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Formulae for Calculating the Camber Surfaces of Thin Swept-Back Wings of Arbitrary Plan-form with Subsonic Leading Edges, and Specified Load Distribution

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Summary. Formulae for calculating the gradients and ordinates of the camber surfaces of swept-back wings of arbitrary plan-form with subsonic leading edges, and specified load distribution, are given, including those which have been programmed and used for DEUCE calculations for some swept-back and M-wings with curved leading edges.

Some methods for the numerical calculation of singular integrals are given.

For polygonal wings with simple load distributions, the equation of the camber surface is given in closed form. This is useful for obtaining approximate results for more general plan-forms.

1. *Introduction.* This paper is a brief account of a method for calculating the gradients and ordinates of thin swept-back wings of arbitrary plan-form, with subsonic leading edges. No claim is made that anything is new except perhaps the suggested treatment of singularities for numerical integration, and some of the formulae and integrations produced.

Integrals, based on linearised supersonic theory, for calculating the upwash or the streamwise gradients of a wing surface for a given load distribution (the direct problem—Ref. 3) have been known for some time. But until fairly recently, when large scale calculations with the aid of electronic computers became possible, little use had been made of these integrals for wing design. The need of a method for designing camber surfaces to produce specified load distributions has arisen in connection with a research programme on problems associated with flight at low supersonic speeds (Refs. 5, 6).

In this paper will be found:

(1) the basic integrals and formulae for calculating the streamwise gradients and ordinates of a camber surface to support a given load distribution, the wing plan-form being arbitrary, with subsonic leading edges and any trailing edge;

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(2) formulae for calculating the camber surface for some special forms of load distribution such that a first integration can be performed analytically;

(3) formulae in integral form (which have been programmed and used for calculations on DEUCE) for calculating the camber surface of two particular wings: a swept-back wing and an M-wing, with curved subsonic leading edges and straight subsonic trailing edges;

(4) some approximate formulae, in closed form, for the ordinates of the camber surfaces of wings of arbitrary plan-form with subsonic leading edges and uniform chord loading; (these have been used for some check calculations);

(5) some methods for dealing with singularities.

The chief difficulty in programming for the numerical integrations lies in dealing with singularities which do not exist in real flow, but which arise because of the approximate linear theory used. These singularities are of two kinds:

(1) those which arise because of the mathematical form of the problem. These singularities cannot be avoided and can be dealt with by using the concepts of 'finite part', 'Cauchy principal part', or 'generalised principal part' of an integral in the regions of the singularities before numerical integration is attempted. This method is given in Section 5, and has been used in existing DEUCE programmes;

(2) those which arise from the type of load distribution chosen, such as logarithmic singularities in the gradients at leading or trailing edges, or in gradients and ordinates at centre or 'kink' sections (where the gradient of leading and/or trailing edge is discontinuous). These singularities could be avoided if certain load distributions were chosen (e.g., a loading coefficient of the form $f(\xi, \eta) \, . \, (\xi - F(\eta))^{1/2}$). But, in practice, the load distributions required seem to be such that some of these singularities must occur.

A method for calculating the ordinates at points on leading or trailing edges, where there are integrable logarithmic singularities, is given in Section 5.

When the load distribution and plan-form are such that logarithmic singularities in both the streamwise gradients and the ordinates occur at centre or kink sections on a wing (due to the fact that the linearised boundary conditions no longer apply), an iteration method could be used to calculate the camber surface near these sections, including both incidence and thickness. Some calculations done for a particular wing showed that convergence could be obtained, but nothing will be written of this here. Some other methods, which have been used for the particular wings mentioned in Section 3 of this R. & M., are given in Refs. 7, 8.

2. *Calculation of the Shape of the Camber Surface of a Wing of Given Plan-form for a Given Load Distribution.* Considering incidence effects only, and using the linear small perturbation theory of supersonic flow, Volterra (Ref. 1) has shown that the acceleration potential Ω can be expressed in the form

$$\Omega(x, y, z) = \frac{1}{2\pi}\frac{\partial}{\partial x}\int_{\tau}\int \frac{\Delta\Omega(x-\xi)(z-\zeta)\,dS}{[(y-\eta)^2+(z-\zeta)^2]\{(x-\xi)^2-\beta^2[(y-\eta)^2+(z-\zeta)^2]\}^{1/2}}, \tag{1}$$

where $\Omega = V\dfrac{\partial\phi}{\partial x}$, V is the free stream velocity, parallel to the x-axis, x, y, z are a right-handed system of rectangular Cartesian co-ordinates, z being measured upwards, and ϕ is the perturbation velocity potential. M is the free stream Mach number and $\beta^2 = M^2 - 1$. $\Delta\Omega$ is the jump in the value of Ω across the surface of the wing, and integration is over the part, τ, of the wing surface for which $(x-\xi)^2 - \beta^2[(y-\eta)^2 + (z-\zeta)^2] > 0$ and $x - \xi > 0$.

$\Delta\Omega$ is given in terms of the loading coefficient, ΔC_P, by the linearised Bernoulli equation:

$$\Delta C_P = \frac{2}{V^2} \Delta\Omega. \tag{2}$$

The streamwise gradient of the camber surface is given by: (according to the linear theory)

$$\frac{\partial z}{\partial x} = \frac{1}{V} \frac{\partial \phi}{\partial z} = \frac{1}{V^2} \frac{\partial}{\partial z} \int_{-\infty}^{x} \Omega(x, y, z)\, dx. \tag{3}$$

Taking the wing to lie approximately in the plane $\zeta = 0$, the streamwise gradient at point (x, y, z) of the surface is given by

$$\frac{\partial z}{\partial x} = \frac{1}{4\pi} \frac{\partial}{\partial z} \int_\tau \int \frac{\Delta C_P z(x-\xi)\, d\xi\, d\eta}{[(y-\eta)^2 + z^2]\,[(x-\xi)^2 - \beta^2\{(y-\eta)^2 + z^2\}]^{1/2}}\,. \tag{4}$$

The spanwise gradient is given by

$$\frac{\partial z}{\partial y} = \frac{\partial \phi}{\partial z} \bigg/ \frac{\partial \phi}{\partial y}\,. \tag{5}$$

It can be shown that, if terms of order z^2 are neglected, (Ref. 3)

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{1}{4\pi} \overset{\mathrm{P}}{\int} d\eta \underset{\tau}{\overset{\mathrm{P}}{\int}} \frac{\Delta C_P(x-\xi)\, d\xi}{(y-\eta)^2\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}} \\ &= -\frac{\beta}{4} \Delta C_P(x, y) + \frac{1}{4\pi} \overset{\mathrm{P}}{\int} d\xi \underset{\tau}{\overset{\mathrm{P}}{\int}} \frac{\Delta C_P(x-\xi)\, d\eta}{(y-\eta)^2\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}}\,, \end{aligned} \tag{6}$$

where $\overset{\mathrm{P}}{\int}$ denotes the 'generalised principal part of the integral'. (*See* Appendix II.)

Relations (6) are equivalent to

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{\beta^2}{\pi V} \overset{*}{\int} d\eta \underset{\tau}{\overset{*}{\int}} \frac{\phi_u(\xi, \eta)\, d\xi}{\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{3/2}} \\ &= -\frac{\beta}{4} \Delta C_P(x, y) - \frac{\beta^2}{\pi V} \overset{*}{\int} d\xi \underset{\tau}{\overset{*}{\int}} \frac{\phi_u(\xi, \eta)\, d\eta}{\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{3/2}}\,, \end{aligned} \tag{7}$$

where $\overset{*}{\int}$ denotes the 'finite part of the integral', and $\phi_u(\xi, \eta)$ is the velocity potential on the upper surface of the wing. (*See* Appendix II.)

In general, the gradient must be evaluated numerically from (6) or (7). The integrand becomes singular along the Mach lines through the point (x, y) and, in Equations (6), also along the line $\eta = y$. Also, for some forms of ΔC_P, the value of $\partial z/\partial x$ given by Equations (6) or (7) may become infinite along the leading or trailing edges, or along certain lines $\eta =$ constant. Methods of dealing with these singularities are discussed in Section 5.

For some forms of ΔC_P, it may be more convenient to use characteristic co-ordinates (r, s). Taking the Mach lines through the origin as axes of co-ordinates, the transformation formulae are:

$$\begin{aligned} x &= \frac{\beta}{M}(r+s), & r &= \frac{M}{2\beta}(x-\beta y), \\ y &= \frac{1}{M}(s-r), & s &= \frac{M}{2\beta}(x+\beta y), \end{aligned} \tag{8}$$

and the formula for the streamwise gradient is

$$\frac{\partial z}{\partial x} = \frac{\beta}{4\pi} \overset{\mathrm{P}}{\int}_{\tau} \overset{\mathrm{P}}{\int} \frac{\Delta C_P(\overline{r-r_1}+\overline{s-s_1})\, dr_1\, ds_1}{\{\overline{s-s_1}-\overline{r-r_1}\}^2 \{(r-r_1)(s-s_1)\}^{1/2}}, \tag{9}$$

where $r_1 = \dfrac{M}{2\beta}(\xi - \beta\eta)$, $s_1 = \dfrac{M}{2\beta}(\xi + \beta\eta)$, and the region of integration τ is that part of the wing plan-form for which $r - r_1 > 0$ and $s - s_1 > 0$. Relation (9) is equivalent to

$$\frac{\partial z}{\partial x} = -\frac{M}{4\pi V} \overset{*}{\int}_{\tau} \overset{*}{\int} \frac{\phi_u(r_1, s_1)\, dr_1\, ds_1}{\{(r-r_1)(s-s_1)\}^{3/2}}, \tag{10}$$

(which can also be put into other forms).

The ordinates z of the camber surface are found by numerical integration from

$$z = \int \frac{\partial z}{\partial x}\, dx + f(y), \tag{11}$$

where $f(y)$ is a small arbitrary function of y, or

$$z - z_0 = \int_{x0}^{x} \frac{\partial z}{\partial x}\, dx, \tag{12}$$

where, for each value of y, z_0 and x_0 are arbitrary constants.

If ΔC_P is chosen so that a first integration with respect to ξ or η can be performed analytically, $\partial z/\partial x$ can be found by numerical integration with respect to a single variable (η or ξ).

Some special forms of ΔC_P are discussed below. (The y-axis is taken through the foremost point, or points, of the leading edge.)

(1) If

$$\Delta C_P = H_0(\eta) + \sum_n [\xi^n H_n(\eta)], \tag{13}$$

where $H_0(\eta)$, $H_n(\eta)$ are functions of η only, and n is a positive integer,

$$\frac{\partial z}{\partial x} = \frac{1}{4\pi}\left[\overset{\mathrm{P}}{\int_{\eta_1}^{\eta_2}} \frac{H_0(\eta)}{(y-\eta)^2} X_{0,1}\, d\eta + \sum_n \overset{\mathrm{P}}{\int_{\eta_1}^{\eta_2}} \frac{H_n(\eta)}{(y-\eta)^2} X_{n,1}\, d\eta\right], \tag{14}$$

where

$$X_{0,1} = \int_{\xi_1(\eta)}^{\xi_2(\eta)} \frac{(x-\xi)\, d\xi}{\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}} = \Big[-\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}\Big]_{\xi_1}^{\xi_2}, \tag{15a}$$

$$X_{n,1} = \int_{\xi_1(\eta)}^{\xi_2(\eta)} \frac{\xi^n(x-\xi)\, d\xi}{\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}}. \tag{15b}$$

The limits of integration are given by:

$$\left.\begin{aligned} &\xi_1(\eta) = f(\eta); \\ &\xi_2(\eta) = x - \beta|y-\eta| \leqslant g(\eta), \\ \text{or}\quad &\xi_2(\eta) = g(\eta) \leqslant x - \beta|y-\eta|; \end{aligned}\right\} x - f(\eta) \geqslant \beta|y-\eta|$$

$$\xi_1 = \xi_2 = 0 \text{ for } x - f(\eta) < \beta|y-\eta|, \tag{16}$$

where $\xi = f(\eta)$, $\xi = g(\eta)$ are the equations of the leading and trailing edges respectively. η_1, η_2 are given by the equations

$$\begin{aligned} x - f(\eta_1) &= \beta(y-\eta_1), \\ x - f(\eta_2) &= \beta(\eta_2 - y). \end{aligned} \tag{17}$$

η_1, η_2 are the extreme limits of integration for η. Intermediate limits (due to the bounding of the region of integration by other leading or trailing edges) are given by similar equations with the appropriate functions replacing $f(\eta_1)$, $f(\eta_2)$.

Reduction formulae for the calculation of the integrals $X_{n,1}$ are given in Appendix I.

(2) If

$$\Delta C_P = G_0(\xi) + \sum_n [\eta^n G_n(\xi)], \tag{18}$$

where $G_0(\xi)$, $G_n(\xi)$ are functions of ξ only, and n is a positive integer,

$$\frac{\partial z}{\partial x} = -\frac{\beta}{4}\Delta C_P(x, y) + \frac{1}{4\pi}\left[\overset{\mathrm{P}}{\int_0^x} G_0(\xi)(x-\xi)Y_{0,2}\,d\xi + \right.$$

$$\left. + \sum_n \overset{\mathrm{P}}{\int_0^x} G_n(\xi)(x-\xi)Y_{n,2}\,d\xi\right], \tag{19}$$

where

$$Y_{0,2} = \overset{\mathrm{P}}{\int_{\eta_1(\xi)}^{\eta_2(\xi)}} \frac{d\eta}{(y-\eta)^2\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}} = \left[\frac{\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}}{(x-\xi)^2(y-\eta)}\right]_{\eta_1}^{\eta_2}, \tag{20a}$$

$$Y_{n,2} = \int_{\eta_1(\xi)}^{\eta_2(\xi)} \frac{\eta^n\,d\eta}{(y-\eta)^2\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}}. \tag{20b}$$

The limits of integration are given by:

$$\left.\begin{aligned} &\eta_1(\xi) = f^{-1}(\xi) \geqslant y - \frac{x-\xi}{\beta}, \\ \text{or}\quad &\eta_1(\xi) = y - \frac{x-\xi}{\beta} \geqslant f^{-1}(\xi); \\ &\eta_2(\xi) = f^{-1}(\xi) \leqslant y + \frac{x-\xi}{\beta}, \\ \text{or}\quad &\eta_2(\xi) = y + \frac{x-\xi}{\beta} \leqslant f^{-1}(\xi); \end{aligned}\right\} \xi < x$$

$$\eta_1 = \eta_2 = 0 \text{ for } \xi > x, \tag{21}$$

where $\eta = f^{-1}(\xi)$ is the equation of the leading edge. (For a yawed wing η_1 or η_2 might be a point on a trailing edge.)

Reduction formulae for the calculation of the integrals $Y_{n,2}$ are given in Appendix I.

(3) Uniform (or variable) chord loading and load varying linearly along the chord:

$$\begin{aligned} \Delta C_P &= A - \frac{B\,(\text{chordwise distance from L.E.})}{\text{local chord}} \\ &= A - \frac{B\{\xi - f(\eta)\}}{g(\eta) - f(\eta)}, \end{aligned} \tag{22}$$

where $\xi = f(\eta)$, $\xi = g(\eta)$ are the equations of the leading and trailing edges respectively and A, B are constants (or functions of η).

This is a special case of (1) with

$$-H_1(\eta) \equiv +H(\eta) = B/\{g(\eta) - f(\eta)\}$$

and

$$H_0(\eta) = A + f(\eta) \,.\, H(\eta) \equiv h(\eta). \tag{23}$$

It can be shown that the integrated chord loading is given by

$$C_L(y) = A - \tfrac{1}{2}B, \tag{24}$$

and hence the total lift coefficient, when A, B are constant is

$$\bar{C}_L = A - \tfrac{1}{2}B. \tag{25}$$

The chordwise gradient of the camber surface at the point (x, y, z) can be written in the form: (neglecting terms of order z^2)

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\beta}{4\pi}\Sigma \,\mathrm{P}\!\!\int_\eta \left[\frac{H_0(\eta) - xH(\eta)}{|y-\eta|}\sinh u_r + \frac{\beta}{2}H(\eta)(u_r + \tfrac{1}{2}\sinh 2u_r)\right] d\eta \qquad r = 1, 2,\\ &\equiv \Sigma \left[\mathrm{P}\!\!\int_{\eta_a}^{\eta_b} F(x, y, \eta, u_1)d\eta - \mathrm{P}\!\!\int_{\eta_c}^{\eta_d} F(x, y, \eta, u_2)d\eta\right]\\ &\equiv \Sigma\,(G_{a,b})_1 - \Sigma\,(G_{c,d})_2;\end{aligned} \tag{26}$$

the summation is over the different regions of integration (cf. Section 3(b) and Fig. 4), and

$$u_1 = \cosh^{-1}\frac{x - f(\eta)}{\beta\,|y-\eta|}, \qquad u_2 = \cosh^{-1}\frac{x - g(\eta)}{\beta\,|y-\eta|}, \tag{27}$$

$$\eta_a,\ \eta_b \text{ are given by } x - f(\eta_{\substack{a\\b}}) = \beta\,|y - \eta_{\substack{a\\b}}|, \tag{28}$$

$$\eta_c,\ \eta_d \text{ are given by } x - g(\eta_{\substack{c\\d}}) = \beta\,|y - \eta_{\substack{c\\d}}|, \tag{29}$$

that is by the points of intersection of the fore Mach lines through the point (x, y) with the leading and trailing edges respectively. The integrands in Equation (26) are of the form

$$\frac{P(x, \eta)\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}}{(y-\eta)^2} + \frac{\beta}{2}H(\eta)\cosh^{-1}\left\{\frac{x-\xi}{\beta\,|y-\eta|}\right\}, \tag{30}$$

where $\xi = f(\eta)$ or $g(\eta)$, and $P(x, \eta)$ is a function of x and η, and thus become singular along the line $\eta = y$ in the region of integration.

Methods for dealing with these singularities and those which occur at the leading and trailing edges are given in Section 5.

3. *Formulae for Calculating the Shape of the Camber Surface of Two Particular Swept-back Wings with Uniform (or Variable) Chord Loading, and Load Varying Linearly along the Chord.* Formulae for calculating the shape of the camber surface of two particular swept-back wings (*see* Refs. 5, 6), with load varying linearly along the chord, are given.

For each wing:—

The equation of the leading edge is given by $\xi = f(\eta)$, and the equation of the trailing edge by $\xi = g(\eta)$.

The loading coefficient is

$$\Delta C_P = A - \frac{B\{\xi - f(\eta)\}}{g(\eta) - f(\eta)}, \tag{31}$$

where A, B are constants.

(The formulae given below also apply if A, B are functions of η.)

All lengths are measured in semi-span lengths.

Formulae for calculating the gradient, $\partial z/\partial x$, for two swept-back wings, are given below. The ordinate, z, is calculated from Equation (11) or (12).

(*a*) Swept-back wing with partly curved subsonic leading edges and straight subsonic trailing edges (Fig. 1):

The plan-form of the wing is given by

$$\left.\begin{array}{l} 0 \leqslant |\eta| \leqslant \tfrac{1}{2}, f(\eta) = k\,|\eta|; \\ \tfrac{1}{2} \leqslant |\eta| \leqslant 1, f(\eta) = k\,|\eta| + c\,[1 - \{2(1-|\eta|)\}^{1/2}]^2; \end{array}\right\} g(\eta) = c + k\,|\eta|. \tag{32}$$

The length of the root chord is c and the leading edge sweep-back angle is $\tan^{-1}k$.

The streamwise gradient of the camber surface at the point (x, y, z) is given by $(y > 0)$

$$\frac{\partial z}{\partial x} = \frac{\beta}{4\pi}\,\mathrm{P}\!\int_{\eta_1}^{\eta_2}\left[\frac{h(\eta) - xH(\eta)}{|y-\eta|}\sinh u_1 + \frac{\beta}{2}H(\eta)(u_1 + \tfrac{1}{2}\sinh 2u_1)\right]d\eta$$

$$= \mathrm{P}\!\int_{\eta_1}^{\eta_2} F(x, y, \eta, u_1)\,d\eta \equiv (G_{1,2})_1, \quad \text{if } x - \beta y \leqslant c; \tag{33}$$

and

$$\frac{\partial z}{\partial x} = (G_{1,2})_1 - (G_{3,4})_2, \quad \text{if } x - \beta y \geqslant c, \tag{34}$$

where $H(\eta) = B/\{g(\eta) - f(\eta)\}$, $h(\eta) = A + f(\eta)\,.\,H(\eta)$.

The variables u_r and the limits η_r are given by

$$u_1 = \cosh^{-1}\frac{x - f(\eta)}{\beta\,|y-\eta|}, \qquad u_2 = \cosh^{-1}\frac{x - g(\eta)}{\beta\,|y-\eta|}; \tag{35}$$

$$\begin{array}{ll} x - f(\eta_1) = \beta(y-\eta_1), & x - f(\eta_2) = \beta(\eta_2 - y), \\ x - g(\eta_3) = \beta(y-\eta_3), & x - g(\eta_4) = \beta(y-\eta_4), \end{array} \tag{36}$$

η_1, η_3 being < 0 and η_2, $\eta_4 > 0$.

(*b*) The M-wing (Fig. 3):

The plan-form of the wing is given by

$$\begin{array}{lll} 0 \leqslant |\eta| \leqslant \tfrac{1}{2}, & f(\eta) = k_1(\tfrac{1}{2} - |\eta|), & g(\eta) = c_1 + k_2(\tfrac{1}{2} - |\eta|); \\ \tfrac{1}{2} \leqslant |\eta| \leqslant 1, & f(\eta) = k(|\eta| - \tfrac{1}{2}) + c_1[1 - \{2(1-|\eta|)\}^{1/2}]^2, & \\ & g(\eta) = c_1 + k(|\eta| - \tfrac{1}{2}). & \end{array} \tag{37}$$

The leading edges are swept forward at an angle $\tan^{-1}k_1$ (for $0 \leqslant |\eta| \leqslant \tfrac{1}{2}$) and swept-back at angle $\tan^{-1}f'(\eta)\,(\tfrac{1}{2} \leqslant |\eta| \leqslant 1)$; and the trailing edges swept-back at angles $\tan^{-1}k_2(0 \leqslant |\eta| \leqslant \tfrac{1}{2})$ and $\tan^{-1}k(\tfrac{1}{2} \leqslant |\eta| \leqslant 1)$. The length of the 'kink' chord (at $|\eta| = \tfrac{1}{2}$) is c_1, and is such that $c_1 + k/2 < 3\beta/2$ (so that the wing tips at $\eta = +1$ and $\eta = -1$ are outside the Mach lines from the leading edge tips at $\eta = -\tfrac{1}{2}$, $\eta = +\tfrac{1}{2}$ respectively). The length of the root chord is c.

The streamwise gradient of the camber surface at the point (x, y, z) is of the form $(y > 0)$

$$\frac{\partial z}{\partial x} = \Sigma \; \mathrm{P}\!\!\int_{\eta_a}^{\eta_b} F(x, y, \eta, u_r)\, d\eta \equiv \Sigma\, (G_{a,b})_r \qquad (38)$$

$$r = 1 \text{ or } 2$$

where

$$u_1 = \cosh^{-1} \frac{x - f(\eta)}{\beta\,|y - \eta|}, \qquad u_2 = \cosh^{-1} \frac{x - g(\eta)}{\beta\,|y - \eta|}, \qquad (39)$$

and $f(\eta)$, $g(\eta)$ are given by (37).

The formulae for calculating $\partial z/\partial x$ in the different regions of the wing are given below: (*see* Figs. 3, 4)

	Region		$\dfrac{\partial z}{\partial x}$
	$\frac{1}{2} < y < 1$		
1.	$x - \beta y \leqslant c_1 - \frac{1}{2}\beta$		$(G_{1,2})_1$
2.	$x - \beta y \geqslant c_1 - \frac{1}{2}\beta$		$(G_{1,2})_1 - (G_{3,4})_2$
	$0 < y < \frac{1}{2}$		
3.	$x - \beta y \leqslant \frac{1}{2}\beta$,	$x + \beta y \leqslant c_1 + \frac{1}{2}\beta$	$(G_{1,2})_1$
4.	$x - \beta y \leqslant \frac{1}{2}\beta$,	$x + \beta y \geqslant c_1 + \frac{1}{2}\beta$	$(G_{1,2})_1 - (G_{3,4})_2$
5.	$\frac{1}{2}k_1 \geqslant x - \beta y \geqslant \frac{1}{2}\beta$,	$x + \beta y \leqslant c_1 + \frac{1}{2}\beta$	$(G_{1,2})_1 + (G_{5,6})_1$
6.	$\frac{1}{2}k_1 \geqslant x - \beta y \geqslant \frac{1}{2}\beta$,	$x + \beta y \geqslant c_1 + \frac{1}{2}\beta$	$(G_{1,2})_1 + (G_{5,6})_1 - (G_{3,4})_2$
7.	$x - \beta y \leqslant \frac{1}{2}k_1$,	$x + \beta y \leqslant c_1 + \frac{1}{2}\beta$	$(G_{5,2})_1$
8.	$\frac{1}{2}k_1 \leqslant x - \beta y \leqslant c_1 + \frac{1}{2}\beta$,	$x + \beta y \geqslant c_1 + \frac{1}{2}\beta$	$(G_{5,2})_1 - (G_{3,4})_2$
9.	$c + \frac{1}{2}k_1 \geqslant x - \beta y \geqslant c_1 + \frac{1}{2}\beta$		$(G_{5,2})_1 - (G_{3,4})_2 - (G_{7,8})_2$

η_r $(r = 1, 2, \ldots 8)$ are given by the equations

$$\begin{aligned} x - f(\eta_r) &= \beta\,|y - \eta_r|, & r &= 1, 6 \quad (0 \leqslant |\eta_r| \leqslant \tfrac{1}{2}) \\ & & r &= 2, 5 \quad (\tfrac{1}{2} \leqslant |\eta_r| \leqslant 1) \\ x - g(\eta_r) &= \beta\,|y - \eta_r|, & r &= 3, 8 \quad (0 \leqslant |\eta_r| \leqslant \tfrac{1}{2}) \\ & & r &= 4, 7 \quad (\tfrac{1}{2} \leqslant |\eta_r| \leqslant 1) \end{aligned} \qquad (40)$$

The formulae given above for calculating the camber surface of wings (*a*), (*b*) have been programmed and used for calculations on a DEUCE computer.

4. *Approximate Formulae for the Calculation of the Ordinates of the Camber Surfaces of Wings with Subsonic Leading Edges and Subsonic or Supersonic Trailing Edges, with Uniform Chord Loading.* An approximate method for the calculation of the ordinates of the camber surface of a wing of any plan-form with subsonic leading edges and any trailing edge is suggested, whereby curved leading and trailing edges are approximated by polygons. It is also useful in some cases to use an approximate formula for the prescribed load distribution, so that formulae for the gradients and ordinates at points on the wing can be obtained in closed form.

The formulae can be used for calculations on a desk machine or for Interpretive Scheme calculations on a DEUCE computer. This method was used for some preliminary calculations for both the swept-back wing and the M-wing mentioned in Section 3; these served as a partial check on the DEUCE calculations for which the integrals given in that section were programmed.

The resulting approximate formulae for any wing with prescribed uniform chord loading are given below. Similar formulae could be derived for other loadings.

Using the notation of Section 3, the loading coefficient is

$$\Delta C_P = A - \frac{B(\xi - f(\eta))}{\text{local chord}}, \tag{41}$$

where A and B are constants.

The vertices of the approximating leading-edge and trailing-edge polygons are on the edges at points $y = y_r$ $(r = 0, 1, 2, \ldots)$, and the slope of a leading or trailing-edge segment is given by

$$k_r = (x_{r+1} - x_r)/(y_{r+1} - y_r), \quad (x_{r+1} > x_r), \tag{42}$$

where $x_r = f(y_r)$ on a leading edge,
and $x_r = g(y_r)$ on a trailing edge. (*See* Fig. 2.)

The local chord of the segment of the wing defined by $y_r \lessgtr y \lessgtr y_{r+1}$ is approximated by the average chord

$$c_r = \tfrac{1}{2}\{g(y_r) + g(y_{r+1}) - f(y_r) - f(y_{r+1})\}. \tag{43}$$

[Alternatively, circumscribing polygons could be taken, the points of tangency being (say) at $y = \frac{1}{2}(y_{r+1} + y_r)$.]

Writing

$$\lambda_r = |k_r|/\beta,$$

$$\vartheta_r = \beta(y - y_r)/(x - x_r), \qquad u_r = \cosh^{-1}\frac{1}{|\vartheta_r|}, \tag{44}$$

$$P_r = \left(\frac{\lambda_{r-1} - 1}{\lambda_{r-1} + 1}\right)^{1/2}\left(\frac{1 + \vartheta_r}{1 - \vartheta_r}\right)^{1/2}, \qquad Q_r = \left(\frac{\lambda_r - 1}{\lambda_r + 1}\right)^{1/2}\left(\frac{1 + \vartheta_r}{1 - \vartheta_r}\right)^{1/2},$$

$$p_r = \tfrac{1}{2}\log\left|\frac{1 + P_r}{1 - P_r}\right|, \qquad q_r = \tfrac{1}{2}\log\left|\frac{1 + Q_r}{1 - Q_r}\right|, \tag{45}$$

the formula for the ordinate z at the point (x, y) of the plan-form can be written in the form

$$z = \frac{\beta}{4\pi}\sum_r\left[A(x - x_r)R_1(\vartheta_r) + \frac{B}{2}(x - x_r)^2 R_2(\vartheta_r)\right]_L$$

$$- \frac{\beta}{4\pi}\sum_r\left[(A - B)(x - x_r)R_1(\vartheta_r) + \frac{B}{2}(x - x_r)^2 R_2(\vartheta_r)\right]_T, \tag{46}$$

with $x_r = f(y_r)$ in the first sum $[L]$, and $x_r = g(y_r)$ in the second sum $[T]$. $R_1(\vartheta_r)$, $R_2(\vartheta_r)$ are functions of ϑ_r given below. The summation of each sum is for values of r for which both $x_r - \beta y_r < x - \beta y$ and $x_r + \beta y_r < x + \beta y$.

For points (x, y) on the wing plan-form for which $x \pm \beta y \leqslant g(y_r) \pm \beta y_r$, or for a wing with all supersonic trailing edges, the second sum $[T]$ does not appear.

The functions R_1, R_2 are given by

$$R_1(\vartheta_r) = (a) + (b) \tag{47}$$

$$R_2(\vartheta_r) = \left(\frac{1}{c_r} - \frac{1}{c_{r-1}}\right)(c) + \left(\frac{(d)}{c_{r-1}} - \frac{(e)}{c_r}\right), \tag{48}$$

where, for $k_r > 0$, $k_{r-1} > 0$:

$$(a) = -(\lambda_r - \lambda_{r-1})\{u_r - \sqrt{(1-\vartheta_r^2)}\} + (\lambda_r^2 - \lambda_{r-1}^2)\vartheta_r u_r$$

$$(b) = -2\sqrt{(\lambda_{r-1}^2 - 1)}\,.\,(1-\lambda_{r-1}\vartheta_r)p_r + 2\sqrt{(\lambda_r^2-1)}\,.\,(1-\lambda_r\vartheta_r)q_r$$

$$(c) = \vartheta_r u_r + \frac{(1-\vartheta_r^2)^{3/2}}{3\vartheta_r}$$

$$(d) = \frac{\lambda_{r-1}}{2}(3-2\lambda_{r-1}\vartheta_r)\sqrt{(1-\vartheta_r^2)} - \frac{\lambda_{r-1}u_r}{2}\{(2\lambda_{r-1}^2-1)\vartheta_r^2 - 4\lambda_{r-1}\vartheta_r + 2\} + 2\sqrt{(\lambda_{r-1}^2-1)}\,.\,(1-\lambda_{r-1}\vartheta_r)^2 p_r$$

$$(e) = \frac{\lambda_r}{2}(3-2\lambda_r\vartheta_r)\sqrt{(1-\vartheta_r^2)} - \frac{\lambda_r u_r}{2}\{(2\lambda_r^2-1)\vartheta_r^2 - 4\lambda_r\vartheta_r + 2\} + 2\sqrt{(\lambda_r^2-1)}\,.\,(1-\lambda_r\vartheta_r)^2 q_r. \tag{49}$$

When $k_r < 0$, replace λ_r by $-\lambda_r$, $\sqrt{(\lambda_r^2-1)}$ by $-\sqrt{(\lambda_r^2-1)}$, and $\left(\frac{\lambda_r-1}{\lambda_r+1}\right)^{1/2}$ by $\left(\frac{\lambda_r+1}{\lambda_r-1}\right)^{1/2}$, with similar replacements (with suffix $r-1$) for $k_{r-1} < 0$.

A sufficient number of points y_r might be chosen so that the points (x, y) at which the ordinates are to be calculated lie on the middle chords of the segments of the wing. Then

$$y = \tfrac{1}{2}(y_r + y_{r+1}), \quad \text{and} \quad \vartheta_r = \frac{\beta(y_{r+1}-y_r)}{2(x-x_r)}.$$

Also the approximation to the local chord might be taken as

$$c_r = g\left(\frac{y_{r+1}+y_r}{2}\right) - f\left(\frac{y_{r+1}+y_r}{2}\right).$$

If *point* (x, y) lies *on a leading-edge or trailing-edge* segment of slope $\pm k_r$, $\vartheta_r = \pm 1/\lambda_r$, $u_r = \cosh^{-1}\lambda_r$, and the last terms in (b) and (e), $[(1-\lambda_r\vartheta_r)q_r, (1-\lambda_r\vartheta_r)^2 q_r]$, tend to zero.

(Point (x_r, y_r) on an edge would be taken as a point on segment of slope k_{r-1}.)

If $\lambda_{r-1} = \lambda_r$ and $c_{r-1} \neq c_r$:

$$(a) = (b) = 0; \qquad (e) = (d),$$

and

$$R_1(\vartheta_r) = 0; \qquad R_2(\vartheta_r) = \left(\frac{1}{c_r} - \frac{1}{c_{r-1}}\right)((\mathrm{c}) - (\mathrm{d})).$$

If $\lambda_{r-1} = -\lambda_r$ and $c_{r-1} = c_r$

$$\text{(a)} = -2\lambda_r\{u_r - \sqrt{(1-\vartheta_r^2)}\}$$

$$\text{(b)} = 2\sqrt{(\lambda_r^2-1)}\,(v_r - \lambda_r\vartheta_r w_r)$$

$$\left(\frac{1}{c_r} - \frac{1}{c_{r-1}}\right)\text{(c)} = 0$$

$$R_1(\vartheta_r) = 2\left[-\lambda_r\{u_r - \sqrt{(1-\vartheta_r^2)}\} + \sqrt{(\lambda_r^2-1)}\,(v_r - \lambda_r\vartheta_r w_r)\right] \tag{50}$$

$$R_2(\vartheta_r) = \frac{2}{c_r}\left[\frac{\lambda_r u_r}{2}\{2 + (2\lambda_r^2-1)\vartheta_r^2\} - \frac{3\lambda_r}{2}\sqrt{(1-\vartheta_r^2)} + \sqrt{(\lambda_r^2-1)}\{2\lambda_r\vartheta_r w_r - (1+\lambda_r^2\vartheta_r^2)v_r\}\right], \tag{51}$$

where

$$v_r = \tfrac{1}{2}\log\left|\frac{1+V_r}{1-V_r}\right|, \qquad w_r = \tfrac{1}{2}\log\left|\frac{1+W_r}{1-W_r}\right|;$$

$$V_r = \frac{\sqrt{(\lambda_r^2-1)}}{\lambda_r}\cdot\frac{1}{\sqrt{(1-\vartheta_r^2)}}, \qquad W_r = \frac{\vartheta_r\sqrt{(\lambda_r^2-1)}}{\sqrt{(1-\vartheta_r^2)}}.$$

When point (x, y) lies on the leading or trailing-edge segment (slope k_r), (50), (51) become

$$R_1(\vartheta_r) = 2\{\sqrt{(\lambda_r^2-1)} - \lambda_r\cosh^{-1}\lambda_r\} \tag{52}$$

$$R_2(\vartheta_r) = \frac{1}{c_r}\left[\frac{(4\lambda_r^2-1)}{\lambda_r}\cosh^{-1}\lambda_r - 3\sqrt{(\lambda_r^2-1)}\right] \tag{53}$$

For the swept-back wing (Figs. 1, 2) for which $g(\eta) = c + k|\eta|$ and, for $0 \leqslant \eta \leqslant \frac{1}{2}$, $f(\eta) = k|\eta|$, for all points (x, y) on the wing such that $x \pm \beta y \leqslant \frac{1}{2}(k+\beta)$, (*i.e.*, all points upstream of the after Mach lines from leading-edge points $y = \pm\frac{1}{2}$) the *exact solution* of the linearised equation is obtained from (46) with (50)–(53) by putting $r = 0$, $x_0 = y_0 = 0$, $y_1 = \frac{1}{2}$, $k_r = k = -k_{r-1}$ in sum $[L]$ when $x \pm \beta y < c$, and in both sums $[L]$, $[T]$ when $x \pm \beta y > c$.

It is not suggested that it would be better to use approximate formulae than to use the integral forms, except perhaps for parts of the wing where the approximate formulae become exact. The above formulae were used to obtain some preliminary results fairly quickly before a DEUCE programme had been made. They were also found useful later for checking.

5. *The Numerical Evaluation of some Singular Integrals.* Analytical methods for evaluating the 'finite part' of an integral, the 'Cauchy principal part' and the 'generalised principal part' of an integral are well known (*see* Refs. 3, 4 for example) and details will not be given here. For purposes of reference, definitions and relevant formulae are given in Appendix II.

For the integrals in Equation (7) in Section 2, one of the usual methods for evaluating the finite part of an integral can be used (*see* Appendix II). Equations (6) are derived from (7) by an integration by parts and evaluation of the finite part.

Integrals discussed here are those which arise from Equations (6):

(1) when integration with respect to ξ is performed analytically;
(2) when integration with respect to η is performed analytically;
(3) when the double integral is evaluated numerically;
(4) when point (x, y) is on a leading or trailing edge.

(1) If $\Delta C_P(\xi, \eta)$ is of such a form that a first integration with respect to ξ can be performed analytically, in general, integrals of the form

$$\overset{\mathrm{P}}{\int_{\eta_1}^{\eta_2}} \frac{P(x, \eta)\{(x-F(\eta))^2 - \beta^2(y-\eta)^2\}^{1/2}}{(y-\eta)^2}\, d\eta$$

or

$$\overset{\mathrm{P}}{\int_{\eta_1}^{\eta_2}} P(x, \eta) \cosh^{-1} \frac{x - F(\eta)}{\beta\,|y-\eta|}\, d\eta$$

have to be evaluated numerically. $\xi = F(\eta)$ is the equation of a leading or trailing edge, and the functions $P(x, \eta)$, $F(\eta)$ are continuous in the range $\eta_1 \leqslant \eta \leqslant \eta_2$.

For the numerical evaluation of these integrals when $\eta_1 < y < \eta_2$, write

$$\overset{\mathrm{P}}{\int_{\eta_1}^{\eta_2}} \equiv \int_{\eta_1}^{a} + \int_{b}^{\eta_2} + \overset{\mathrm{P}}{\int_{a}^{b}},$$

where a, b are suitably chosen, and $\eta_1 < a < y < b < \eta_2$. (It is usually not convenient to take $a = \eta_1$ or $b = \eta_2$, one reason being that the integrand may diverge for a different reason near $\eta = \eta_1$ or η_2.) The integrands in the first and second integrals are finite and cause no difficulty. The third integral can be put in the form

$$\begin{aligned}
&\overset{\mathrm{P}}{\int_{a}^{b}} \frac{P(x, \eta) \,.\, R(x, y, \eta)}{(y-\eta)^2}\, d\eta \\
&\quad = -\frac{1}{b-y} P(x, b) \,.\, R(x, y, b) - \frac{1}{y-a} P(x, a) \,.\, R(x, y, a) \\
&\qquad + [P'(x, y)(x-F(y)) - P(x, y) \,.\, F'(y)] \log \frac{b-y}{y-a} - \beta^2 \int_a^b \frac{P(x, \eta)}{R(x, y, \eta)}\, d\eta \\
&\qquad - \int_a^b \frac{P'(x, \eta) \,.\, R(x, y, \eta) - P'(x, y)(x-F(y))}{y-\eta}\, d\eta \\
&\qquad + \int_a^b \left[\frac{P(x, \eta)(x-F(\eta))F'(\eta)}{R(x, y, \eta)} - P(x, y) \,.\, F'(y)\right] \frac{1}{y-\eta}\, d\eta,
\end{aligned} \tag{54}$$

where $R(x, y, \eta) \equiv \{(x-F(\eta))^2 - \beta^2(y-\eta)^2\}^{1/2}$, and the dash indicates differentiation with respect to η or y.

The integrands of the three integrals in (54) are finite at $\eta = y$ provided that $P(x, \eta)$ and $P'(x, \eta)$ exist and are single-valued at $\eta = y$. A formula similar to (54) has been used in a DEUCE programme.

Another form (useful if the expansion converges sufficiently rapidly near $\eta = y$) can be obtained by expanding $R(x, y, \eta)$ as a power series in $(y-\eta)$. Thus

$$\begin{aligned}
&\overset{\mathrm{P}}{\int_{a}^{b}} \frac{P(x, \eta) \,.\, R(x, y, \eta)}{(y-\eta)^2}\, d\eta \\
&\quad = \overset{\mathrm{P}}{\int_{a}^{b}} \frac{P(x, \eta)(x-F(\eta))}{(y-\eta)^2} \left[1 - \frac{1}{2}\frac{\beta^2(y-\eta)^2}{(x-F(\eta))^2} - \frac{1}{8}\frac{\beta^4(y-\eta)^4}{(x-F(\eta))^4} - \cdots\right] d\eta.
\end{aligned} \tag{55}$$

The first integral

$$\overset{\mathrm{P}}{\int_a^b} \frac{P(x, \eta)(x - F(\eta))}{(y-\eta)^2}\, d\eta \equiv \overset{\mathrm{P}}{\int_a^b} \frac{P_1(x, \eta)}{(y-\eta)^2}\, d\eta$$

is equal to

$$-\frac{P_1(x, b)}{b-y} - \frac{P_1(x, a)}{y-a} + P_1'(x, y) \log \frac{b-y}{y-a} - \int_a^b \frac{P_1'(x, \eta) - P_1'(x, y)}{y-\eta}\, d\eta. \tag{56}$$

The integrands in (56) and the remaining terms of the series in (55) are finite or zero at $\eta = y$. The numerical evaluation of the integral

$$\overset{\mathrm{P}}{\int_a^b} P(x, \eta) \cosh^{-1} \frac{x - F(\eta)}{\beta\, |y-\eta|}\, d\eta$$

is fairly simple (provided that ΔC_P and $F(\eta)$ are suitably chosen). One method is to write the integral in the form

$$\begin{aligned}
&\overset{\mathrm{P}}{\int_a^b} P(x, \eta) \cosh^{-1} \frac{x - F(\eta)}{\beta\, |y-\eta|}\, d\eta \\
&\quad = (b-y)\, P(x, b) \cosh^{-1} \frac{x - F(b)}{\beta(b-y)} + (y-a)\, P(x, a) \cosh^{-1} \frac{x - F(a)}{\beta(y-a)} \\
&\quad\quad + \int_a^b \frac{P(x, \eta)\,[x - F(\eta) - F'(\eta)(y-\eta)]}{R(x, y, \eta)}\, d\eta \\
&\quad\quad + \int_a^b P'(x, \eta)(y-\eta) \cosh^{-1} \frac{x - F(\eta)}{\beta\, |y-\eta|}\, d\eta,
\end{aligned} \tag{57}$$

in which the integrands are finite at $\eta = y$.

(2) If a first integration with respect to η is possible by analytical methods, integrals of the form

$$\overset{\mathrm{P}}{\int_0^x} \frac{Q(x, y, \xi)}{x - \xi}\, d\xi$$

occur, the function $Q(x, y, \xi)$ being continuous in the range $0 \leqslant \xi \leqslant x$, and generally (but not always) of such a form that the integral converges near $\xi = x$. The treatment of this integral depends on the particular forms of $\Delta C_P(\xi, \eta)$ and $F(\eta)$.

(3) If the double integral in Equation (6) is evaluated numerically:—

The double integral to be evaluated is

$$\overset{\mathrm{P}}{\int}_\tau \overset{\mathrm{P}}{\int} \frac{\Delta C_P(\xi, \eta)\,(x-\xi)\, d\xi\, d\eta}{(y-\eta)^2\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}},$$

where the region of integration, τ, is the region on the wing plan-form for which $x - \xi \geqslant \beta\, |y-\eta|$. This integral can be written in the form

$$\begin{aligned}
&\overset{\mathrm{P}}{\int_{\eta_1}^{\eta_2}} \left[- \{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}\, \Delta C_P(\xi, \eta) \right]_{\xi_1}^{\xi_2} \frac{1}{(y-\eta)^2}\, d\eta \\
&\quad + \overset{\mathrm{P}}{\int}_\tau \overset{\mathrm{P}}{\int} \frac{\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2} \dfrac{\partial}{\partial \xi} \Delta C_P(\xi, \eta)}{(y-\eta)^2}\, d\xi\, d\eta.
\end{aligned} \tag{58}$$

$\xi_1 = f(\eta)$ and $\xi_2 = x - \beta\,|y-\eta\,|$ or $g(\eta)$, and the first integral is of the form

$$\mathrm{P}\int_{\eta_1}^{\eta_2} \frac{P(x, y, \eta) \,.\, R(x, y, \eta)}{(y-\eta)^2}\, d\eta, \qquad \eta_1 < y < \eta_2,$$

the same form as given in (54) and (55).

The integrand of the second integral in (58) is finite in the region of integration except near $\eta = y$, ($\Delta C_P(\xi, \eta)$ being suitably chosen). For integration over a strip $a \leqslant \eta \leqslant b$, the integral could be written in the form

$$\int_a^b \int_{f(\eta)}^{x-\beta\,|\,y-\eta\,|} \left[\frac{\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2} \frac{\partial}{\partial\xi}\Delta C_P(\xi, \eta) - (x-\xi)\frac{\partial}{\partial\xi}\Delta C_P(\xi, \eta)}{(y-\eta)^2}\right] d\xi\, d\eta$$

$$+ \int_{\xi_1}^{x} (x-\xi)\left(\mathrm{P}\int_{a(\xi)}^{b(\xi)} \frac{\frac{\partial}{\partial\xi}\Delta C_P(\xi, \eta)}{(y-\eta)^2}\, d\eta\right) d\xi, \tag{59}$$

where $\xi_1 = f(\mathrm{a})$ or $f(\mathrm{b})$ (whichever is least), and in the second integral $a(\xi)$, $b(\xi)$ vary over part of the range $\xi_1 \leqslant \xi \leqslant x$ for which the integral is to be evaluated. Before numerical evaluation, the second integral in (59) should be put in a form similar to that given in (54).

(4) When the point (x, y) is on a leading or trailing edge (Equation: $\xi = F(\eta)$), $x = F(y)$ and η_1 or η_2 is equal to y; η_1, η_2 are given by

$$x - F(\eta_r) = \beta\,|y-\eta_r\,| \qquad r = 1, 2.$$

Integrals of the form

$$\mathrm{P}\int_{\eta_1}^{\eta_2} P(x, \eta) \cosh^{-1}\frac{x - F(\eta)}{\beta\,|y-\eta\,|}\, d\eta$$

remain finite provided that $P(x, \eta)$ is finite in the region of integration. But integrals of the form

$$\mathrm{P}\int_{\eta_1}^{\eta_2} \frac{P(x, \eta)\{(x-F(\eta))^2 - \beta^2(y-\eta)^2\}^{1/2}}{(y-\eta)^2}\, d\eta$$

become infinite unless $\Delta C_P(\xi, \eta)$ is so chosen that

$$\lim_{\eta\to\eta_r} \frac{P[F(\eta_r), \eta]\; R(x, y, \eta)}{(\eta_r - \eta)^2} \qquad (r = 1 \text{ or } 2)$$

exists.

If this limit does not exist, the integral has a logarithmic singularity when $\eta_r = y$, although, in general, an analytical integration of the singular expression with respect to x is possible (and finite) at $x = F(y)$. This causes some difficulty when numerical methods are used, especially with respect to the camber shape near the wing tips.

Extrapolation near leading and trailing edges has been used in some existing DEUCE programmes. This is not very satisfactory. A programme for calculating the shape of the surface at the wing tips, where the distances between leading and trailing edges are small is then not possible. It may not even be possible to obtain the shape near the tip by extrapolation, since the ordinates extrapolated along different lines through the tip may not converge to a unique point there.

Some research on methods of programming for such singularities could be done. One method of dealing with the difficulty is suggested below. (The method is given for an integration with respect to η, but a similar method could be used for numerical calculation of the double integral.)

If the point (x, y) is on an edge $\xi = F(\eta)$, so that the limit $\eta_2 = y$, writing

$$\int_{\eta_1}^{\eta_2} \equiv \int_{\eta_1}^{a} + \int_{a}^{\eta_2}$$

(for small local chord, *e.g.*, near the wing tips, a could be taken equal to η_1), the contribution to the ordinate z at point (x, y) from an integral of the form

$$\text{P}\int_{\eta_1}^{\eta_2} \frac{P(x, \eta) \, . \, R(x, y, \eta)}{(y-\eta)^2} \, d\eta$$

could be written

$$\Delta z = (\Delta z)_1 + (\Delta z)_2,$$

where

$$(\Delta z)_1 = \int_{x_0}^{x} \left[\int_{\eta_1}^{a} \frac{P(x, \eta)\{(x-F(\eta))^2 - \beta^2(y-\eta)^2\}^{1/2}}{(y-\eta)^2} \, d\eta \right.$$
$$\left. + \int_{a}^{y} \left[P(x, \eta) \left\{ \left(\frac{x - F(\eta)}{y - \eta} \right)^2 - \beta^2 \right\}^{1/2} - P(x, y)\{(F'(y))^2 - \beta^2\}^{1/2} \right] \frac{d\eta}{y - \eta} \right] dx \quad (60)$$

and

$$(\Delta z)_2 = -\{(F'(y))^2 - \beta^2\}^{1/2} \lim_{x \to F(y)} \int_{x}^{x_0} P(x', y) \left(\int_{a}^{\eta_2} \frac{d\eta}{y - \eta} \right) dx'. \quad (61)$$

The two integrals in (60) should cause no difficulty, and (61) can be put in the form:

$$(\Delta z)_2 = -\{(F'(y))^2 - \beta^2\}^{1/2} \left[\log (y-a) \int_{F(y)}^{x_0} P(x', y) \, dx' \right.$$
$$- P(x_0, y)(x_0 - F(y)) \log \{\eta_2(x_0, y) - y\}$$
$$+ \int_{F(y)}^{x_0} \left\{ \frac{\partial}{\partial x'} P(x', y)(x' - F(y)) \log (x' - F(\eta_2)) \right.$$
$$\left. \left. + P(x', y) \frac{x' - F(y)}{\eta_2 - y} \cdot \frac{1}{\beta + F'(\eta_2)} \right\} dx' \right]. \quad (62)$$

It must be remembered that in the integrand, $\eta_2 \equiv \eta_2(x', y)$, a function of x', y, given by

$$x' - F(\eta_2) = \beta(\eta_2 - y),$$

and that

$$\eta_2(x', y) = y \qquad \text{when} \quad x' = F(y),$$

$$\eta_2(x', y) > y \qquad \text{when} \quad x' > F(y). \quad (63)$$

The integrand in (62) converges at $x' = F(y)$ and it should be possible to programme (62) for DEUCE, or other electronic computers, and thus avoid the need for extrapolation near the edges and a certain amount of guessing at the wing tips.

The formula for the total ordinate z at a point (x, y) on an edge would be of the form

$$z = z_0 + \int_{x_0}^{x} \frac{\partial z_1}{\partial x}\, dx + \Sigma\,(\Delta z), \tag{64}$$

where $\partial z_1/\partial x$ is the contribution to the gradient from other integrals which can be evaluated at the edge, and x_0, z_0 are arbitrary constants or functions of y.

A similar formula could be derived if $\eta_1 = y$.

6. *Conclusion.* A method has been given for calculating the shapes of camber surfaces of swept-back wings of arbitrary plan-form, with subsonic leading edges and specified load distribution, with particular reference to some of the difficulties encountered in the numerical integrations. The method is primarily intended for programming for calculations on an automatic digital computer. It is obviously not possible to produce perfectly general formulae or programmes (beyond the basic integrals from which all such formulae would be derived) to suit all plan-forms and all load distributions. The particular methods used must depend partly on the type of load distribution chosen and (to a lesser degree) on the plan-form of the wing.

NOTATION

A		A constant coefficient in Equations (22), (31)
$A(y, \eta)$		A function of η, y (Appendix II)
a		cf. (54), (59), (60)
B		A constant coefficient in Equation (22)
$B(y, \eta)$	=	$\sum_{r=0}^{n} \left[\frac{A^{(r)}(y)}{r!} (\eta - y)^r \right]$ (Appendix II)
b		cf. Equations (54), (59)
$C(y, \eta)$		$\sum_{r=0}^{n} \left[\frac{(-1)^r}{r!} A^{(r)}(y)(y-\eta)^r \right]$ (Appendix II)
$C_L(y)$		Integrated chord loading coefficient
$\bar{C}_L$		Lift coefficient
c		Root chord of wing
c_1		'Kink' chord of wing
$F(\eta)$		Value of ξ on a leading or trailing edge
$F(x, y, \eta, u_r)$	=	$\frac{\beta}{4\pi} \left\{ \frac{h(\eta) - xH(\eta)}{\lvert y - \eta \rvert} \sinh u_r + \frac{\beta}{2} H(\eta)(u_r + \tfrac{1}{2} \sinh 2u_r) \right\}$
$f(\eta)$		Value of ξ on a leading edge
$f^{-1}(\xi)$		Value of η on a leading edge
$(G_{a,b})_r$	=	$\mathrm{P}\int_{\eta_a}^{\eta_b} F(x, y, \eta, u_r)\, d\eta$, $r = 1$ or 2
$G_n(\xi)$		cf. Equation (18)
$g(\eta)$		Value of ξ on a trailing edge
$g^{-1}(\xi)$		Value of η on a trailing edge
$H_n(\eta)$		cf. Equation (13)
$H(\eta)$	=	$B/(g(\eta) - f(\eta))$
$h(\eta)$	=	$A + f(\eta) \,.\, H(\eta)$
k		Tan (sweep-back angle)
k_r		cf. Equation (42)
M		Free-stream Mach number
$P(x, \eta)$		A function of (x, η). cf. Equation (54), etc.
$P_1(x, \eta)$	=	$P(x, \eta)(x - F(\eta))$

P_r	=	$\left(\dfrac{\lambda_{r-1} - 1}{\lambda_{r-1} + 1}\right)^{1/2} \left(\dfrac{1 + \vartheta_r}{1 - \vartheta_r}\right)^{1/2}$
p_r	=	$\frac{1}{2} \log \left\| \dfrac{1 + P_r}{1 - P_r} \right\|$
$Q(x, y, \xi)$		A function of (x, y, ξ). cf. Section 5, (2)
Q_r	=	$\left(\dfrac{\lambda_r - 1}{\lambda_r + 1}\right)^{1/2} \left(\dfrac{1 + \vartheta_r}{1 - \vartheta_r}\right)^{1/2}$
q_r	=	$\frac{1}{2} \log \left\| \dfrac{1 + Q_r}{1 - Q_r} \right\|$
R_1, R_2		cf. Equation (46)
$R(x, y, \eta)$	=	$\{(x - F(\eta))^2 - \beta^2(y - \eta)^2\}^{1/2}$
r	=	$\dfrac{M}{2\beta}(x - \beta y)$ (a characteristic co-ordinate)
dS		Element of wing surface (Equation (1))
s	=	$\dfrac{M}{2\beta}(x + \beta y)$ (a characteristic co-ordinate)
u	=	$\cosh^{-1} \left\| \dfrac{x - \xi}{\beta(y - \eta)} \right\|$
u_1, u_2		cf. (27), (35)
u_r	=	$\cosh^{-1} \dfrac{1}{\|\vartheta_r\|}$ (cf. (44))
V		Free-stream velocity
V_r	=	$\dfrac{1}{\lambda_r} \left(\dfrac{\lambda_r^2 - 1}{1 - \vartheta_r^2}\right)^{1/2}$
v_r	=	$\frac{1}{2} \log \left\| \dfrac{1 + V_r}{1 - V_r} \right\|$
W_r	=	$\vartheta_r \left(\dfrac{\lambda_r^2 - 1}{1 - \vartheta_r^2}\right)^{1/2}$
w_r	=	$\frac{1}{2} \log \left\| \dfrac{1 + W_r}{1 - W_r} \right\|$
x		Chordwise co-ordinate (measured in the free stream direction)
y		Spanwise co-ordinate (positive to starboard)
z		Normal co-ordinate (positive upwards)
β		$(M^2 - 1)^{1/2}$

ΔC_P		Loading coefficient
$\Delta\Omega$		Jump in the value of Ω across the surface of the wing
θ	$=$	$\cos^{-1}\dfrac{\beta(y-\eta)}{x-\xi}$
ϑ	$=$	$\beta(y-y_r)/(x-x_r)$
λ_r	$=$	$\lvert k_r \rvert/\beta$
ξ		Chordwise co-ordinate—cf. (1), etc. (variable of integration)
ζ		Normal co-ordinate—cf. (1), etc. (variable of integration)
η		Spanwise co-ordinate—cf. (1), etc. (variable of integration)
τ		Region of wing surface for which $(x-\xi)^2 - \beta^2[(y-\eta)^2 + (z-\zeta)^2] \geqslant 0,\ x-\xi > 0$
ϕ		Velocity potential
Ω		Acceleration potential

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APPENDIX I

Evaluation of the integrals:

$$\text{(i)} \qquad X_{n,1} \equiv \int_{\xi_1(\eta)}^{\xi_2(\eta)} \frac{\xi^n(x-\xi)\,d\xi}{\{(x-\xi)^2-\beta^2(y-\eta)^2\}^{1/2}}$$

$$\text{(ii)} \qquad Y_{n,2} \equiv \mathrm{P}\!\!\int_{\eta_1(\xi)}^{\eta_2(\xi)} \frac{\eta^n\,d\eta}{(y-\eta)^2\,\{(x-\xi)^2-\beta^2(y-\eta)^2\}^{1/2}}$$

(i) Writing

$$X_{n,r} \equiv \int_{\xi_1(\eta)}^{\xi_2(\eta)} \frac{\xi^n(x-\xi)^r\,d\xi}{\{(x-\xi)^2-\beta^2(y-\eta)^2\}^{1/2}}$$

$$X_{n,r} = xX_{n,r-1} - X_{n+1,r-1}, \qquad n \geqslant 0,\ r \geqslant 1,$$

and thus

$$X_{n,1} = xX_{n,0} - X_{n+1,0} \tag{65}$$

for all values of $n \geqslant 0$.

For $r = 0$:

$$\begin{aligned} nX_{n,0} &= (2n-1)xX_{n-1,0} - (n-1)\{x^2-\beta^2(y-\eta)^2\}X_{n-2,0} \\ &\quad + \Big[\xi^{n-1}\{(x-\xi)^2-\beta^2(y-\eta)^2\}^{1/2}\Big]_{\xi_1}^{\xi_2} \\ &\equiv (2n-1)xX_{n-1,0} - (n-1)\{x^2-\beta^2(y-\eta)^2\}X_{n-2,0} \\ &\quad + \beta(y-\eta)\Big[\sinh u\,\{x-\beta(y-\eta)\cosh u\}^{n-1}\Big]_{u_1}^{u_2}, \qquad n \geqslant 2. \end{aligned} \tag{66}$$

$$X_{0,0} = \left[-\cosh^{-1}\left|\frac{x-\xi}{\beta(y-\eta)}\right|\right]_{\xi_1}^{\xi_2} \equiv -u_2+u_1, \tag{67}$$

$$\begin{aligned} X_{1,0} &= xX_{0,0} + \Big[\{(x-\xi)^2-\beta^2(y-\eta)^2\}^{1/2}\Big]_{\xi_1}^{\xi_2} \\ &\equiv xX_{0,0} + \beta(y-\eta)\Big[\sinh u\Big]_{u_1}^{u_2}, \end{aligned} \tag{68}$$

where

$$u = \cosh^{-1}\left|\frac{x-\xi}{\beta(y-\eta)}\right|.$$

Also

$$X_{0,1} = \Big[-\{(x-\xi)^2-\beta^2(y-\eta)^2\}^{1/2}\Big]_{\xi_1}^{\xi_2} \equiv -\beta(y-\eta)\Big[\sinh u\Big]_{u_1}^{u_2}. \tag{69}$$

(ii) Writing

$$Y_{n,r} \equiv \int_{\eta_1(\xi)}^{\eta_2(\xi)} \frac{\eta^n\,d\eta}{(y-\eta)^r\{(x-\xi)^2-\beta^2(y-\eta)^2\}^{1/2}}$$

$$Y_{n,r} = yY_{n-1,r} - Y_{n-1,r-1}, \qquad n \geqslant 1,\ r \geqslant 1,$$

and thus

$$Y_{n,2} = yY_{n-1,2} - Y_{n-1,1}, \qquad n \geqslant 1, \tag{70}$$

and

$$Y_{n,1} = yY_{n-1,1} - Y_{n-1,0}, \qquad n \geqslant 1. \tag{71}$$

For $n = 0$

$$Y_{0,2} = \left[\frac{\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{1/2}}{(x-\xi)^2\,(y-\eta)}\right]_{\eta_1}^{\eta_2}$$

$$\equiv \frac{\beta}{(x-\xi)^2}\left[\tan\theta\right]_{\theta_1}^{\theta_2}, \tag{72}$$

$$Y_{0,1} = \left[\frac{1}{x-\xi}\left[\log\frac{x-\xi+\{(x-\xi)^2-\beta^2(y-\eta)^2\}^{1/2}}{\beta(y-\eta)}\right]_{\eta_1}^{\eta_2}\right.$$

$$\equiv \frac{1}{x-\xi}\left[\log\frac{1+\sin\theta}{\cos\theta}\right]_{\theta_1}^{\theta_2}. \tag{73}$$

For $r = 0$:

$$nY_{n,0} = (2n-1)yY_{n-1,0} + \frac{n-1}{\beta^2}\{(x-\xi)^2 - \beta^2y^2\}Y_{n-2,0}$$

$$-\frac{1}{\beta^2}\left[\eta^{n-1}\{(x-\xi)^2-\beta^2(y-\eta)^2\}^{1/2}\right]_{\eta_1}^{\eta_2}$$

$$\equiv (2n-1)yY_{n-1,0} + \frac{n-1}{\beta^2}\{(x-\xi)^2-\beta^2y^2\}Y_{n-2,0}$$

$$-\frac{1}{\beta^2}(x-\xi)\left[\sin\theta\left\{y-\frac{x-\xi}{\beta}\cos\theta\right\}^{n-1}\right]_{\theta_1}^{\theta_2}, \tag{74}$$

$$Y_{1,0} = yY_{0,0} - \frac{1}{\beta^2}\left[\{(x-\xi)^2-\beta^2(y-\eta)^2\}^{1/2}\right]_{\eta_1}^{\eta_2}$$

$$\equiv yY_{0,0} - \frac{1}{\beta^2}(x-\xi)\left[\sin\theta\right]_{\theta_1}^{\theta_2}, \tag{75}$$

$$Y_{0,0} = \frac{1}{\beta}\left[\cos^{-1}\frac{\beta(y-\eta)}{x-\xi}\right]_{\eta_1}^{\eta_2} \equiv \frac{1}{\beta}(\theta_2-\theta_1), \tag{76}$$

where

$$\theta = \cos^{-1}\frac{\beta(y-\eta)}{x-\xi}.$$

Singular Integrals—Definitions and Formulae

The 'Finite Part' of an Integral

The symbol $\overset{*}{\int}$ denotes 'finite part of an integral', and is equivalent to Hadamard's symbol $\left|\overline{\underline{\int}}\right.$ for single integrals, but not for double integrals. (*See* Refs. 3, 4.)

The order of integrations $\overset{*}{\int}$ cannot be reversed, each definite integral being independent of succeeding operations. In using the symbol $\left|\overline{\underline{\int}}\right.$ all singularities for which the order of integration is irreversible are excluded from the area of integration and are treated separately. This symbol is not used in this report.

By definition:

$$\overset{*}{\int_a^y} \left(\frac{\partial}{\partial y}\right)^n \frac{A(\eta)}{(y-\eta)^{1/2}}\, d\eta = \left(\frac{\partial}{\partial y}\right)^n \int_a^y \frac{A(\eta)}{(y-\eta)^{1/2}}\, d\eta, \tag{77}$$

and hence

$$\overset{*}{\int_a^y} \frac{A(\eta)}{(y-\eta)^{n+1/2}}\, d\eta = \frac{(-1)^n 2^{2n} n!}{(2n)!} \left(\frac{\partial}{\partial y}\right)^n \int_a^y \frac{A(\eta)}{(y-\eta)^{1/2}}\, d\eta, \qquad n \geqslant 1. \tag{78}$$

A sufficient condition for convergence is that the function $A(\eta)$ is continuous at $\eta = y$ and integrable elsewhere in the region of integration.

In particular,

$$\overset{*}{\int_a^y} \frac{d\eta}{(y-\eta)^{n+1/2}} = \frac{(-1)^n 2^{2n} n!}{(2n)!} \left(\frac{\partial}{\partial y}\right)^n \int_a^y \frac{d\eta}{(y-\eta)^{1/2}}$$

$$= \frac{-2}{(2n-1)(y-a)^{n-1/2}} \qquad n \geqslant 0. \tag{79}$$

Also

$$\int_a^y \frac{A(y,\eta)}{(y-\eta)^{n+1/2}}\, d\eta = \frac{(-1)^n 2^{2n} n!}{(2n)!} \left(\frac{\partial}{\partial y}\right)^n \int_a^y \frac{A(y,\eta)}{(y-\eta)^{1/2}}\, d\eta, \tag{80}$$

provided that

$$\operatorname*{limit}_{\eta \to y} \left[(y-\eta)^{1/2} \left(\frac{\partial}{\partial y}\right)^n A(y,\eta)\right] = 0.$$

Writing

$$\overset{*}{\int_a^y} \frac{A(\eta)}{(y-\eta)^{n+1/2}}\, d\eta \equiv \int_a^y \frac{A(\eta) - C(y,\eta)}{(y-\eta)^{n+1/2}}\, d\eta + \overset{*}{\int_a^y} \frac{C(y,\eta)}{(y-\eta)^{n+1/2}}\, d\eta,$$

where

$$C(y,\eta) = \sum_{r=0}^{n} \left[\frac{(-1)^r}{r!} A^{(r)}(y)\,(y-\eta)^r\right],$$

it can be shown that

$$\overset{*}{\int_a^y} \frac{A(\eta)}{(y-\eta)^{n+1/2}}\, d\eta = \int_a^y \frac{A(\eta) - C(y, \eta)}{(y-\eta)^{n+1/2}}\, d\eta$$

$$+ \sum_{r=0}^{n} \left[\frac{(-1)^{n-r+1}}{(n-r)!}\, \frac{2}{(2r-1)}\, A^{(n-r)}(y)\, \frac{1}{(y-a)^{r-1/2}}\right], \qquad (81)$$

where $A^{(r)}(y)$ denotes $\left(\dfrac{\partial}{\partial y}\right)^r A(y)$.

Similarly

$$\overset{*}{\int_a^y} \frac{A(y, \eta)}{(y-\eta)^{n+1/2}}\, d\eta = \int_a^y \frac{A(y, \eta) - C(y; y, \eta)}{(y-\eta)^{n+1/2}}\, d\eta$$

$$+ \sum_{r=0}^{n} \left[\frac{(-1)^{n-r+1}}{(n-r)!}\, \frac{2}{(2r-1)}\, A^{(n-r)}(y, y)\, \frac{1}{(y-a)^{r-1/2}}\right], \qquad (82)$$

where

$$C(y; y, \eta) = \sum_{r=0}^{n} \left[\frac{(-1)^r}{r!}\, A^{(r)}(y, y)(y-\eta)^r\right]$$

and $A^{(r)}(y, y)$ denotes $\left[\left(\dfrac{\partial}{\partial \eta}\right)^r A(y, \eta)\right]_{\eta=y}$.

In both (81) and (82) the integrand on the right-hand side is finite at $\eta = y$.

It is also useful to remember that if the indefinite integral $\displaystyle\int \frac{f(y, \eta)}{(y-\eta)^{n+1/2}}\, d\eta \equiv F(y, \eta)$ exists, and if a $< y < b$ and $f(y, \eta)$ is real in the interval $a \leqslant \eta \leqslant b$, then

$$\overset{*}{\int_a^y} \frac{f(y, \eta)}{(y-\eta)^{n+1/2}}\, d\eta = \mathrm{Re} \overset{*}{\int_a^b} \frac{f(y, \eta)}{(y-\eta)^{n+1/2}}\, d\eta$$

$$= \mathrm{Re}\, [F(y, b) - F(y, a)]. \qquad (83)$$

The methods shown above can also be used to evaluate an integral of the form

$$\overset{**}{\iint} \frac{A(x, y, \xi, \eta)\, d\xi\, d\eta}{\{(x-\xi)^2 - \beta^2(y-\eta)^2\}^{n+1/2}},$$

where the region of integration is defined by $(x-\xi)^2 \geqslant \beta^2(y-\eta)^2$.

The 'Generalised Principal Part' of an Integral

The symbol $\overset{\mathrm{P}}{\int}$ denotes 'generalised principal part of an integral'.

By definition:

$$\overset{\mathrm{P}}{\int_a^b} \frac{A(\eta)}{(\eta-y)^{n+1}}\, d\eta = -\frac{1}{n!}\left(\frac{\partial}{\partial y}\right)^{n+1} \int_a^b A(\eta) \log |y-\eta|\, d\eta$$

$$= \frac{1}{n!}\left(\frac{\partial}{\partial y}\right)^{n} -\!\!\!\!\!\!\int_a^b \frac{A(\eta)}{\eta - y}\, d\eta, \qquad (84)$$

$$a < y < b, \qquad n > 0.$$

When $n = 0$, the above integral becomes the Cauchy principal part

$$\overset{C}{\int_a^b} \frac{A(\eta)}{\eta - y}\, d\eta = -\frac{\partial}{\partial y} \int_a^b A(\eta) \log |\eta - y|\, d\eta. \tag{85}$$

For convergence it is sufficient to assume that $A(\eta)$ and its first n derivatives exist and are single-valued at $\eta = y$, and that elsewhere, $A(\eta)$ is continuous or with integrable singularities.

In particular

$$\overset{P}{\int_a^b} \frac{d\eta}{(\eta - y)^{n+1}} = \frac{1}{n!}\left(\frac{\partial}{\partial y}\right)^n \overset{C}{\int_a^b} \frac{d\eta}{\eta - y}$$

$$= -\frac{1}{n}\left[\frac{1}{(b-y)^n} - \frac{1}{(a-y)^n}\right], \qquad n \geqslant 1. \tag{86}$$

Also

$$\overset{P}{\int_a^b} \frac{A(y, \eta)}{(\eta - y)^{n+1}}\, d\eta = \frac{1}{n!}\left(\frac{\partial}{\partial y}\right)^n \int_a^b \frac{A(y, \eta)}{\eta - y}\, d\eta, \qquad n > 0 \tag{87}$$

provided that $A(y, \eta)$ and its first n derivatives with respect to y and η exist at $\eta = y$.

Writing

$$\overset{P}{\int_a^b} \frac{A(\eta)}{(\eta - y)^{n+1}}\, d\eta = \int_a^b \frac{A(\eta) - B(y, \eta)}{(\eta - y)^{n+1}} + \overset{P}{\int_a^b} \frac{B(y, \eta)}{(\eta - y)^{n+1}}\, d\eta,$$

where

$$B(y, \eta) = \sum_{r=0}^{n} \left[\frac{A^{(r)}(y)}{r!} (\eta - y)^r\right],$$

it can be shown that

$$\overset{P}{\int_a^b} \frac{A(\eta)}{(\eta - y)^{n+1}}\, d\eta = \int_a^b \frac{A(\eta) - B(y, \eta)}{(\eta - y)^{n+1}} + \frac{A^{(n)}(y)}{n!} \log \frac{b - y}{y - a}$$

$$- \sum_{r=1}^{n} \left[\frac{A^{(n-r)}(y)}{(n-r)!\, r} \left\{\frac{1}{(b-y)^r} - \frac{1}{(a-y)^r}\right\}\right] \tag{88}$$

Similarly

$$\overset{P}{\int_a^b} \frac{A(y, \eta)}{(\eta - y)^{n+1}}\, d\eta = \int_a^b \frac{A(y, \eta) - B(y; y, \eta)}{(\eta - y)^{n+1}}\, d\eta + \frac{A^{(n)}(y, y)}{n!} \log \frac{b - y}{y - a}$$

$$- \sum_{r=1}^{n} \left[\frac{A^{(n-r)}(y, \eta)}{(n-r)!\, r} \left\{\frac{1}{(b-y)^r} - \frac{1}{(a-y)^r}\right\}\right]. \tag{89}$$

In both (88) and (89) the integrand on the right-hand side is finite at $\eta = y$.

It can be proved, by induction, that if the indefinite integral $\int \frac{A(y, \eta)}{(\eta - y)^{n+1}}\, d\eta \equiv H(\eta, y)$ exists, and $y \neq a, b$, then

$$\overset{P}{\int_a^b} \frac{A(y, \eta)}{(\eta - y)^{n+1}} = H(b, y) - H(a, y). \tag{90}$$

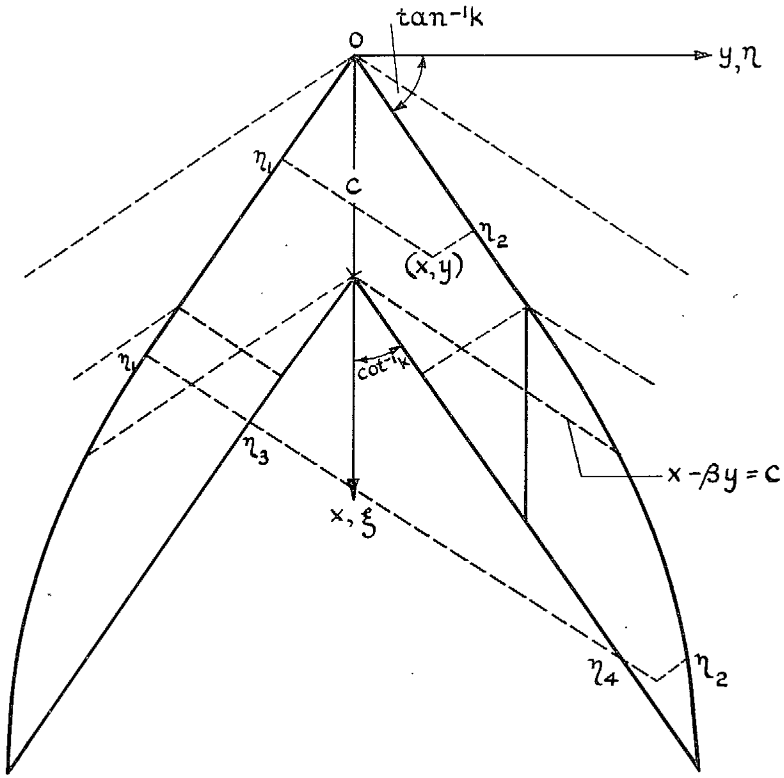

FIG. 1. Swept-back wing (*a*), plan-form and regions of integration.

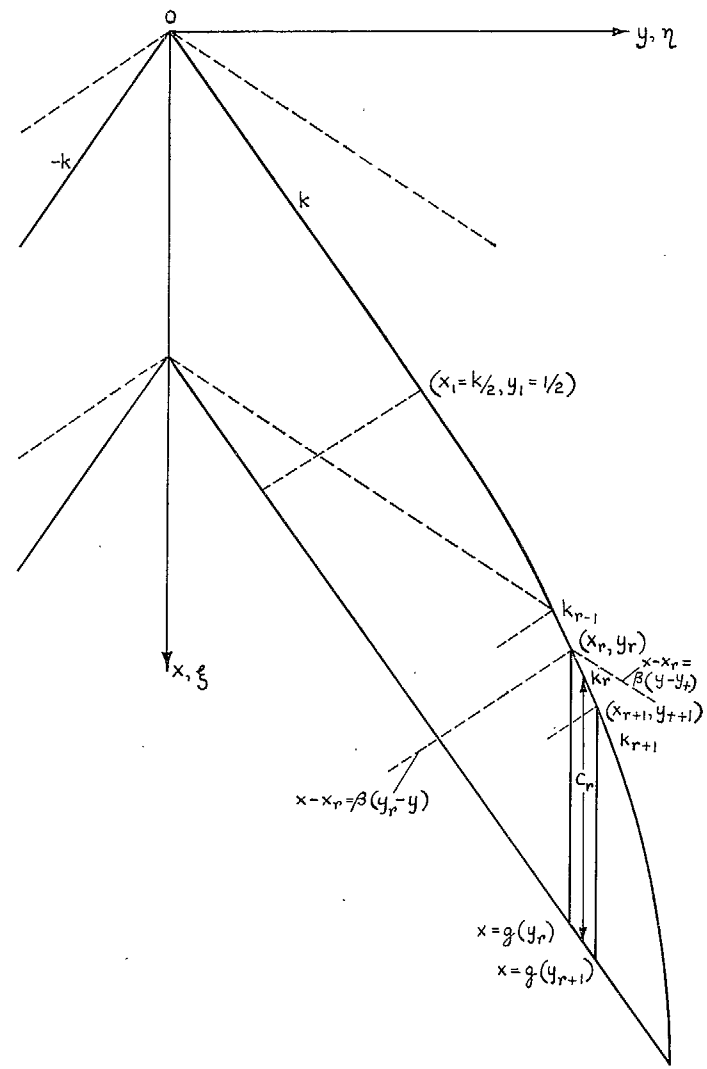

FIG. 2. Swept-back wing (*a*) (leading edge approximated by a polygon).

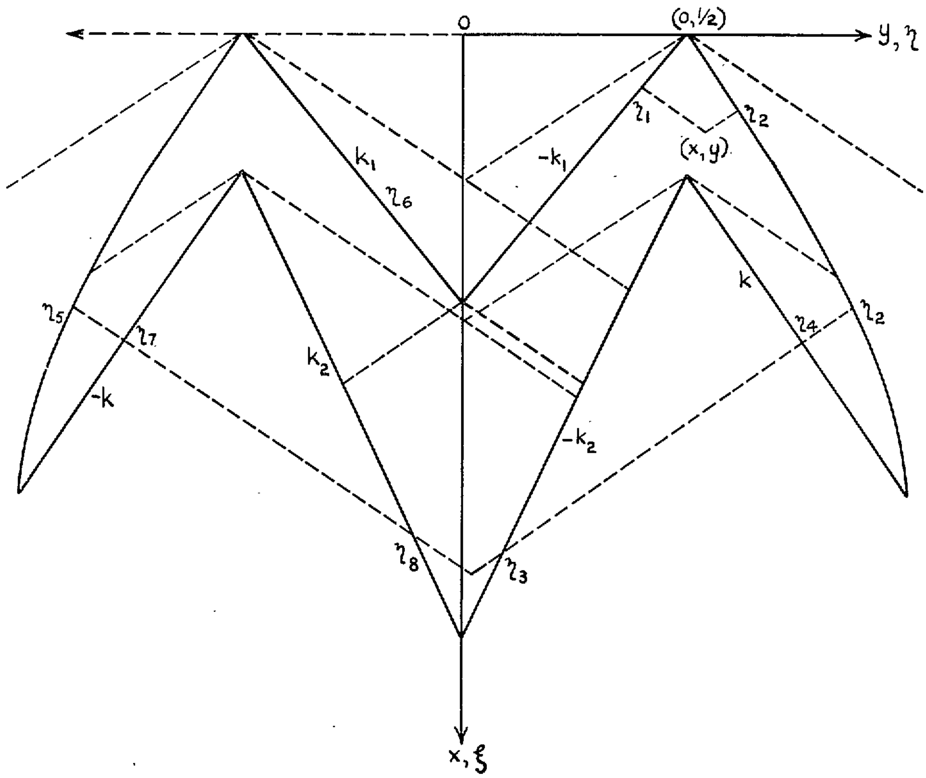

FIG. 3. M-wing (b) (Plan-form and regions of integration).

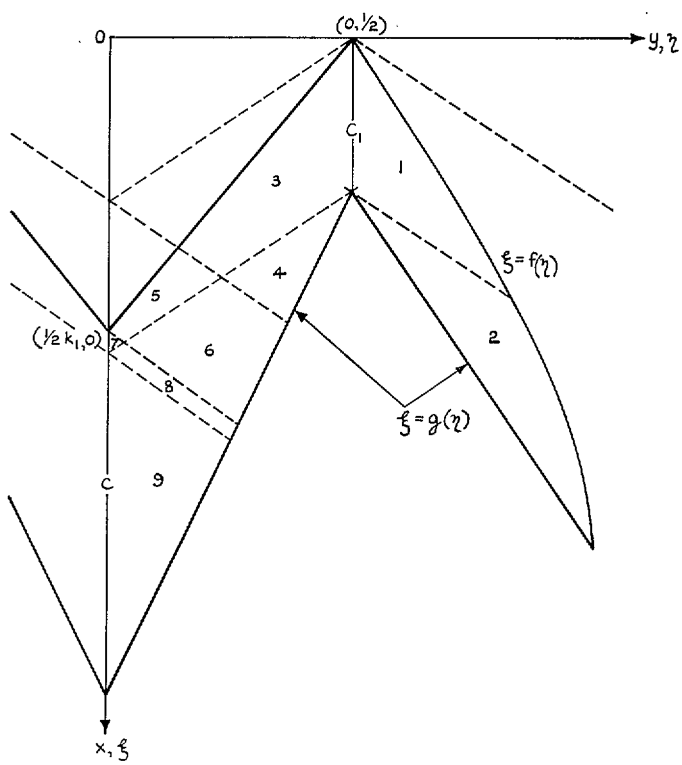

FIG. 4. M-wing (b), showing the different regions.

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