R. & M. No. 3191
(21,144)
A.R.C. Technical Report

R. & M. No. 3191

MINISTRY OF AVIATION

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

The Growth of Compressible Turbulent Boundary Layers on Isothermal and Adiabatic Walls

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LONDON: HER MAJESTY'S STATIONERY OFFICE
1961
PRICE 6s. 6d. NET

SECTION V

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The Growth of Compressible Turbulent Boundary Layers on Isothermal and Adiabatic Walls

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MINISTRY OF SUPPLY

*Reports and Memoranda No. 3191**
June, 1959

Summary.—A recent study of velocity profiles in the turbulent boundary layer on a flat plate at high Mach numbers has suggested the formula

$$H = \frac{T_w}{T_e} H_i + \frac{T_r}{T_e} - 1$$

for relating the form parameter H to its incompressible value H_i (T_w, T_e, T_r are the wall, free-stream and recovery temperatures). This formula is used in conjunction with a generalisation of the Stewartson–Illingworth transformation to reduce the left-hand side of the integral momentum equation (including pressure gradients) to incompressible form, for arbitrary values of the flat-plate recovery factor (or of the turbulent Prandtl number), which is about 0·89 for air. The intermediate temperature formula of Eckert is used to relate the skin friction to its incompressible value, for which a $1/n$th power law is used. The momentum equation is then integrable when H_i is given a constant value, which may be chosen to secure agreement in the incompressible case with Maskell's quadrature for the momentum thickness θ, or with the author's modification of the latter. The integration is carried out for the cases of zero heat transfer and of a constant-temperature wall, the details of the Stewartson-type transformation being slightly different in the two cases, but the final forms for

$$\Theta = \frac{\rho_e}{\rho_m}\left(\frac{\rho_e u_e \theta}{\mu_m}\right)^{1/n} \theta$$

are the same:

$$\Theta M^B F(M) = (\text{const}) \int M^B F(M)\, dx.$$

The constant B depends in the second case on the ratio T_w/T_0 of wall to total stream temperature, but $F(M)$ does not. This is of the same form as the equation obtained by Reshotko and Tucker in the special case of unit recovery factor. A numerical example shows results very close to those obtained by Young's method, which involves two quadratures, in a particular case at zero heat transfer. Some calculations have also been made to illustrate the effect of cooling the wall on the boundary-layer growth in a pressure gradient.

1. *Introduction.*—A recent study[1] of the velocity profiles measured by Lobb, Winkler and Persh[2] in turbulent boundary layers on a flat plate at Mach numbers up to 8 has shown that it is very largely possible to represent the effect of compressibility by writing

$$\frac{u}{u_e} = f\left(\frac{\eta}{\Delta}\right), \tag{1}$$

* R.A.E. Report Aero. 2619, received 15th October, 1959.

where

$$\eta = \int_0^y \frac{\rho}{\rho_e}\, dy, \qquad \Delta = \int_0^\delta \frac{\rho}{\rho_e}\, dy, \tag{2}$$

u_e and ρ_e being velocity and density at the outer edge of the boundary layer where $y = \delta$. The distributions of shear stress and thence of temperature across the layer can then be inferred, and it is found that provided the turbulent Prandtl number $\alpha = \epsilon C_p/\kappa$ (ϵ = eddy viscosity, κ = eddy conductivity) is between about 0·6 and 1, the temperature is very nearly a quadratic function of the velocity*. A value of about 0·85 for α would be consistent with the observed value of about 0·9 for the recovery factor r, which is defined by

$$r = (T_r - T_e)/(T_0 - T_e) \tag{3}$$

(Here suffixes $_0$, $_e$ and $_r$ refer respectively to an isentropic stagnation point in the free stream, to the edge of the boundary layer, and to wall conditions at zero heat transfer. $_w$ will be used for wall conditions in general.)

It is shown in Appendix I that equation (1), together with the quadratic temperature-velocity relationship, results in the following expression for the form parameter $H = \delta^*/\theta$:

$$H = \frac{T_w}{T_e} H_i + \frac{T_r}{T_e} - 1. \tag{4}$$

This expression for H and a generalisation of the Stewartson–Illingworth transformation are used in Section 2 to reduce the left-hand side of the integral momentum equation to its incompressible form. The momentum thickness θ and the velocity u_e are replaced by

$$\left.\begin{aligned} \bar{\theta} &= \left(\frac{T_e}{T_0}\right)^k \theta = \xi\theta \\ \bar{u}_e &= \left(\frac{a_0}{a_e}\right)^l u_e = \left(\frac{T_e}{T_0}\right)^{-l/2} u_e \end{aligned}\right\}, \tag{5}$$

where k and l depend on the flat-plate recovery factor r, and for $r = 1$ take the values used by Stewartson[3], namely,

$$k = \tfrac{1}{2} + 1/(\gamma - 1), \qquad l = 1. \tag{6}$$

The latter values reduce the integral momentum equation of the laminar boundary layer, at zero heat transfer and unit Prandtl number, to its incompressible form, and, as has been pointed out by Culick and Hill[4], would do the same in the turbulent case if the temperature dependence of H were still given by

$$H = \frac{T_0}{T_e}(H_i + 1) - 1. \tag{7}$$

Culick and Hill[4] have integrated the momentum equation with the aid of (5) and (6) on the assumption that equation (7) does in fact hold. But (7) is the equation to which (4) reduces for zero heat transfer ($T_w = T_r$) only when the Prandtl number or recovery factor is unity, i.e., when $T_r = T_0$, and this is not found to be the case in air. The present paper contains the generalisation to Prandtl

* In this respect the turbulent boundary layer is considerably different from the laminar layer, in which the shear-stress distribution is such that the temperature-velocity relation can be treated as quadratic only when the Prandtl number is close to 1.

numbers different from unity of the integration by Culick and Hill and of a similar one by Reshotko and Tucker[5].

The intermediate temperature formula of Eckert[6] is used in Section 3 to relate the skin-friction coefficient to its incompressible value for which a flat-plate expression of the form $2C(R_\theta)^{-1/n}$ is assumed. The momentum equation is then integrated in Section 4 with H_i constant. The justification for this procedure is that, as pointed out in Section 5, it leads to precisely the same form of equation in the incompressible case as was found by Maskell[7] from a comprehensive examination of experimental results, and which is known to provide an accurate quadrature for θ when pressure gradients are present. The solution is given for the two cases in which it is particularly simple, those of zero heat transfer and of constant wall temperature.

No attempt has been made to go further and calculate the way in which H rises near separation, and the consequential departures of skin friction and heat transfer from the assumed flat-plates values. To do this one would require an auxiliary equation for dH/dx, the derivation of which from incompressible theories would involve much more speculative assumptions than those required to deal with the momentum equation. Maskell's work has, however, shown that such departures are insufficient to affect at all seriously the values of θ obtained by ignoring them.

2. *Transformation of the Integral Momentum Equation.*—The momentum equation is (Young[8])

$$\frac{d\theta}{dx}+(H+2-M^2)\frac{\theta}{u_e}\frac{du_e}{dx}=\frac{\tau_w}{\rho_e u_e^2} \tag{8}$$

where $M = u_e/a_e$ is the local Mach number at the edge of the boundary layer. By differentiating the relations in (5), and using the isentropic relations

$$T_0 = \text{const} = T_e+u_e^2/2C_p = T_e\left\{1+\frac{\gamma-1}{2}M^2\right\} \tag{9}$$

which hold in the free stream, we obtain

$$\xi\frac{d\theta}{dx}=\frac{d\bar{\theta}}{dx}-\frac{\bar{\theta}}{\xi}\frac{d\xi}{dx}=\frac{d\bar{\theta}}{dx}+(\gamma-1)\,kM^2\frac{\bar{\theta}}{u_e}\frac{du_e}{dx},$$

and

$$\frac{1}{\bar{u}_e}\frac{d\bar{u}_e}{dx}=\left\{1+\frac{l}{2}(\gamma-1)\,M^2\right\}\frac{1}{u_e}\frac{du_e}{dx},$$

on substitution of which (8) becomes

$$\frac{d\bar{\theta}}{dx}+\left[\frac{H+2-M^2+(\gamma-1)\,kM^2}{1+\frac{1}{2}l(\gamma-1)\,M^2}\right]\frac{\bar{\theta}}{\bar{u}_e}\frac{d\bar{u}_e}{dx}=\xi\frac{\tau_w}{\rho_e u_e^2}. \tag{10}$$

The choice of k, l to simplify this equation is slightly different in the adiabatic and isothermal wall cases:

2.1. *Zero Heat Transfer* ($T_w = T_r$).—In this case (4) becomes

$$H+1=\frac{T_r}{T_e}(H_i+1). \tag{11}$$

The recovery temperature T_r is found from (3) and (9) as

$$T_r = T_e+r(T_0-T_e) = T_e\left\{1+\frac{r}{2}(\gamma-1)\,M^2\right\}. \tag{12}$$

$1 + 0{\cdot}148\,M^2$

Substituting from (11), the numerator of the expression in square brackets in (10) is

$$H+2-M^2+(\gamma-1)kM^2 = \frac{T_r}{T_e}(H_i+1)+1+\left(2k-\frac{2}{\gamma-1}\right)\frac{\gamma-1}{2}M^2 = \frac{T_r}{T_e}(H_i+2) \tag{13}$$

if

$$2k-\frac{2}{\gamma-1} = r. \tag{14}$$

And if $l = r$ the denominator of the same expression is T_r/T_e. Thus if

$$\left.\begin{aligned} T_w &= T_r \\ k &= \frac{r}{2}+\frac{1}{\gamma-1} \\ l &= r \end{aligned}\right\}, \tag{15}$$

equation (10) reduces to

$$\frac{d\bar{\theta}}{dx}+(H_i+2)\frac{\bar{\theta}}{\bar{u}_e}\frac{d\bar{u}_e}{dx} = \xi\frac{\tau_w}{\rho_e u_e^2}. \tag{16}$$

If $T_w \neq T_r$, the term H_i+2 in this equation becomes $(T_w/T_r)H_i+2$. Integration is therefore possible by the method described in Section 4 whenever T_w/T_r has a constant value, and is not limited to the zero heat-transfer case.

2.2. *Wall at a Constant Temperature* T_w.—It is convenient to write (4) in this case as

$$H = \frac{T_0}{T_e}\frac{T_w}{T_0}H_i+\frac{T_r}{T_e}-1 = \left(1+\frac{\gamma-1}{2}M^2\right)\frac{T_w}{T_0}H_i+\frac{r}{2}(\gamma-1)M^2. \tag{17}$$

Then

$$\begin{aligned} H+2-M^2+(\gamma-1)kM^2 &= \frac{T_w}{T_0}H_i+2+\left(\frac{T_w}{T_0}H_i+2k+r-\frac{2}{\gamma-1}\right)\frac{\gamma-1}{2}M^2 \\ &= \left(\frac{T_w}{T_0}H_i+2\right)\left(1+\frac{\gamma-1}{2}M^2\right) \end{aligned} \tag{18}$$

if

$$2k+r-\frac{2}{\gamma-1} = 2. \tag{19}$$

And if $l = 1$, the denominator of the expression in square brackets is

$$1+\frac{\gamma-1}{2}M^2.$$

Thus if

$$\left.\begin{aligned} T_w &= \text{const} \\ k &= 1-\frac{r}{2}+\frac{1}{\gamma-1} \\ l &= 1 \end{aligned}\right\}, \tag{20}$$

equation (10) reduces to

$$\frac{d\bar{\theta}}{dx}+\left(\frac{T_w}{T_0}H_i+2\right)\frac{\bar{\theta}}{\bar{u}_e}\frac{d\bar{u}_e}{dx} = \xi\frac{\tau_w}{\rho_e u_e^2}, \tag{21}$$

which is of the same form as (16).

3. *Expression for the Skin Friction.*—The intermediate temperature method enables us to write the wall shearing stress at a distance x from the virtual origin of the turbulence on a flat plate as

$$\tau_w = \rho_m u_\tau^2, \tag{22}$$

where the suffix $_m$ indicates evaluation of physical quantities at an empirically defined reference temperature T_m, and $(u_\tau/u_e)^2$ is the same function of $u_e x/\nu_m$ as in incompressible flow. The accepted definition of the reference temperature, due to Eckert[6], is

$$T_m = 0{\cdot}5(T_w+T_e)+0{\cdot}22(T_r-T_e). \tag{23}$$

With $\gamma = 1{\cdot}4$, $r = 0{\cdot}89$, this becomes

$$\frac{T_m}{T_e} = 1+0{\cdot}128M^2 \tag{24}$$

in the case of zero heat transfer ($T_w = T_r$), and

$$\frac{T_m}{T_e} = 0{\cdot}5(1+0{\cdot}078M^2)+0{\cdot}5\frac{T_w}{T_0}(1+0{\cdot}2M^2) \tag{25}$$

when the wall temperature T_w is constant.

It is shown in Appendix II that the local momentum-thickness Reynolds number which corresponds to $u_e x/\nu_m$ is $\rho_e u_e \theta/\mu_m$. Referred to the density and viscosity at an isentropic stagnation point this is

$$\frac{\rho_e u_e \theta}{\mu_m} = \left(\frac{T_e}{T_0}\right)^{\frac{1}{\gamma-1}-\omega}\left(\frac{T_m}{T_e}\right)^{-\omega}\frac{u_e\theta}{\nu_0}, \tag{26}$$

in which the viscosity-temperature relation has been taken as $\mu \propto T^\omega$. The dependence of $(u_\tau/u_e)^2$ on this will be taken as an inverse $1/n$th power law, so that from (22)

$$\frac{\tau_w}{\rho_e u_e^2} = \frac{\rho_m}{\rho_e}C\left(\frac{\rho_e u_e \theta}{\mu_m}\right)^{-1/n} = C\left(\frac{T_m}{T_e}\right)^{-1+\frac{\omega}{n}}\left(\frac{T_e}{T_0}\right)^{-(\frac{1}{\gamma-1}-\omega)/n}\left(\frac{u_e\theta}{\nu_0}\right)^{-1/n}. \tag{27}$$

With $C = 0{\cdot}0088$, $n = 5$, this agrees within about 5 per cent with the Kármán–Schoenherr line in the range $500 < R_\theta < 10^5$ in the incompressible case (for which $T_m = T_e = T_0$).

4. *Integration of the Momentum Equation.*—The right-hand sides of (16) and (21) are both equal to $\xi\tau_w/\rho_e u_e^2$, which may be written in terms of the transformed variables as

$$\frac{\xi\tau_w}{\rho_e u_e^2} = \left(\frac{T_e}{T_0}\right)^k \frac{\tau_w}{\rho_e u_e^2} = C\left(\frac{T_e}{T_0}\right)^\alpha\left(\frac{T_m}{T_e}\right)^{-\beta}\left(\frac{\bar{u}_e\bar{\theta}}{\nu_0}\right)^{-1/n}, \tag{28}$$

where

$$\left.\begin{aligned}\alpha &= k+\left(k-\frac{1}{\gamma-1}-\frac{l}{2}+\omega\right)\Big/ n\\ \beta &= 1-\frac{\omega}{n}\end{aligned}\right\}, \tag{29}$$

k and l being defined by (15) and (20) in the two cases. Then, introducing

$$\bar{\Theta} = \bar{\theta}\left(\frac{\bar{u}_e\bar{\theta}}{\nu_0}\right)^{1/n} \tag{30}$$

both (16) and (21) may be written

$$\frac{d\bar{\Theta}}{dx}+B\frac{\bar{\Theta}}{\bar{u}_e}\frac{d\bar{u}_e}{dx} = \left(1+\frac{1}{n}\right)C\left(\frac{T_e}{T_0}\right)^{\alpha}\left(\frac{T_m}{T_e}\right)^{-\beta}, \tag{31}$$

where for zero heat transfer, from (16),

$$B = \left(1+\frac{1}{n}\right)(H_i+2)-\frac{1}{n} \tag{32}$$

and for a constant temperature wall, from (21),

$$B = \left(1+\frac{1}{n}\right)\left(\frac{T_w}{T_0}H_i+2\right)-\frac{1}{n}. \tag{33}$$

If H_i is constant, (31) may be integrated to

$$\bar{\Theta}\bar{u}_e{}^B = \left(1+\frac{1}{n}\right)C\int\left(\frac{T_e}{T_0}\right)^{\alpha}\left(\frac{T_m}{T_e}\right)^{-\beta}\bar{u}_e{}^B\,dx. \tag{34}$$

In transforming back to the original variables, it is convenient to introduce also

$$\Theta = \frac{\rho_e}{\rho_m}\left(\frac{\rho_e u_e \theta}{\mu_m}\right)^{1/n}\theta = \left(\frac{T_e}{T_0}\right)^{-\alpha}\left(\frac{T_m}{T_e}\right)^{\beta}\bar{\Theta}. \tag{35}$$

In passing it may be noted from (27) that

$$\frac{\Theta}{\theta} = \frac{2C}{c_f} = \left(\frac{T_m}{T_e}\right)^{\beta}\left(\frac{T_e}{T_0}\right)^{\left(\frac{1}{\gamma-1}-\omega\right)/n}\left(\frac{u\theta}{\nu_0}\right)^{1/n}, \tag{36}$$

where c_f is the local skin-friction coefficient $\tau_w/\rho_e u_e^2$.

Equation (34) then becomes

$$\Theta u_e{}^B\left(\frac{T_e}{T_0}\right)^{\alpha-Bl/2}\left(\frac{T_m}{T_e}\right)^{-\beta} = \left(1+\frac{1}{n}\right)C\int\left(\frac{T_e}{T_0}\right)^{\alpha-Bl/2}\left(\frac{T_m}{T_e}\right)^{-\beta}u_e{}^B\,dx. \tag{37}$$

Clearly T_0 could be replaced on both sides of (37) by a different reference temperature if desired. The equation can also be put in terms of the Mach number M at the edge of the boundary layer, in the form

$$\Theta M^B F(M) = \left(1+\frac{1}{n}\right)C\int M^B F(M)\,dx, \tag{38}$$

where

$$F(M) = \left(\frac{T_e}{T_0}\right)^{\alpha+(1-l)B/2}\left(\frac{T_m}{T_e}\right)^{-\beta}. \tag{39}$$

T_e/T_0 is $(1+\frac{1}{5}M^2)^{-1}$, and T_m/T_e is given in terms of Mach number by (24) or (25).

The constants of integration on the right-hand sides of (37) and (38) are to be found from a laminar boundary-layer calculation, for which a number of methods are available including that recently published by Luxton and Young[9]. This will give a value of θ_T, say, for the initial momentum thickness, which may be assumed continuous at the transition point x_T.

Non-dimensionally, (38) may be written in terms of a reference length c and the Reynolds number $R = a_0 c/\nu_0$ based on the viscosity and speed of sound at an isentropic stagnation point in the free stream, as

$$\left(\frac{\theta}{c}\right)^{1+1/n}M^{B+1/n}G(M) = \left(1+\frac{1}{n}\right)CR^{-1/n}\int_{x_T/c}^{x/c}M^B F(M)\,d\left(\frac{x}{c}\right)+K, \tag{40}$$

where it is found with the aid of (36) that

$$G(M) = \left(\frac{T_e}{T_0}\right)^{(1+1/n)\,k+(B+1/n)\,(1-l)/2} \tag{41}$$

and K is the value of the left-hand side at transition.

Although B depends on T_w/T_0 (by equation (33)) in the constant wall-temperature case, both $F(M)$ and $G(M)$ are independent of this ratio, for since $l = 1$ (equation (20)), the terms involving B disappear from the indices in equations (39) and (41).

5. *Numerical Values.*—We shall take $\gamma = 1{\cdot}4$, $r = 0{\cdot}89$, $\omega = 8/9$ for air, and give two alternative sets of values for the remaining constants, between which there is little to choose.

5.1. *1/5th Power Law for Skin Friction* (*Young*[8]).—In order to be consistent with an earlier report[10] we may take

$$C = 0{\cdot}0088, \qquad n = 5, \qquad H_i = 1{\cdot}5. \tag{42}$$

(40) then becomes

$$\left(\frac{\theta}{c}\right)^{1\cdot2} M^{B+0\cdot2}\,G(M) = 0{\cdot}0106R^{-0\cdot2}\int_{x_T/c}^{x/c} M^B\,F(M)\,d\left(\frac{x}{c}\right) + K, \tag{43}$$

where B, $F(M)$ and $G(M)$ in the two cases are:

	Zero heat transfer	Constant wall temperature
B	4	$1{\cdot}8\dfrac{T_w}{T_0}+2{\cdot}2$
$F(M)$	$\left(\dfrac{T_e}{T_0}\right)^{3\cdot343}\left(\dfrac{T_m}{T_e}\right)^{-0\cdot822}$	$\left(\dfrac{T_e}{T_0}\right)^{3\cdot244}\left(\dfrac{T_m}{T_e}\right)^{-0\cdot822}$
$G(M)$	$\left(\dfrac{T_e}{T_0}\right)^{3\cdot765}$	$\left(\dfrac{T_e}{T_0}\right)^{3\cdot666}$
$\dfrac{T_m}{T_e}$	$1+0{\cdot}128M^2$	$\dfrac{1}{2}\left\{1+0{\cdot}078M^2+\dfrac{T_w}{T_0}(1+0{\cdot}2M^2)\right\}$
$\dfrac{T_e}{T_0}$	$(1+0{\cdot}2M^2)^{-1}$ in both cases	

(44)

5.2.—*Maskell's Constants*[7].—To make (40) agree with Maskell's formula in incompressible flow, the constants in which were chosen to secure the best-fitting linear relation between $d\Theta/dx$ and $(\Theta/u_e)\,du_e/dx$ for a wide range of experimental data, we should require the values

$$\frac{1}{n} = 0{\cdot}2155, \qquad C = 0{\cdot}00965, \qquad H_i = 1{\cdot}633. \tag{45}$$

The values of $c_f = 2CR_\theta^{-1/n}$ corresponding to (42) and (45) are the same at $R_\theta = 1000$, and differ by at most a few per cent over the whole range $500 < R_\theta < 10{,}000$; thus both sets of constants lead to very much the same value of θ. Using the values (45), (43) and (44) must be replaced by

$$\left(\frac{\theta}{c}\right)^{1\cdot2155} M^{B+0\cdot2155}\,G(M) = 0{\cdot}01173R^{-0\cdot2155}\int_{x_T/c}^{x/c} M^B\,F(M)\,d\left(\frac{x}{c}\right) + K, \tag{46}$$

TABLE 1

x/c	c_p	$\theta^{1\cdot 2} \times 10^4$		$\dfrac{\tau_w}{\rho_\infty u_\infty^2}$	
		Young	Present method	Young	Present method
0·2	+0·050	0·116	0·116	3·54	3·70
0·3	+0·000	0·412	0·420	2·70	2·77
0·4	−0·031	0·703	0·712	2·33	2·43
0·5	−0·048	0·993	1·010	2·18	2·23
0·6	−0·058	1·287	1·304	2·06	2·10
0·7	−0·062	1·578	1·605	1·97	2·01
0·8	−0·060	1·874	1·907	1·92	1·97
0·9	−0·057	2·172	2·219	1·89	1·93
1·0	−0·060	2·466	2·512	1·84	1·87

favourable (i.e., falling) pressure gradient, in which the Mach number rises linearly with distance from 2 to 6. Equation (43) was used, with a value of R and the Mach-number gradient such that

$$0{\cdot}0106/R^{0\cdot 2}\frac{dM}{d\left(\dfrac{x}{c}\right)} = 0{\cdot}001,$$

and the starting value of θ/c is 0·00177 (for which $M^{4\cdot 2}G(M)(\theta/c)^{1\cdot 2} = 0{\cdot}001$ when $M = 2$). The results are plotted in Fig. 3, and it is seen that the lower the wall to stagnation-temperature ratio, the more rapid is the boundary-layer growth.

7. *Concluding Remarks*.—7.1. *Accuracy of Results*.—There is at present insufficient data to test the accuracy of the value of θ found from the present calculation in a pressure gradient at high Mach number. However, the assumptions on which it rests are well accepted, and it is unlikely to be seriously wrong. The final form of equation (40) is similar to that found by Reshotko and Tucker[5], but the present derivation is considerably more straightforward. The close agreement with Young's method shown in the example is reassuring, but does not necessarily mean the answers are correct.

7.2. *Axisymmetric Flow*.—Professor Cooke has pointed out to the author that, with the same assumptions as used by Young in dealing with axisymmetric flow about a body of radius $r_0(x)$, namely, that the skin friction and form parameter are the same as in two dimensions, the equation (34) may be replaced by

$$\bar{\Theta}\bar{u}_e^{\,B} r_0^{\,1+1/n} = C\left(1+\frac{1}{n}\right)\int r_0^{\,1+1/n}\left(\frac{T_e}{T_0}\right)^{\alpha}\left(\frac{T_m}{T_0}\right)^{-\beta}\bar{u}_e^{\,B}\,dx. \qquad (51)$$

7.3. *Steps in Calculation of θ and c_f*.—The procedure for carrying out the calculation may briefly be recapitulated:

Given a distribution of Mach number, and the boundary-layer momentum thickness at the transition point, one finds $F(M)$ and $G(M)$ from (44), then

$$\int_{x_T/c}^{x/c} F(M)\, d\left(\frac{x}{c}\right)$$

by quadrature and hence $(\theta/c)^{1\cdot 2}$ from (43); to find c_f also, Θ/c is found from (38), and by (36)

$$c_f = 0{\cdot}0176\frac{\theta}{\Theta}. \qquad (52)$$

NOTATION

a	Velocity of sound
B	Defined by (32) or (33)
c	Reference length
C	Constant in skin-friction law, equation (27)
c_f	Local skin-friction coefficient $\tau_w/\frac{1}{2}\rho_e u_e^2$
C_p	Specific heat at constant pressure
$F(M)$, $G(M)$	Defined by (39), (41)
H	Form parameter δ_1/θ
k, l	Indices in generalised Stewartson transformation (*see* equation (5))
K	Constant of integration
M	Local Mach number at the edge of the boundary layer
n	Index in skin-friction law, equation (27)
r	Recovery factor defined by (3)
R	Reynolds number $a_0 c/\nu_0$
t	η/Δ used in Appendix I
T	Temperature (for suffixes *see* below)
u	Velocity
u_τ	Friction velocity defined by (22)
x	Distance measured along the surface
y	Distance measured normal to the surface
α, β	Defined by (29). α is also used for the turbulent Prandtl number
γ	Ratio of specific heats
δ, δ_1, θ	Boundary-layer thickness, displacement thickness, momentum thickness (for definitions *see* Appendix I)
Δ, η	Defined by (2)
Θ	Defined by (35)
μ, ν, ρ	Viscosity, kinematic viscosity, density
ξ	$= (T_e/T_0)^k$
τ_w	Wall shearing stress
ω	Defined by relation $\mu \propto T^\omega$
κ	Eddy conductivity
Suffixes $_0$	refers to a stagnation point in the free stream
$_e$	the outer edge of the boundary layer
$_m$	to the intermediate temperature, defined by (23)
$_r$	to the recovery (zero heat transfer) temperature
$_i$	to the value in incompressible flow
$_w$	to the wall
$_\infty$	to the undisturbed stream
$_T$	to the transition point

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APPENDIX I

Variation of H with Stream Temperature

Using equations (6) and (7) of Section 1, the displacement thickness is obtained as

$$\delta_1 = \int_0^\delta \left(1-\frac{\rho u}{\rho_e u_e}\right) dy = \delta - \int_0^\Delta \frac{u}{u_e}\, d\eta = \delta - C_1 \Delta, \tag{53}$$

where
$$C_1 = \int_0^1 f(t)\, dt.$$

Similarly the momentum thickness is

$$\theta = \int_0^\delta \frac{\rho u}{\rho_e u_e}\left(1-\frac{u}{u_e}\right) dy = (C_1 - C_2)\Delta, \tag{54}$$

where
$$C_2 = \int_0^1 \{f(t)\}^2\, dt.$$

Thus
$$H = \frac{\delta_1}{\theta} = (\delta/\Delta - C_1)/(C_1 - C_2). \tag{55}$$

Now
$$\delta = \int_0^\delta dy = \int_0^\Delta \frac{\rho_e}{\rho}\, d\eta = \int_0^\Delta \frac{T}{T_e}\, d\eta. \tag{56}$$

If now we use the quadratic temperature-velocity relationship

$$T = T_w + (T_r - T_w)\frac{u}{u_e} - (T_r - T_e)\left(\frac{u}{u_e}\right)^2, \tag{57}$$

which can be shown to hold very closely for turbulent boundary layers, without restriction to unit Prandtl number, we obtain from (56)

$$\frac{\delta}{\Delta} = \int_0^1 \left[\frac{T_w}{T_e} + \left(\frac{T_r - T_w}{T_e}\right) f(t) - \left(\frac{T_r}{T_e} - 1\right)\{f(t)\}^2\right] dt = \frac{T_w}{T_e} + \frac{T_r - T_w}{T_e} C_1 - \left(\frac{T_r}{T_e} - 1\right) C_2. \tag{58}$$

On substitution of δ/Δ in (55), there results

$$(C_1 - C_2)H = (1 - C_1)\frac{T_w}{T_e} + (C_1 - C_2)\left(\frac{T_r}{T_e} - 1\right). \tag{59}$$

But in incompressible flow $\delta = \Delta$; thus from (55)

$$H_i = (1 - C_1)/(C_1 - C_2) \tag{60}$$

and (59) becomes
$$H = \frac{T_w}{T_e} H_i + \frac{T_r}{T_e} - 1, \tag{61}$$

the value quoted in equation (8) of Section 1.

APPENDIX II

Intermediate Temperature Formula for Skin Friction

The intermediate temperature formula is usually expressed in terms of the Reynolds number based on x; thus if in incompressible flow the skin friction is given by

$$\tau_w = \rho u_e^2 f\left(\frac{u_e x}{\nu}\right), \tag{62}$$

ρ and $\nu = \mu/\rho$ being constant everywhere, then in compressible flow τ_w is obtained from the above expression by evaluating density and viscosity at the reference temperature T_m given by equation (19); thus

$$\tau_w = \rho_m u_e^2 f\left(\frac{u_e x}{\nu_m}\right). \tag{63}$$

The integral momentum equation in flat plate flow is then

$$\frac{d\theta}{dx} = \frac{\rho_m}{\rho_e} f\left(\frac{u_e x}{\nu_m}\right), \tag{64}$$

i.e.,

$$\frac{d(\rho_e u_e \theta/\mu_m)}{d(u_e x/\nu_m)} = f\left(\frac{u_e x}{\nu_m}\right), \tag{65}$$

whence on integration, and putting $\theta = 0$ when $x = 0$ it is seen that $\rho_e u_e \theta/\mu_m$ is a function of $u_e x/\nu_m$, and thus by (63) that

$$\tau_w = \rho_m u_e^2 g\left(\frac{\rho_e u_e \theta}{\mu_m}\right), \tag{66}$$

where g is the function which expresses the dependence of $\tau_w/\rho u_e^2$ on $\rho u_e \theta/\mu$ in incompressible flow.

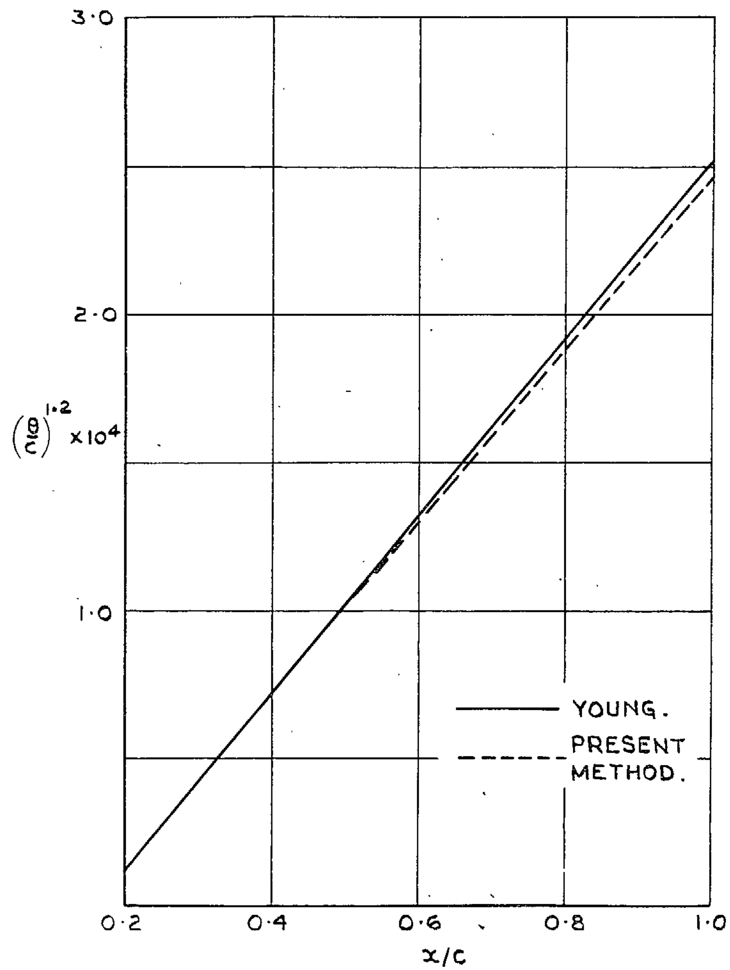

FIG. 1. Momentum thickness as calculated by two methods, in a particular case at zero heat transfer.

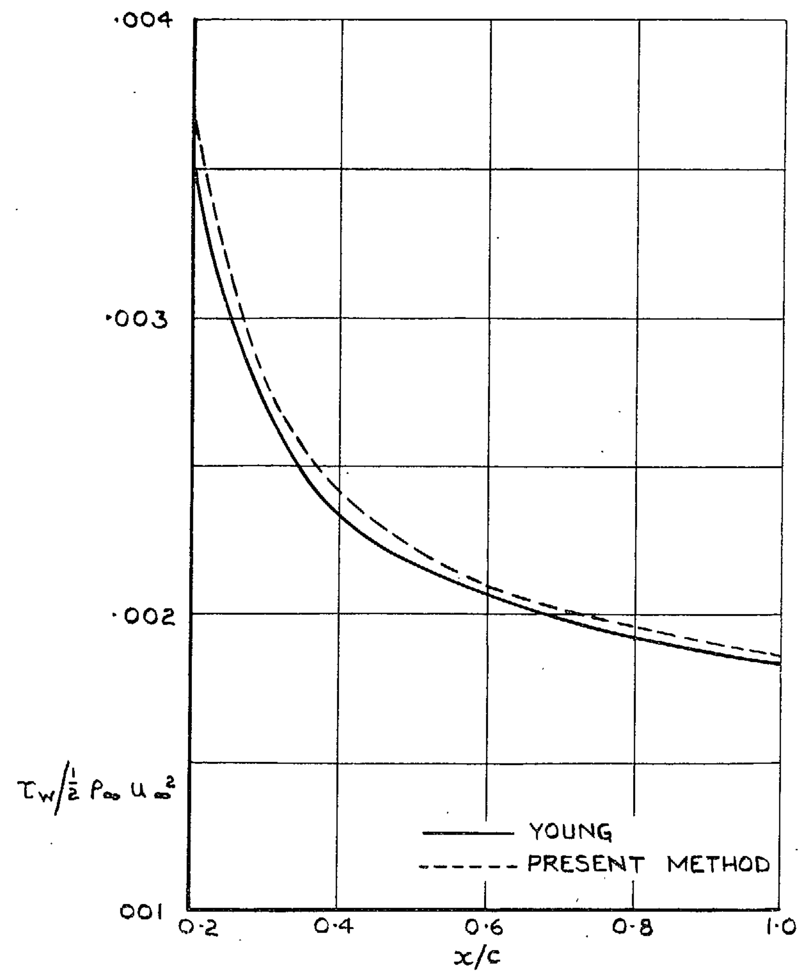

FIG. 2. Skin friction as calculated by two methods, in a particular case at zero heat transfer.

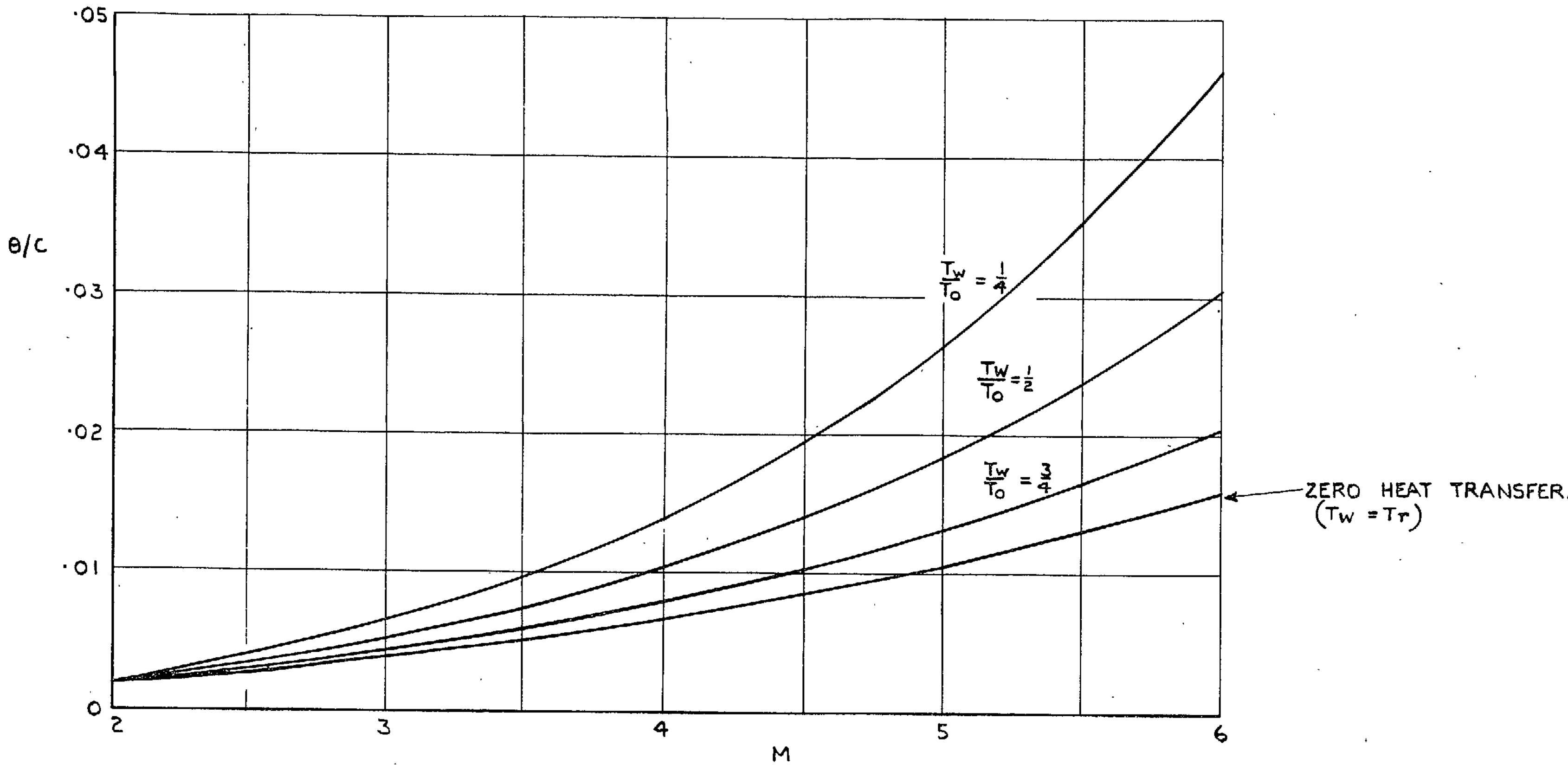

FIG. 3. Calculated growth of momentum thickness in a favourable pressure gradient at different wall temperatures.

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