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The Stability of Rotor Blade Flapping Motion

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Summary. An improved method of solving the rotor blade flapping equation is described which avoids the laborious computation of those at present available. Successive approximations to the solution are made but the rapid convergence makes the second approximation sufficient for all practical purposes. The effect of inclining the flapping hinge is considered and an analytical solution including these effects is given in an Appendix. The results show good agreement with an exact solution.

1. *Introduction.* In this paper a study is made of the transient flapping motion of a rotor blade in forward flight. This problem, which is of importance in the determination of the control response of a helicopter, is of further interest in that the equation of motion suggests that flapping might be unstable at high forward speeds. For this reason there have been many attempts in the past to obtain an exact solution of the blade flapping equation. With one or two exceptions these solutions indicate that flapping is stable for all practical values of the tip-speed ratio. Therefore, since no cases of instability have been reported, and since an exact solution is already available, it would seem that there is little point in presenting yet another solution. But in none of the solutions so far presented are the physical processes made clear, nor is the convergence rapid. As a result, the different methods yield differing numerical results and the exact solution can only be obtained after laborious calculation. Again, most of the previous analyses were concerned only with the stability of the flapping oscillations and little attention has been paid to the problem of calculating the actual blade motion.

Hitherto, apart from the possibility that the blade might strike the droop stops, an exact knowledge of the blade motion following a disturbance has been of little importance. But recent strain-gauge measurements of the stresses in rotor blades have revealed that these cannot be correctly predicted on the basis of simple aerodynamic theory. Therefore, as part of a wider experimental programme, it is intended in the near future to make measurements of the aerodynamic forces on a rotor blade which is in forced oscillation. In order to estimate the influence of frequency on the aerodynamic forces it will be necessary to separate the changes in the blade motion which are due simply to a change in exciting frequency from those which are caused by variations in the aerodynamic forces. In other words, we must be able to calculate the flapping frequency response when the aerodynamic forces are given and to do this the complete solution of the flapping equation must be known. Thus it will be seen that there are good reasons why flapping should be more fully investigated, especially if a method of solution can be found which can be extended to deal with the motion of elastic blades.

Since the derivation of the flapping equation is straightforward, the details are omitted here, but are given in Appendix I. For simplicity, it is assumed that the blade is untwisted, of constant chord, and that it is stiff against both bending and twisting. The flapping hinge is assumed to be on the axis of rotation* and perpendicular to the spanwise axis of the blade, i.e., $\delta_3 = 0$ (Results for the case when $\delta_3 \neq 0$ are quoted in Appendix II). The lift curve is assumed to be a straight line and the effects of stalling and reversed flow are ignored. Then the flapping equation is

$$\frac{d^2\beta}{dt^2} + 2C(t)\frac{d\beta}{dt} + P^2(t)\,\beta = E(\omega t)\,, \tag{1}$$

where

$$2C(t) = n\omega\left[1 + \frac{4\mu}{3}\sin\omega t\right], \tag{2}$$

$$P^2(t) = \omega^2\left[1 + \frac{4n\mu}{3}\cos\omega t + n\mu^2\sin 2\omega t\right], \tag{3}$$

$$E(\omega t) = \text{aerodynamic forcing function (independent of } \beta)\,. \tag{4}$$

Equation (1) is a linear second-order differential equation with periodic coefficients and therefore its solution consists of a complementary function (transient) and a particular integral (steady state). At this stage we are concerned only with the transient motion, *i.e.*, with the solution of

$$\frac{d^2\beta}{dt^2} + 2C(t)\frac{d\beta}{dt} + P^2(t)\,\beta = 0\,. \tag{5}$$

The first attempt at a solution to (5) was given by Glauert and Shone[1], who neglected the second derivative and concluded that flapping is unstable for large values of μ. Bennett[2] later showed that this approximation was invalid and obtained a numerical solution to (5) for the particular values $n = 1{\cdot}5$, $\mu = 1{\cdot}0$, $\delta_3 = 0$. From his results he concluded that flapping is stable for all $\mu < 1$.

Another approximate solution was developed by Sissingh[3], who assumed that β could be expressed in the form

$$\beta = \sum_r A_r \cos r\omega t + \sum_r B_r \sin r\omega t\,, \tag{6}$$

where the A_r and B_r are functions of time. Substitution of (6) in (5) leads to a set of simultaneous second-order differential equations with constant coefficients. For the cases considered it was found that flapping is heavily damped for small values of μ. The principle disadvantage of this method is the heavy labour required, even for the first approximation.

By far the most rigorous and precise solution was given by Horvay[4], who expressed the coefficients of (5) in complex form and then made use of the substitution

$$\beta = e^{\gamma\omega t}\sum_{-\infty}^{+\infty} C_k\, e^{ik\omega t}\,, \tag{7}$$

in which γ and the C_k were unknown. This led to an infinite set of simultaneous differential equations in the C_k, and γ was determined from the condition that these equations should be consistent. The disadvantages of this method are:

(i) The number of equations to be solved is very large and therefore it is necessary to expand a determinant of very high order

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* The case of offset hinges presents no difficulty but is omitted here for the sake of convenience.

(ii) After expanding the determinant, γ must be determined from the roots of a complex transcendental equation.

The labour involved in these computations, whilst not prohibitive, is considerable and the method does not lend itself to routine calculation, especially if 'δ_3' effects are taken into account.

In a later paper, Horvay and Yuan[5] described an approximate method of solving (5) based on the assumption that the periodic coefficients could be replaced by constant time averages throughout each of the four quadrants of a revolution. Once again this technique suffers from the disadvantage that a determinant of large order has to be expanded. It is very difficult to justify the expenditure of a great deal of effort on a solution which can be only approximate.

Parkus[6] solved (5) by means of a substitution of the type

$$\beta = \beta_0 + \mu\beta_1 + \mu^2\beta_2 + \ldots + \mu^r\beta_r + \ldots \tag{8}$$

in which the β_r are functions of time. This method, which is a standard technique for dealing with both linear and non-linear equations, is very well suited to this problem, but Parkus did not use it to the best advantage. The convergence of his solutions was poor and the numerical results did not agree with those of Horvay.

A much simplified approach has been suggested by Owen[7]. The procedure is to assume that, at any azimuth position, the periodic coefficients are constants corresponding to that azimuth position. This suggests that the blade first tends to become unstable on the advancing side and it is shown that μ must be less than a certain value to prevent the onset of instability. Because it is so simple this method has much to recommend it, especially since it may be used for blades with more than one degree of freedom, but detailed comparison with an exact solution will be necessary before its wider use can be justified.

In the present paper an alternative method is described which avoids many of the difficulties of the previous solutions, particularly the expansion of determinants of high order. A method of successive approximations is used but in the early stages of the analysis it is necessary to make certain substitutions which, if one is not familiar with the theory of differential equations with periodic coefficients, may appear to be quite without purpose. Therefore, in order to demonstrate the necessity for these substitutions it has been decided to discuss the more salient features of the results before setting out the theory. From these results it will appear that the solution of the flapping equation may take one of several forms and it follows that the convergence of the process of successive approximations will be most rapid if a solution of the correct type is chosen for the first approximation. In general it is found that this cannot be done without introducing two additional parameters into the flapping equation. It is their presence which may lead to some doubt and confusion, but it is possible to avoid this if it is clearly understood from the outset that the sole purpose of these parameters is to improve the convergence of the solution. In particular, no attempt should be made to endow them with physical significance.

2. *Preliminary Discussion of the Results.* To avoid the complication of too many variables attention will at first be confined to the case where $\delta_3 = 0$; there is no loss of generality in this but a more complete discussion including the effects of δ_3 is given in Appendix II and Section 6.

In hovering (i.e., $\mu = 0$) the flapping equation reduces to

$$\beta'' + n\beta' + \beta = 0\,, \tag{9}$$

where dashes represent differentiation with respect to ψ ($= \omega t$) and $\psi = \omega t$ has been substituted into (5). The solution is

$$\beta(\psi) = e^{-n\psi/2} \left[C \cos \left(1 - \frac{n^2}{4}\right)^{1/2} \psi + D \sin \left(1 - \frac{n^2}{4}\right)^{1/2} \psi \right] . \tag{10}$$

The oscillations following a disturbance are therefore damped and the rate of decay and the period of oscillation increase with the inertia number, n (In practice, $0 < n < 2$ usually, but if $n > 2$ the motion is, of course, a subsidence).

When $\mu \neq 0$ it can be shown that[8]

$$\beta(\psi) = C_1 e^{\gamma_1 \psi} P_1(\psi) + C_2 e^{\gamma_2 \psi} P_2(\psi) , \tag{11}$$

where C_1, C_2, are arbitrary constants, $P_1(\psi)$, $P_2(\psi)$ are periodic functions, and γ_1, γ_2 are real constants. For certain values of μ and n it is found that forward flight tends to make the flapping less stable, *i.e.*,

$$\gamma_1(\text{say}) = -\frac{n}{2} + \lambda , \tag{12}$$

where λ is positive. When presenting the results it is convenient to express this reduced rate of decay as an apparent reduction in the value of n, i.e., we put

$$\gamma_1 = -\frac{n_{app}}{2} , \tag{13}$$

where

$$-n_{app} = -n + 2\lambda . \tag{14}$$

The degree of instability is then given by the ratio n_{app}/n, where

$$\frac{n_{app}}{n} = 1 - \frac{2\lambda}{n} . \tag{15}$$

If $n_{app}/n = 1$, $\lambda = 0$ and forward flight has no effect on the stability, and if $n_{app}/n = 0$, the disturbed flapping motion is an oscillation of constant amplitude. If $n_{app}/n < 0$ the flapping is unstable. The actual values of n_{app}/n for the case $\delta_3 = 0$, $0 < n < 2$, $0 < \mu < 0{\cdot}4$ are shown in Fig. 1. It will be seen that $\lambda > 0$ in only two regions of the μ, n plane. Elsewhere, outside these regions, λ is zero and forward speed has no effect on the stability. For a blade with $n = \sqrt{3}$, $\mu = 0{\cdot}5$, the amplitude of the transient reduces by 93 per cent per revolution so we conclude that flapping is stable under most practical conditions. For the same blade, when $\mu = 0{\cdot}75$, the transient reduces by 87 per cent per revolution, so that the motion is still stable although the damping coefficient of equation (1) becomes zero at some azimuthal position ($\psi = 3\pi/2$). Although these results are of considerable importance in themselves, from the point of view of the method of solution, there are others of even greater value.

The most important result is that within the regions of reduced stability the period of the functions $P_1(\psi)$, $P_2(\psi)$ is a constant, whatever the values of μ and n. In the lower region, *i.e.*, for small values of n, this period is one revolution and in the upper region ($n \sim \sqrt{3}$) the period is two revolutions (frequencies of one and one half cycle per revolution respectively). Between these two regions the period varies continuously.

With the exception of the periods of oscillation, the numerical values in Fig. 1 apply only to this problem, but the fact that the period of oscillation is constant in the less stable regions applies to any blade configuration. Indeed, this result appears to hold for any linear differential equation with periodic coefficients. The above results provide the starting point for the method of solution described in Section 3.

Because the period remains constant for $\lambda > 0$, the solution is more easily obtained in the less stable regions of the μ, n plane. We shall therefore demonstrate the procedure by looking for solutions which have periods of one and two revolutions of the rotor.

3. *Determination of the Solution when* $\lambda > 0, \delta_3 = 0$. 3.1. *General.* The first step in the analysis is to reduce the flapping equation to a more convenient form. When $\delta_3 = 0$, the left-hand side of the flapping equation becomes

$$\beta'' + n(1 + \tfrac{4}{3}\mu \sin \psi)\, \beta' + (1 + \tfrac{4}{3}n\mu \cos \psi + n\mu^2 \sin 2\psi)\beta = 0 . \tag{16}$$

If we put

$$\beta(\psi) = \exp\left[-\frac{n}{2}\psi + \frac{2}{3}n\mu \cos \psi\right]\nu(\psi) , \tag{17}$$

then (16) reduces to

$$\nu'' + \left(1 - \frac{n^2}{4} - \frac{2}{9}n^2\mu^2 + \frac{2}{3}n\mu \cos \psi - \frac{2}{3}n^2\mu \sin \psi + n\mu^2 \sin 2\psi + \frac{2}{9}n^2\mu^2 \cos 2\psi\right)\nu = 0 . \tag{18}$$

This equation for $\nu(\psi)$ contains no first derivative (ν'). Now (18) itself is a linear differential equation with periodic coefficients and therefore its solution is of the form

$$\nu(\psi) = e^{\lambda\psi}\, \phi(\psi) , \tag{19}$$

where $\phi(\psi)$ is periodic and λ is a constant. The substitution of (19) in (18) now yields

$$\phi'' + 2\lambda\phi' + \ldots$$
$$\ldots + \left(1 - \frac{n^2}{4} - \frac{2}{9}n^2\mu^2 + \lambda^2 + \frac{2}{3}n\mu \cos \psi - \frac{2}{3}n^2\mu \sin \psi + n\mu^2 \sin 2\psi\right.$$
$$\left. + \frac{2}{9}n^2\mu^2 \cos 2\psi\right)\phi = 0 . \tag{20}$$

At first sight this may seem to be a retrograde step since the first derivative has reappeared, but if we put $\mu = 0$ then (18) reduces to

$$\nu'' + \left(1 - \frac{n^2}{4}\right)\nu = 0 , \tag{21}$$

the solution of which is periodic for $n < 2$. Hence, when $\mu = 0$, $\lambda = 0$, and $\nu(\psi) = \phi(\psi)$. Thus if the exponent λ does exist it can only do so in forward flight ($\mu > 0$) and this is the case in which we are interested. Therefore, if we solve (20) for ϕ and λ the effect of forward flight on stability will be immediately evident.

In principle the method of solution of (20) is straightforward. At this stage only solutions whose period is one or two revolutions are sought and therefore we require $\phi(\psi)$ to have one or other of these periods. Also the equation for $\phi(\psi)$ must reduce to (21) when $\mu = 0$. λ is to be positive and must vanish when $\mu = 0$.

Now μ is small, therefore a possible approach is to expand $\phi(\psi)$ and λ in ascending powers of μ, i.e., we put

$$\phi = \phi_0 + \mu\phi_1 + \mu^2\phi_2 + \ldots, \tag{22}$$

$$\lambda = \mu\lambda_1 + \mu^2\lambda_2 + \ldots, \tag{23}$$

where the ϕ_i are periodic functions of ψ and the λ_i are constants. This substitution satisfies the conditions that λ vanishes when $\mu = 0$ and that $\phi = \phi_0$ when $\mu = 0$. If $\phi(\psi)$ is to have the necessary period then either

$$\left.\begin{aligned} \phi_0 &= E_1 \cos\frac{\psi}{2} + F_1 \sin\frac{\psi}{2} \\ \text{or} \qquad \phi_0 &= E_2 \cos\psi + F_2 \sin\psi \end{aligned}\right\}. \tag{24}$$

But equation (21) shows that this is only possible when $n = 0$ or $n = \sqrt{3}$. We shall consider the case $n \doteq \sqrt{3}$ first, since the results are of greater practical interest and the less stable region is wider and more easily determined.

3.2. *Solution when* $p/\omega = \frac{1}{2}$. When $\mu = 0$, the period of oscillation is determined by the constant part $(1 - n^2/4)$ of the periodic coefficient in (20). In this case $n = \sqrt{3}$ when $\mu = 0$ and $1 - n^2/4 = \frac{1}{4}$.

When $\mu > 0$ the constant part becomes $1 - \frac{1}{4}n^2 - \frac{2}{9}n^2\mu^2$ and for a given value of μ there is only one positive value of n for which $1 - \frac{1}{4}n^2 - \frac{2}{9}n^2\mu^2 = \frac{1}{4}$. Because μ is small the period must still be dependent upon this constant part but the form of the dependence is as yet unknown. Therefore the constant term in (20) will be replaced by

$$1 - \frac{n^2}{4} - \frac{2}{9}n^2\mu^2 = \frac{1}{4} + \mu a_i + \mu^2 a_2 + \ldots \tag{25}$$

where the a_i are constants to the determined. When $\mu = 0$, equation (25) is satisfied by $n = \sqrt{3}$. In other words, whatever the values of μ and n, the constants a_i must be adjusted in such a way that (25) is always satisfied. (25) is the first of the substitutions discussed in the Introduction.

In order to solve (20) we substitute (22), (23) and (25) in (20) and obtain

$$\begin{aligned} &\phi_0'' + \mu\phi_1'' + \mu^2\phi_2'' + \ldots + 2(\mu\lambda_1 + \mu^2\lambda_2 + \ldots)(\phi_0' + \mu\phi_1' + \mu^2\phi_2' + \ldots) + \ldots \\ &\ldots + [\tfrac{1}{4} + \mu a_1 + \mu^2 a_2 + \ldots + \mu^2\lambda_1^2 + 2\mu^3\lambda_1\lambda_2 + \ldots + \tfrac{2}{3}n\mu\cos\psi - \tfrac{2}{3}n^2\mu\sin\psi + \\ &\quad + n\mu^2\sin 2\psi + \tfrac{2}{9}n^2\mu^2\cos 2\psi]\,(\phi_0 + \mu\phi_1 + \mu^2\phi_2 + \ldots) = 0\,. \end{aligned} \tag{26}$$

The terms may be regrouped as coefficients of the ascending powers of μ:

$$\begin{aligned} &\phi_0'' + \frac{\phi_0}{4} + \left[\phi_1'' + \frac{\phi_1}{4} + 2\lambda_1\phi_0' + a_1\phi_0 + \frac{2}{3}n\cos\psi\phi_0 - \frac{2}{3}n^2\sin\psi\phi_0\right]\mu + \ldots \\ &\ldots + \left[\phi_2'' + \frac{\phi_2}{4} + 2\lambda_2\phi_0' + 2\lambda_1\phi_1' + a_1\phi_1 + a_2\phi_0 + \lambda_1^2\phi_0 + \frac{2}{3}n\cos\psi\phi_1 - \frac{2}{3}n^2\sin\psi\phi_1 + \ldots\right. \\ &\left.\ldots + n\sin 2\psi\phi_0 + \tfrac{2}{9}n^2\cos 2\psi\phi_0\right]\mu^2 + \ldots = 0\,. \end{aligned} \tag{27}$$

If this equation is to be satisfied for all values of μ, then the coefficients of individual powers of μ must vanish separately, *i.e.*,

$$\phi_0'' + \frac{\phi_0}{4} = 0 \tag{28}$$

$$\phi_1'' + \frac{\phi_1}{4} = -2\lambda_1\phi_0' - a_1\phi_0 - \frac{2}{3}n\cos\psi\phi_0 + \frac{2}{3}n^2\sin\psi\phi_0 \tag{29}$$

$$\phi_2'' + \frac{\phi_2}{4} = -2\lambda_2\phi_0' - a_2\phi_0 - 2\lambda_1\phi_1' - a_1\phi_1 - \lambda_1^2\phi_0 - \frac{2}{3}n\cos\psi\phi_1 + \dots$$

$$\dots + \frac{2}{3}n^2\sin\psi\phi_1 - n\sin 2\psi\phi_0 - \frac{2}{9}n^2\cos 2\psi\phi_0 \tag{30}$$

$$\phi_3'' + \frac{\phi_3}{4} = \dots, \text{ etc.}$$

Thus the original differential equation with periodic coefficients has been reduced to a set of successive equations with constant coefficients. The general solution of (28) is,

$$\phi_0 = E_1\cos\frac{\psi}{2} + F_1\sin\frac{\psi}{2}. \tag{31}$$

But since (20) is linear its solution is only determined to within an arbitrary constant; therefore we can, without loss of generality, assume that

$$\phi_0 = \sin\left(\frac{\psi}{2} - \sigma\right). \tag{32}$$

The 'phase' angle σ is the second of the parameters mentioned in the Introduction; the reason for introducing it here is explained below. If we now substitute (32) in (29), (30), etc., (29) becomes

$$\phi_1'' + \frac{\phi_1}{4} = -\lambda_1\cos\left(\frac{\psi}{2} - \sigma\right) - a_1\sin\left(\frac{\psi}{2} - \sigma\right) - \frac{n}{3}\sin\left(\frac{3\psi}{2} - \sigma\right) - \frac{n^2}{3}$$

$$\cos\left(\frac{3\psi}{2} - \sigma\right) + \frac{n}{3}\sin\left(\frac{\psi}{2} + \sigma\right) + \frac{n^2}{3}\cos\left(\frac{\psi}{2} + \sigma\right) + \dots. \tag{33}$$

Now ϕ_1 must be periodic, but the solution of (33) is not periodic unless the coefficients of $\sin(\frac{1}{2}\psi - \sigma)$, $\cos(\frac{1}{2}\psi - \phi)$ in the right-hand side are zero, *i.e.*, unless

$$\lambda_1 = \frac{n}{3}(\sin 2\sigma + n\cos 2\sigma), \tag{34}$$

$$a_1 = \frac{n}{3}(\cos 2\sigma - n\sin 2\sigma). \tag{35}$$

Note. In arriving at (34) and (35) use has been made of

$$\left.\begin{aligned}\sin\left(\frac{\psi}{2} + \sigma\right) &= \sin\left(\overline{\frac{\psi}{2} - \sigma} + 2\sigma\right)\\ \cos\left(\frac{\psi}{2} + \sigma\right) &= \cos\left(\overline{\frac{\psi}{2} - \sigma} + 2\sigma\right)\end{aligned}\right\}. \tag{36}$$

It is now possible to explain the introduction of σ in (32). The solution of (33) would be non-periodic if a term in either $\cos(\frac{1}{2}\psi - \sigma)$ or $\sin(\frac{1}{2}\psi - \sigma)$ appeared in the right-hand side. Therefore, since these two functions are independent, at least two arbitrary unknowns must be made available to ensure that the coefficients of both $\cos(\frac{1}{2}\psi - \sigma)$ and $\sin(\frac{1}{2}\psi - \sigma)$ are zero. But the a_i are not completely arbitrary since equation (25) must also be satisfied. Thus if it had been assumed that $\sigma = 0$, (25) could only have been satisfied for one value of μ. But σ may take any value, even complex, without affecting the form of the solution. Therefore it is possible to satisfy (25) for all μ if a_i is given by (35).

With a_1 and λ_1 given by (34) and (35), (33) becomes

$$\phi_1'' + \frac{\phi_1}{4} = -\frac{n}{3}\sin\left(\frac{3\psi}{2} - \sigma\right) - \frac{n^2}{3}\cos\left(\frac{3\psi}{2} - \sigma\right) \tag{37}$$

so that

$$\phi_1 = \frac{n}{6}\left[\sin\left(\frac{3\psi}{2} - \sigma\right) + n\cos\left(\frac{3\psi}{2} - \sigma\right)\right]. \tag{38}$$

If we now substitute for ϕ_0, ϕ_1, (30) can be written in the form

$$\begin{aligned}\phi_2'' + \frac{\phi_2}{4} = &-\lambda_2\cos\left(\frac{\psi}{2} - \sigma\right) - \left[a_2 + \lambda_1^2 + \frac{n^2}{18}(1 + n^2)\right]\sin\left(\frac{\psi}{2} - \sigma\right) + \dots \\ &- \left[\frac{n}{2}\lambda_1 + \frac{n}{2}\cos 2\sigma - \frac{n^2}{9}\sin 2\sigma\right]\cos\left(\frac{3\psi}{2} - \sigma\right) \\ &+ \left[\frac{n^2}{2}\lambda_1 + \frac{n}{2}\sin 2\sigma + \frac{n^2}{9}\cos 2\sigma\right]\sin\left(\frac{3\psi}{2} - \sigma\right) \\ &+ \left[\frac{n^2}{18}(n^2 - 3)\right]\sin\left(\frac{5\psi}{2} - \sigma\right) - \left[\frac{n^3}{9} - \frac{n}{2}\right]\cos\left(\frac{5\psi}{2} - \sigma\right).\end{aligned} \tag{39}$$

ϕ_2 will not be periodic unless the coefficients of $\cos(\frac{1}{2}\psi - \sigma)$ and $\sin(\frac{1}{2}\psi - \sigma)$ are zero. Hence, we must have

$$\lambda_2 = 0 \tag{40}$$

$$a_2 = -\lambda_1^2 - \frac{n^2}{18}(1 + n^2). \tag{41}$$

Then

$$\begin{aligned}\phi_2 = &\left[\frac{n}{4}\lambda_1 + \frac{a_1}{12}n^2 + \frac{n}{4}\cos 2\sigma - \frac{n^2}{18}\sin 2\sigma\right]\cos\left(\frac{3\psi}{2} - \sigma\right) + \dots \\ &+ \left[\frac{n}{2}a_1 - \frac{n^2}{4}\lambda_1 - \frac{n}{4}\sin 2\sigma - \frac{n^2}{18}\cos 2\sigma\right]\sin\left(\frac{3\psi}{2} - \sigma\right) \\ &+ \left[\frac{n^3}{54} - \frac{n}{12}\right]\cos\left(\frac{5\psi}{2} - \sigma\right) - \left[\frac{n^2}{108}(n^2 - 3)\right]\sin\left(\frac{5\psi}{2} - \sigma\right).\end{aligned} \tag{42}$$

This process of solution and substitution may now be continued indefinitely, depending upon the degree of accuracy required, but for small μ it has been found to be sufficient to stop at the second approximation. Then to the second order in μ we have, from equations (25), (35) and (41)

$$1 - \frac{n^2}{4} - \frac{2}{9}n^2\mu^2 = \frac{1}{4} + \mu\frac{n}{3}(\cos 2\sigma - n\sin 2\sigma) - \mu^2\left[\lambda_1^2 + \frac{n^2}{18}(1 + n^2)\right] \tag{43}$$

and from equations (23), (34) and (40),

$$\lambda = \mu\lambda_1 = \mu\frac{n}{3}(\sin 2\sigma + n\cos 2\sigma). \tag{44}$$

The method used for the computation of stability boundaries (*i.e.*, the determination of λ for given μ and n) is given in Section 4.1.

3.3. *Solution when* $p/\omega = 1$. The analysis for this case is exactly similar to that described in Section 3.2. The substitution corresponding to (25) is

$$1 - \frac{n^2}{4} - \frac{2}{9}n^2\mu^2 = 1 + \mu a_1 + \mu^2 a_2 + \dots \qquad (45)$$

Using (22), (23) and (45), equation (20) becomes

$$\phi_0'' + \mu\phi_1'' + \mu^2\phi_2'' + \dots + 2(\mu\lambda_1 + \mu^2\lambda_2 + \dots)(\phi_0' + \mu\phi_1' + \mu^2\phi_2' + \dots) + \dots$$
$$\dots + [1 + \mu a_1 + \mu^2 a_2 + \dots + \mu^2\lambda_1^2 + 2\mu^3\lambda_1\lambda_2 + \dots + \tfrac{2}{3}n\mu\cos\psi - \tfrac{2}{3}n^2\mu\sin\psi + \dots$$
$$\dots + n\mu^2\sin 2\psi + \tfrac{2}{9}n^2\mu^2\cos 2\psi]\,(\phi_0 + \mu\phi_1 + \mu^2\phi_2 + \dots) = 0. \qquad (46)$$

Regrouping the terms as coefficients of ascending powers of μ:

$$\phi_0'' + \phi_0 + [\phi_1'' + \phi_1 + 2\lambda_1\phi_0' + a_1\phi_0 + \tfrac{2}{3}n\cos\psi\phi_0 - \tfrac{2}{3}n^2\sin\psi\phi_0]\mu + \dots$$
$$\dots + [\phi_2'' + \phi_2 + 2\lambda_2\phi_0' + 2\lambda_1\phi_1' + a_1\phi_1 + a_2\phi_0 + \lambda_1^2\phi_0 + \tfrac{2}{3}n\cos\psi\phi_1 - \dots$$
$$\dots - \tfrac{2}{3}n^2\sin\psi\phi_1 + n\sin 2\psi\phi_0 + \tfrac{2}{9}n^2\cos 2\psi\phi_0]\mu^2 + \dots = 0. \qquad (47)$$

As in Section 3.2, on equating individual powers of μ to zero,

$$\phi_0'' + \phi_0 = 0, \qquad (48)$$
$$\phi_1'' + \phi_1 = -2\lambda_1\phi_0' - a_1\phi_0 - \tfrac{2}{3}n\cos\psi\phi_0 + \tfrac{2}{3}n^2\sin\psi\phi_0 \qquad (49)$$
$$\phi_2'' + \phi_2 = -2\lambda_2\phi_0' - a_2\phi_0 - 2\lambda_1\phi_1' - a_1\phi_1 - \lambda_1^2\phi_0 - \tfrac{2}{3}n\cos\psi\phi_1 + \dots$$
$$+ \tfrac{2}{3}n^2\sin\psi\phi_1 - n\sin 2\psi\phi_0 - \tfrac{2}{9}n^2\cos 2\psi\phi_0\,. \qquad (50)$$
$$\phi_3'' + \phi_3 = \dots, \text{ etc.}$$

The general solution of (48) is

$$\phi_0 = E_1\cos\psi + F_1\sin\psi\,.$$

As in Section 3.2 we assume that

$$\phi_0 = \sin(\psi - \sigma)\,. \qquad (51)$$

Then (49) becomes

$$\phi_1'' + \phi_1 = -2\lambda_1\cos(\psi - \sigma) - a_1\sin(\psi - \sigma) - \tfrac{2}{3}n\cos\psi\sin(\psi - \sigma) + \dots$$
$$\dots + \tfrac{2}{3}n^2\sin\psi\sin(\psi - \sigma)\,. \qquad (52)$$

But ϕ_1 must be periodic, so that

$$\lambda_1 = 0\,, \qquad (53)$$
$$a_1 = 0\,. \qquad (54)$$

(52) may now be written

$$\phi_1'' + \phi_1 = \frac{n}{3}(n\cos\sigma + \sin\sigma) - \frac{n^2}{3}\cos(2\psi - \sigma) - \frac{n}{3}\sin(2\psi - \sigma)\,, \qquad (55)$$

which has the solution

$$\phi_1 = \frac{n}{3}(n\cos\sigma + \sin\sigma) + \frac{n}{9}[n\cos(2\psi - \sigma) + \sin(2\psi - \sigma)]\,. \qquad (56)$$

(50) now becomes

$$\phi_2'' + \phi_2 = \left[- 2\lambda_2 + \frac{n^4}{9} \sin 2\sigma - n \left(\frac{1}{2} + \frac{2}{9} n^2\right) \cos 2\sigma \right] \cos (\psi - \sigma) + \dots$$

$$\dots + \left[- a_2 + \frac{n^4}{9} \cos 2\sigma + n \left(\frac{1}{2} + \frac{2}{9} n^2\right) \sin 2\sigma + \frac{2}{27} n^2 (1 + n^2) \right] \sin (\psi - \sigma) + \dots$$

$$\dots + \frac{n^2}{27} (n^2 - 4) \sin (3\psi - \sigma) + n \left(\frac{1}{2} - \frac{2}{27} n^2\right) \cos (3\psi - \sigma) . \tag{57}$$

Since ϕ_2 must be periodic,

$$\lambda_2 = \frac{n^4}{18} \sin 2\sigma - \frac{n}{2} \left(\frac{1}{2} - \frac{2}{9} n^2\right) \cos 2\sigma , \tag{58}$$

$$a_2 = \frac{n^4}{9} \cos 2\sigma + n \left(\frac{1}{2} + \frac{2}{9} n^2\right) \sin 2\sigma + \frac{2}{27} (1 + n^2) n^2 , \tag{59}$$

and hence

$$\phi_2'' + \phi_2 = \frac{n^2}{27} (n^2 - 4) \sin (3\psi - \sigma) + n \left(\frac{1}{2} - \frac{2}{27} n^2\right) \cos (3\psi - \sigma) . \tag{60}$$

The equations determining λ to the second order in μ are:

$$\lambda = \mu^2 \lambda_2 = \mu^2 \left[\frac{n^4}{18} \sin 2\sigma - \frac{n}{2} \left(\frac{1}{2} + \frac{2}{9} n^2\right) \cos 2\sigma \right] \tag{61}$$

$$1 - \frac{n^2}{4} - \frac{2}{9} n^2 \mu^2 = 1 + \mu^2 \left[\frac{n^4}{9} \cos 2\sigma + n \left(\frac{1}{2} + \frac{2}{9} n^2\right) \sin 2\sigma + \frac{2}{27} n^2 (1 + n^2) \right] . \tag{62}$$

A discussion of the results, together with methods of computation, are given in the next Section.

4. *Calculation of Degree of Instability and Transients.* 4.1. *Stability when $p/\omega = \frac{1}{2}$.* It was shown in Section 2 that the degree of instability is given by

$$\frac{n_{\text{app}}}{n} = 1 - \frac{2\lambda}{n} . \tag{63}$$

In other words, for a given blade and forward speed the stability is determined by λ. For the case where the period of oscillation is two revolutions the relationship between λ, μ and n is obtained by eliminating σ from (43) and (44). Since both (43) and (44) contain first-order terms in μ, and μ is small, it is possible to obtain an approximate solution by ignoring μ^2, i.e., the equations

$$1 - \frac{n^2}{4} = \frac{1}{4} + \mu \frac{n}{3} (\cos 2\sigma - n \sin 2\sigma) , \tag{64}$$

$$\lambda = \mu \frac{n}{3} (\sin 2\sigma + n \cos 2\sigma) , \tag{65}$$

are solved simultaneously.

Noting that

$$\cos 2\sigma - n \sin 2\sigma = A \cos (2\sigma + \epsilon) , \tag{66}$$

where $A^2 = 1 + n^2$ and $\tan \epsilon = n$, it follows that

$$A \sin (2\sigma + \epsilon) = \sin 2\sigma + n \cos 2\sigma . \tag{67}$$

Therefore (64) becomes

$$\frac{3 - n^2}{4} = \mu \frac{n}{3} A \cos(2\sigma + \epsilon) \tag{68}$$

and

$$\lambda = \mu \frac{n}{3} A \sin(2\sigma + \epsilon). \tag{69}$$

Then if $n_{\text{app}}/n = N$, (68) and (69) give

$$\sin(2\sigma + \epsilon) = \frac{3}{2\mu A}(1 - N), \tag{70}$$

and finally,

$$\mu_1^2 = \frac{9}{16A^2}\left[\frac{(3 - n^2)^2}{n^2} + 4(1 - N)^2\right], \tag{71}$$

where μ_1 is the first approximation to μ.

On including terms in μ^2 and eliminating σ from (43) and (44) a quartic equation in μ is obtained. Newton's method for finding roots of polynomials has been used to find μ to the second approximation (μ_2), i.e.,

$$\mu_2 = \mu_1 - \frac{f(\mu_1)}{f'(\mu_1)}, \tag{72}$$

where $f(\mu) = 0$ is the quartic in μ. The resulting expression for μ is

$$\mu_2 = \mu_1\left[1 - \frac{1}{4}\frac{C^2\mu_1^4 + (2BC - D)\mu_1^2 + (B^2 + E)}{C^2\mu_1^4 + \frac{1}{2}(2BC - D)\mu_1^2}\right], \tag{73}$$

where

$$B = \frac{3 - n^2}{4} + E$$

$$C = -\frac{n^2}{18}(3 - n^2)$$

$$D = \frac{n^2}{9}(1 + n^2)$$

$$E = \frac{n^2}{4}(1 - N)^2.$$

The stability diagram (Fig. 1) is obtained by choosing N and solving (71) and then (73) for μ over a range of n. It will be seen that (73) provides a direct estimate of the difference between first and second approximations to μ. Fig. 2 indicates the error for corresponding values of μ and n, and suggests that the first approximation is sufficient for most practical purposes. For this reason the transient motions derived in Section 4.3 are to the first order in μ.

4.2. *Calculation of Blade Transient Motion when* $p/\omega = \frac{1}{2}$. In Section 2 the transient flapping motion is given as

$$\beta(\psi) = C_1 e^{\gamma_1\psi} P_1(\psi) + C_2 e^{\gamma_2\psi} P_2(\psi) \tag{74}$$

$$= C_1\beta_1(\psi) + C_2\beta_2(\psi), \tag{75}$$

where C_1 and C_2 are arbitrary constants. The two solutions $\beta_1(\psi)$ and $\beta_2(\psi)$ are determined by the two values (σ_1 and σ_2) of σ which satisfy (64) for a given n and μ. From (64),

$$\cos(2\sigma + \epsilon) = \frac{3 - n^2}{4}\frac{3}{\mu n A}, \tag{76}$$

so that σ_1 and σ_2 are easily determined for inclusion in $\phi_0(\psi)$, $\phi_1(\psi)$ in $P(\psi)$. The exponents γ_1, γ_2 are given by (12) as

$$\left.\begin{aligned} \gamma_1 &= -\frac{n}{2} + \lambda(\sigma_1) \\ \gamma_2 &= -\frac{n}{2} + \lambda(\sigma_2) \end{aligned}\right\}. \tag{77}$$

But from (65),

$$\lambda = \mu\frac{n}{3}A\sin(2\sigma + \epsilon), \tag{78}$$

so that on eliminating σ from (76) and (78) we see that

$$\lambda(\sigma_1) = -\lambda(\sigma_2) \tag{79}$$

for all μ and n in this region ($p/\omega = \frac{1}{2}$). The complete solution is therefore

$$\beta(\psi) = \exp[\tfrac{2}{3}n\mu\cos\psi]\,[C_1\,e^{(-\frac{1}{2}n+\lambda)\psi}\,\phi(\psi, \sigma_1) + C_2\,e^{(-\frac{1}{2}n-\lambda)\psi}\,\phi(\psi, \sigma_2)]\,. \tag{80}$$

The constants C_1 and C_2 are determined by the initial conditions. For example, Fig. 3 shows the resulting motion for

(i) $\beta(0) = 0$, $\beta'(0) = 1$

(ii) $\beta(0) = 1$, $\beta'(0) = 0$

with $n = 1{\cdot}6$ and $\mu = 0{\cdot}3$.

Fig. 4 gives the corresponding results for $\delta_3 = \pm 5$ deg. The analysis for this latter case appears in Appendix II.

4.3. *Stability when $p/\omega = 1$.* The analysis for determining the range of instability in this region closely follows that described in Section 4.1. Since both a_1 and λ_1 are zero, the method of successive approximations is not necessary. However, since for this region $0 < n < 0{\cdot}4$, it is possible to ignore powers of n higher than the second. The equations to be solved are (61) and (62). From (61) and (63) with $n_{\text{app}}/n = N$,

$$\cos(2\sigma + \epsilon) = -\frac{n}{A\mu^2}(1 - N), \tag{81}$$

where

$$A^2 = \left(\frac{n^4}{9}\right)^2 + n^2\left(\frac{1}{2} + \frac{2}{9}n^2\right)^2$$

or, ignoring powers of n higher than the fourth,

$$A^2 = \frac{n^2}{4}\,.$$

Substituting for $(2\sigma + \epsilon)$ from (78) in (62) yields the following quadratic in μ^2:

$$(\alpha - \delta^2)\mu^4 - 2\alpha\delta\mu^2 - (\beta + \alpha^2) = 0\,, \tag{82}$$

where $\alpha = A^2$, $\beta = n^2(1 - N)^2$, and $\delta = 8n^2/27$. The curves corresponding to $N = 1{\cdot}0$, $N = 0{\cdot}9$, are given in Fig. 1. The motion becomes unstable at $\mu = \sqrt{2}$.

Although most helicopters have blades with n in the range $1 \cdot 6 < n < 2 \cdot 0$, the solution for $n \sim 0$ is expected to be of value in connection with wind-tunnel experiments on model rotors. No blade motions have been calculated and no attention has been paid to 'δ_3' effects but it should be borne in mind that full-size machines with tip-jets would also have a low value of n. These extensions would present no difficulty either in analysis or computation.

In order to complete the solution of the flapping equation, the solution for the region in which $p/\omega \neq \frac{1}{2}$ or 1 is discussed in Section 5.

5. *Determination of the Solution when $p/\omega \neq \frac{1}{2}$ or 1.* 5.1. *Solution when $\frac{1}{2} < p/\omega < 1$.* Between the $p/\omega = \frac{1}{2}$ and $p/\omega = 1$ regions there exists a region of complete stability, i.e., $\lambda = 0$ for all μ and n. The analysis is therefore much simplified because it is only necessary to look for periodic solutions of equation (18). For a solution of frequency p, (25) becomes

$$1 - \frac{n^2}{4} - \frac{2}{9}n^2\mu^2 = p'^2 + \mu a_1 + \mu^2 a_2 + \ldots \tag{83}$$

and $\phi(\psi)$ may be written

$$\phi(\psi) = \sin(p'\psi - \sigma) + \mu\phi_1 + \mu^2\phi_2 + \ldots, \tag{84}$$

where $p' = p/\omega$. Substituting (83) and (84) in (20) with $\lambda = 0$ and following the methods of Sections 3.2 and 3.3, the results

$$a_1 = 0, \tag{85}$$

$$a_2 = \frac{2n^2(1 + n^2)}{9(4p'^2 - 1)}, \tag{86}$$

are obtained. Then from (83), (85) and (86), the variation of frequency ratio ($p/\omega = p'$), with n and μ may be expressed analytically by

$$\mu^2 = \frac{9}{n^2}\left[\frac{(4p'^2 - 1)\left(1 - \frac{n^2}{4} - p'^2\right)}{2(4p'^2 - 1) + 2(1 + n^2)}\right]. \tag{87}$$

A few curves along which the frequency of the solution remains constant are shown on Fig. 1.

5.2. *Solution for $n > 2$.* On examination of equation (10) it will be seen that the flapping frequency becomes zero when $n = 2$, $\mu = 0$. When forward flight is considered for $n > 2$ it is found that a region of reduced stability exists throughout which the flapping frequency remains zero. Since it is of little practical interest the analysis for this region of the stability diagram is omitted.

6. *Conclusions and Further Developments.* An analytic solution of the rotor blade flapping equation has been obtained by a method of successive approximations. The necessity of approximating to the flapping equation has been eliminated and the excessive labour of computation of previous exact methods avoided. The convergence of the solution is rapid and the first approximation is correct, within the accuracy of the basic assumptions, for $\mu \leqslant 0 \cdot 3$. These improvements have made it possible to take into account the effects of the δ_3 flapping hinge inclination.

The results confirm that flapping motion is stable over the normal range of advance ratio (μ) and inertia number (n), and that there exists a region of reduced stability in which the frequency is constant (half the rotational frequency). Of considerable interest, however, is the discovery of a second region in which stability is reduced with increasing forward speed and in which the frequency remains constant at one oscillation per revolution. This reveals the necessity of obtaining the correct inertia number in experimental work employing model blades, more especially as the aerodynamic forces have been shown[9] to depend acutely upon p/ω.

For a typical blade ($n = 1 \cdot 6$) a small positive δ_3 has a considerable effect on the stability. For example, when $\mu = 0 \cdot 3$ the percentage reductions in the transient in each revolution for corresponding values of δ_3 are (*see* Figs. 3 and 4):

$\delta_3 = -5$ deg, reduction of $94 \cdot 0$ per cent per revolution

$\delta_3 = 0$ deg, reduction of $96 \cdot 2$ per cent per revolution

$\delta_3 = +5$ deg, reduction of $99 \cdot 1$ per cent per revolution.

Although this paper has been concerned with a single-degree-of-freedom oscillation the method can be extended to two or more degrees of freedom. Combined flapping-bending or flapping-pitch oscillations or flutter can be considered in this way and it might have some application to the ground resonance problem especially if friction dampers are used.

NOTATION

a_i	Constants
c	Blade chord
C_i, D_i, etc.	Constants
I_F	Blade moment of inertia about the flapping hinge
n	Inertia number, $n = \left(\rho \dfrac{dC_L}{d\alpha} cR^4\right) \Big/ 8I_F$
n_{app}	Apparent value of n, $\mu > 0$
N	$\dfrac{n_{\text{app}}}{n}$
p	Flapping frequency
p'	Frequency ratio (p/ω)
$P_i(\psi)$	Periodic functions of ψ
R	Tip radius
t	Time
$\nu(\psi)$	$e^{\lambda\psi} \phi(\psi)$
V	Forward speed of aircraft
α	Angle of attack
$\beta(\psi)$	Flapping angle measured from undeflected position
$\beta'(\psi)$, $\beta''(\psi)$	$\dfrac{d\beta}{d\psi}, \dfrac{d^2\beta}{d\psi^2}$
$\beta_1(\psi)$, $\beta_2(\psi)$	First and second solutions to flapping equation
γ	Exponent in Horvay's solution
γ_1, γ_2	Constant exponents
$\pi/2 - \delta_3$	Angle that the flapping hinge forms with the blade span axis
λ	Exponent in $\nu(\psi)$
λ_i	Coefficient of μ^i in λ
μ	Advance ratio $\mu = V/\omega R$
σ	Phase angle
$\phi(\psi)$	Periodic function of ψ
$\phi_i(\psi)$	Coefficient of μ^i in $\phi(\psi)$
ψ	Azimuth position (measured from aft) $= \omega t$
ω	Rotational frequency

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APPENDIX I

Derivation of Flapping Equation of Motion

To avoid continual reference to previous work, the flapping equation of motion is derived in this Appendix. When the blades are rotating with angular velocity ω, the centrifugal force experienced by each is

$$C = \int_0^R dC = \int_0^R r\omega^2 \, dm \, , \tag{A.1}$$

where dm is the mass of the blade element at radius r. The blades also experience lift forces

$$\int_0^R dL = \tfrac{1}{2} \int_0^R \rho c \frac{dC_L}{d\alpha} (\phi + \vartheta) U^2 \, dr \, , \tag{A.2}$$

where ρ is the air density, c the chord, $dC_L/d\alpha$ the slope of the lift curve, ϑ the blade pitch angle, $\vartheta + \phi$ the angle of attack, and U is the resultant air velocity (see Fig. 5a).

In the course of forward flight the tangential component of air velocity at element dm is

$$U_T = r\omega + \mu\omega R \sin \psi \tag{A.3}$$

and the perpendicular component is

$$U_P = \lambda' \omega R - \beta \frac{d\beta}{dt} - \mu\omega R\beta \cos \psi \, . \tag{A.4}$$

In (A.4), $\lambda'\omega R$ is the difference between the sinking speed of the helicopter and induced velocity through the rotor disc; it is called the inflow ratio. The flapping motion of the blade is governed by (*see* Fig. 5b):

$$I_F \frac{d^2\beta}{dt^2} = \int_0^R r(dL - g \, dm - \beta \, dC) \, . \tag{A.5}$$

Since ϑ and ϕ are small,

$$(\vartheta + \phi) U^2 \simeq U_T U_P + \vartheta U_T^2 \, . \tag{A.6}$$

Substituting (A.1), (A.2) and (A.6) into (A.5), and performing the integration (A.5), we arrive at the equation of flapping motion,

$$\frac{d^2\beta}{dt^2} + 2C(t) \frac{d\beta}{dt} + P^2(t)\beta = E(\omega t) \, , \tag{A.7}$$

with

$$2C(t) = n\omega[1 + \tfrac{4}{3}\mu \sin \omega t] \tag{A.8}$$

$$P^2(t) = \omega^2[1 + \tfrac{4}{3}\mu \cos \omega t + n\mu^2 \sin 2\omega t] \, . \tag{A.9}$$

$E(\omega t)$ is a forcing function containing periodic terms which are independent of β. On replacing ωt by ψ, (A.7) becomes

$$\frac{d^2\beta}{d\psi^2} + n \, [1 + \tfrac{4}{3}\mu \sin \psi] \frac{d\beta}{d\psi} + [1 + \tfrac{4}{3}n\mu \cos \psi + n\mu^2 \sin 2\psi] \, \beta = E(\psi) \, . \tag{A.10}$$

APPENDIX II

Consideration of 'δ_3' Effects

In Ref. 5, Horvay and Yuan derive the flapping equation for the case when the flapping hinge makes an angle $(\frac{1}{2}\pi - \delta_3)$ with the blade span axis. The equation for the transient motion then becomes

$$\frac{d^2\beta}{d\psi^2} + 2C(\psi)\frac{d\beta}{d\psi} + P^2(\psi)\beta = 0\,, \tag{B.1}$$

where

$$2C(\psi) = n[1 + \tfrac{4}{3}\mu \sin\psi] \tag{B.2}$$

$$P^2(\psi) = [1 + n\mu(\tfrac{4}{3}\cos\psi + \mu\sin 2\psi) + n\tan\delta_3(1 + \mu^2 + \tfrac{8}{3}\mu\sin\psi - \mu^2\cos 2\psi)]\,. \tag{B.3}$$

The procedure for determining stability and transient solutions is no more difficult than that described in the main text and so only the results will be given. These results have been derived for the $p/\omega = \frac{1}{2}$ region and are as follows (it has been found convenient to abbreviate $n - 4\tan\delta_3 = n_1$):

$$a_1 = \frac{n}{3}[\cos 2\sigma - n_1 \sin 2\sigma] \tag{B.4}$$

$$a_2 = -\frac{n^2}{18} - \frac{n^2}{18}n_1^2 - \lambda_1^2 \tag{B.5}$$

$$\lambda_1 = \frac{n}{3}[\sin 2\sigma + n_1\cos 2\sigma] \tag{B.6}$$

$$\lambda_2 = 0 \tag{B.7}$$

and the equations to be solved by means of (B.4), (B.5), (B.6) and (B.7) are:

$$\lambda = \mu\lambda_1 \tag{B.8}$$

$$\frac{3 - n^2}{4} - \tfrac{2}{9}n^2\mu^2 + n(1 + \mu^2)\tan\delta_3 = \mu a_1 + \mu^2 a_2 \tag{B.9}$$

to the second order in μ.

Ignoring μ^2 and eliminating σ from (B.8) and (B.9) yields the first approximation to μ in the form

$$\mu_1^2 = \frac{9}{16(1 + n^2)}\left[\frac{(3 - nn_1)^2}{n^2} + 4(1 - N)^2\right] \tag{B.10}$$

(*cf.* equation (71)).

The second approximation is then obtained by Newton's formula,

$$\mu^2 = \mu_1\left[1 - \frac{1}{4}\frac{C^2\mu_1^4 + (2BC - D)\mu_1^2 + (B^2 + E)}{C^2\mu_1^4 + \frac{1}{2}(2BC - D)\mu_1^2}\right], \tag{B.11}$$

where

$$B = \frac{3 - nn_1}{4} + E$$

$$C = \tfrac{1}{18}\,[n^2 n_1{}^2 - 3n(n - 6 \tan \delta_3)]$$

$$D = \frac{n^2}{9}(1 + n^2)$$

$$E = \frac{n^2}{4}(1 - N)^2$$

(*cf.* equation (73)). The solution for $\phi_1(\psi)$ is (*cf.* equation (38)),

$$\phi_1(\psi) = \frac{n}{6}\left[\sin\left(\frac{3\psi}{2} - \sigma\right) + n_1 \cos\left(\frac{3\psi}{2} - \sigma\right)\right]. \tag{B.12}$$

The transient motions for $n = 1{\cdot}6$, $\mu = 0{\cdot}3$ and $\delta_3 = -5$ deg are shown on Fig. 4.

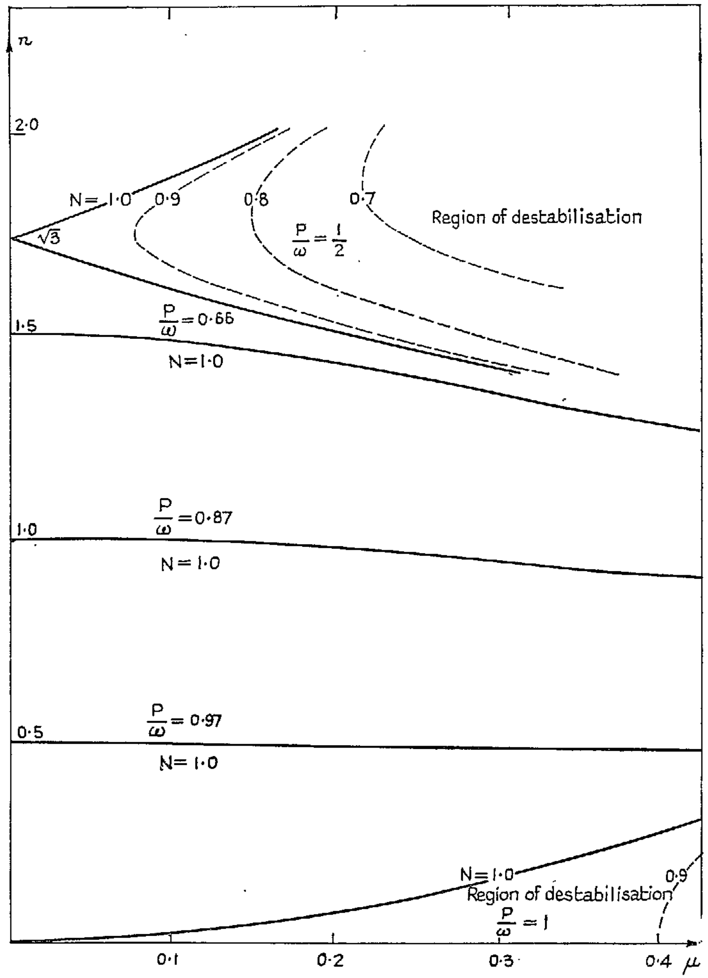


FIG. 1. Regions of destabilisation, $\delta_3 = 0$.

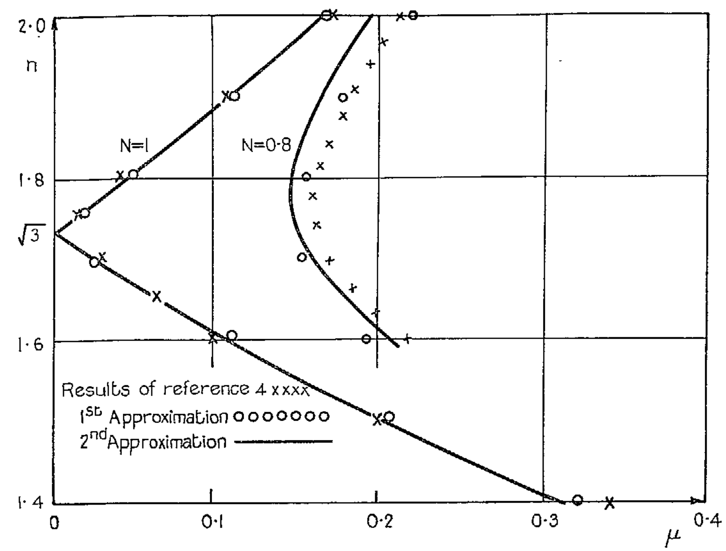


FIG. 2. First and second approximations to stability diagram ($P/\omega = \frac{1}{2}$) compared with results of Ref. 4.

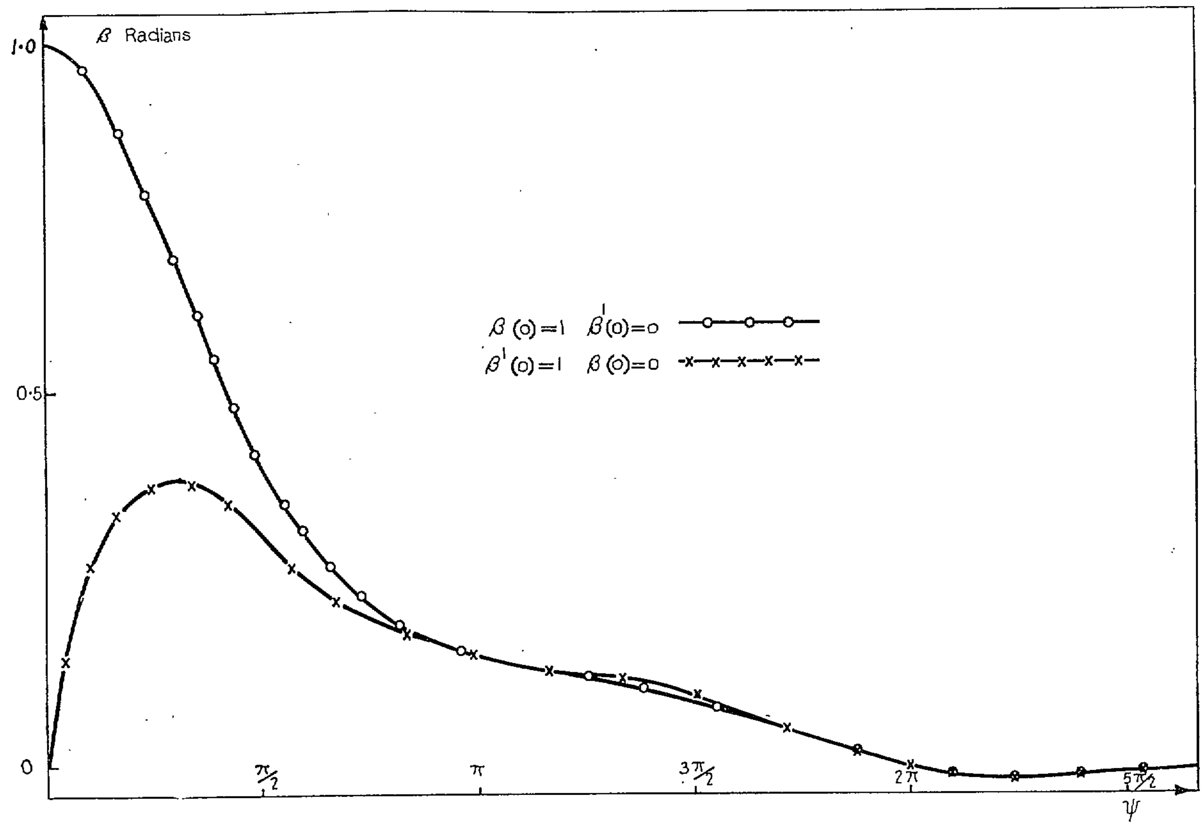


FIG. 3. Flapping transients $\mu = 0 \cdot 3$, $n = 1 \cdot 6$, $\delta_3 = 0$ deg.

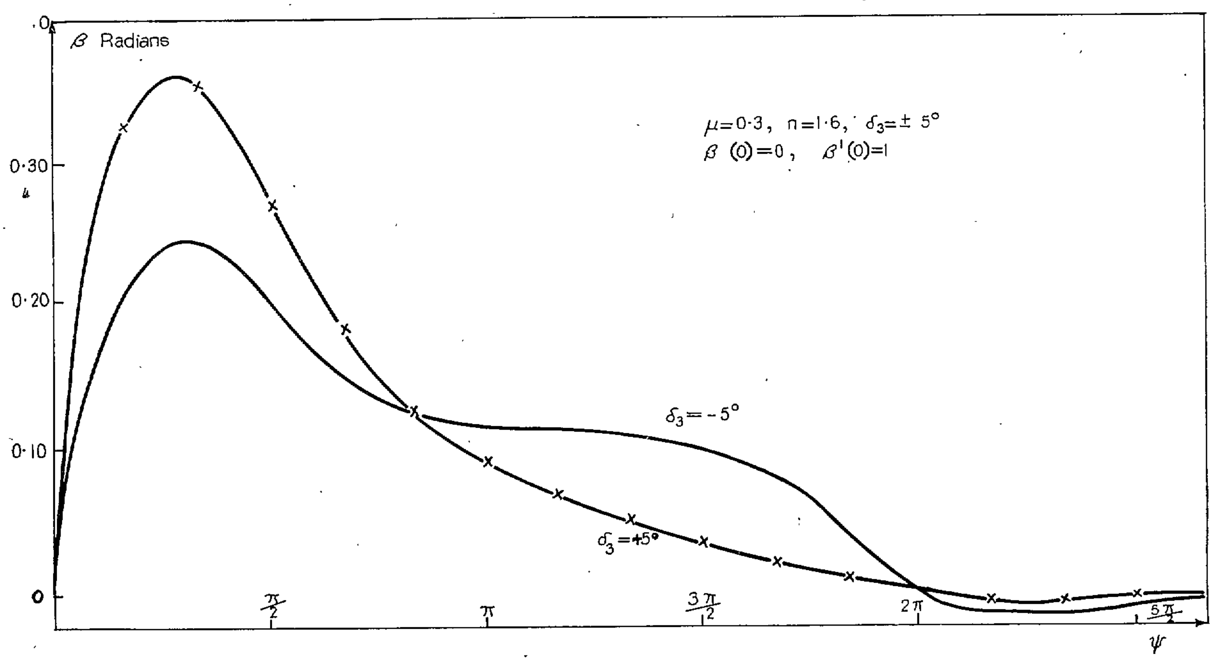
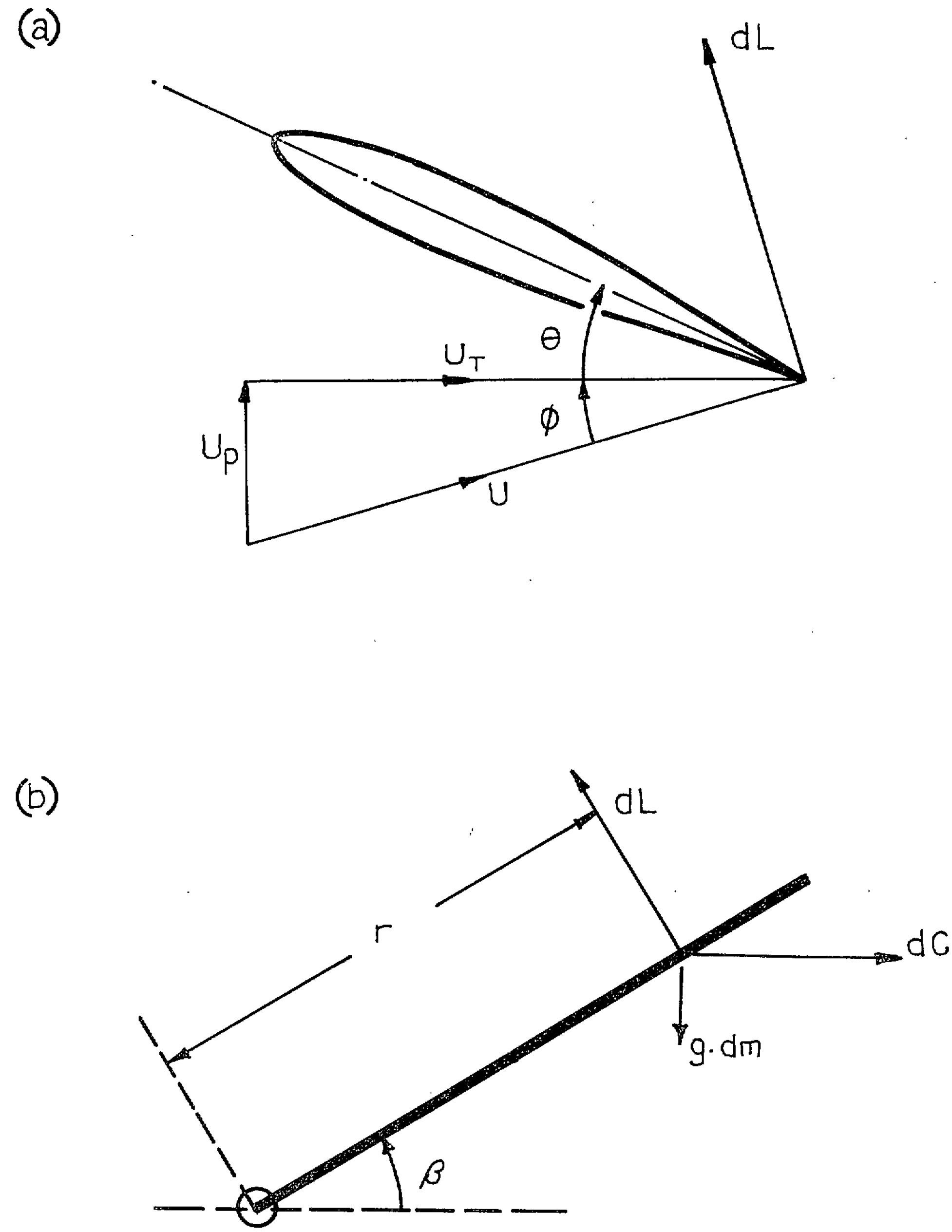


FIG. 4. Flapping transient, $\mu = 0 \cdot 3$, $n = 1 \cdot 6$, $\delta_3 = \pm 5$ deg.



FIGS. 5a and 5b. Velocities and forces at a blade element.

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