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Entrainment in the Turbulent Boundary Layer

By M. R. HEAD, D.S.O., D.F.C., M.A., Ph.D.

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Entrainment in the Turbulent Boundary Layer

By M. R. HEAD, D.S.O., D.F.C., M.A., Ph.D.*

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Summary. By considering the entrainment properties of the turbulent boundary layer, a new and simple form of auxiliary equation has been derived for the calculation of the form parameter H. Comparison with experimental results shows very fair agreement and it should be possible to apply the method in circumstances where normal forms of the auxiliary equation would fail. Further experiment or analysis of existing experimental data is required to establish the accuracy of the assumptions on which the method is based.

1. *Introduction.* In fluid mechanics many situations occur where a turbulent region of flow is bounded by a flow which is non-turbulent and substantially irrotational. In the interaction between the two flow regimes the turbulence spreads with time into the neighbouring fluid, and due to turbulent mixing the region into which it spreads partakes of the general motion of the turbulent flow. This process is generally referred to as entrainment (of the non-turbulent by the turbulent flow) and is, perhaps, most obvious in the case of the turbulent jet, where quite substantial flows may be generated towards the jet due to its entrainment properties.

In the present paper consideration is given to entrainment as a controlling factor in the development of the turbulent boundary layer. The growth of the layer is obviously dependent upon its entraining fluid from the irrotational flow outside it, and if the laws governing the entrainment process can be adequately formulated in terms of the parameters normally used in turbulent boundary-layer calculations then a new equation is made available for predicting turbulent boundary-layer development. Here it is tentatively suggested that the entrainment process should be substantially independent of Reynolds number, and that the quantity entrained per unit area of surface will depend only upon the boundary-layer thickness, the velocity outside the boundary layer and the distribution of velocity in the outer part of the layer. A new and simple form of the auxiliary equation is derived on the basis of these assumptions and is used in conjunction with rather scanty empirical data to calculate the development of the form parameter H for a variety of experimental cases. The agreement with experiment is, in general, very satisfactory. In one or two cases, however, systematic discrepancies do occur, and although these may be accounted for by inadequacies of the experimental data, it is also possible that they may reflect some shortcomings in the basic assumptions. Further experiment is required either to establish the validity of the present assumptions or to provide accurate experimental data on which more sophisticated assumptions may be based. Whether or not the present method will need substantial modifications in the light of subsequent experiment, it is hoped that sufficient has been done to establish at least the potential usefulness of the entrainment concept in turbulent boundary-layer calculations.

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2. *Derivation of Auxiliary Equation.* 2.1. *Dimensional Analysis.* Consider unit width of a two-dimensional turbulent boundary layer. Then, if the quantity flow in the boundary layer per unit time is Q, and the distance along the surface x, dQ/dx represents the amount of fluid entrained by the boundary layer per unit length (assuming no inflow or outflow through the surface). Now, it may be expected, to a first approximation, that the entrainment parameter dQ/dx will be fixed at any position if we specify the shape of the mean velocity profile, the local velocity outside the boundary layer, and some measure of the boundary-layer thickness. The neglect of viscosity in the present connexion is justified by reference to the analogous case of turbulent jets and wakes, where the rate of spread and hence the entrainment is evidently substantially independent of Reynolds number. Thus we may write

$$\frac{dQ}{dx} = f\,(\text{profile shape, external velocity and some measure of boundary-layer thickness}).$$

Now

$$Q = \int_0^\delta u\,dy = \int_0^\delta U\,dy - \int_0^\delta U\left(1 - \frac{u}{U}\right) dy = U(\delta - \delta^*)\,,$$

so that

$$\frac{dQ}{dx} = \frac{d}{dx}[U(\delta - \delta^*)]\,.$$

It will be found convenient to choose the quantity $\delta - \delta^*$ as the measure of boundary-layer thickness, and to take the ratio $(\delta - \delta^*)/\theta$, which we shall call $H_{\delta-\delta^*}$, as the parameter specifying the profile shape. Thus we may write

$$\frac{d}{dx}[U(\delta - \delta^*)] = f(H_{\delta-\delta^*},\ U,\ \delta - \delta^*)\,,$$

or, in non-dimensional terms,

$$\frac{1}{U}\frac{d}{dx}[U(\delta - \delta^*)] = F(H_{\delta-\delta^*})\,. \tag{1}$$

This equation forms the basis of the present method of calculation.

It will be noted that

$$\frac{1}{U}\frac{d}{dx}[U(\delta - \delta^*)] = \frac{d(\delta - \delta^*)}{dx} + \frac{\delta - \delta^*}{U}\frac{dU}{dx},$$

so that (1) may be written

$$\frac{d(\delta - \delta^*)}{dx} = F(H_{\delta-\delta^*}) - \frac{\delta - \delta^*}{U}\frac{dU}{dx}\,. \tag{1a}$$

This alternative form of equation (1) may sometimes be more convenient in use, particularly where the velocity derivative is required in any case for the calculation of momentum thickness.

2.2. *Further Considerations.* If we assume the existence of a one-parameter family of turbulent boundary-layer velocity profiles, as has been done implicitly in the previous Section, we may express $H_{\delta-\delta^*}$ as a function of the conventional form parameter H. Let us say

$$H_{\delta-\delta^*} = G(H)\,. \tag{2}$$

If functions F and G are known, equation (1) may be used in the same way as the more conventional forms of the auxiliary equation; that is, either to calculate the development of H for a given distribution of momentum thickness and external velocity, or in conjunction with the momentum equation to calculate simultaneously the development of H and θ.

2.3. *Determination of Functions F and G.* For this purpose the experimental data of Newman[2] and of Schubauer and Klebanoff[3] have been used†. In each case values of δ were obtained from tables of the measured profiles, δ being arbitrarily defined as the value of y for which $u/U = 0 \cdot 995$. From the values of δ and the corresponding values of H, θ, U and x, the quantities $\frac{1}{U}\frac{d}{dx}[U(\delta - \delta^*)]$ and $H_{\delta-\delta^*}$ were obtained and are shown plotted in Figs. 1 and 2. If the assumptions made in the previous Sections had been correct, and if both the analysis and the experimental data had been entirely free from error then, of course, the points obtained from the two sets of results should have coincided with common curves defining the two functions. In fact, however, as will be seen from the Figures there is considerable scatter of the points, and in Fig. 1 there is a fairly marked and consistent discrepancy between the two sets of results which makes the drawing of a hypothetical common curve, representing the function $F(H_{\delta-\delta^*})$, a somewhat arbitrary procedure. However, such a curve has been drawn, its justification being found *a posteriori*, in the accuracy with which it has enabled the form-parameter development to be predicted in the cases considered below. The curve relating $H_{\delta-\delta^*}$ to the normal form parameter H is rather more accurately defined, although here also there is some discrepancy between the two sets of results, and the values of H given by Schubauer and Klebanoff for the region where the pressure gradient was favourable appear somewhat high.

The curves shown in Figs. 1 and 2 were used for the calculations described in the following Sections.

3. *Calculations and Comparison with Experiment.* Considering the various cases for which the development of the turbulent boundary layer has been measured it would have been possible, as indicated in Section 2.2, to calculate simultaneously both the θ and H development from initial experimental values, and to compare these with the measured values. For the present purpose this seemed a somewhat unjustified procedure, and all the calculations of H described below, with one exception, were based on the measured growth of momentum thickness. This of course considerably reduced the labour of computation.

3.1. *Calculation Procedure.* In each case initial values of H, θ and U were used to obtain starting values of $H_{\delta-\delta^*}$, $\delta - \delta^*$ and $U(\delta - \delta^*)$. From the value of $H_{\delta-\delta^*}$ the corresponding value of $\frac{d}{dx}[U(\delta - \delta^*)]$ was obtained by multiplying the appropriate value of F by the local velocity U. The increment in $U(\delta - \delta^*)$ over the first step was obtained by extrapolating $\frac{d}{dx}[U(\delta - \delta^*)]$ to the middle of the interval and using the finite difference relation $\Delta[U(\delta - \delta^*)] = \Delta x \times (UF)_{\text{extrap}}$, where $(UF)_{\text{extrap}}$ was taken to represent the mean value of $\frac{d}{dx}[U(\delta - \delta^*)]$ over the interval Δx. At the end of the step the known values of θ and U and the increment in $U(\delta - \delta^*)$ enabled the new values of $\delta - \delta^*$ and $H_{\delta-\delta^*}$ to be obtained, so that the procedure could be repeated for a further step. Where it was necessary to calculate the simultaneous development of H and θ, the distribution of external velocity was first differentiated graphically for use in the momentum equation, and it was then more convenient in the calculation of H to use the alternative form of equation (1) given at the end of Section 2.1.

† The author is indebted to Dr. Newman for access to his original measurements.

3.2. *Modification to Equation* (1) *for Suction or Blowing.* There is no reason to suppose that conditions at the inner boundary will have any appreciable effect on the entrainment process, except in so far as they may modify the profile shape, the influence of which is, we assume, already taken into account in the present method. There is some justification, therefore, for applying the curves given in Figs. 1 and 2 to cases where suction or blowing is applied through the surface; it will, however, be necessary to modify equation (1) by the inclusion of a term representing the continuous addition or removal of fluid through the surface.
Thus

$$\frac{1}{U}\frac{d}{dx}[U(\delta - \delta^*)] = F(H_{\delta-\delta^*}) \pm \frac{v_0}{U}, \tag{3}$$

where v_0 is the velocity at the surface in the direction of y, so that the positive and negative signs are appropriate to blowing and suction respectively.

3.3. *Cases for which Calculations have been Performed.* The development of the form parameter H has been calculated by the present method for the following cases:

(1) The flat plate without suction or blowing

(2) The NACA 65(216)–222 aerofoil tested by Doenhoff and Tetervin[4] for the following conditions:

(*a*) Incidence 10·1 deg, $R = 2 \cdot 64 \times 10^6$,
(*b*) Incidence 8·1 deg, $R = 2 \cdot 67 \times 10^6$,
(*c*) Incidence 8·1 deg, $R = 0 \cdot 92 \times 10^6$.

(3) The boundary layer measured by Schubauer and Klebanoff[3]

(4) The boundary layer measured by Newman[2]

(5) The flat plate with uniform blowing (Mickley and Davis[5])

(6) The flat plate with uniform suction following a solid entry (Dutton[6]).

The results are shown compared with experiment in Figs. 3 to 8 and 11, and are discussed below. It will be recognised that cases (3) and (4) were those used to obtain the functions $F(H_{\delta-\delta^*})$ and $G(H)$, so that it might be thought that best agreement would be obtained with these cases. In fact this was not found to be so, since apparently each represented a fairly extreme condition.

3.4. *Discussion of the Individual Cases.* 3.4.1. *The flat plate without suction or blowing.* The calculation was based on values of θ given by the Schoenherr line as presented in Ref. 7, the starting value of H being taken as $1 \cdot 42$ at $R_x = 10^6$. In Fig. 3 the results obtained are compared with the experimental values given by Smith and Walker[7] and the curve representing the variation of H for the ideal turbulent boundary layer, due to Coles[8] and given by Spence in Ref. 1. It is surprising that the somewhat arbitrarily chosen curves representing the functions $F(H_{\delta-\delta^*})$ and $G(H)$ should give such close agreement with what may be accepted as standard values. It would of course have been possible, in any case, to adjust either of the curves to obtain satisfactory agreement but in fact no such adjustment was necessary.

3.4.2. *The N.A.C.A. aerofoil tested by Doenhoff and Tetervin*[4]. The starting values of H and the distributions of U and θ were obtained from Ref. 4, and the comparison of the calculated variation of H with observed values is shown in Figs. 4 and 5. No comment appears to be necessary beyond the statement that the results of the calculations appear to be very satisfactory in all three cases.

3.4.3. *The boundary layer measured by Schubauer and Klebanoff*[3]. Two sets of calculations were performed for this case, the first starting at 1 ft from the leading edge, and the second at 18 ft, *i.e.*, at the beginning of the adverse pressure gradient. For the first set of calculations the starting value of H was taken as $1 \cdot 44$ and for the second $1 \cdot 37$. In both cases the measured values of θ were used in the calculation of H. Fig. 6 shows the results compared with the measurements of Schubauer and Klebanoff. It will be seen that the agreement is quite fair, with the values of H calculated from near the leading edge lying consistently below the measured values by a small amount, and those calculated from 18 ft agreeing closely right up to separation. A trial was made with a different starting value of $H(1 \cdot 37)$ in the calculation proceeding from 1 ft. Here it was found that beyond the 3 ft station the results were unaffected by the change. Evidently the choice of initial values is very much less important when the pressure gradient is favourable than when it is unfavourable.

3.4.4. *The boundary layer measured by Newman*[2]. Three sets of calculations were performed here, the first based on measured values of θ, the second on values of θ calculated by the use of the momentum equation without the additional terms proposed by Newman, and the third on values of θ calculated with approximate account taken of these terms where they were known to be significant. In using the momentum equation, values of the skin-friction coefficient were obtained from the Ludwieg-Tillmann relation[13], and values of H by the simultaneous application of the present method of calculation. The results of the three sets of calculations are shown in Figs. 7 and 8. It will be seen that calculated values of H based on the measured values of θ are in rather poor agreement with experiment but that where calculated values of θ have been used the agreement is very satisfactory. It will be noted that the inclusion, near the trailing edge, of the turbulence and static-pressure terms proposed by Newman has an appreciable effect on the values of calculated momentum thickness and hence on the calculated values of H. From the results shown in the Figures it is difficult to draw any very positive conclusion, except that the use of measured values of θ in the calculation of H is certainly no guarantee that better agreement will be obtained with measured H values. As in the other cases considered, the possibility of small departures from two-dimensional flow being responsible for the observed discrepancies cannot be ruled out, but it would be rather remarkable if such secondary flow were responsible for the apparently increased rate of growth of momentum thickness while leaving the H development unchanged.

3.4.5. *The flat plate with uniform suction following a solid entry.* In experiments performed by Dutton[6] in zero pressure gradient a particular rate of suction was found to produce a turbulent layer which remained constant in thickness and in profile shape over the major part of the test surface. The boundary layer, which was fully turbulent at the beginning of suction, rapidly settled down to the velocity profile shown in Fig. 9, the momentum thickness being close to the initial value.

For this calculation, the curve of $F(H_{\delta-\delta^*})$ was considerably extrapolated, as shown in Fig. 10, on the assumption of an exponential approach to zero as in the case of the turbulent wake far downstream. The initial value of H was taken as $1 \cdot 48$, appropriate to the rather low Reynolds number, and it was assumed that the momentum thickness remained constant at the initial value of $0 \cdot 020$ in. As the calculation proceeded, using equation (3) and the measured value of v_0/U $(= -0 \cdot 0044)$, $H_{\delta-\delta^*}$ steadily increased, and it was evident that, at a sufficient distance downstream, the value of approximately 20, representing the asymptotic condition where $d(\delta - \delta^*)/dx = 0$, would be closely approached. At 20 in. from the beginning of suction the value of $H_{\delta-\delta^*}$ had increased from 6·25 to approximately 12. This represented a rather slower approach to asymptotic conditions than was observed in the experiments, where asymptotic conditions were apparently achieved at 24 in. from the start of suction. The measurements were, however, not particularly precise, and in the calculation no account was taken of the initial reduction of θ following the commencement of suction.

It is very satisfactory that the value of $H_{\delta-\delta^*}$ for the asymptotic suction profile shown in Fig. 9 should agree with the value of $H_{\delta-\delta^*}$ for which the rate of entrainment was equal to the rate of withdrawal through the surface, (*i.e.*, for which $F(H_{\delta-\delta^*}) = -v_0/U$), but in view of the manner in which the curve of Fig. 10 was obtained the very close agreement must be regarded as largely fortuitous.

3.4.6. *Flat plate with uniform blowing.* In Ref. 5 Mickley and Davis present the results of a large number of experiments with substantially uniform blowing applied through a porous surface. Calculations were performed for the highest ratio of blowing to stream velocity (0·010) and the results obtained are shown compared with experiment in Fig. 11. Faired experimental values of θ were used in the calculations and the initial value of H was chosen as 1·7, this being roughly the mean experimental value over the surface. The values of H tabulated by Mickley and Davis do not agree in all cases with the values obtained from the corresponding tabulated values of δ^* and θ, and where the values differ both are shown on the Figures. The experimental values of $H_{\delta-\delta^*}$ shown were obtained from the tabulated values of δ, δ^* and θ given in the report. It will be seen that there is considerable scatter of the experimental points, particularly in the plot of H and it is found that corresponding experimental values of H and $H_{\delta-\delta^*}$ do not line up at all closely with the curve of Fig. 2. Moreover, the values of H given for the zero blowing case all lie in the region of 1·33, very much lower than the accepted values for the flat plate at the low Reynolds numbers of the tests. For these reasons the accuracy of the experimental data must be considered limited and the measure of agreement between the calculated and experimental results shown in the Figures is probably as good as could be expected.

4. *Comparison with Existing Forms of the Auxiliary Equation.* It is interesting to note that the present entrainment equation

$$\frac{1}{U}\frac{d}{dx}[U(\delta - \delta^*)] = F(H_{\delta-\delta^*}) \tag{4}$$

can, with some manipulation, be put in a form essentially similar to that of several existing empirical auxiliary equations†. The procedure may be outlined as follows: To simplify the writing $H_{\delta-\delta^*}$ is replaced by H_1.

Now, since

$$\theta\frac{dH}{dx} = \theta\frac{dH}{dH_1}\frac{dH_1}{dx} = \frac{dH}{dH_1}\left[\theta\frac{d}{dx}\left(\frac{\delta - \delta^*}{\theta}\right)\right]$$

we obtain, using equation (4),

$$\theta\frac{dH}{dx} = \frac{dH}{dH_1}\left[F - H_1\frac{\theta}{U}\frac{dU}{dx} - H_1\frac{d\theta}{dx}\right].$$

† It will be noted that there is no practical advantage in doing this; the method becomes more difficult to use and its physical basis is obscured. However, the analysis serves to show that the existing equations are not incompatible with the basic equation given here.

Substitution for $d\theta/dx$ from the momentum equation then gives, with some rearrangement,

$$\theta \frac{dH}{dx} = - \frac{dH}{dH_1} H_1 (H + 1) \left[- \frac{\theta}{U} \frac{dU}{dx} + \frac{1}{H + 1} \frac{\tau_0}{\rho U^2} - \frac{F}{H_1(H + 1)} \right] . \tag{5}$$

In Ref. 1 Spence points out that his own auxiliary equation and those of Garner[9], Zaat[10], Maskell[11] and Schuh[12] may be put in the form

$$\Theta \frac{dH}{dx} = \Phi(H)\Gamma - \Psi(H) ,$$

where

$$\Theta = \theta \left(\frac{U\theta}{\nu}\right)^n , \tag{6}$$

$c_f = 0{\cdot}246\, R_\theta^{-0{\cdot}268}\, 10^{-0{\cdot}678H}$

$$\Gamma = - \frac{\Theta}{U} \frac{dU}{dx} .$$

If now, in equation (5), we substitute for $\tau_0/\rho U^2$ in terms of H and c_f by the use of the Ludwieg-Tillmann relation[13] we obtain

$$\Theta \frac{dH}{dx} = - \frac{dH}{dH_1} H_1 (H + 1)\Gamma - \left[- \frac{dH}{dH_1} H_1 (H + 1) \right] \times$$
$$\times \left[\frac{F}{H_1(H + 1)} \left(\frac{U\theta}{\nu}\right)^{0{\cdot}268} - \frac{0{\cdot}123 \times 10^{-0{\cdot}678H}}{H + 1} \right] ,$$

where

$$\Theta = \theta \left(\frac{U\theta}{\nu}\right)^{0{\cdot}268} .$$

Thus, functions Φ and Ψ are given in the present method by

$$\Phi(H) = - \frac{dH}{dH_1} H_1 (H + 1) \tag{7}$$

and

$$\Psi(H, R_\theta) = \Phi(H) \left[\frac{F}{H_1(H + 1)} \left(\frac{U\theta}{\nu}\right)^{0{\cdot}268} - \frac{0{\cdot}123 \times 10^{-0{\cdot}678H}}{H + 1} \right] . \tag{8}$$

It will be seen that there is now some dependence of Ψ on R_θ and that $\Phi(H)$ is a simple geometrical property of the family of turbulent boundary-layer velocity profiles, in fact, of the curve relating H_1 to H (Fig. 2). The function $\Phi(H)$ of equation (7) can be compared directly with the corresponding functions given by Schuh, Maskell, Spence and Garner as presented in Ref. 1. In Fig. 12 these functions are compared graphically. It will be seen that they differ very considerably among themselves at high values of H, but that the curve representing equation (7) above lies well within the spread of the other curves and agrees most closely with the functions proposed by Maskell. Direct comparison of equation (8) with the functions $\Psi(H)$ given by other authors is scarcely possible because of the dependence of the present expression upon R_θ and because different values have in some cases been chosen for the exponent n in the definition of Θ. However, in Table 1 a comparison is given for two values of R_θ with the functions of Maskell and Schuh, who both use values of $n = 0{\cdot}268$ as in the present analysis. Comparison with the functions of other authors on a similar basis could of course be made by using for each case the appropriate skin-friction law instead of that of Ludwieg and Tillmann.

The comparisons given here indicate that the present auxiliary equation should give results which are at least as good as the generally accepted empirical equations, since the function $\Phi(H)$ indicating the dependence on pressure gradient agrees closely with the corresponding functions given by Spence and Maskell and considerable support for the function $\Psi(H)$ is provided by the close agreement with flat-plate results indicated in Section 3.4.1 and Fig. 3.

5. *General Discussion.* The present analysis was suggested by the fact that in a wake the velocity defect at the centre exerts a controlling effect on the rate of entrainment, the greater this defect the greater then being the entrainment (in appropriate non-dimensional terms). It can be simply shown that, sufficiently far downstream, the entrainment will in fact be proportional to this defect, *i.e.*,

$$\frac{d(\delta - \delta^*)}{dx} \propto \frac{u_1}{U_0}$$

where u_1 is the velocity defect. This leads to the suggestion made here that the distribution of mean velocity in the boundary layer, in particular the velocity defect in its outer part (measured roughly by form parameters such as H or $H_{\delta-\delta^*}$), should control, or at least stand in close quantitative relationship with, the entrainment process. In view of the way in which this tentative conclusion was reached it might perhaps appear more logical to base the analysis on the wake function proposed by Coles, who does in fact briefly introduce the topic of entrainment in a discussion of the equilibrium turbulent boundary layer[14]. This point may be worth further investigation in the present connection, along with other proposed velocity defect laws and the equilibrium layers measured by Clauser[15].

6. *Advantages of the Present Method and Possible Applications.* In view of the somewhat arbitrary nature of the assumptions made in the present analysis, and the reservations with which certain of the comparisons with experiment have been treated, it may be worth drawing attention at this stage to certain important advantages which the method possesses. First, of course, in its present form it is very simple and easy to use. Second, it can be applied with reasonable confidence to cases involving suction or blowing through the surface, provided this is more or less continuously porous, and possibly to other cases where the normal auxiliary equations would fail. Third, the method is capable of further development as more becomes known of the particular physical process involved. Compared, for example, with the measurement of such quantities as the shear-stress distribution in the layer, the measurement of entrainment quantity in various experimental situations should present a relatively simple problem, particularly if, as seems likely, an absence of sensitivity to Reynolds number makes it possible to carry out the experiments at a conveniently small aerodynamic scale.

The most promising immediate application of the present method is to cases involving suction or blowing through the surface where, in conjunction with skin-friction laws proposed by Black and Sarnecki[16], it will provide a method of calculation where none at present exists. An extension of the method which may also be of interest is to turbulent boundary layers in supersonic flow, where it may be expected that conditions in the outer part of the layer will not be too much influenced by high Mach number, the major changes in the physical properties of the fluid being confined to a region close to the surface. Consideration may also be given to the possibility of applying entrainment considerations to the laminar boundary layer.

7. *Conclusions.* The method developed in this paper represents a first attempt at utilising the principle of entrainment in the calculation of turbulent boundary layers. In spite of the relatively crude assumptions on which the method is based, the results obtained in the limited number of cases treated show very fair agreement with experiment, probably at least as good as would be obtained by the use of the more conventional forms of the auxiliary equation. The method is simple to apply and has potentially a very wide field of application. However, further experimental evidence is required either to confirm the basic assumptions of the present analysis or to suggest less simple alternatives which may be more in keeping with the evidently complex physical situation.

8. *Acknowledgements.* The author wishes to acknowledge helpful discussions with Dr. B. G. Newman in the course of this work.

NOTATION

ρ		Fluid density
ν		Kinematic viscosity
x		Distance along the surface
y		Distance normal to the surface
U_0		Free-stream velocity
U		Local velocity outside the boundary layer
u		Velocity in the boundary layer in the direction of x
v_0		Normal velocity at the surface in the direction of y
δ		Boundary-layer thickness
δ^*	$=$	$\int_0^\delta \left(1 - \frac{u}{U}\right) dy$ (Displacement thickness)
θ	$=$	$\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$ (Momentum thickness)
c		Length of chord
Q	$=$	$\int_0^\delta u\, dy$ (Quantity flow in the boundary layer per unit time per unit span)
H	$=$	$\frac{\delta^*}{\theta}$ (Conventional form parameter)
$H_{\delta-\delta^*}$	$=$	$\frac{\delta - \delta^*}{\theta}$ (Form parameter used in present analysis)
H_1	$\equiv$	$H_{\delta-\delta^*}$
R_x	$=$	$\frac{U_0 x}{\nu}$ (Reynolds number based on x)
R	$=$	$\frac{U_0 c}{\nu}$ (Reynolds number based on c)
R_θ	$=$	$\frac{U\theta}{\nu}$ (Reynolds number based on θ)
τ_0		Shearing stress at the surface
c_f	$=$	$\frac{\tau_0}{\frac{1}{2}\rho U^2}$ (Skin friction coefficient)
n		Index occurring in skin-friction law $\frac{\tau_0}{\rho U^2} = \frac{\alpha}{R_\theta^{\ n}}$ where α is in general some function of H
Θ	$=$	$\theta\left(\frac{U\theta}{\nu}\right)^n$
Γ	$=$	$-\frac{\Theta}{U}\frac{dU}{dx}$
Φ and Ψ		functions of H used by Spence[1] and discussed in Section 4.

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TABLE 1

Comparison of Functions Ψ

H	Maskell		Schuh	Present method	
	Γ small	Γ large		$R_\theta = 2000$	$R_\theta = 20{,}000$
1·4	0·0005	0·03	0·001	−0·0004	0·0063
1·8	0·0176	0·09	0·0039	+0·0495	0·1010
2·2	0·0392	0·15	0·016	0·2823	0·5370
2·6	0·0609	0·21	0·066	+1·042	1·950

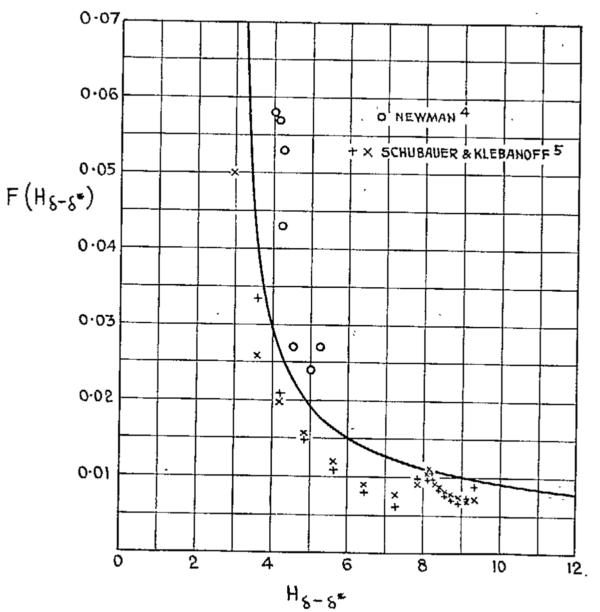

FIG. 1. $\frac{1}{U}\frac{d}{dx}[U(\delta - \delta^*)]$ as a function of $H_{\delta-\delta^*}$.

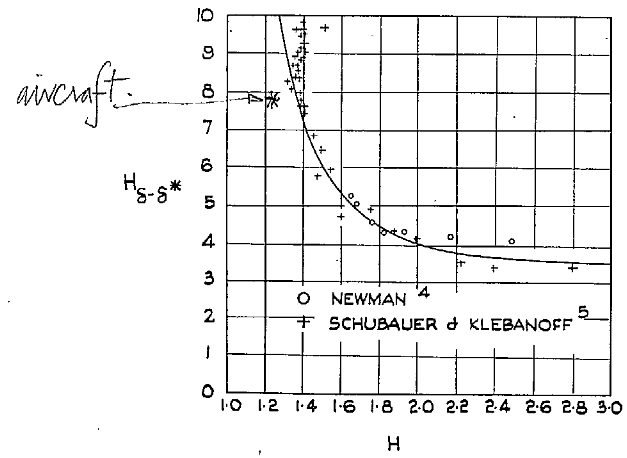

FIG. 2. Variation of $H_{\delta-\delta^*}$ with conventional form parameter H.

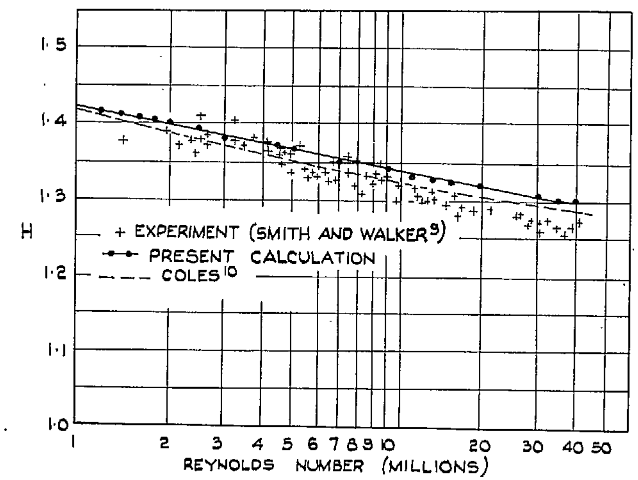

FIG. 3. Flat-plate results compared with experiment.

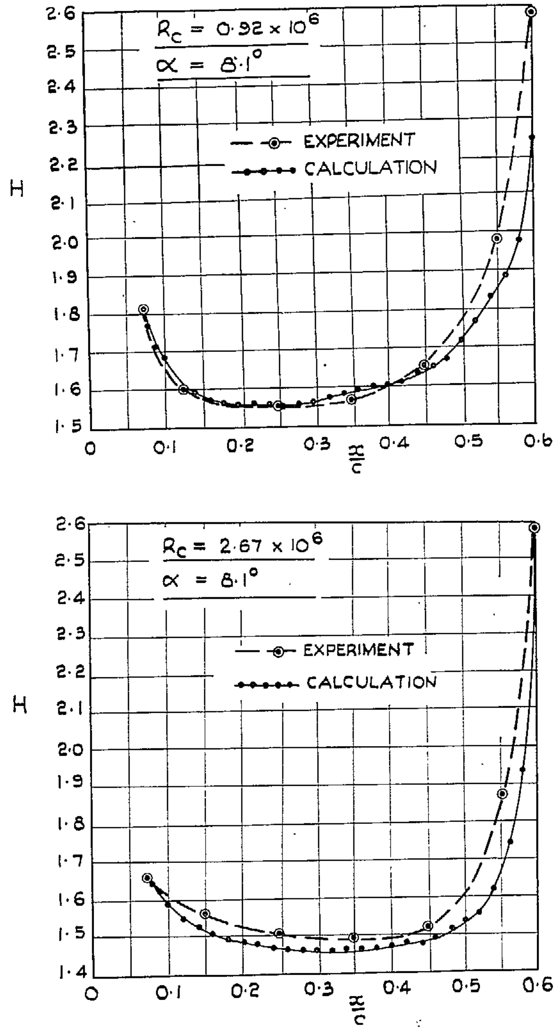

FIG. 4. Comparison with measurements of von Doenhoff and Tetervin[6] for aerofoil section NACA 65(216)–222.

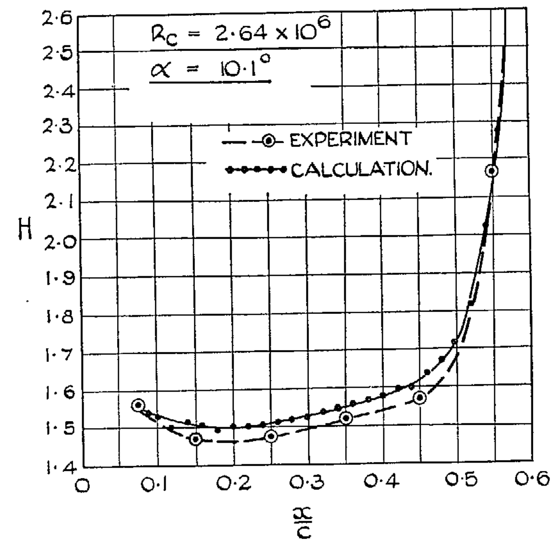

FIG. 5. Comparison with measurements of von Doenhoff and Tetervin[6] for aerofoil section NACA 65(216)–222.

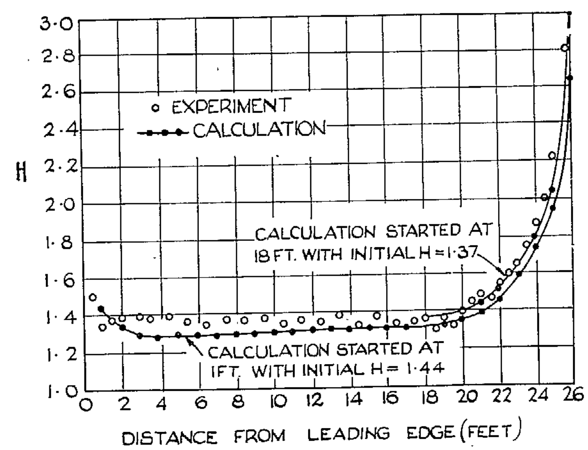

FIG. 6. Comparison with measurements of Schubauer and Klebanoff[5].

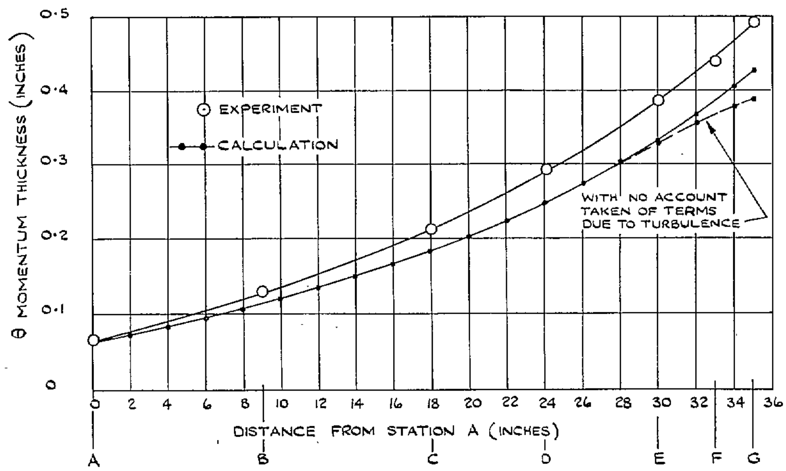

FIG. 7. Comparison between measured and calculated development of θ.

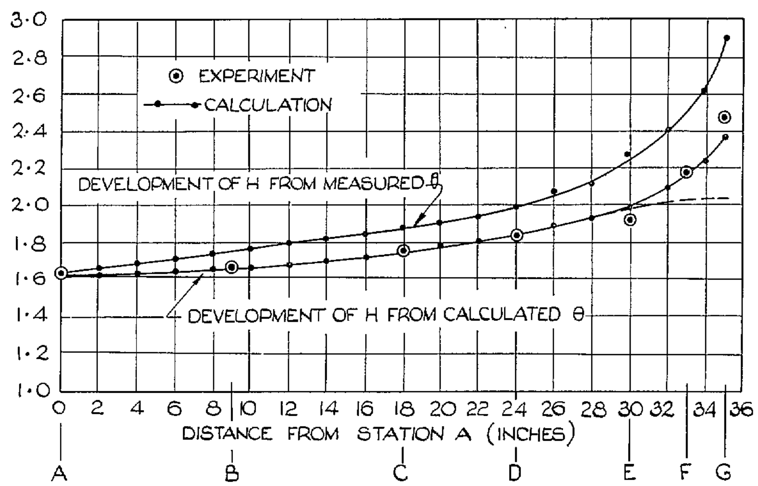

FIG. 8. Comparison with measurements of Newman.

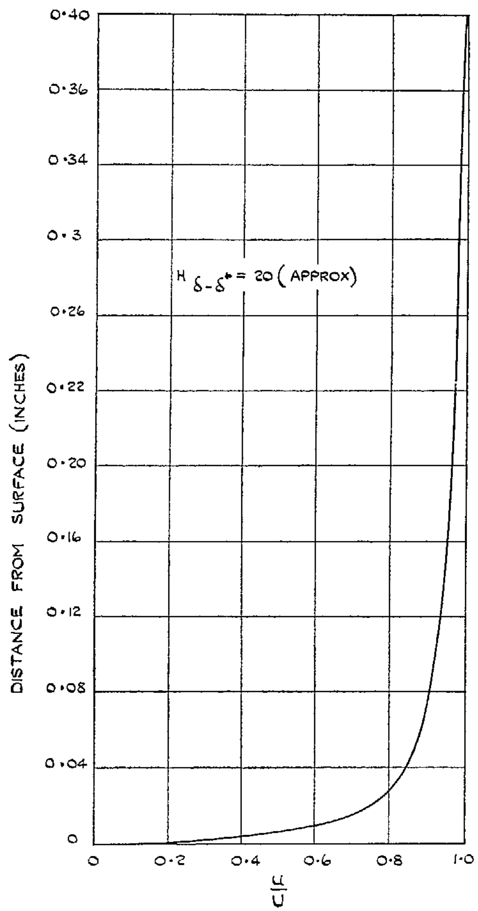

FIG. 9. Asymptotic suction profile measured by Dutton[8].

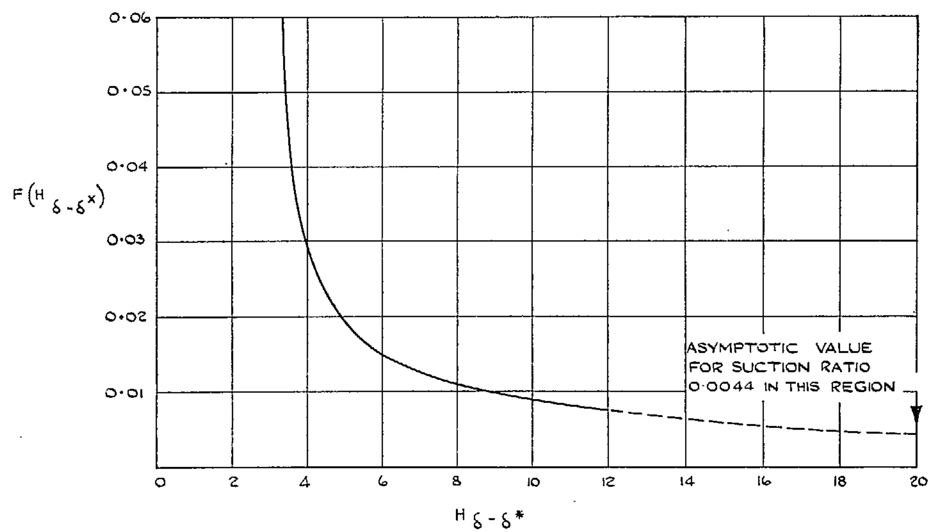

FIG. 10. Extrapolation of $F(H_{\delta-\delta^*})$ curve for calculation of form-parameter development for flat plate with uniform suction.

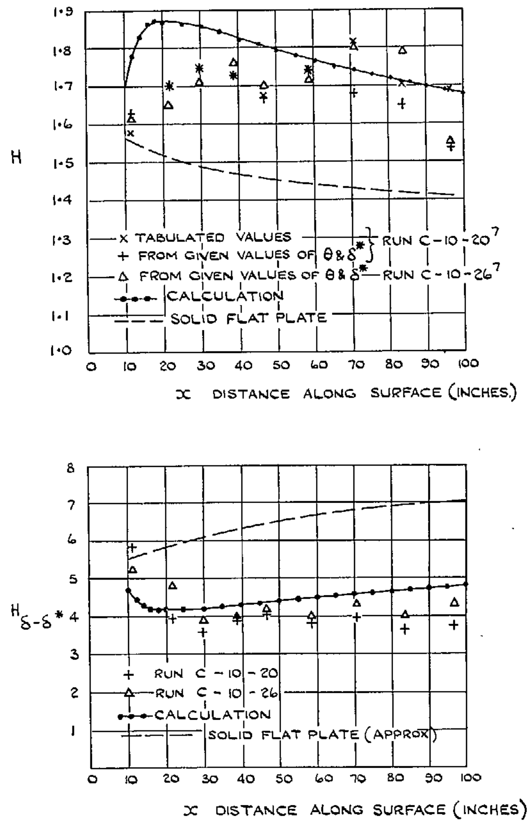

FIG. 11. Variation of H and $H_{\delta-\delta^*}$ with x for flat plate with uniform blowing (Mickley and Davis[7]).

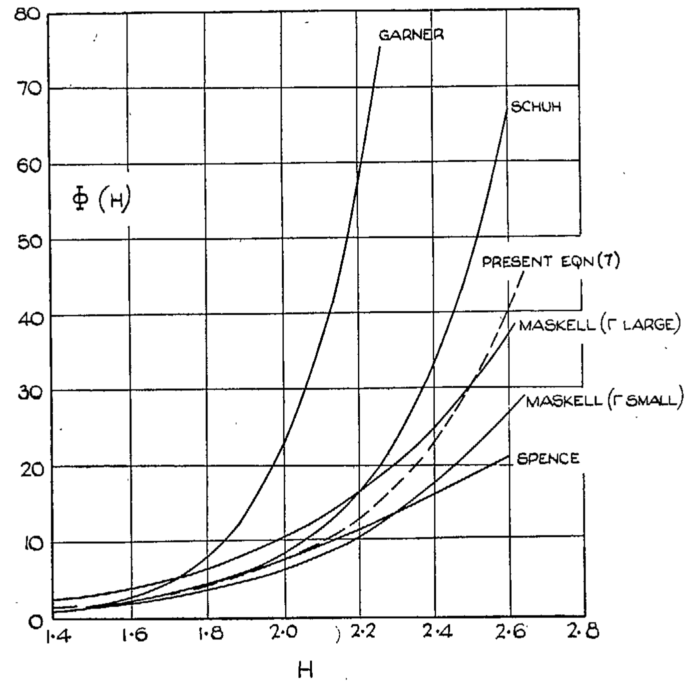

FIG. 12. Comparison of functions $\Phi(H)$ as given by different authors.

(78640) Wt. 54/8210 K.5 9/60 Hw.

$$\frac{1}{U}\frac{dU}{dx} = -\frac{1}{2}\,\frac{1}{1-C_p}\,\frac{dC_p}{dx}.$$

$$\frac{d\theta}{dx} = C_f/2 - \frac{\theta}{U}\frac{dU}{dx}(H+2)$$

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