \sum_{n=1}^{\infty} n a_n \cos n\vartheta' \right] \log |\cos \vartheta - \cos \vartheta'| \, d\vartheta \, d\vartheta'. \end{aligned}$$

Using the Relations (3) to (5) of the Appendix and the value of a_1 from Equation (7), we obtain:

$$\begin{aligned} I_1 &= \frac{1}{\pi} [S'(1)]^2 \log 2 + \frac{\pi}{4} \sum_{n=1}^{\infty} n a_n^2 \\ &= \frac{1}{\pi} [S'(1)]^2 \log 2 + \frac{4}{\pi} \left[S(1) - S(o) - \frac{S'(1)}{2} \right]^2 + \frac{\pi}{4} \sum_{n=2}^{\infty} n a_n^2. \end{aligned} \tag{9}$$

We determine the coefficients a_n such that the function $S(x)$, defined by Equation (8), has the prescribed values at $x = 0$, $x = 1$ and at N positions x_i, has the prescribed derivative $S'(1)$ and is such that the integral I_1 has a minimum value for the specified conditions. We determine, therefore, the coefficients a_2, a_3, . . . such that $\sum_{n=2}^{\infty} na_n^2$ has a minimum value and that the equations:

$$\frac{1}{4}\sum_{n=2}^{\infty} a_n \left[\frac{\sin (n-1)\vartheta_i}{n-1} - \frac{\sin (n+1)\vartheta_i}{n+1}\right] =$$
$$S(x_i) - S(o) - \frac{S(1) - S(o)}{\pi}(\vartheta_i - \sin \vartheta_i \cos \vartheta_i) +$$
$$+ \frac{S'(1)}{2\pi}(1+\cos \vartheta_i)(\vartheta_i - \sin \vartheta_i) \,, \tag{10}$$

where $\vartheta_i = \cos^{-1} (1 - 2x_i)$, are satisfied for $i = 1, 2, \ldots N$. A necessary condition for this is the existence of N constant Lagrange multipliers λ_j such that

$$a_n = \frac{1}{n}\sum_{j=1}^{N} \lambda_j \left[\frac{\sin (n-1)\vartheta_j}{n-1} - \frac{\sin (n+1)\vartheta_j}{n+1}\right]. \tag{11}$$

The constants λ_j are determined by inserting the a_n from Equation (11) into the system of Equations (10):

$$\frac{1}{4}\sum_{n=2}^{\infty}\frac{1}{n}\sum_{j=1}^{N} \lambda_j \left[\frac{\sin (n-1)\vartheta_j}{n-1} - \frac{\sin (n+1)\vartheta_j}{n+1}\right]\left[\frac{\sin (n-1)\vartheta_i}{n-1} - \frac{\sin (n+1)\vartheta_i}{n+1}\right] =$$
$$S(x_i) - S(o) - \frac{S(1) - S(o)}{\pi}(\vartheta_i - \sin \vartheta_i \cos \vartheta_i)$$
$$+ \frac{S'(1)}{2\pi}(1+\cos \vartheta_i)(\vartheta_i - \sin \vartheta_i).$$

Applying Relation (6) of the Appendix (derived in Ref. 3) we obtain the following N linear equations for the constants λ_1, λ_2, . . . λ_N:

$$\sum_{j=1}^{N} \lambda_j \left[-\frac{1}{8}(\cos \vartheta_i - \cos \vartheta_j)^2 \log \frac{1 - \cos (\vartheta_i + \vartheta_j)}{1 - \cos (\vartheta_i - \vartheta_j)} + \right.$$
$$\left. + \frac{1}{4}\sin \vartheta_i \sin \vartheta_j (1 - \cos \vartheta_i\vartheta_j)\right] =$$
$$S(x_i) - S(o) - \frac{S(1) - S(o)}{\pi}(\vartheta_i - \sin \vartheta_i \cos \vartheta_i) +$$
$$+ \frac{S'(1)}{2\pi}(1+\cos \vartheta_i)(\vartheta_i - \sin \vartheta_i)$$

or:

$$\sum_{j=1}^{N} \lambda_j \left[-\frac{1}{2}(x_i - x_j)^2 \log \frac{x_i + x_j - 2x_ix_j + 2\sqrt{\{x_ix_j(1-x_i)(1-x_j)\}}}{x_i + x_j - 2x_ix_j - 2\sqrt{\{x_ix_j(1-x_i)(1-x_j)\}}}\right.$$
$$\left. + 2(x_i + x_j - 2x_ix_j)\sqrt{\{x_ix_j(1-x_i)(1-x_j)\}}\right] =$$
$$S(x_i) - S(o) - \frac{S(1) - S(o)}{\pi}\left[\cos^{-1}(1-2x_i) - 2(1-x_i)\sqrt{\{x_i(1-x_i)\}}\right]$$
$$+ \frac{S'(1)}{\pi}(1-x_i)\left[\cos^{-1}(1-2x_i) - 2\sqrt{\{x_i(1-x_i)\}}\right]. \tag{12}$$

Inserting the a_n from Equation (11) into Equation (9) and applying again Relation (6) of the Appendix, we obtain for the approximate value of I_1 the equation:

$$I_1 = \frac{1}{\pi}[S'(1)]^2 \log 2 + \frac{4}{\pi}\left[S(1) - S(o) - \frac{S'(1)}{2}\right]^2$$
$$+ \pi \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i\lambda_j \left[-\frac{1}{8}(\cos\vartheta_i - \cos\vartheta_j)^2 \log\frac{1-\cos(\vartheta_i+\vartheta_j)}{1-\cos(\vartheta_i-\vartheta_j)}\right.$$
$$\left. + \frac{1}{4}\sin\vartheta_i \sin\vartheta_j(1-\cos\vartheta_i\cos\vartheta_j)\right]$$
$$= \frac{1}{\pi}[S'(1)]^2 \log 2 + \frac{4}{\pi}\left[S(1) - S(o) - \frac{S'(1)}{2}\right]^2$$
$$+ \pi \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i\lambda_j \left[-\tfrac{1}{2}(x_i-x_j)^2 \log\frac{x_i + x_j - 2x_ix_j + 2\sqrt{\{x_ix_j(1-x_i)(1-x_j)\}}}{x_i + x_j - 2x_ix_j - 2\sqrt{\{x_ix_j(1-x_i)(1-x_j)\}}}\right.$$
$$\left. + 2(x_i+x_j-2x_ix_j)\sqrt{\{x_ix_j(1-x_i)(1-x_j)\}}\right]. \tag{13}$$

The calculation of the double integral I_1 is thus reduced to solving a system of N linear Equations, (12), and computing a double sum of N^2 terms, Equation (13).

With the notation:

$$u_i = u(x_i) = \frac{1}{\pi}[\cos^{-1}(1-2x_i) - 2(1-2x_i)\sqrt{\{x_i(1-x_i)\}}] \tag{14}$$

$$v_i = v(x_i) = \frac{1}{\pi}(1-x_i)[\cos^{-1}(1-2x_i) - 2\sqrt{\{x_i(1-x_i)\}}] \tag{15}$$

$$p_{ij} = p(x_i, x_j) =$$
$$-\tfrac{1}{2}(x_i-x_j)^2 \log\frac{x_i + x_j - 2x_ix_j + 2\sqrt{\{x_ix_j(1-x_i)(1-x_j)\}}}{x_i + x_j - 2x_ix_j - 2\sqrt{\{x_ix_j(1-x_i)(1-x_j)\}}}$$
$$+ 2(x_i+x_j-2x_ix_j)\sqrt{\{x_ix_j(1-x_i)(1-x_j)\}} \tag{16}$$

$$c_i = c(x_i) = S(x_i) - S(o) - [S(1) - S(o)]u_i + S'(1)v_i \tag{17}$$

the integral I_1 is given by the relation:

$$I_1 = \frac{1}{\pi}[S'(1)]^2 \log 2 + \frac{4}{\pi}\left[S(1) - S(o) - \frac{S'(1)}{2}\right]^2$$
$$+ \pi \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i\lambda_j p_{ij},$$

where the λ_j are determined by the linear system of equations:

$$\sum_{j=1}^{N} \lambda_j p_{ij} = c_i. \tag{18}$$

If $\{f_{ij}\}$ is the inverted matrix of $\{p_{ij}\}$ the solution of Equation (18) is:

$$\lambda_j = \sum_{j=1}^{N} f_{ij}c_i. \tag{19}$$

The final result reads:

$$I_1 = -\frac{1}{2\pi}\int_0^1\int_0^1 S''(x)S''(x')\log|x-x'|\,dx\,dx'$$

$$= \frac{1}{\pi}[S'(1)]^2\log 2 + \frac{4}{\pi}\left[S(1) - S(o) - \frac{S'(1)}{2}\right]^2$$

$$+ \pi \sum_{i=1}^{N}\sum_{j=1}^{N} f_{ij}c_ic_j\,. \tag{20}$$

The coefficients f_{ij} for $x_i = i/20$ ($i = 1, 2 \ldots 19$) are tabulated in Ref. 3. The values of f_{ij} are reproduced from Ref. 3 in Table 1. Table 2 gives the values of u_i and v_i, which are required for calculating c_i from Equation (17). The computation of the double sum is easily done on an automatic computer such as the DEUCE at the Royal Aircraft Establishment, for which a standard programme has been written.

2.3. *Numerical Calculation of the Integral* $\int_0^1 S''(x)\log(1-x)\,dx$. Equation (1) for the zero-lift wave drag contains in addition to the double integral I_1 a single integral:

$$I_2 = \frac{S'(1)}{\pi}\int_0^1 S''(x)\log(1-x)\,dx. \tag{21}$$

One might think of determining I_2 by means of the Fourier series for $S'(x)$, Equation (6), and applying the above minimisation procedure to the sum $I_1 + I_2$ of the double integral I_1 and the single integral I_2 (*i.e.*, approximate the given area distribution by one which has the given values $S(x_i)$, the given $S'(1)$, and which gives the minimum value of the sum $I_1 + I_2$). Such a procedure is however not possible since it leads to non-convergent infinite series in the relations between λ_j and $S(x_i)$.

One may further think of using the Fourier series for $S'(x)$, Equation (6), with the coefficients a_n from Equation (11), and determine I_2 from:

$$I_2 = -\frac{2}{\pi}[S'(1)]^2\log 2 + \frac{4}{\pi}S'(1)\left[S(1) - S(o) - \frac{S'(1)}{2}\right]$$

$$- S'(1)\sum_{n=2}^{\infty}(-1)^n a_n$$

$$= -\frac{2}{\pi}[S'(1)]^2\log 2 + \frac{4}{\pi}S'(1)\left[S(1) - S(o) - \frac{S'(1)}{2}\right]$$

$$- 2\pi S'(1)\sum_{i=1}^{N}\sum_{j=1}^{N} f_{ij}v_ic_j\,. \tag{22}$$

However, the area distribution which approximates to the given area distribution and gives the minimum value of I_1 has infinitely large values of the second derivative at the points x_i where the values of $S(x_i)$ are specified. This property of the approximating area distribution does not affect the accuracy of the approximate value for I_1 (being the minimum under the given conditions) but it may impair the accuracy of the approximate value for I_2 derived from Equation (22). It is important that I_2 should be found with sufficient accuracy since in many cases the value of I_2 is of the same order as that of I_1 but of opposite sign, so that the percentage total error of the drag $\frac{D}{q}$ is much larger than the percentage error of I_1 or I_2.

It seems therefore advisable to determine I_2 by a different method. It is sufficient to deal with area distributions for which the first and second derivative at the trailing edge are finite (slender theory is not applicable to configurations for which the second derivative at the trailing edge is infinite). Such area distributions can be written in the form:

$$\begin{aligned} S(x) = S(o) &+ [3S(1) - 3S(o) - S'(1)]x^2 \\ &- [2S(1) - 2S(o) - S'(1)]x^3 + \Delta S(x) \end{aligned} \tag{23}$$

where the function $\Delta S(x)$ and its first derivative $\Delta' S(x)$ are zero at $x = 0$ and $x = 1$. As a consequence of these properties:

$$\begin{aligned} \int_0^1 \Delta'' S(x) \log(1-x)\, dx &= \int_0^1 \frac{\Delta' S(x)}{1-x}\, dx \\ &= -\int_0^1 \frac{\Delta S(x)}{(1-x)^2}\, dx, \end{aligned} \tag{24}$$

and

$$I_2 = \frac{S'(1)}{\pi} \left\{ 3S(1) - 3S(o) - \frac{5}{2} S'(1) - \int_0^1 \frac{\Delta S(x)}{(1-x)^2}\, dx \right\}. \tag{25}$$

The integrand is finite in the whole range of integration. At $x = 1$:

$$\frac{\Delta S(x)}{(1-x)^2} = 3S(1) - 3S(o) - 2S'(1) + \frac{1}{2} S''(1).$$

The integral can, therefore, be evaluated by the usual numerical methods.

It may be pointed out that for numerically given values of $S(x)$ reliable values of the wave drag can be obtained by determining the first derivative $S'(x)$ by graphical or numerical means and applying Equations (39) and (45) of the following Section with $L(x)$ replaced by $S'(x)$.

3. *The Numerical Calculation of the Lift-Dependent Wave Drag According to the Not-So-Slender-Wing Theory of Adams and Sears*[5]. 3.1. *The Drag Formula.* Applying the so-called 'not-so-slender' wing theory, Adams and Sears[5] derived the following formula for the lift-dependent wave drag:

$$\begin{aligned} \frac{D}{q} = \frac{\beta^2}{8} \Bigg\{ &-\frac{1}{2\pi} \int_0^1 \int_0^1 L'(x) L'(x') \log|x - x'|\, dx\, dx' \\ &+ \frac{L(1)}{\pi} \int_0^1 L'(x) \log(1-x)\, dx \\ &- \frac{1}{2\pi} \int_{-1}^{+1} \int_{-1}^{+1} l(\eta) l(\eta') \log|\eta - \eta'|\, d\eta\, d\eta' \\ &+ \frac{[L(1)]^2}{2\pi} \left[\frac{1}{2} + \log 2 - \log \beta s\right] \Bigg\} \end{aligned} \tag{26}$$

where

$$L(x) = \int_{-s(x)}^{s(x)} l(x, y)\, dy \tag{27}$$

is the cross load,

$$l(x, y) = -\Delta C_p(x, y) \tag{28}$$

is the local load coefficient,

$$l(\eta) = l(x = 1, y) \tag{29}$$

semi

is the load coefficient at the trailing edge, $s(x)$ the local semi-span and s the span at the trailing edge.

In those cases where the distribution of the local total chord load

$$\bar{L}(x) = \int_0^x L(x)\,dx \tag{30}$$

is known, Equations (20) and (25) can be applied, if $S(x)$ is replaced by $\bar{L}(x)$. (Such a case arises when slender-thin-wing theory is applied to design cambered wings with the attachment line along the leading edge. In Ref. 6 it was suggested that an estimate for the lift-dependent wave drag of the wings designed by slender-thin-wing theory might be obtained by inserting the load distribution resulting from slender-thin-wing theory into Equation (26). Within slender theory $\bar{L}(x)$ depends only on the downwash distribution at the station x = const and can thus be determined by a simpler relation than the one for $L(x)$.)

3.2. *A Numerical Method for Determining the Double Integral*

$$-\frac{1}{2\pi}\int_0^1\int_0^1 L'(x)L'(x')\log|x-x'|\,dx\,dx'.$$

In some cases $L(x)$ is a numerically given function and the task is to determine numerically the value of

$$I_3 = -\frac{1}{2\pi}\int_0^1\int_0^1 L'(x)L'(x')\log|x-x'|\,dx\,dx'. \tag{31}$$

We consider $L(x)$ distributions with $L(o) = 0$ and $L(1) \neq 0$. $L(x)$ can be expressed in the same form as $S'(x)$ in Equation (6):

$$L(x) = \frac{L(1)}{\pi}\vartheta + \sum_{n=1}^{\infty} a_n \sin n\vartheta \tag{32}$$

where ϑ is again defined by Equation (5) and

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left[L(x) - L(1)\frac{\vartheta}{\pi}\right]\sin n\vartheta\,d\vartheta\,. \tag{33}$$

By Equation (32) and Relations (3) to (5) of the Appendix we obtain the relation

$$I_3 = \frac{1}{\pi}[L(1)]^2\log 2 + \frac{\pi}{4}\sum_{n=1}^{\infty} n a_n^2 \tag{34}$$

which corresponds to Equation (9).

It is not possible to apply a similar procedure as in Section 2.2 and determine an $L(x)$ distribution which has specified values $L(x_i)$ at given points x_i and which gives a minimum value of the integral I_3 since a non-convergent infinite series does occur if one tries such a procedure.

We refrain therefore from using the infinite Fourier series, Equation (32), for determining the value of I_3. Instead we approximate $L(x)$ by a finite Fourier series of degree $N-1$ (N being an even integer) which has the specified values

$$L_\mu = L(x_\mu) \tag{35}$$

at the positions

$$x_\mu = \frac{1-\cos\vartheta_\mu}{2} \text{ with } \vartheta_\mu = \frac{\mu\pi}{N}\,. \tag{36}$$

This series is given by the relation (*see* for example Ref. 7):—

$$L(x) = \frac{L(1)}{\pi}\vartheta$$

$$+ \sum_{n=1}^{N-1} \frac{2}{N} \sum_{\mu=1}^{N-1} \left(L_\mu - \frac{L(1)}{\pi}\vartheta_\mu\right) \sin n\vartheta_\mu \sin n\vartheta. \tag{37}$$

By Equations (34) and (37) we obtain as an approximate value of I_3:—

$$I_3 = \frac{1}{\pi}[L(1)]^2 \log 2$$

$$+ \frac{\pi}{N^2} \sum_{n=1}^{N-1} n \left\{ \sum_{\mu=1}^{N-1} \left(L_\mu - \frac{L(1)}{\pi}\vartheta_\mu\right) \sin n\vartheta_\mu \right\}^2$$

$$= \frac{1}{\pi}[L(1)]^2 \log 2$$

$$+ \frac{\pi}{N^2} \sum_\mu \sum_\nu \left(L_\mu - \frac{L(1)}{\pi}\vartheta_\mu\right)\left(L_\nu - \frac{L(1)}{\pi}\vartheta_\nu\right) \sum_{n=1}^{N-1} n \sin n\vartheta_\mu \sin n\vartheta_\nu. \tag{38}$$

Now

$$\sum_{n=1}^{N-1} n \cos n\vartheta = \frac{(N-1)\cos N\vartheta - N\cos(N-1)\vartheta + 1}{2(\cos\vartheta - 1)}$$

and

$$\cos N(\vartheta_\mu + \vartheta_\nu) = \cos N(\vartheta_\mu - \vartheta_\nu) = (-1)^{\mu-\nu}$$

$$\sin N(\vartheta_\mu + \vartheta_\nu) = \sin N(\vartheta_\mu - \vartheta_\nu) = 0,$$

so that for $\mu \neq \nu$

$$\sum_{n=1}^{N} n \sin n\vartheta_\mu \sin n\vartheta_\nu$$

$$= \sum_n \frac{n}{2}\left[\cos n(\vartheta_\mu - \vartheta_\nu) - \cos n(\vartheta_\mu + \vartheta_\nu)\right]$$

$$= \frac{1-(-1)^{\mu-\nu}}{4}\left[\frac{1}{\cos(\vartheta_\mu - \vartheta_\nu) - 1} - \frac{1}{\cos(\vartheta_\mu + \vartheta_\nu) - 1}\right]$$

$$= -\frac{1-(-1)^{\mu-\nu}}{2} \frac{\sin\vartheta_\mu \sin\vartheta_\nu}{(\cos\vartheta_\mu - \cos\vartheta_\nu)^2}$$

and for $\mu = \nu$

$$\sum_{n=1}^{N-1} n \sin n\vartheta_\mu \sin n\vartheta_\nu$$

$$= \sum_{n=1}^{N-1} \frac{n}{2}\left[1 - \cos n(2\vartheta_\nu)\right]$$

$$= \frac{N^2}{4}.$$

Thus

$$I_3 = -\frac{1}{2\pi}\int_0^1\int_0^1 L'(x)L'(x')\log|x-x'|\,dx\,dx'$$

$$= \frac{1}{\pi}[L(1)]^2\log 2$$

$$+\pi\sum_{\mu=1}^{N-1}\sum_{\nu=1}^{N-1} f_{\mu\nu}\left(L_\mu - \frac{L(1)}{\pi}\vartheta_\mu\right)\left(L_\nu - \frac{L(1)}{\pi}\vartheta_\nu\right) \tag{39}$$

with

$$f_{\mu\nu} = -\frac{1-(-1)^{\mu-\nu}}{2N^2}\,\frac{\sin\vartheta_\mu\sin\vartheta_\nu}{(\cos\vartheta_\mu-\cos\vartheta_\nu)^2}\quad\text{for}\quad \mu \neq \nu \tag{40}$$

and

$$f_{\nu\nu} = \frac{1}{4}. \tag{41}$$

The coefficients $f_{\mu\nu}$ are tabulated in Table 3 for $N = 36$.

3.3. *Numerical Calculation of the Integral* $\int_0^1 L'(x)\log(1-x)\,dx$. Equation (26) for the lift-dependent wave drag contains in addition to the integral I_3 the term

$$I_4 = \frac{L(1)}{\pi}\int_0^1 L'(x)\log(1-x)\,dx. \tag{42}$$

Since we are only concerned with load distributions for which $L'(1)$ is finite, the integral $\int_0^1 L'(x)\log(1-x)\,dx$ can be written in the form

$$\int_0^1 L'(x)\log(1-x)\,dx = \int_0^1 \frac{L(x)-L(1)}{1-x}\,dx. \tag{43}$$

The integrand $\{L(x)-L(1)\}/(1-x)$ is finite in the interval $0 \leqslant x \leqslant 1$; at $x = 1$ the integrand is equal to $-L'(1)$. Therefore, the usual numerical methods for evaluating the integral can be applied. These require however the knowledge of the function $L(x)$ at positions $x_i = i/N$ at equal distances. For the evaluation of the double integral I_3, we are however using the values of $L(x)$ at the positions x_μ (which are not at equal intervals). It is possible to use the same $L(x_\mu)$ for determining I_4.

By means of Equation (32) and Equations (1) and (2) of the Appendix, we obtain for I_4 the relation:

$$I_4 = -\frac{2}{\pi}[L(1)]^2\log 2 - L(1)\sum_{n=1}^{\infty}(-1)^n a_n. \tag{44}$$

With the approximate series, Equation (37), for $L(x)$:

$$I_4 = -\frac{2}{\pi}[L(1)]^2\log 2$$

$$-L(1)\frac{2}{N}\sum_{\mu=1}^{N-1}\left(L_\mu - \frac{L(1)}{\pi}\vartheta_\mu\right)\sum_{n=1}^{N-1}(-1)^n\sin n\vartheta_\mu.$$

Since

$$\sum_{n=1}^{N-1} \sin n\vartheta = \frac{\sin (N-1)\vartheta - \sin N\vartheta + \sin \vartheta}{2(1-\cos \vartheta)},$$

and N is even

$$\sum_{n=1}^{N-1} (-1)^n \sin n\vartheta_\mu = \Sigma \sin n\,(\pi+\vartheta_\mu)$$

$$= -\frac{1-(-1)^\mu}{2} \frac{\sin \vartheta_\mu}{1+\cos \vartheta_\mu}.$$

Therefore:

$$I_4 = -\frac{2}{\pi}[L(1)]^2 \log 2 + L(1) \sum_{\mu=1}^{N-1} g_\mu \left(L_\mu - \frac{L(1)}{\pi}\vartheta_\mu\right) \tag{45}$$

where

$$g_\mu = \frac{[1-(-1)^\mu] \sin \vartheta_\mu}{N(1+\cos \vartheta_\mu)}. \tag{46}$$

The coefficients g_μ and the positions x_μ are tabulated in Table 4 for $N = 36$.

4. *A Numerical Method for Determining the Double Integral* $\int_{-1}^{+1}\int_{-1}^{+1} f(\eta)f(\eta') \log |\eta - \eta'| \, d\eta \, d\eta'$.

When calculating the zero-lift wave drag by Equations (1) and (2) and the lift-dependent wave drag by Equation (26), we require the value of the double integral

$$I_5 = \int_{-1}^{+1}\int_{-1}^{+1} f(\eta)f(\eta') \log |\eta - \eta'| \, d\eta \, d\eta'. \tag{47}$$

where $f(\eta)$ is a given function. Though in this case, the numerical methods of Refs. 1 and 2 can be applied, we consider here also the calculation of I_5 by means of a Fourier series. We consider here only cases for which

$$f(\eta) = f(-\eta), \tag{48}$$

and for which $f(\eta)$ is finite (only finite values of $\epsilon(\eta)$ in Equation (2) are permissible since Equation (1) is derived from a small-perturbation theory; on lifting wings with attached flow the load at the leading edge must be zero).

We introduce the angular co-ordinate φ by

$$\eta = \cos \varphi. \tag{49}$$

The function

$$g(\eta) = f(\eta) \sin \varphi \tag{50}$$

can be written as a cosine series:

$$g(\eta) = \sum_{\nu=0}^{\infty} b_\nu \cos \nu\varphi. \tag{51}$$

Due to the symmetry of $g(\eta)$, Equation (48), only terms with even values of ν occur. It follows from

$$g(\eta = 1) = 0$$

that

$$b_0 = -\sum_{\nu=2}^{\infty} b_\nu. \tag{52}$$

With Equations (49) to (51) and Equations (3) to (5) of the Appendix, the integral reads:

$$I_5 = \int_0^\pi \int_0^\pi \sum_{\nu=0}^{\infty} b_\nu \cos \nu\varphi \sum_{\mu=0}^{\infty} b_\mu \cos \mu\varphi' \log |\cos \varphi - \cos \varphi'| \, d\varphi \, d\varphi'$$

$$= b_0^2 \int_0^\pi \int_0^\pi \log |\cos \varphi - \cos \varphi'| \, d\varphi \, d\varphi'$$

$$+ 2b_0 \sum_{\nu=2}^{\infty} b_\nu \int_0^\pi \int_0^\pi \cos \nu\varphi \log |\cos \varphi - \cos \varphi'| \, d\varphi \, d\varphi'$$

$$+ \sum_{\nu=2}^{\infty} b_\nu \sum_{\mu=2}^{\infty} b_\mu \int_0^\pi \int_0^\pi \cos \nu\varphi \cos \mu\varphi' \log |\cos \varphi - \cos \varphi'| \, d\varphi \, d\varphi'$$

$$= - b_0^2 \pi^2 \log 2 - \frac{\pi^2}{2} \sum_{\nu=2}^{\infty} \frac{b_\nu^2}{\nu} . \tag{53}$$

A comparison of this equation with the corresponding relations for I_1 and I_3, Equations (9) and (34), shows that for I_5 one can expect a more rapid convergence of the infinite sum than for I_1 and I_3.

Instead of using the infinite Fourier series of Equation (51), it is again appropriate to use an approximate finite Fourier series. (The spanwise load distribution at the trailing edge, $l(\eta)$ in Equation (26), behaves near $\eta = \pm 1$ as $\sqrt{(1-\eta^2)}$ multiplied by a polynomial in η. It seems appropriate to approximate such a function by a finite Fourier series.)

A finite Fourier series which has given values of an even function

$$g_\mu = g(\eta_\mu) \tag{54}$$

at the $N + 1$ positions

$$\eta_\mu = \cos \varphi_\mu, \qquad \varphi_\mu = \frac{\mu\pi}{N}, \qquad 0 \leqslant \mu \leqslant N \tag{55}$$

(N being an even integer) is given by the relation (*see* for example Ref. 8):

$$g(\varphi) = \frac{2}{N} \left\{ \sum_{\mu=1}^{N-1} g_\mu \left[\sum_{\nu=1}^{N-1} \cos \nu\varphi_\mu \cos \nu\varphi + \frac{1 + \cos N\varphi_\mu \cos N\varphi}{2} \right] \right.$$

$$+ \frac{g_0}{2} \left[\sum_{\nu=1}^{N-1} \cos \nu\varphi + \frac{1 + \cos N\varphi}{2} \right]$$

$$\left. + \frac{g_N}{2} \left[\sum_{\nu=1}^{N-1} \cos \nu\pi \cos \nu\varphi + \frac{1 + \cos N\varphi}{2} \right] \right\} . \tag{56}$$

In the present case

$$g_0 = g_N = 0 \tag{57}$$

and by Equation (48)

$$g_\mu = g_{N-\mu} . \tag{58}$$

Thus an approximation to $g(\eta)$ is given by:

$$g(\varphi) = \sum_{\nu=0}^{N} b_\nu \cos \nu\varphi , \qquad \nu \text{ even} \tag{59}$$

where

$$b_0 = \frac{1}{N}\sum_{\mu=1}^{N-1} g_\mu = \frac{2}{N}\sum_{\mu=1}^{(N/2)-1} g_\mu + \frac{1}{N}g(\eta = 0), \tag{60}$$

$$\begin{aligned} b_\nu &= \frac{2}{N}\sum_{\mu=1}^{N-1} g_\mu \cos \nu\varphi_\mu \\ &= \frac{2}{N}\left\{2\sum_{\mu=1}^{(N/2)-1} g_\mu \cos \nu\varphi_\mu + (-1)^{\nu/2} g(\eta = 0)\right\} \\ &\qquad \nu \text{ even}, \quad \neq 0, \quad \neq N \end{aligned} \tag{61}$$

$$\begin{aligned} b_N &= \frac{1}{N}\sum_{\mu=1}^{N-1} (-1)^\mu g_\mu \\ &= \frac{1}{N}\left\{2\sum_{\mu=1}^{(N/2)-1} (-1)^\mu g_\mu + (-1)^{N/2} g(\eta = 0)\right\}. \end{aligned} \tag{62}$$

An approximate value of I_5 is thus:

$$I_5 = -b_0{}^2\pi^2 \log 2 - \frac{\pi^2}{2}\sum_{\nu=2}^{N} \frac{b_\nu{}^2}{\nu} \tag{63}$$

with b_ν from Equations (60)–(62).

The sum $\sum_{\nu=2}^{N-2} \frac{b_\nu{}^2}{\nu}$ could be written as a double sum:

$$\begin{aligned} \sum_{\nu=2}^{N-2} \frac{b_\nu{}^2}{\nu} &= \frac{4}{N^2}\sum_{\mu=1}^{N-1}\sum_{m=1}^{N-1} g_\mu g_m \sum_{\nu=2}^{N-2} \frac{\cos \nu\varphi_\mu \cos \nu\varphi_m}{\nu} \\ &= \sum_{\mu=1}^{N-1}\sum_{m=1}^{N-1} g_\mu g_m c_{\mu m}, \end{aligned}$$

but since there is no short formula for the sum

$$\sum_{n=2}^{N-2} \frac{\cos n\vartheta}{n}$$

the coefficients $c_{\mu m}$ cannot be expressed by explicit formulae. We have therefore not determined numerical values of the coefficients $c_{\mu m}$.

We can draw an interesting consequence of Equations (49) to (51). It follows from Equations (49) to (51) that

$$\int_{-1}^{+1} f(\eta)d\eta = \int_0^\pi f(\eta) \sin \varphi d\varphi = \pi b_0. \tag{64}$$

For the value of

$$k = \log 2 - \frac{\int_{-1}^{+1}\int_{-1}^{+1} f(\eta)f(\eta') \log |\eta - \eta'|\, d\eta\, d\eta'}{\left[\int_{-1}^{+1} f(\eta)d\eta\right]^2} \tag{65}$$

we obtain by Equations (53) and (64):—

$$k = 2\log 2 + \left[\sum_{\nu=2}^{\infty} \frac{b\nu^2}{\nu}\right] \Big/ 2b_0{}^2 \tag{66}$$

2 log 2 is therefore a lower bound for k.

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APPENDIX

List of Formulae Used

$$\int_0^\pi \log|\cos\vartheta - \cos\vartheta'|\, d\vartheta' = -\pi \log 2 \tag{1}$$

$$\int_0^\pi n \cos n\vartheta' \log|\cos\vartheta - \cos\vartheta'|\, d\vartheta' = -\pi \cos n\vartheta \tag{2}$$

$$\int_0^\pi \int_0^\pi \log|\cos\vartheta - \cos\vartheta'|\, d\vartheta\, d\vartheta' = -\pi^2 \log 2 \tag{3}$$

$$\int_0^\pi \int_0^\pi n \cos n\vartheta' \log|\cos\vartheta - \cos\vartheta'|\, d\vartheta\, d\vartheta' = 0 \tag{4}$$

$$\int_0^\pi \int_0^\pi nm \cos n\vartheta \cos m\vartheta' \log|\cos\vartheta - \cos\vartheta'|\, d\vartheta\, d\vartheta' = \begin{cases} 0 \text{ for } m \neq n \\ -\dfrac{\pi^2 n}{2} \text{ for } m = n \end{cases} \tag{5}$$

$$\sum_{n=2}^{\infty} \frac{1}{n} \left[\frac{\sin(n-1)\vartheta_i}{n-1} - \frac{\sin(n+1)\vartheta_i}{n+1}\right] \left[\frac{\sin(n-1)\vartheta_j}{n-1} - \frac{\sin(n+1)\vartheta_j}{n+1}\right]$$
$$= -\frac{1}{2}(\cos\vartheta_i - \cos\vartheta_j)^2 \log\frac{1-\cos(\vartheta_i+\vartheta_j)}{1-\cos(\vartheta_i-\vartheta_j)}$$
$$+ \sin\vartheta_i \sin\vartheta_j (1 - \cos\vartheta_i \cos\vartheta_j) \tag{6}$$

TABLE 1

Coefficients f_{ij} in Equation (20) for $x_i = i/20$ (from Ref. 3)

$$f_{ij} = f_{ji} = f_{20-i,\ 20-j}$$

i \ j	1	2	3	4	5	6	7	8	9	10
1	373·95407									
2	−232·67229	349·20580								
3	59·11797	−228·71539	348·59266							
4	− 6·21210	58·56236	−228·68881	348·65889						
5	1·71557	− 6·12524	58·54661	−228·65259	348·53644					
6	− 0·05310	1·64837	− 6·01561	58·41615	−228·51462	348·44872				
7	0·13417	0·04282	1·52228	− 5·93491	58·37679	−228·53114	348·54097			
8	0·08192	0·06019	0·10337	1·52384	− 5·96591	58·45101	−228·63818	348·59256		
9	0·04876	0·05356	0·06568	0·10359	1·52772	− 6·01177	58·49852	−228·62191	348·58151	
10	− 0·09495	0·11674	0·06586	0·02471	0·10920	1·55578	− 6·01087	58·47954	−228·66922	348·65563
11	0·10798	− 0·07223	0·03153	0·11296	0·07566	0·04202	1·57587	− 6·03869	58·53355	
12	− 0·01183	0·06499	− 0·01238	0·01928	0·05718	0·10362	0·04774	1·59750		
13	0·03950	− 0·07615	0·10275	− 0·01667	− 0·00419	0·13977	0·03849			
14	− 0·06627	0·11247	− 0·11227	0·08243	0·05096	− 0·08652				
15	0·03265	− 0·04182	0·09922	− 0·08761	0·04439					
16	0·01205	0·00956	− 0·06664	0·11193						
17	0·04012	− 0·04716	0·06888							
18	− 0·09825	0·10347								
19	0·09212									

TABLE 2

Coefficients u_i and v_i in Eqvation (17) for $x_i = i/20$

i	u_i	v_i
1	0·018693	0·004577
2	0·052044	0·012462
3	0·094060	0·021985
4	0·142378	0·032415
5	0·195501	0·043251
6	0·252316	0·054092
7	0·311919	0·064587
8	0·373530	0·074416
9	0·436444	0·083271
10	0·500000	0·090845
11	0·563556	0·096826
12	0·626470	0·100886
13	0·688081	0·102668
14	0·747684	0·101776
15	0·804499	0·097751
16	0·857622	0·090037
17	0·905940	0·077925
18	0·947956	0·060418
19	0·981307	0·035884

TABLE 3

Coefficients $N^2 f_{\mu\nu}$ in Equation (39) for $N = 36$ $\qquad f_{\mu\nu} = f_{\nu\mu} = f_{N-\mu,\, N-\nu}$

$\mu \backslash \nu$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	324·000000																	
2	−116·721748	324·000000																
3	0	−126·058997	324·000000															
4	− 9·337247	0	−128·630866	324·000000														
5	0	− 11·909117	0	−129·688512	324·000000													
6	− 2·571869	0	− 12·966763	0	−130·223076	324·000000												
7	0	− 3·629517	0	− 13·501327	0	−130·529641	324·000000											
8	− 1·057647	0	− 4·164080	0	− 13·807892	0	−130·721020	324·000000										
9	0	− 1·592210	0	− 4·470645	0	− 13·999271	0	−130·847881	324·000000									
10	− 0·534565	0	− 1·898775	0	− 4·662024	0	− 14·126132	0	−130·935703	324·000000								
11	0	− 0·841129	0	− 2·090146	0	− 4·788884	0	− 14·213954	0	−130·998422	324·000000							
12	− 0·306565	0	− 1·032508	0	− 2·217015	0	− 4·876707	0	− 14·276672	0	−131·044154	324·000000						
13	0	− 0·497944	0	− 1·159368	0	− 2·304838	0	− 4·939425	0	− 14·322405	0	−131·077871	324·000000					
14	− 0·191379	0	− 0·624804	0	− 1·247191	0	− 2·367556	0	− 4·985158	0	− 14·356123	0	−131·102726	324·000000				
15	0	− 0·318239	0	− 0·712627	0	− 1·309909	0	− 2·413289	0	− 5·018875	0	− 14·380977	0	−131·120766	324·000000			
16	− 0·126860	0	− 0·406063	0	− 0·775345	0	− 1·355642	0	− 2·447006	0	− 5·043730	0	− 14·399017	0	−131·133324	324·000000		
17	0	− 0·214683	0	− 0·468781	0	− 0·821078	0	− 1·389366	0	− 2·471851	0	− 5·061769	0	− 14·411583	0	−131·141286	324·000000	
18	− 0·087823	0	− 0·277401	0	− 0·514514	0	− 0·854796	0	− 1·414214	0	− 2·489900	0	− 5·074336	0	− 14·419537	0	−131·145143	324·000000
19	0	− 0·150541	0	− 0·323135	0	− 0·548231	0	− 0·879650	0	− 1·432258	0	− 2·502466	0	− 5·082290	0	− 14·423394	0	
20	− 0·062718	0	− 0·196274	0	− 0·356852	0	− 0·573085	0	− 0·897691	0	− 1·444820	0	− 2·510420	0	− 5·086146	0		
21	0	− 0·108451	0	− 0·229991	0	− 0·381706	0	− 0·591125	0	− 0·910256	0	− 1·452774	0	− 2·514277	0			
22	− 0·045733	0	− 0·142168	0	− 0·254846	0	− 0·399746	0	− 0·603691	0	− 0·918210	0	− 1·456630	0				
23	0	− 0·079450	0	− 0·167023	0	− 0·272886	0	− 0·412313	0	− 0·611645	0	− 0·922067	0					
24	− 0·033717	0	− 0·104304	0	− 0·185063	0	− 0·285451	0	− 0·420267	0	− 0·615502	0						
25	0	− 0·058571	0	− 0·122344	0	− 0·197629	0	− 0·293406	0	− 0·424123	0							
26	− 0·024854	0	− 0·076611	0	− 0·134911	0	− 0·205583	0	− 0·297263	0								
27	0	− 0·042894	0	− 0·089177	0	− 0·142865	0	− 0·209440	0									
28	− 0·018040	0	− 0·055461	0	− 0·097132	0	− 0·146721	0										
29	0	− 0·030606	0	− 0·063414	0	− 0·100988	0											
30	− 0·012566	0	− 0·038560	0	− 0·067271	0												
31	0	− 0·020520	0	− 0·042417	0													
32	− 0·007954	0	− 0·024377	0														
33	0	− 0·011811	0															
34	− 0·003857	0																
35	0																	

TABLE 4

Coefficients g_μ in Equation (45) and Positions x_μ for $N = 36$

μ	x_μ	g_μ
1	0·00190	0·002426
2	0·00760	0
3	0·01704	0·007314
4	0·03015	0
5	0·04685	0·012316
6	0·06699	0
7	0·09042	0·017517
8	0·11698	0
9	0·14645	0·023012
10	0·17861	0
11	0·21560	0·028920
12	0·25000	0
13	0·28869	0·035393
14	0·32899	0
15	0·37059	0·042629
16	0·41318	0
17	0·45642	0·050907
18	0·50000	0
19	0·54358	0·060628
20	0·58682	0
21	0·62941	0·072401
22	0·67101	0
23	0·71131	0·087205
24	0·75000	0
25	0·78440	0·106721
26	0·82139	0
27	0·85355	0·134123
28	0·88302	0
29	0·90958	0·176200
30	0·93301	0
31	0·95315	0·250595
32	0·96985	0
33	0·98296	0·421986
34	0·99240	0
35	0·99810	1·272431

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