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Methods for Estimating Distributions and Intensities of Sonic Bangs

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Summary.—Methods recently developed for estimating distributions and intensities of sonic bangs are described. They are applied to several interesting flight manoeuvres and the results discussed in detail. The effect on sonic-bang distributions and intensities of refraction (caused by the temperature gradient existing in the actual atmosphere) is also considered.

1. *Introduction.*—This paper has been written to meet a variety of needs. First, it replaces the earlier work of Warren[1], parts of which have now become outdated due to the appearance of Rao's papers[2,3]. The opportunity is taken to correct one or two errors which crept into Ref. 1. Secondly, a detailed discussion of the distributions and intensities of bangs produced during various flight manoeuvres is given. Thirdly, the phenomenon of refraction of sonic bangs, which is caused by the non-homogeneity of the atmosphere, receives closer consideration than in Ref. 1. Finally, it is hoped that the elementary explanation of Section 2 will be of use to those who wish to brush up their sonic bangs.

2. *The Nature of Sonic Bangs.*—The controversy which raged a few years ago over the cause of sonic bangs has now died down and few aerodynamicists would disagree with the explanation of the phenomenon given here. For simplicity, linear theory is used at first, but the introduction of a non-linear theory does not alter the basic ideas.

When a disturbance is produced in a homogeneous atmosphere at a time taken to be zero then, after a time t, the disturbance is confined to a sphere with its centre at the origin of the disturbance and its radius equal to at, a being the speed of sound. It follows that the disturbance is propagated along straight lines which all pass through the origin of the disturbance. These lines are called 'rays'.

If an object (*e.g.*, an aircraft) causing disturbances is moving in a straight line at a constant speed V, where $V > a$, then the fronts of the disturbances at any time have an envelope which is a cone with its axis along the direction of motion and its vertex at the aircraft, the vertex angle being $2 \sin^{-1} (a/V)$. As the aircraft moves along, the cone moves with it and, whatever the length of time that has elapsed since the motion began, the disturbances are all confined within this cone. This statement assumes that the motion started impulsively. When the cone

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passes through any point there is a sudden change in pressure at the point, and it is this which causes a sonic bang. If the aircraft is in straight and level flight the effect on the ground is confined to a hyperbola moving over the ground. The successive positions of the hyperbola form a family of curves and all points lying on one of these curves receive a bang at the same instant of time.

Fig. 2 shows that this cone, the envelope of the disturbances, is propagated along straight lines making an angle $\frac{1}{2}\pi - \sin^{-1}(a/V)$ with the flight direction. Thus, the rays along which the envelope is propagated are a sub-family of the rays along which each separate disturbance is propagated, and the component of the aircraft's velocity along them is sonic. The rays of the envelope which emanate from a particular point on the flight path form a cone, which faces the opposite way to the cone described above and which has a vertex angle of $\pi - 2\sin^{-1}(a/V)$. Each point on the flight path has such a cone associated with it and these cones again intersect the ground in hyperbolae, which form another family of curves on the ground. All points lying on one of these hyperbolae receive bangs which were created at the same time but they receive them, in general, at different times. Thus, there are two families of curves on the ground to be considered in sonic bang problems : the first consists of curves connecting points at which bangs are received at the same time ; the second consists of curves connecting points which receive bangs created at the same time. The first family might seem the more suitable physically but, since it is easier to perform calculations on sonic bangs by starting from the flight path, the second family of curves is obtained in practice. If the first family is required it can be obtained from the second without difficulty.

When the aircraft is not moving in the extremely simple manner of the above example, the resulting patterns of curves become more complex. In accelerated flight some regions on the ground may be covered more than once by the curves connecting points at which bangs are received at the same time, so that in these regions multiple bangs are received. In such a case there are lines on the ground on and near which the pressure jumps causing the bangs are larger than elsewhere ; the line forming the boundary of the area subjected to bangs is always one of these lines although there may be others inside the area. Since the bangs received on and near such lines are more intense than those elsewhere in the locality they are called 'super-bangs'. The second family of curves, each consisting of points receiving bangs created at the same time, are always conics, although, if the aircraft dives during its flight, some of these conics may be ellipses. When the flight path is known, together with the Mach number at each point, this second family of curves is obtained exactly as in the simple special case described above.

In reality, the successive positions of the curves joining points which receive bangs at the same time correspond to shock waves moving over the ground, while the sudden change in pressure associated with the passage of the curves is replaced by the phenomenon of an 'N-wave'. Fig. 1 is a sketch of such a wave. As this wave passes a point there is a sudden increase in pressure

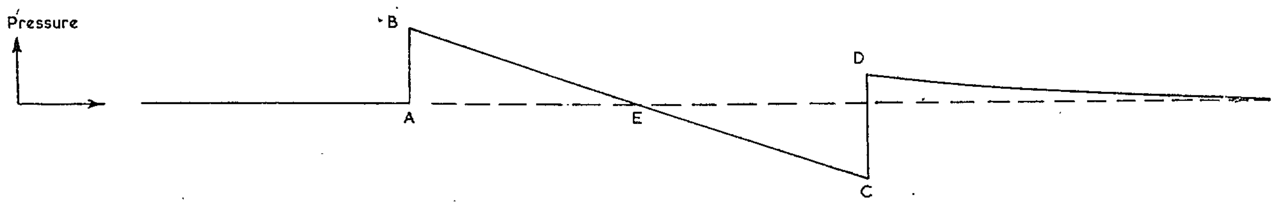

Fig. 1. An N-Wave

(A to B), an approximately linear fall to a pressure lower than the initial value (B to C), a second sudden rise in pressure to a value slightly higher than the initial value (C to D) and then a tail in which the pressure gradually falls to the initial value. The jump AB is called the 'intensity' of the wave and is approximately equal to CD. The 'impulse' of the wave is defined to be the area of ABE. The distance between A and D is of the order of the aircraft's length and, in horizontal flight, the wave moves with the speed of the aircraft, which is at least equal to the speed of sound. The time interval between the arrival of AB and that of CD is, therefore, of the order of the time taken to travel a distance of 50 ft with a speed of 1,000 ft/sec, *i.e.*, a time interval of the order of 0·05 sec. This is sufficiently long for the ear to be able to distinguish the pressure jumps A to B and C to D. Hence, the single bang described above is to be replaced by a double bang, both bangs being roughly of the same intensity. The first pressure jump, A to B, is called the 'bow shock wave' and the second pressure jump, C to D, the 'stern shock wave'.

3. *The Estimation of Sonic Bangs.*—The first estimates of distributions and intensities of sonic bangs were obtained from linear theory[4, 5]. These estimates were then improved by applying the technique of Whitham[6, 7]. Briefly, Whitham assumes that linear theory yields the correct values of required quantities along any particular characteristic but he recognises that the linear-theory characteristics are incorrect and replaces them by characteristics derived from a non-linear theory. Rao[2, 3] applies this technique to the linear theory of sonic bangs as developed by Warren[4]. No further description of Rao's theory is given here and the interested reader is referred to the original papers. Rao's results, which apply to a homogeneous atmosphere only, are quoted and briefly discussed in the next Section.

3.1. *Sonic-Bang Distributions on the Ground.*—Concentrating on a particular point P of the flight path it is assumed that the Mach number, M, and dM/dt are known at P; t is the time, measured from a suitable origin, when the aircraft is at P. The cone with its vertex at P, its axis along the direction of motion at P and its vertex angle equal to $\pi - 2\sin^{-1}(1/M)$ cuts the ground in a conic C and, if Q is any point on C, then the component of the aircraft's velocity along PQ is sonic. Each point P has a conic associated with it and these conics form the second family of curves described in Section 2, those joining points which receive bangs created at the same time. (If the aircraft is climbing, it is possible that cones from some points never intersect the ground and the conics C do not exist in this case. This means that no bangs created at such points reach the ground).

It is convenient here to introduce three geometrical quantities, θ, s and $\varkappa$. θ is the angle between the plane containing PQ and the tangent to the flight path at P and the instantaneous plane of motion at P. s is the distance between P and Q, and $\varkappa$ is numerically equal to the curvature of the flight path at P. $\varkappa$ is positive if the angle between the lines PQ and PO (O being the centre of curvature) is acute, and negative if this angle is obtuse. The time taken for a bang to travel from P to Q is s/a where a is the speed of sound, and so the time when a bang arrives at Q is $s/a + t$. The curves along which bangs are received at the same time are obtained, therefore, by drawing the conics C corresponding to points on the flight path, working out values of $s/a + t$ at points on these conics and drawing curves through points having the same value of $s/a + t$. It is clear that the two families of curves described in Section 2 can be obtained by elementary means, although some entertaining three-dimensional geometrical problems may have to be solved.

3.2. *Intensities of Sonic Bangs on the Ground.*—A formula for the intensity of the bow shock wave is given in Section 4 of Ref. 3; as stated in Section 2 of this note, the intensity of the stern shock wave is approximately the same as that of the bow shock wave. The formula applies for all s, but a considerable simplification occurs when s is large compared with the length of the aircraft. Since this is almost always the case, the simplified formula only is quoted here. It is

$$\frac{\Delta p}{p_0} = \frac{2^{1/4}M^{3/4}\gamma}{(\gamma+1)^{1/2}(M^2-1)^{1/4}}\left[\frac{\int_0^{\eta_0} F(\eta)\,d\eta}{B(s,\lambda)\,s\{1-(s/\lambda)\}}\right]^{1/2} \quad \ldots \quad (1)$$

Here, p_0 is the pressure of the undisturbed air, Δp the intensity of the bang (*i.e.*, the jump in pressure which causes the bang), γ the ratio of the specific heat of air at constant pressure to that at constant volume and M the Mach number. γ is taken to be 1·4. The remaining symbols are explained in the following Sub-sections, but it can be said now that the numerator of the term in brackets is a measure of the effect of the aircraft's geometry and that the denominator is a measure of the effect of the aircraft's motion.

3.2.1. *The effect of the aircraft's geometry.*—The function $F(\eta)$ in equation (1) is given by

$$F(\eta) = \frac{1}{2\pi} \int_0^{\eta} \frac{S''(\eta_1)\, d\eta_1}{(\eta - \eta_1)^{1/2}}, \quad \ldots \quad (2)$$

where $S(\eta_1)$ is the cross-sectional area of the aircraft at a distance η_1 from the nose. η_0 in equation (1) is the smallest solution of

$$F(\eta) = 0 . \qquad (\eta_0 > 0) \quad \ldots \quad (3)$$

The area distribution of a parabolic body of revolution of length l and maximum cross-sectional area S_m is

$$S = 16 S_m \left(\frac{\eta}{l}\right)^2 \left(1 - \frac{\eta}{l}\right)^2 \quad \ldots \quad (4)$$

and the corresponding distribution of $F(\eta)$ is

$$F(\eta) = \frac{32}{\pi} \frac{S_m}{l^{3/2}} \left(\frac{\eta}{l}\right)^{1/2} \left(1 - 4\frac{\eta}{l} + 3 \cdot 2 \frac{\eta^2}{l^2}\right), \quad \ldots \quad (5)$$

giving $\eta_0 = 0 \cdot 3455 l$. It follows that

$$\int_0^{\eta_0} F(\eta)\, d\eta = 0 \cdot 461 \frac{S_m}{l^{1/2}} . \quad \ldots \quad (6)$$

These distributions of $S(\eta)$ and $F(\eta)$ are shown in Fig. 3.

For large s the detailed geometry of the aircraft is of little consequence and a knowledge of the length and maximum cross-sectional area is probably sufficient for estimating the effect of the aircraft's geometry. The area distribution of equation (4) is typical of conventional aircraft and so the numerical constant in equation (6) is taken as representative and is used throughout the rest of the work. If a particular aircraft has an area distribution differing considerably from equation (4) the determination of $\int_0^{\eta_0} F(\eta)\, d\eta$ is straightforward, although tedious.

The constant factor in equation (1) is now given by

$$\frac{2^{1/4} \gamma}{(\gamma + 1)^{1/2}} \left[\int_0^{\eta_0} F(\eta)\, d\eta\right]^{1/2} = \frac{0 \cdot 73 S_m^{1/2}}{l^{1/4}} . \quad \ldots \quad (7)$$

3.2.2. *The effect of the aircraft's motion.*—The function $B(s, \lambda)$ in equation (1) takes different forms depending on the sign of λ and the relative magnitudes of s and λ. λ is defined by

$$\lambda = \frac{(M^2 - 1) a^2}{\frac{a}{M} \frac{dM}{dt} + M (M^2 - 1)^{1/2} a^2 \varkappa \cos \theta} . \quad \ldots \quad (8)$$

The denominator is the component of the aircraft's acceleration along the line PQ (*see* Section 3.1). B has the following forms

$$B = (-\lambda)^{1/2} \sinh^{-1} (s/-\lambda)^{1/2}, \qquad \lambda < 0 \quad \ldots \quad (9a)$$

$$B = (\lambda)^{1/2} \sin^{-1} (s/\lambda)^{1/2}, \qquad \lambda > 0, s < \lambda \quad \ldots \quad (9b)$$

$$B = -(\lambda)^{1/2} \left[\frac{\pi}{2} + \cosh^{-1} (s/\lambda)^{1/2}\right], \qquad \lambda > 0, s > \lambda \quad \ldots \quad (9c)$$

$$B = s^{1/2}, \qquad \lambda = \infty \quad \ldots \quad (9d)$$

s and λ appear only in the combination $[Bs\{1 - (s/\lambda)\}]^{1/2}$ in the denominator of equation (1), and this is plotted against λ for various values of s in Fig. 4.

3.3. *A Discussion of Rao's Formula.*—Combining equations (1) and (7),

$$\frac{\Delta p}{p_0} = \frac{0 \cdot 73 S_m^{1/2} M^{3/4}}{l^{1/4}(M^2 - 1)^{1/4}\,[Bs\{1 - (s/\lambda)\}]^{1/2}} \,. \qquad \qquad (10)$$

A few remarks on this equation must be made. At first sight it appears that Δp becomes infinite when M tends to 1 and also when λ tends to s. The former conjecture is not true, since the presence of $\{1 - (s/\lambda)\}^{1/2}$ in the denominator means that $(M^2 - 1)^{1/2}$ appears in the numerator (from equation (8)). Therefore, Δp tends to zero as M tends to 1, a result which is physically evident. There is, however, a real difficulty when λ tends to s. The following remarks are made without proof, although their validity is established in Refs. 2 and 3 and in Section 4.4. of Ref. 1. In regions on the ground where $s > \lambda$, the phenomenon described in Section 2 takes place and multiple bangs are received. The rays of the envelope converge as s approaches λ and, at points where $s = \lambda$ they cross, while the curves connecting points which receive bangs at the same time have two (or more) branches in these regions ; s is greater than λ on the rear branches and less than λ on the front branches. The branches meet at a cusp at which $\lambda = s$ and at these cusps the theory leading to equation (10) breaks down. When $\lambda = s$, $d^2s/dt^2 = 0$, and this is the case discussed in Section 4.4 of Ref. 1 (where the notation for d^2s/dt^2 is $\ddot{r}_0$). A method for use when d^2s/dt^2 vanishes is given in that Section, but it is somewhat difficult to apply and assumes that the flight path is known in great detail. If d^3s/dt^3 is known at points where $\lambda = s$, and higher derivatives can be ignored, then, using the theory of Ref. 1, the formula replacing (10) is

$$\frac{\Delta p}{p_0} = \frac{1 \cdot 05 a S_m^{1/2} M^{11/12}(M^2 - 1)^{1/4}}{l^{5/12} s^{5/4} \left|\dfrac{d^3 s}{dt^3}\right|^{1/3}} \qquad \qquad (11)$$

(This result is derived by a method similar to that described in Section 4 of the Appendix to the present paper.) The formula is expected to possess the same degree of accuracy as the other formulae of Ref. 1. A comparison of these with Rao's formula, where possible, suggests that equation (11) certainly gives the correct order of magnitude of Δp, and it should be adequate for estimating the intensity of sonic bangs at points for which $d^2s/dt^2 = 0$, where high accuracy is not expected any way.

3.4. *Propagation of Bangs in a Non-homogeneous Atmosphere.*—The effect of a non-homogeneous atmosphere (in which pressure and temperature vary with altitude) is discussed in a later Section.

The variation in pressure can be crudely allowed for by replacing $\Delta p/p_0$ in equations (10) and (11) by $\Delta p/(p_g p_a)^{1/2}$. p_g is the pressure at the ground and p_a that at the altitude of the aircraft when it created the bang. a in the quantity $(s/a) + t$ is similarly replaced by $(a_g a_a)^{1/2}$, where a_g is the speed of sound at the ground and a_a that at the altitude of the aircraft.

4. *Examples of Sonic-Bang Patterns Produced during Various Manoeuvres.*—The work described in the previous Section is now illustrated by considering the distributions and intensities of sonic bangs produced during various flight manoeuvres. The aircraft is assumed to have a length of 50 ft and a maximum cross-sectional area of 30 sq ft.

4.1. *Steady Level Flight.*—The simplest manoeuvre is that of an aircraft moving with constant velocity at a constant altitude ; this is referred to as 'steady level flight'. In this case both dM/dt and $\varkappa$ are zero and so $\lambda = \infty$. Allowing for a non-homogeneous atmosphere, equation (10) becomes

$$\frac{\Delta p}{\sqrt{(p_a p_g)}} = \frac{0 \cdot 73 S_m^{1/2} M^{3/4}}{l^{1/4}(M^2 - 1)^{1/4} s^{3/4}} = \frac{1 \cdot 50 M^{3/4}}{(M^2 - 1)^{1/4} s^{3/4}} \,. \qquad \qquad (12)$$

Fig. 5 shows how the intensity of the bangs received at points on the track varies with the altitude of the aircraft and with its Mach number. The variation with Mach number is, in general, small, particularly for Mach numbers sufficiently large for $M^{1/2}$ to be a satisfactory approximation to $(M^2 - 1)^{1/4}$. s is then approximately equal to the altitude and so Δp varies as $M^{1/4}$. The variation with altitude is more marked than the power $\frac{3}{4}$ in equation (12), since p_a increases as the altitude decreases. Fig. 7 shows how the intensities fall off as the distance away from the track increases. The curves are drawn for various altitudes and for a Mach number of 1·414. Figs. 6 and 8 give similar information about the impulses of bangs produced during steady level flight. To calculate the impulse of a bang it is necessary to know the value of AE in Fig. 1 of Section 2. Whitham's result for the distance between the two shock waves of an N-wave produced during steady level flight provides a reasonable estimate.

In all four of these Figures the curves are cut off by broken lines. To one side of these lines no bangs reach the ground because of refraction (in the standard atmosphere). The method by which these lines are obtained is described in Section 5.

Although the Figures are strictly true for steady level flight only, they provide a measure of the average intensity of the bangs produced during any flight. If the dimensions of an aircraft differ from those used here, this can be easily allowed for ; the factor $S_m^{1/2}/l^{1/4}$ in equation (12) shows that the intensity varies as the linear dimensions of the aircraft to the power $\frac{3}{4}$ or roughly as the all-up weight to the power $\frac{1}{4}$. Figs. 5 to 8 inclusive were given in Ref. 1, but were slightly in error.

4.2. *A Horizontal Circular Turn.*—An instructive example of the effect of accelerated flight on distributions and intensities of sonic bangs is that of an aircraft moving in a circle at constant altitude and at constant Mach number. Figs. 9 and 10 apply to an aircraft performing this manoeuvre at an altitude of 30,000 ft and at a Mach number of 1·5. The radius of the circle is 24,600 ft, so that the indicated acceleration is $3g$ (*i.e.*, the usual normal accelerometer would read $3g$, but the centripetal acceleration is $2\sqrt{2}g$). In both diagrams of Fig. 9 the abscissa is distance measured from the point O vertically beneath the centre of the circle. The upper diagram shows the intensities of the bangs. Inside a circle with centre O and of radius 31,400 ft no bangs are received, while outside this circle two double bangs are received. The circle itself is the locus of points for which $s = \lambda$ (*see* Section 3.3). The lower diagram shows the times of reception of bangs, the origin of time being the time at which a bang first reaches a point on the radius being considered. Fig. 10 shows one of the curves along which bangs are received at the same time ; it moves with the same angular velocity about O as the aircraft, and has a cusp at the point B where $s = \lambda$. As stated in Section 3.3, equation (10) is not valid at such a point, but equation (11) provides an estimate for the intensity of the super-bang received there and this estimate is indicated by a cross. It is clear from Fig. 10 that only one double bang is received at B, while for points a little further out than B two double bangs very close together are received and the resulting pressure variation is complex. This can be seen from Fig. 11 which is a sketch depicting the interaction of two N-waves.

The broken line in the upper diagram of Fig. 9 comes from using equation (12) to calculate the intensities, so that the effect of acceleration has been ignored in deriving it. It ceases to be even qualitatively correct near the points where $s = \lambda$. The intensities there are two to three times as large as those obtained when acceleration is ignored, and this factor is typical.

4.3. *Sinusoidal Variation of Altitude.*—The next example is that of an aircraft moving in a vertical plane at a constant Mach number of 1·1 and performing sinusoidal oscillations of amplitude 100 ft about a mean altitude of 15,000 ft. The length of each oscillation is chosen so that the maximum normal acceleration produced is $0·5g$. Altitude and normal acceleration are plotted against time in Fig. 12.

The upper diagrams of Figs. 13 and 14 show the intensities of the bangs received along the track and along a line 15,000 ft to one side of the track respectively. The lower diagrams show the corresponding times of reception. Over most of the distance one double bang only is received but there are very small portions, not visible on the Figures, where three double bangs are received. The extremities of these portions, marked with crosses, are points at which $s = \lambda$ and once again equation (11) has been used to estimate the intensities of the super-bangs received at such points.

The intensities obtained when acceleration is neglected are shown by dotted lines in the upper diagrams of Figs. 13 and 14. These lines provide a good approximation except in the region of points where $s = \lambda$.

4.4. *Sinusoidal Variation of Forward Speed.*—The previous Section contained an example of varying normal acceleration; in this Section an example of varying forward acceleration is discussed. The aircraft is assumed to be flying in a straight line and at a constant altitude but with its Mach number and forward acceleration varying sinusoidally. The altitude is 10,000 ft, the mean Mach number is 1·10 with an amplitude of 0·03 and the period of oscillation is 20 sec. The amplitude of the acceleration becomes $0 \cdot 317g$. Mach number and forward acceleration are plotted against time in Fig. 15.

The upper diagrams of Figs. 16 and 17 show the intensities of the bangs received along the track and along a line 10,000 ft to one side of the track respectively. The lower diagrams show the corresponding times of reception. Over most of the distance one double bang only is received but there are small portions where three double bangs are received. The extremities of these portions, marked with crosses, are points at which $s = \lambda$ and again equation (11) has been used to estimate the intensities at these points. The intensities obtained when acceleration is neglected are shown by dotted lines in the upper diagrams of Figs. 16 and 17. These lines provide a good approximation except in the region of points where $s = \lambda$.

The distribution of bangs is very similar to that obtained in the previous Section (sinusoidal variation of altitude). In both these cases the distributions in the neighbourhood of the points at which $s = \lambda$ are much more complex than is suggested by Figs. 13, 14, 16 and 17; the reason for this is given in Section 4.2.

5. *The Effect of Refraction on the Propagation of Sonic Bangs.*—In Section 3.4 a crude procedure was described for estimating the changes (due to variation in atmospheric pressure), in intensities of sonic bangs when a non-homogeneous atmosphere is considered instead of a homogeneous one. Further changes caused by the non-homogeneity are now considered. The speed of sound, a, is assumed to vary with altitude according to the equation

$$a = a_g - kh, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where a_g is the speed of sound at the ground, k is a positive constant and h is altitude. That this is a close approximation to the variation in the I.C.A.O. standard atmosphere[8] (where temperature and not speed of sound varies linearly) can be seen by examining Fig. 18. This is a plot of speed of sound against altitude in the I.C.A.O. standard atmosphere. From ground level to an altitude of 36,089 ft (the 'tropopause'), the curve is almost indistinguishable from a straight line. Above this altitude the speed of sound remains constant and, if bangs are produced at an altitude above the tropopause, they propagate as described in Section 2 until they reach the tropopause. Below the tropopause, bangs are not propagated along straight lines but are refracted away from the ground in a manner to be described below. All results quoted without proof are derived in the Appendix.

A disturbance created in an atmosphere in which the speed of sound varies according to equation (13) is propagated along rays which are not straight lines but circles. The circles all pass through the point of origin, O, of the disturbance and all have their centres lying in the plane $h = a_g/k$ where the speed of sound in the fictitious atmosphere now being considered falls to zero. At any instant of time the disturbance is confined to a sphere with its centre on the vertical line through O. The centre moves downwards as time increases, while the highest point of the sphere approaches the plane $h = a_g/k$ asymptotically. If an aircraft is moving with supersonic speed, the disturbances produced by it have an envelope and, just as in the case of a homogeneous atmosphere, the rays along which the envelope is propagated form a sub-family of the rays along which each separate disturbance is propagated. The rays of the envelope which emanate from a particular point on the flight path are again those rays of the disturbance created at that point which make the complement of the Mach angle with the flight direction. The rays of the envelope are, in general, circles and so, even if they start with a downward slope, they eventually become horizontal and finally move up away from the ground. It follows that if the aircraft is at a sufficiently great altitude, the bangs it produces are refracted upwards without reaching the ground.

These remarks are illustrated by Fig. 19 which depicts an aircraft at an altitude of 25,000 ft and flying at a Mach number of 1·1. The variation of speed of sound is approximately that in the real atmosphere, except that the tropopause has been ignored (this diagram should be compared with Fig. 2). The altitude and Mach number are such that a bang reaches the ground just as the ray along which it is travelling becomes horizontal. The full lines are the portions of rays and of the envelope (or shock wave) which are actually formed. They bear a considerable resemblance to the rays and the envelope in Fig. 2. If the altitude is increased slightly the rays no longer hit the ground but continue along the portions drawn with a broken line. A second branch of the envelope (also drawn with a broken line) now appears, the two branches meeting at a cusp. Rays in the vicinity of the cusp converge and, at the cusp, they cross, so that there is a strong resemblance to the behaviour described in Section 3.3 near points at which $s = \lambda$. For this reason there is little point in extending Rao's technique for determining the intensity of sonic bangs to a non-homogeneous atmosphere since it would break down in the region of most interest. Instead, the method described in Section 4.4 of Ref. 1 is extended to cover the case of a non-homogeneous atmosphere. First, however, the problem of the effect of refraction on the distribution of sonic bangs is considered.

The equation of the flight path is assumed to be $x = f(t)$, $y = g(t)$, $z = h(t)$, t being the time and x, y, z being rectangular Cartesian co-ordinates. The origin of t, x and y is of no importance but z is measured vertically upwards from the ground. f', g' and h' are written for df/dt, dg/dt and dh/dt respectively. Fixing on the point P of the flight path corresponding to time t, the curve on the ground connecting points which receive bangs created at time t is found as follows : Let P_1 be the point on the ground vertically below P. The required curve is given by

$$r = \frac{1}{k}\left\{(A^2 - a_a^2)^{1/2} - (A^2 - a_g^2)^{1/2}\right\}, \qquad \ldots \quad (14)$$

where a_g is the speed of sound at the ground and a_a that at an altitude h (so that $a_a = a_g - kh$) and

$$A = \frac{a_a\{a_a(f'^2 + g'^2)^{1/2} \pm h'[(f'^2 + g'^2) - (a_a^2 - h'^2)\sec^2 \Omega]^{1/2}\}}{(a_a^2 - h'^2)\sec \Omega}.$$

r and Ω are polar co-ordinates with P_1 as pole and the tangent to the track at P as axis. This result comes from equations (I.20) and (I.15) in the Appendix. If α is the angle of dive at time t it may be written as

$$A = \frac{Ma_a\{\cos \alpha \cos \Omega \pm \sin \alpha[M^2 \cos^2 \alpha \cos^2 \Omega + M^2 \sin^2 \alpha - 1]^{1/2}\}}{(1 - M^2 \sin^2 \alpha)}. \qquad \ldots \quad (15)$$

The two signs arise because there are, in general, two directions in a vertical plane through the aircraft which make the complement of the Mach angle with the direction of flight. The lower sign corresponds to the upper ray. This bang can reach the ground only if the aircraft is diving and the angle of dive is greater than the complement of the Mach angle, and even then it may be refracted away from the ground. The condition for the bang to be refracted away is that

$$A < a_g. \qquad \ldots \quad (16)$$

If Q is a point on the curve of equation (14) with co-ordinates (r, Ω) then, from equation (I.19), the time taken for the bang to travel from P to Q is

$$t_{PQ} = \frac{1}{k}\left(\cosh^{-1}\frac{A}{a_a} - \cosh^{-1}\frac{A}{a_g}\right).$$

Thus, the bang is received at a time t_R, where

$$t_R = t + \frac{1}{k}\left(\cosh^{-1}\frac{A}{a_a} - \cosh^{-1}\frac{A}{a_g}\right) \quad \ldots \quad (17)$$

These formulae simplify when applied to an aircraft moving in supersonic steady level flight at an altitude h and with a speed V. Equation (14) becomes

$$r = \frac{1}{k}\{(V^2 \cos^2 \Omega - a_a^2)^{1/2} - (V^2 \cos^2 \Omega - a_g^2)^{1/2}\} \quad \ldots \quad (18)$$

and equation (17) becomes

$$t_R = t + \frac{1}{k}\left(\cosh^{-1}\frac{V \cos \Omega}{a_a} - \cosh^{-1}\frac{V \cos \Omega}{a_g}\right), \quad \ldots \quad (19)$$

while from (16) the bang never reaches the ground if

$$\cos \Omega < \frac{a_g}{V}. \quad \ldots \quad (20)$$

If d is the distance of Q from the track so that $d = r \sin \Omega$, then no bangs are received at points for which $d < d_m$ where

$$d_m = \frac{1}{k}\left(1 - \frac{a_g^2}{V^2}\right)^{1/2} (a_g^2 - a_a^2)^{1/2} = h\left(\frac{a_g + a_a}{a_g - a_a}\right)^{1/2}\left(1 - \frac{a_g^2}{V^2}\right)^{1/2}. \quad \ldots \quad (21)$$

This formula, which predicts the lateral spread of the sonic bangs produced during steady level flight, is illustrated in Fig. 20. There d_m is plotted against Mach number for six altitudes ; the change in the equation of propagation at the tropopause has been taken into account. No bangs are received anywhere on the ground if

$$V < a_g. \quad \ldots \quad (22)$$

These results apply to level ground only. If the terrain is hilly it may be prudent to consider the propagation of disturbances more carefully. (In a homogeneous atmosphere the region affected by bangs may theoretically extend to an infinite distance to either side of the track. The intensity of the bangs tends to zero as the distance tends to infinity).

The presence of winds in the atmosphere also affects sonic bang distributions. Two simple examples must suffice : a gradient in headwind speed and a constant sidewind. If the headwind speed at altitude h is Δw greater than that at the ground the speed of the aircraft in (22) must be reduced by Δw so that no bangs reach the ground anywhere if

$$V - \Delta w < a_g. \quad \ldots \quad (23)$$

This result is illustrated in Fig. 21 in which the Mach number below which no sonic bangs reach the ground is plotted against altitude for several values of the difference in headwind speeds. The curves have discontinuities in slope at the tropopause. The presence of a sidewind affects the result of equation (21). Suppose the speed of the sidewind is w_s. From (19) and (20) the bang arriving furthest away from the track takes a time $1/k \cosh^{-1} (a_g/a_a)$ to travel from the aircraft to the ground. During this time the wind has blown it a distance $w_s/k \cosh^{-1} (a_g/a_a)$. Hence the lateral spread on the side of the track towards which the wind is blowing should be increased by this amount while that on the other side should be decreased by the same amount. It is noteworthy that this result does not depend on the Mach number of the aircraft.

As an example on the above work the distribution of bangs produced by an aircraft in steady level flight at a Mach number of 1·15 and at an altitude of 30,000 ft is considered. The speed of sound at this altitude is 995 ft/sec, while at the ground it is 1,117 ft/sec. Equation (13) then gives k as $0{\cdot}00407\ \text{sec}^{-1}$. Fig. 22 shows one of the curves on the ground along which bangs are received at the same time. This curve moves with the same velocity as the aircraft and, as explained in Section 2, it corresponds to an N-wave moving over the ground. Also shown is the position at the same time of the curve corresponding to a homogeneous atmosphere in which the constant speed of sound in feet per second is the geometric mean of 995 and 1,117, *i.e.*, 1,054 (*see* Section 3.4). Up to what may be called the 'cut-off point' the two curves lie very close together. This suggests that the effect of refraction on a sonic-bang distribution can be allowed for, simply by finding the cut-off point corresponding to various points on the flight path. In the general case (*i.e.*, not necessarily steady level flight), the value of Ω at a cut-off point is given by $A = a_g$ as follows from (16), A being defined by equation (15). The corresponding value of r is then obtained from equation (14).

The effect of refraction on the intensities of sonic bangs is considered in Section 4 of the Appendix, which deals with a special case only, that of an aircraft in steady level flight at an altitude and speed such that bangs just reach the ground (as in Fig. 19). The results are shown in Fig. 23 and it can be seen that the focusing effect of refraction increases the intensities considerably, by a factor roughly equal to 5. The curve has been drawn only to a point just below the tropopause. To obtain results beyond this altitude would entail considerable work and, since the atmosphere is homogeneous above the tropopause, the factor is likely to remain approximately constant beyond it. For more complicated flight paths the method described in Section 4 of the Appendix can, in principle, be used, although it may well require considerable labour. This case provides an upper limit for the focusing effect of refraction, since it assumes the worst possible combination of altitude and Mach number. For other combinations and, in particular, for Mach numbers greater than about 1·15, the effect is considerably reduced.

On occasions temperature inversions occur in the atmosphere, but they are close to the ground, extending to altitudes of about 4,000 ft only. Their effect is, therefore, small ; this effect is one of defocusing.

6. *Conclusions.*—The phenomenon of sonic bangs is explained in Section 2 of this note and methods for determining the distributions and intensities of sonic bangs are given in Section 3. Section 3.1 deals with the determination of the distributions and Sections 3.2 and 3.3 with the technique developed by Rao for estimating the intensities. Normally equation (10) suffices for this, but at and near points where $d^2s/dt^2 = 0$ (s being the distance from the aircraft to the point on the ground and t the time at the aircraft), this equation breaks down. It is suggested that equation (11) be used in such a case. In practice it is probably best to find the lines along which d^2s/dt^2 vanishes first and then use equation (11) to estimate the intensities on these lines. For the rest of the area subjected to bangs it should be sufficiently accurate to use Figs. 5 and 7 (strictly applicable to steady level flight only). The difficulty is that equation (11) requires a knowledge of d^3s/dt^3 and it may be impossible to obtain this in actual cases. As a rough guide it is usually true that the intensities of super-bangs, *i.e.*, bangs received in areas where d^2s/dt^2 is small, are two to three times as large as the average of the intensities elsewhere, although there is always the possibility of a combination of circumstances leading to much larger intensities.

Examples of manoeuvres producing super-bangs are examined in Section 4 and the results confirm that, whether the aircraft is accelerating by performing a horizontal turn or by oscillating in a vertical plane or by varying its speed, the super-bangs produced have intensities of two to three times the intensities obtained by using Figs. 5 and 7.

A further complication in sonic-bang calculations arises because the actual atmosphere is not homogeneous and so the speed of sound varies with altitude. The most marked effect on a distribution of sonic bangs is the 'cut-off' of bangs at a sufficiently great distance from the track (*see* Fig. 20). It may even happen that no bangs are received anywhere on the ground. This is due to the phenomenon of refraction which is discussed in Section 5. Formulae are given in that Section for determining the cut-off points and a knowledge of these together with an application of the technique described in Section 3.4 (*i.e.*, using the geometric mean of the speeds of sound at the ground and at the aircraft as a constant speed of sound in a fictitious homogeneous atmosphere), should be sufficient for determining a sonic-bang distribution. In Section 3.4 it is also suggested that p_0 in equations (10) and (11) be replaced by the geometric mean of the pressures of the air at the aircraft and at the ground. This probably gives a sufficiently accurate estimate of the change in intensity due to the variation of pressure with altitude but refraction may also increase the intensity of sonic bangs. This is because the rays along which the bangs are propagated are approximately circles in the real atmosphere and when their slopes become nearly horizontal they converge, thus producing an enhancement of the intensity in the same way as in accelerated flight. One example of this is considered in detail in Section 4 of the Appendix, that of an aircraft in steady level flight. The results suggest that, in the worst case, *i.e.*, at points where the rays become horizontal the intensities of the bangs are increased by a factor roughly equal to 5. The area receiving super-bangs (caused by either accelerated flight or refraction effects) is unlikely to extend further than about a quarter of a mile inward from the boundary of the area affected by bangs.

This note deals with thickness effects only. Other work[9] suggests that the results given here would not be significantly affected if incidence effects were included.

LIST OF SYMBOLS

A	Defined by equation (15)
a	Speed of sound
a_a	Speed of sound at altitude h
a_g	Speed of sound at the ground
B	Defined by equations (9)
d	Distance of a point on the ground from the track
d_m	Distance of cut-off point from track
F	Defined by equation (2)
f, g, h	$x = f(t)$, $y = g(t)$, $z = h(t)$ is the equation of the flight path
h	Altitude of aircraft
k	Defined in equation (13)
l	Length of aircraft
M	Mach number of aircraft
p	See Δp
p_0	Pressure of undisturbed air
p_a	Pressure at altitude h
p_g	Pressure at the ground
r	Radial co-ordinate defined after equation (14)
S	Cross-sectional area of aircraft
S_m	Maximum cross-sectional area of aircraft
s	Distance of point of origin of bang to point of reception
t	Time
t_{PQ}	Time taken for bang to travel from P to Q
t_R	Time of reception of bang
V	Speed of aircraft
w	See Δw
w_s	Speed of sidewind
x, y, z	Rectangular Cartesian co-ordinates, z being measured vertically upwards from the ground
α	Angle of dive
γ	Ratio of specific heats of air
Δp	Jump in pressure which causes a sonic bang
Δw	Difference between headwind speed at altitude h and that at the ground
η, η_1	Distance of a section of the aircraft from the nose
η_0	Smallest solution of $F(\eta) = 0$, $\eta_0 > 0$
θ	Defined in second paragraph of Section 3.1.
$\varkappa$	Curvature of flight path
λ	Defined by equation (8)
Ω	Polar angle defined after equation (14)

Dashes denote differentiation with respect to time.

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APPENDIX

The Propagation of Sonic Bangs in a Non-homogeneous Atmosphere

1. *A Stationary Disturbance.*—Let r and z be rectangular co-ordinates in a vertical plane, z being measured vertically upwards. The origin of co-ordinates is of no importance. Let $a(z)$ be the speed of sound, which is assumed to obey equation (13) so that

$$a = a_0 - kz . \qquad \text{(I.1)}$$

a_0 is the speed of sound at $z = 0$, while k is a positive constant. Let ds be an infinitesimally small section of a normal to a disturbance created at time $t = 0$ at the point $r = 0$, $z = 0$ and let dr and dz be the changes in r and z associated with the change ds. On linear theory the disturbance is propagated as a sound wave and, since sound is refracted according to the well-known refraction law,

$$a \frac{ds}{dr} = A ,$$

where A is a constant. Thus,

$$(a_0 - kz) \left\{1 + \left(\frac{dz}{dr}\right)^2\right\}^{1/2} = A$$

and

$$\frac{dz}{dr} = - \frac{\{A^2 - (a_0 - k_z)^2\}^{1/2}}{(a_0 - kz)}$$

or

$$\{A^2 - (a_0 - kz)^2\}^{1/2} = - kr + A_1 ,$$

where A_1 is an arbitrary constant. Since the origin of the disturbance is at the point $r = 0$, $z = 0$, all the rays pass through this point. Hence,

$$(A^2 - a_0^2)^{1/2} = A_1$$

and the equation of the rays is

$$(a_0 - kz)^2 + \{(A^2 - a_0^2)^{1/2} - kr\}^2 = A^2. \qquad \text{(I.2)}$$

The rays are, therefore, circles with centres lying on the line $z = a_0/k$, which is the line where the speed of sound falls to zero. To determine the equation of the wave fronts the orthogonal trajectories of equation (I.2) must be obtained. The slope of the rays is given by

$$\frac{dz}{dr} = - \frac{\{A - (a_0 - kz)^2\}^{1/2}}{(a_0 - kz)}$$

and so the slope of the wave fronts is

$$\frac{dz}{dr} = \frac{(a_0 - kz)}{\{A - (a_0 - kz)^2\}^{1/2}} .$$

The differential equation for the wave fronts is obtained by eliminating A between this equation and (I.2). Using equation (I.2)

$$\frac{dz}{dr} = \frac{(a_0 - kz)}{\{(A^2 - a_0^2)^{1/2} - kr\}}$$

or

$$(A^2 - a_0^2)^{1/2} - kr = (a_0 - kz) \frac{dr}{dz} .$$

Equation (I.2) may be written

$$(a_0 - kz)^2 - 2kr(A^2 - a_0^2) + k^2r^2 = a_0^2$$

and so

$$(a_0 - kz)^2 - 2kr(a_0 - kz)\frac{dr}{dz} - k^2r^2 = a_0^2$$

or

$$-2a_0z + kz^2 - 2r(a_0 - kz)\frac{dr}{dz} - kr^2 = 0\,.$$

Now

$$\frac{d}{dz}\left\{\frac{r^2}{(a_0 - kz)}\right\} = \frac{2r(a_0 - kz)\dfrac{dr}{dz} + kr^2}{(a_0 - kz)^2}\,.$$

Hence

$$\frac{-2a_0z + kz^2}{(a_0 - kz)^2} = \frac{d}{dz}\left\{\frac{r^2}{(a_0 - kz)}\right\}$$

or

$$\frac{d}{dz}\left\{\frac{r^2}{(a_0 - kz)}\right\} = \frac{1}{k} - \frac{a_0^2}{k(a_0 - kz)^2}\,.$$

This gives

$$r^2 = \frac{z}{k}(a_0 - kz) - \frac{a_0^2}{k^2} + A_2(a_0 - kz)$$

or

$$k^2r^2 + \{k^2z^2 - k(a_0 - k^2A_2)z\} = -a_0^2 + k^2A_2a_0$$

and so

$$k^2r^2 + \{kz - \tfrac{1}{2}(a_0 - k^2A_2)\}^2 = -(a_0^2 - a_0k^2A_2) + \tfrac{1}{4}(a_0 - k^2A_2)^2\,.$$

Here, A_2 is an arbitrary constant (or, rather, a function of time) ; writing

$$(a_0 - k^2A_2) = -k^2a_0A_3\,,$$

it follows that

$$r^2 + \left(z + \frac{a_0k}{2}A_3\right)^2 = a_0^2A_3(1 + \tfrac{1}{4}k^2A_3)\,. \qquad \text{(I.3)}$$

There remains the correlation of A_3 with the time, t, that has elapsed since the disturbance was created. The ray with an infinite slope at $z = 0$, $r = 0$ is simply the line $r = 0$, as can be seen by letting A tend to infinity in equation (I.2). Along this ray the wave front is propagated with the speed

$$\frac{dz}{dt} = a_0 - kz\,,$$

so that

$$z = \frac{a_0}{k}(1 - e^{-kt})\,,$$

the constant in the integration having been chosen to make $z = 0$ when $t = 0$. On the other hand, $r = 0$ in equation (I.3) produces

$$z^2 + a_0kA_3z = a_0^2A_3\,.$$

Hence,

$$A_3 = \frac{z^2}{a_0(a_0 - kz)} = \frac{(a_0^2/k^2)(1 - e^{-kt})^2}{a_0^2\,e^{-kt}}$$

$$= \frac{2}{k^2}(\cosh kt - 1)\,.$$

Finally,

$$r^2 + \left\{z + \frac{a_0}{k}(\cosh kt - 1)\right\}^2 = \frac{a_0^2}{k^2}\sinh^2 kt \,. \quad \ldots \quad \text{(I.4)}$$

It follows that the wave fronts are circles with centres lying on the line $r = 0$ and gradually moving down this line as t increases. The maximum value of z for a particular value of t is

$$z_m = \frac{a_0}{k}\{\sinh kt - \cosh kt + 1\} = \frac{a_0}{k}(1 - e^{-kt})$$

and so the circles approach the line, $z = a_0/k$, at which the speed of sound falls to zero asymptotically as t tends to infinity. As k tends to zero, equation (I.4) tends to the familiar form ,

$$r^2 + z^2 = a_0^2 t^2 \,.$$

Putting the results obtained in a general three-dimensional form, the wave fronts of a disturbance created at a time t_0 at a point x_0, y_0, z_0, move according to the equation

$$(x - x_0)^2 + (y - y_0)^2 + \left\{(z - z_0) + \frac{a_0}{k}(\cosh k\,\overline{t - t_0} - 1)\right\}^2 = \frac{a_0^2}{k^2}\sinh^2 k\,\overline{t - t_0}\,, \quad \ldots \quad \text{(I.5)}$$

where a_0 is the speed of sound at $z = z_0$. The lines along which the wave fronts are propagated, *i.e.*, the rays, are given by

$$\{(A^2 - a_0^2)^{1/2} - k(1 + D^2)^{1/2}(x - x_0)\}^2 + \{a_0 - k(z - z_0)\}^2 = A^2\,, \quad \ldots \quad \text{(I.6}a\text{)}$$

$$(y - y_0) = D(x - x_0) \quad \ldots \quad \text{(I.6}b\text{)}$$

A and D are arbitrary constants. This pair of equations expresses the fact that, in any vertical plane through x_0, y_0, z_0 the rays are a family of circles of the form of equation (I.2).

2. *A Moving Source of Disturbances.*—Suppose now that a source of disturbances is moving along a path given by $x = f(\tau)$, $y = g(\tau)$, $z = h(\tau)$, τ being the time (its origin is of no importance). Just as in the case of a homogeneous atmosphere, the disturbances form an envelope if the speed of the source is greater than the speed of sound at the source, and it is the motion of this envelope which causes sonic bangs. In this Section it is now proved that the rays of the envelope are a sub-family of the rays of each separate disturbance and that this sub-family consists of those rays of the disturbances which make initially the complement of the Mach angle to the flight direction.

The position at time t of the wave front originating at time τ is given by

$$\{x - f(\tau)\}^2 + \{y - g(\tau)\}^2 + \{z - h(\tau) + \frac{a}{k}(\cosh k\,\overline{t - \tau}\; 1\}^2 = \frac{a^2}{k^2}\sinh^2 k\,\overline{t - \tau}\,, \quad \ldots \quad \text{(I.7}a\text{)}$$

where a, the speed of sound at the point $x = f$, $y = g$, $z = h$, is

$$a = a_0 - kh(\tau)\,, \quad \ldots \quad \text{(I.7}b\text{)}$$

a_0 being the speed of sound when $z = 0$.

The position of the envelope at time t is obtained by eliminating τ between (I.7a) and this equation differentiated with respect to τ, namely,

$$\{x - f(\tau)\}f'(\tau) + \{y - g(\tau)\}g'(\tau) + \{z - h(\tau) + \frac{a}{k}(\cosh k\,\overline{t - \tau} - 1)\} \times \{h'(\tau)\cosh k\,\overline{t - \tau} +$$

$$+ a\sinh k\,\overline{t - \tau}\} = \frac{a}{k}\sinh k\,\overline{t - \tau}\{h'(\tau)\sinh k\,\overline{t - \tau} + a\cosh k\,\overline{t - \tau}\}\,. \quad \ldots \quad \text{(I.8)}$$

Considering the position of the envelope at time t, the points on the envelope belonging to a disturbance created at time τ form a curve, *i.e.*, there is a single infinity of such points. By the nature of an envelope, the tangent planes of the envelope and the disturbance created at time τ, coincide at each one of these points and so the rays of the envelope and of the disturbance created at time τ have the same direction at these points. Now the constants A and D, defining that particular ray of the disturbance created at time τ which passes through a point x, y, z, are given by equations (I.6) as the solution of

$$[(A^2 - a^2)^{1/2} - k(1 + D^2)^{1/2}\{x - f(\tau)\}]^2 + [a - k\{z - h(\tau)\}]^2 = A^2 \quad \text{(I.9}a\text{)}$$

$$\{y - g(\tau)\} = D\{x - f(\tau)\} . \quad (\tau.9b)$$

If now x, y and z are eliminated between equations (I.7a), (I.8), (I.9a) and (I.9b), the resulting relationship between A and D defines the single infinity of the rays of the disturbance created at time τ which are momentarily tangent (at time t) to rays of the envelope. If this relationship does not contain t it follows that the same single infinity of rays of the disturbance created at time τ are always rays of the envelope.

Putting $k\{x - f(\tau)\} = \xi$, then $k\{y - g(\tau)\} = D\xi$ from equation (I.9b). Putting also $a - k\{z - h(\tau)\} = \zeta$, $\cosh k\overline{t - \tau} = c$, and $\sinh k\overline{t - \tau} = s$, and writing f' for $f'(\tau)$, g' for $g'(\tau)$, and h' for $h'(\tau)$, the following equations are obtained

$$(1 + D^2)\xi^2 + (ac - \zeta)^2 = a^2s^2 ,$$

$$(f' + Dg')\xi + (ac - \zeta)(ch' + as) = as(sh' + ac) ,$$

$$\{(A^2 - a^2)^{1/2} - (1 + D^2)^{1/2}\xi\}^2 + \zeta^2 = A^2 .$$

These become

$$(1 + D^2)\xi^2 + \zeta^2 - 2ac\zeta = - a^2 , \quad \text{(I.10}a\text{)}$$

$$(f' + Dg')\xi - (ch' + as)\zeta = - ah' , \quad \text{(I.10}b\text{)}$$

$$(1 + D^2)\xi^2 + \zeta^2 - 2(A^2 - a^2)^{1/2}(1 + D^2)^{1/2}\xi = a^2 . \quad \text{(I.10}c\text{)}$$

Subtracting equation (I.10a) from equation (I.10c),

$$- (A^2 - a^2)^{1/2}(1 + D^2)^{1/2}\xi + ac\zeta = \xi^2 .$$

Together with equation (I.10b) this yields

$$\xi = \frac{a^3s}{- \delta(ch' + as) + \beta ac} ,$$

$$\zeta = \frac{a(- \delta h' + a\beta)}{- \delta(ch' + as) + \beta ac} ,$$

where

$$\delta = (A^2 - a^2)^{1/2}(1 + D^2)^{1/2} , \quad \text{(I.11}a\text{)}$$

$$\beta = (f' + Dg') . \quad \text{(I.11}b\text{)}$$

Substituting in equation (I.10a), which may be written

$$(1 + D^2)\xi^2 + (ac - \zeta)^2 = a^2s^2 ,$$

it follows that

$$(1 + D^2)a^6s^2 + a^2s^2\{ - (sh' + ac)\delta + as\beta\}^2 = a^2s^2\{ - (ch' + as)\delta + \beta ac\}^2$$

or

$$(h'^2 - a^2)\delta^2 - 2ah'\beta\delta + a^2\{\beta^2 - a^2(1 + D^2)\} = 0 . \quad \text{(I.12)}$$

This provides a relationship between A and D and this relationship is independent of t. Hence, the rays of the envelope are a sub-family of the rays of the disturbances.

The rays of the disturbance created at a point P of the flight path given by $x = f(\tau)$, $y = g(\tau)$, $z = h(\tau)$, which make initially the complement of the Mach angle at P with the direction of motion at P, are now obtained. These rays form a sub-family of those given by equation (I.9) and this sub-family is characterised by a relationship between A and D. It is now proved that this relationship is the same as equation (I.12). From equations (I.9) the initial direction of a ray is

$$dx : dy : dz : : \frac{-a}{(A^2 - a^2)^{1/2}(1 + D^2)^{1/2}} : \frac{-aD}{(A^2 - a^2)^{1/2}(1 + D^2)^{1/2}} : 1,$$

so that the initial direction cosines are

$$\frac{a}{A(1 + D^2)^{1/2}}, \quad \frac{aD}{A(1 + D^2)^{1/2}}, \quad \frac{-(A^2 - a^2)^{1/2}}{A}.$$

The direction cosines of the flight path at P are

$$\frac{f'}{V}, \quad \frac{g'}{V}, \quad \frac{h'}{V},$$

where V, the speed, is equal to $(f'^2 + g'^2 + h'^2)^{1/2}$. The cosine of the angle at P between the ray and the flight direction is, therefore,

$$\frac{a(f' + Dg') - (1 + D^2)^{1/2}(A^2 - a^2)^{1/2}h'}{VA(1 + D^2)^{1/2}}.$$

If this is to equal the cosine of the complement of the Mach angle, *i.e.*, a/V, then ,

$$a(f' + Dg') - (1 + D^2)^{1/2}(A^2 - a^2)^{1/2}h' = aA(1 + D^2)^{1/2}. \quad \quad (I.13)$$

Making the substitutions of equations (I.11), this becomes

$$a\beta - h'\delta = a\{\delta^2 + a^2(1 + D^2)\}^{1/2},$$

so that

$$(h'^2 - a^2)\delta^2 - 2a\beta h\gamma + a^2\{\beta^2 - a^2(1 + D^2)\} = 0$$

and equation (I.12) is, therefore, the same as equation (I.13). This completes the proof of the above statement about the rays along which the envelope is propagated.

3. *Sonic-Bang Distributions.*—Equation (I.13) is a relationship between A and D which determines a sub-family of rays of the disturbance originating at a point P ; the members of this sub-family are all rays of the envelope of disturbances. (I.13) can be written as

$$a^2\{A(1 + D^2)^{1/2} - (f' + Dg')\}^2 = (1 + D^2)(A^2 - a^2)h'^2,$$

i.e.,

$$(1 + D^2)(a^2 - h'^2)A^2 - 2a^2(1 + D^2)^{1/2}(f' + Dg')A + a^2\{(f' + Dg')^2 + (1 + D^2)h'^2\} = 0.$$

Hence

$$A = \frac{a[a(f' + Dg') \pm h'\{(f' + Dg')^2 - (a^2 - h'^2)(1 + D^2)\}^{1/2}]}{(1 + D^2)^{1/2}(a^2 - h'^2)}. \quad \quad (I.14a)$$

Supposing the flight path to be projected on to the plane $z = 0$, and defining Ω as the angle between the vertical plane that is tangent to the projected curve (or 'track') and any other vertical plane $\{y - g(\tau)\} = D\{x - f(\tau)\}$, then D is given by

$$D = \frac{g' + f' \tan \Omega}{f' - g' \tan \Omega}. \quad \quad (I.14b)$$

It follows that equation (I.14a) becomes

$$A = \frac{a[a(f'^2 + g'^2)^{1/2} \pm h'\{(f'^2 + g'^2) - a^2 - h'^2) \sec^2 \Omega\}^{1/2}]}{(a^2 - h'^2) \sec \Omega}. \quad \quad (I.15)$$

The two signs arise because there are two directions in the plane $(y - g) = D(x - f)$ which make the required angle with the direction of motion. If the aircraft is diving so that $h' = - a$, then the negative direction of z is one of the required directions. This, however, corresponds to $A = \infty$, as can be seen by letting A tend to infinity in equations (I.6). $A = \infty$ comes from equation (I.15), when $h' = - a$, by taking the lower sign. Thus, the negative sign in equation (I.15) corresponds to the lower ray.

Writing r for $\{(x - f)^2 + (y - g)^2\}^{1/2}$, the ray to the envelope starting from a point P given by $x = f$, $y = g$, $z = h$, and lying in the vertical plane such that equation (I.14b) holds, is from equations (I.9) :

$$\{(A^2 - a^2)^{1/2} + kr\}^2 + \{a - k(z - h)\}^2 = A^2 \,, \qquad \text{(I.16)}$$

where A is determined from equation (I.15). The time taken for the envelope to travel along this ray until it reaches the plane $z = 0$ is now determined. If s is the arc length along the circle of (I.16) measured downwards from the point $r = 0$, $z = h$, then

$$\frac{ds}{dt} = a - k(z - h) \,, \qquad \text{(I.17)}$$

since the envelope is propagated with the speed of sound. Now,

$$\frac{ds}{dz} = - \left\{1 + \left(\frac{dr}{dz}\right)^2\right\}^{1/2} , \qquad \text{(I.18)}$$

since z decreases as s increases. From equation (I.16)

$$a - k(z - h) = [A^2 - \{(A^2 - a^2)^{1/2} + kr\}^2]^{1/2}$$

and hence

$$- k\frac{dz}{dr} = \frac{- \{(A^2 - a^2)^{1/2} + kr\}k}{a - k(z - h)} \,,$$

or

$$\frac{dz}{dr} = \frac{[A^2 - \{a - k(z - h)\}^2]^{1/2}}{a - k(z - h)} \,.$$

Thus, using equation (I.18),

$$\frac{ds}{dz} = \frac{- A}{[A^2 - \{a - k(z - h)\}^2]^{1/2}} \,.$$

It follows that

$$\frac{dz}{dt} = \frac{dz}{ds}\frac{ds}{dt} = \frac{- \{a - k(z - h)\}[A^2 - \{a - k(z - h)\}^2]^{1/2}}{A} \,,$$

where equation (I.17) has been used. Taking t to be zero when $z = h$,

$$\int_z^h \frac{dz_1}{\{a - k(z_1 - h)\}[A^2 - \{a - k(z_1 - h)\}^2]^{1/2}} = \frac{t}{A} \,.$$

Putting $z = 0$, the required time becomes

$$t = \frac{1}{k} \log \frac{a_0\{A + (A^2 - a^2)^{1/2}\}}{a\{A + (A^2 - a_0{}^2)^{1/2}\}} = \frac{1}{k}\left(\cosh^{-1}\frac{A}{a} - \cosh^{-1}\frac{A}{a_0}\right) . \qquad \text{(I.19)}$$

Here, a is the speed of sound at altitude h and is equal to $a_0 - kh$, where a_0 is the speed of sound at $z = 0$. Hence, the time taken to reach the plane $z = 0$ for a disturbance emanating from the point $x = f(\tau)$, $y = g(\tau)$, $z = h(\tau)$, and travelling in the vertical plane making an angle Ω with the vertical plane tangent to the projected curve is given by equation (I.19).

If P_1 is the point in the plane $z = 0$ vertically below a point P on the flight path and Q is the point in the plane $z = 0$ where the ray of (I.16) arrives, then the distance P_1Q comes from

$$\{(A^2 - a^2)^{1/2} + kr\}^2 + (a + kh)^2 + A^2 \,.$$

r represents P_1Q and this equation comes from (I.16) on putting $z = 0$. It is found that

$$r = \frac{1}{k}\{(A^2 - a^2)^{1/2} - (A^2 - a_0^2)^{1/2}\}. \qquad \text{(I.20)}$$

4. *Sonic-Bang Intensities.*—This Section is devoted to the estimation of the intensities produced along the track by an aircraft in steady level flight, assuming that the bangs just reach the ground (as in Fig. 19). The rays of the envelope converge as their slopes become horizontal and, as in the case of an accelerating aircraft in a homogeneous atmosphere, bangs received in the region of convergence have intensities markedly greater than those predicted by equation (12). The method used can, in principle, be applied to any flight path.

Let Q be a point on the track, P_0 a point on the flight path and O the point on the ground vertically below P_0. Suppose that the aircraft is at P_0 at time $\tau = 0$. Introduce rectangular Cartesian co-ordinates, x and z, with origin at O, z being measured vertically upwards and x being measured along the track in the direction of motion. Let the speed of sound at the ground be a_g, the speed of sound at h, the altitude of the aircraft be a_a, and the speed of the aircraft be a_g. From equation (20) the bangs just reach the ground. To estimate the intensities an extension of the method described in Section 4.4 of Ref. 1 is used. There is an interval of τ, $-\tau_1 \leqslant \tau \leqslant \tau_2$, including $\tau = 0$, such that the disturbances produced by the aircraft during this time all arrive at Q simultaneously. τ_1 and τ_2 are obtained from the condition that, in this interval,

$$|\tau + t - t_0| \leqslant \frac{l}{2a_g},$$

t is the time taken for a disturbance to travel from a point P on the flight path to Q, P being such that the time when the aircraft is at P lies in the interval $-\tau_1$ to τ_2. t_0 is the value of t when P coincides with P_0, and l is the length of the aircraft. The method requires a knowledge of the length of the interval, *i.e.*, a knowledge of both τ_1 and τ_2, and thus involves solving the equation

$$|\tau + t - t_0| = \frac{l}{2a_g}. \qquad \text{(I.21)}$$

As is shown in Ref. 1, the length of this interval is a measure of the intensity of the bang. Normally the length of the interval is small (of the order of l/a_g), but, if P_0 and Q lie on a ray of the envelope, the length of this interval becomes larger (of the order of $\sqrt{l/a_g}$). Since the rays are converging the length of the interval is still further increased.

Assume, therefore, that P_0 and Q do lie on a ray to the envelope. To solve equation (I.21) it is necessary to obtain $t - t_0$ in terms of τ. Now the co-ordinates of P_0 are $x = 0$, $z = h$; of P, $x = a_g\tau$, $z = h$, and of Q, $x = (a_g^2 - a_a^2)^{1/2}/k$, $z = 0$. The x co-ordinate of Q comes from equation (18). The ray of the disturbance from P which passes through Q is a circle with its centre lying on the line $z = a_g/k$, as shown by equation (I.6*a*). Since the circle passes through P it is of the form

$$(K - x)^2 + \left(\frac{a_g}{k} + z\right)^2 = (K - a_g\tau)^2 + \left(\frac{a_g}{k} - h\right)^2.$$

h is given by

$$h = \frac{a_g - a_a}{k}$$

and so the circle is of the form

$$(K - x)^2 + \left(\frac{a_g}{k} - z\right)^2 = (K - a_g\tau)^2 + \frac{a_a^2}{k^2}.$$

K is determined from the condition that the circle pass through Q, and so

$$\left\{K - \frac{1}{k}(a_g^2 - a_a^2)^{1/2}\right\}^2 + \frac{a_g^2}{k^2} = (K - a_g\tau)^2 + \frac{a_a^2}{k^2}.$$

Hence

$$K = \frac{2(a_g^2 - a_a^2) - k^2 a_g^2 \tau^2}{2k\{(a_g^2 - a_a^2)^{1/2} - k a_g \tau\}} . \quad \text{(I.22}a\text{)}$$

Thus, the circle has the equation

$$(K - x)^2 + \left(\frac{a_g}{k} - z\right)^2 = R^2 , \quad \text{(I.22}b\text{)}$$

where

$$R^2 = \frac{\{2(a_g^2 - a_a^2) - 2k a_g (a_g^2 - a_a^2)^{1/2} \tau + k^2 a_g^2 \tau^2\}^2}{4k^2\{(a_g^2 - a_a^2)^{1/2} - k a_g \tau\}^2} + \frac{a_a^2}{k^2} . \quad \text{(I.22}c\text{)}$$

The formula for t, the time taken for the disturbance to travel from P *to* Q, is obtained in the same way as equation (I.19). The result is

$$t = \frac{1}{k}\left(\cosh^{-1}\frac{kR}{a_a} - \cosh^{-1}\frac{kR}{a}\right) .$$

It follows that

$$t_0 = \frac{1}{k} \cosh^{-1}\frac{a_g}{a_a} .$$

Hence

$$t - t_0 = \frac{1}{k}\left(\cosh^{-1}\frac{kR}{a_a} - \cosh^{-1}\frac{kR}{a_g} - \cosh^{-1}\frac{a_g}{a_a}\right) . \quad \text{(I.23)}$$

Equations (I.23) and (I.22c) provide the required relation between $t - t_0$ and τ.

Let $\Delta\tau$ be the length of the interval $-\tau_1$ to τ_2 so that $\Delta\tau = \tau_1 + \tau_2$. Equation (22) of Ref. 1 (in which the constant factor is incorrect), gives (in corrected form) :

$$\frac{\Delta p}{(p_a p_g)^{1/2}} = 0 \cdot 365 \frac{S_m^{1/2}}{l^{3/4}} (a_a a_g)^{1/2} \frac{(M^2 - 1)^{7/8}}{h^{5/4}} \Delta\tau \quad \text{(I.24)}$$

(For p_a and p_g see Section 3.4.)

This formula takes into account the effect of a non-homogeneous atmosphere in two ways : first, the factor $(p_g p_a a_g a_a)^{1/2}$ allows for the changes in pressure and in speed of sound between the altitude of the aircraft and the ground ; secondly, the factor $\Delta\tau$ allows for the convergence of the rays. Letting suffix $_i$ denote the value of a quantity in the standard non-homogeneous atmosphere and suffix $_h$ that in a homogeneous atmosphere in which pressure and speed of sound are geometric means as described in Section 3.4, it follows that

$$\frac{(\Delta p)_i}{(\Delta p)_h} = \frac{(\Delta\tau)_i}{(\Delta\tau)_h} . \quad \text{(I.25)}$$

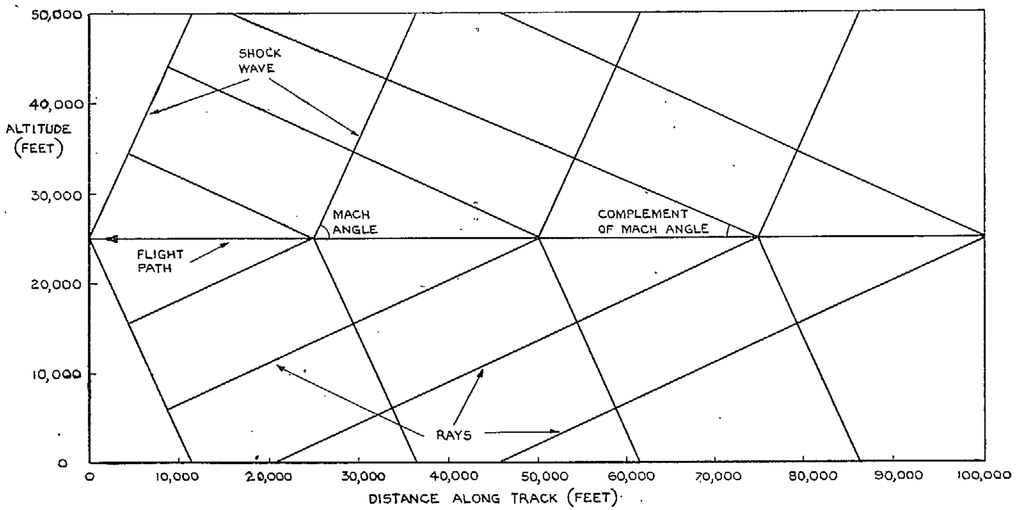

FIG. 2. Motion of the shock wave produced by an aircraft in a homogeneous atmosphere (Straight and steady flight ; $M = 1 \cdot 1$).

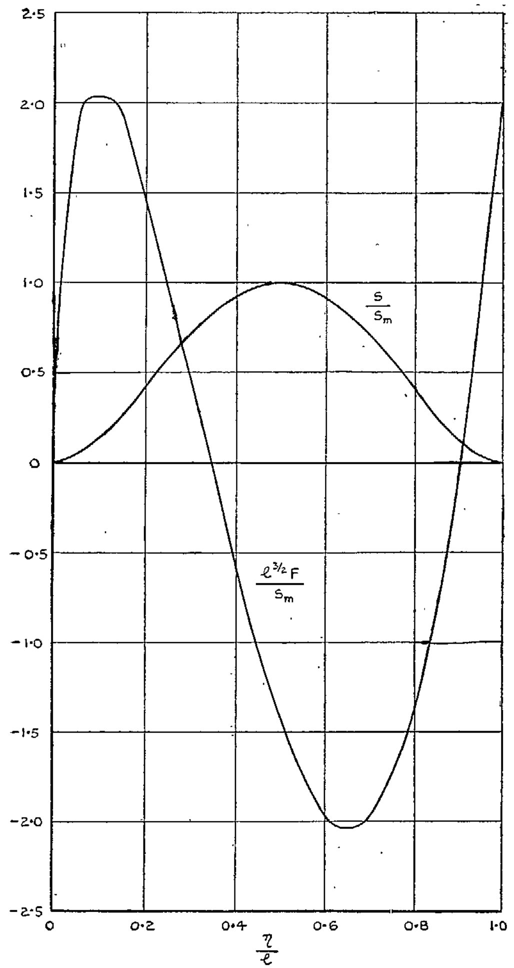

FIG. 3. S/S_m and $l^{3/2}F/S_m$ as functions of η/l.

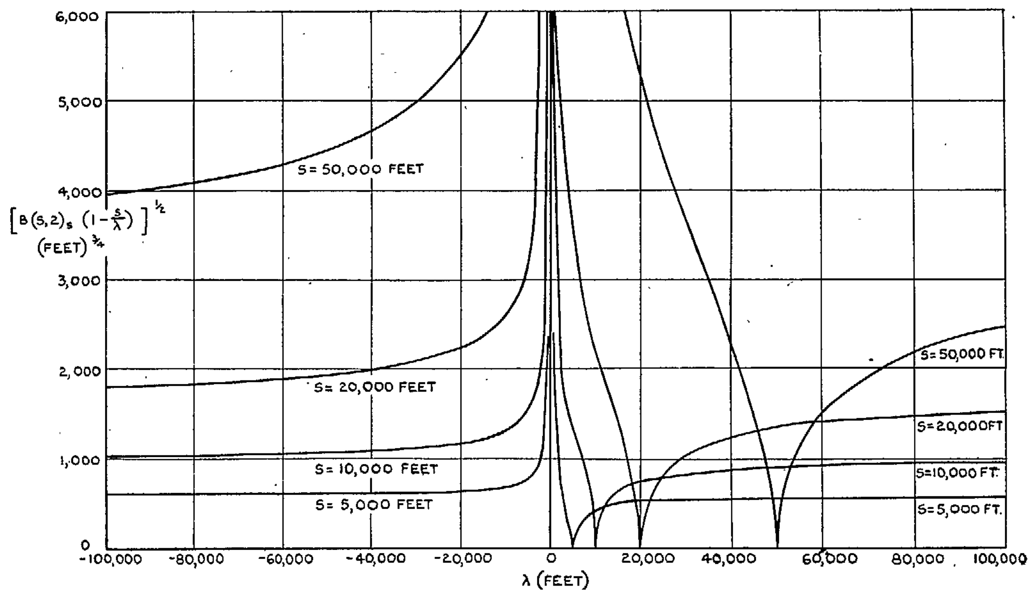

FIG. 4. The function $[Bs\{1 - (s/\lambda)\}]^{1/2}$.

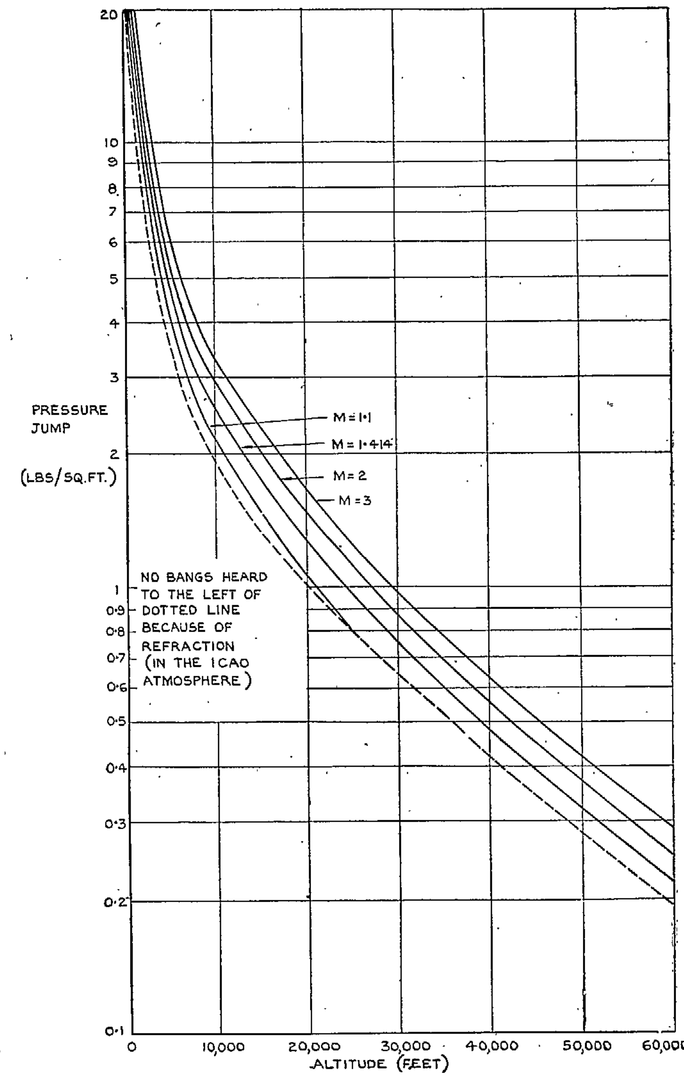

FIG. 5. Pressure jumps produced by an aircraft in steady level flight, along the track.

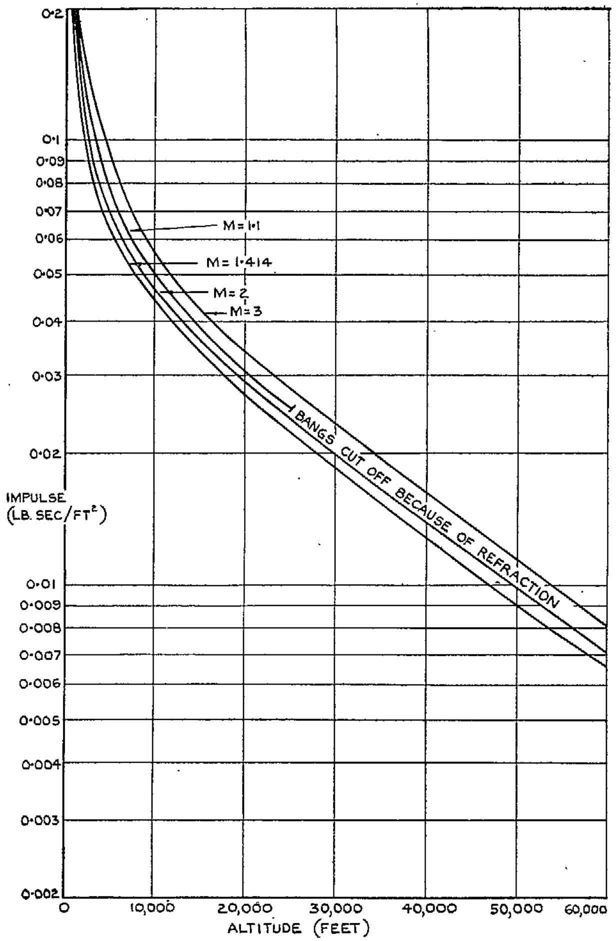

FIG. 6. Impulses produced by an aircraft in steady level flight, along the track.

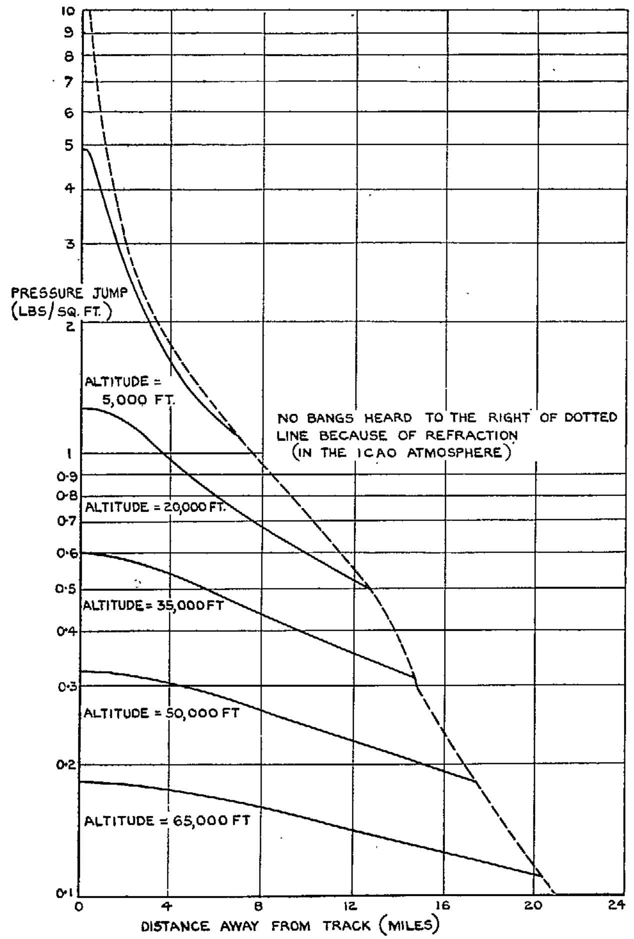

FIG. 7. Pressure jumps produced by an aircraft in steady level flight, off the track. (Mach number = 1·414).

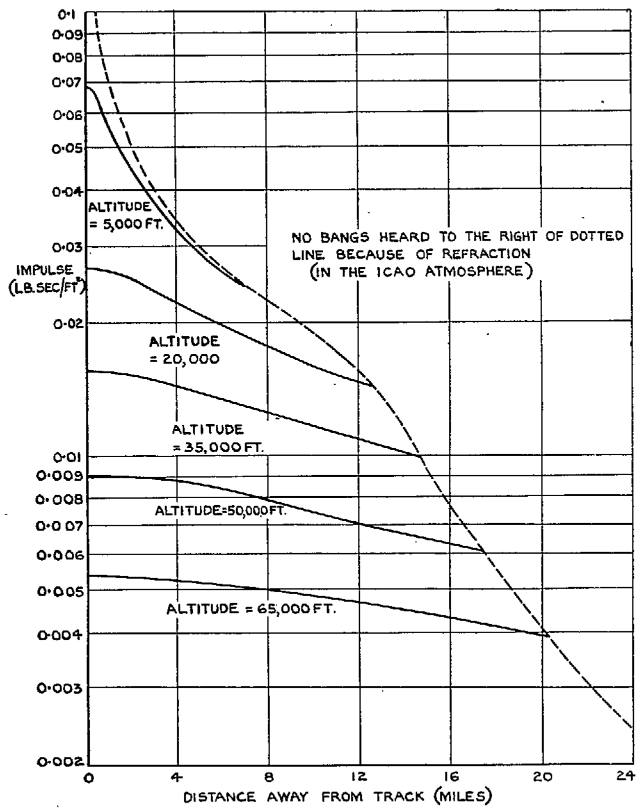

FIG. 8. Impulses produced by an aircraft in steady level flight, off the track. (Mach number = 1·414).

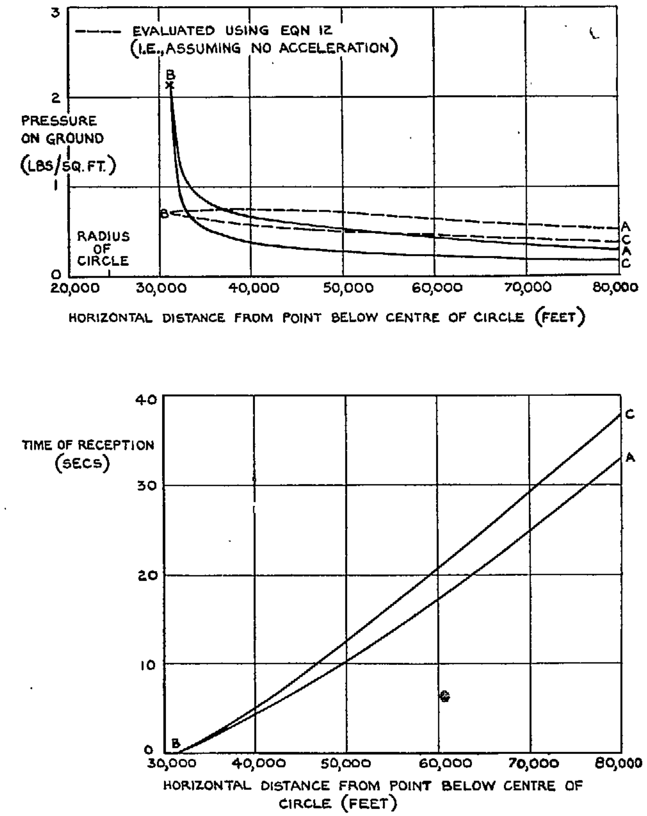

FIG. 9. Pressure jump and time of reception at the ground for an aircraft performing a horizontal circular turn (Height of aircraft = 30,000 ft ; Mach number = 1·5).

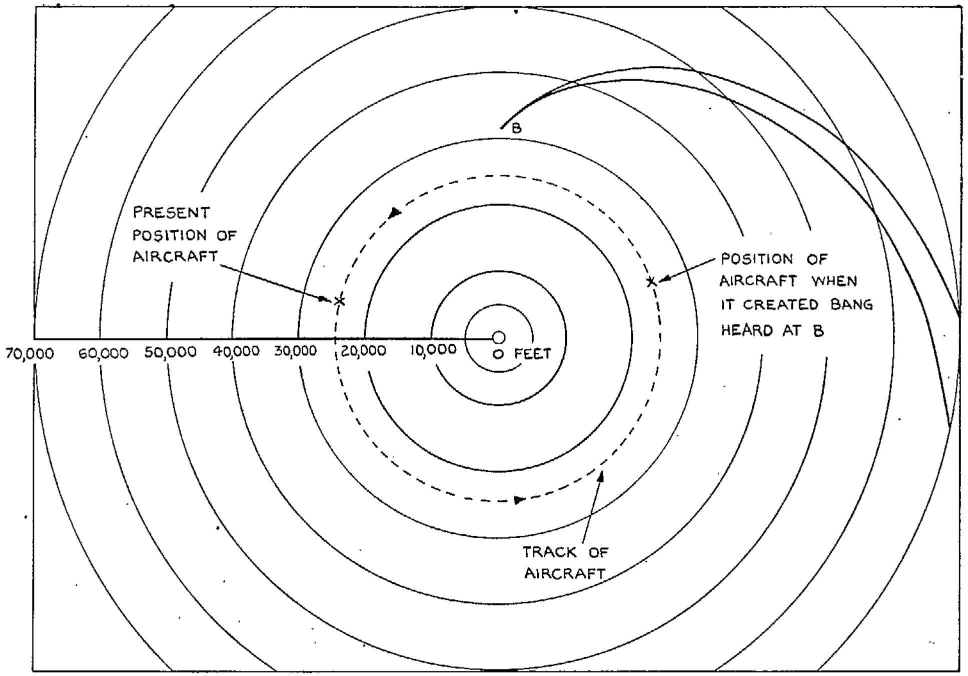

FIG. 10. Shock-wave pattern on ground at a particular instant resulting from the flight path of FIG. 9.

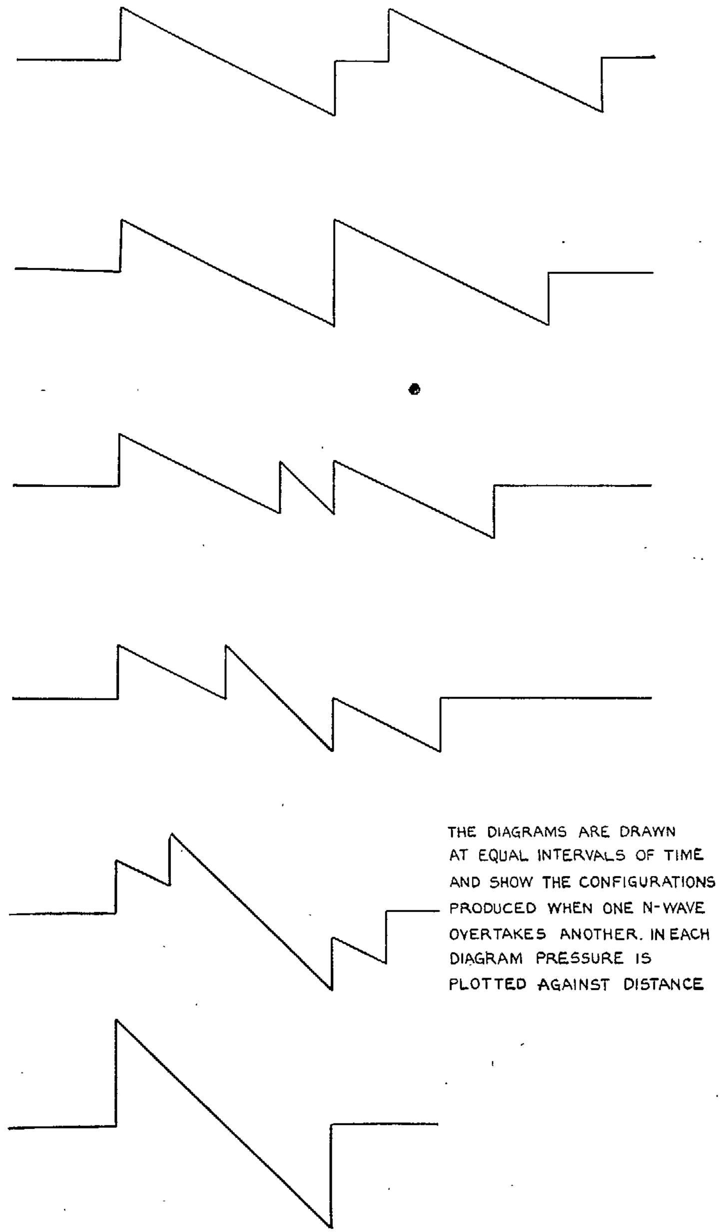

FIG. 11. Superposition of two N-waves.

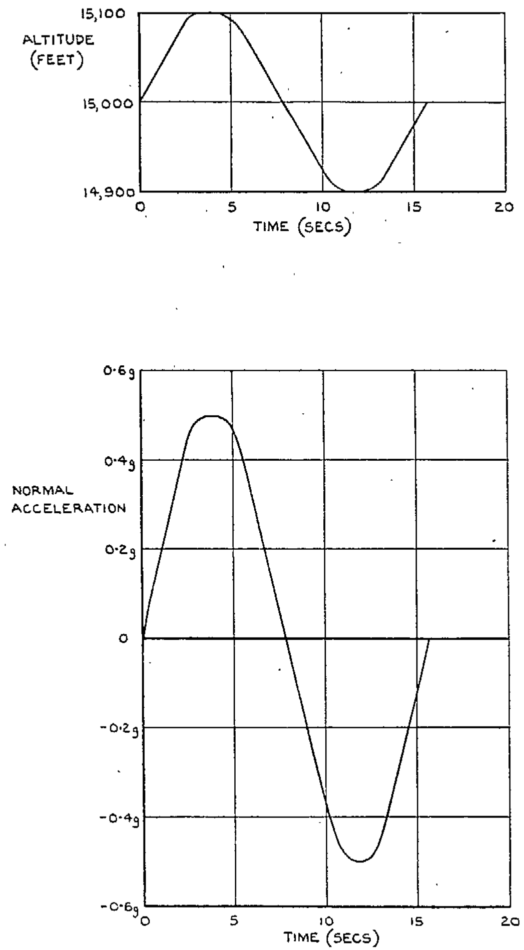

FIG. 12. Altitude and normal acceleration of an aircraft moving in a vertical plane at a Mach number of 1·1.

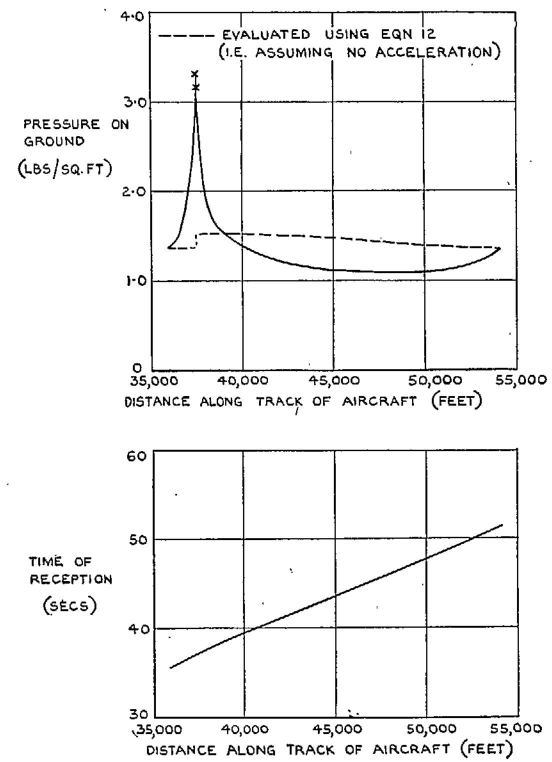

FIG. 13. Pressure jump along track and time of reception of bangs resulting from flight depicted in FIG. 12 (Origin of distance is point on track corresponding to time $t = 0$).

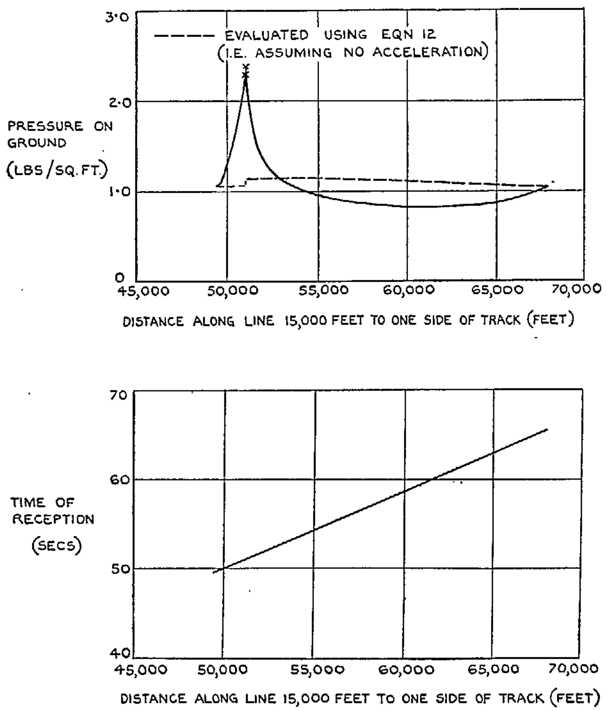

FIG. 14. Pressure jump off track and time of reception of bangs resulting from flight depicted in FIG. 12 (Origin of distance corresponds to that of FIG. 13).

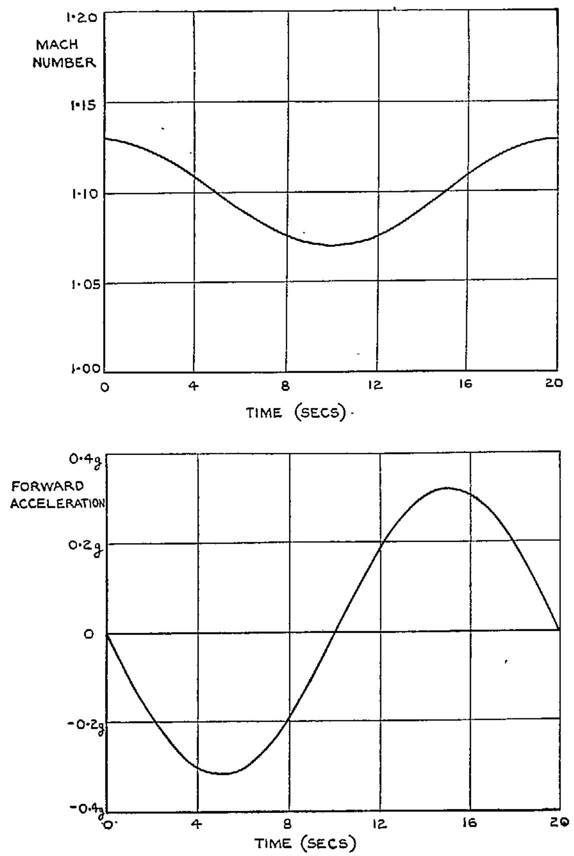

FIG. 15. Mach number and forward acceleration of an aircraft in straight and level flight at an altitude of 10,000 ft.

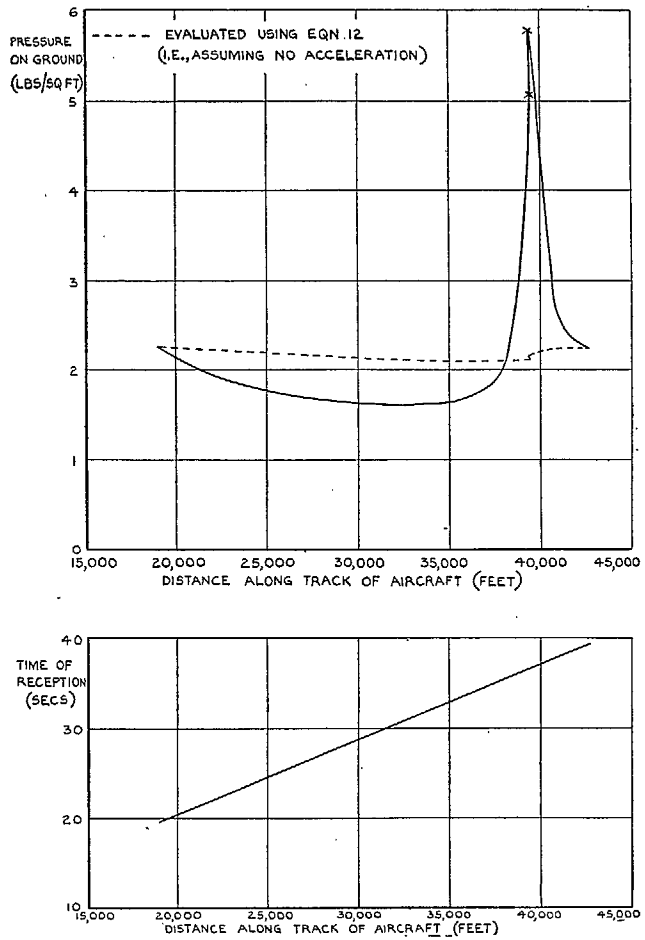

FIG. 16. Pressure jump along track and time of reception of bangs resulting from flight depicted in FIG. 15 (Origin of distance is point on track corresponding to time $t = 0$).

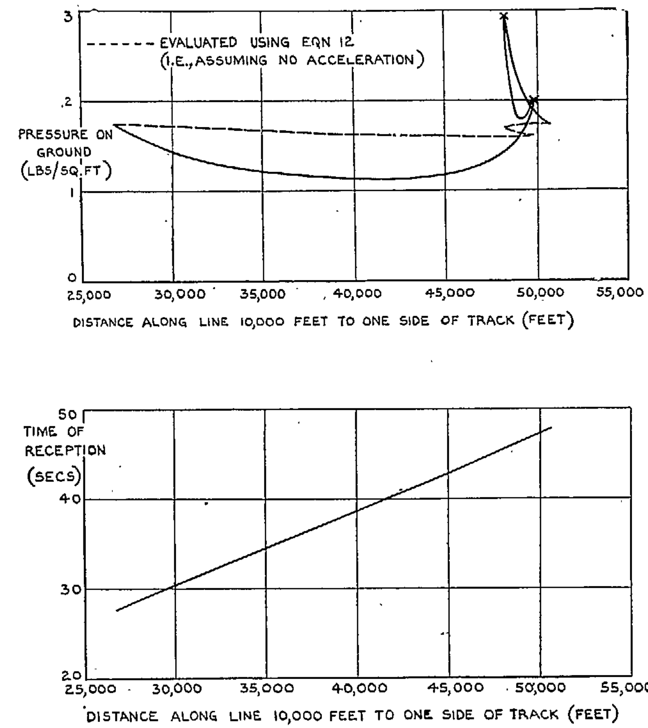

FIG. 17. Pressure jump off track and time of reception of bangs resulting from flight depicted in FIG. 15 (Origin of distance corresponds to that of FIG. 16).

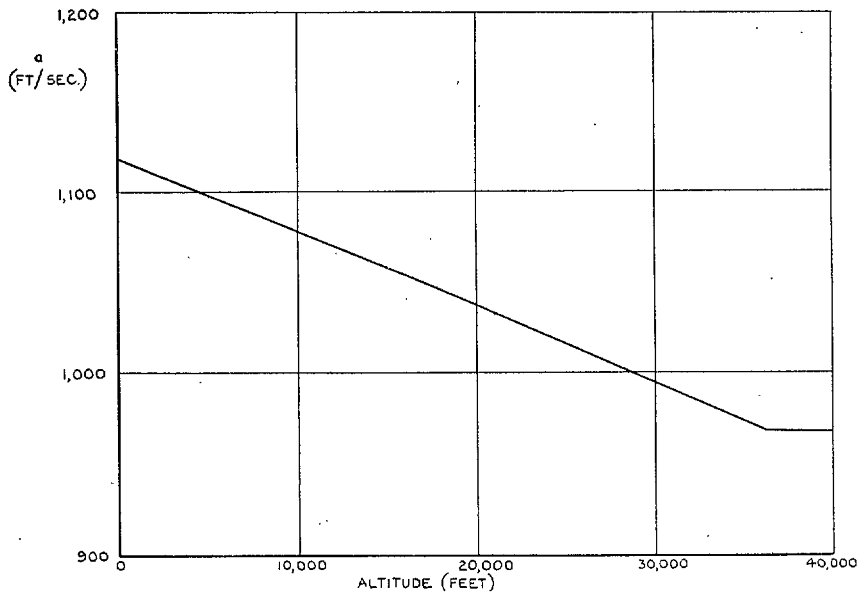

FIG. 18. Variation of speed of sound with altitude in the I.C.A.O. standard atmosphere.

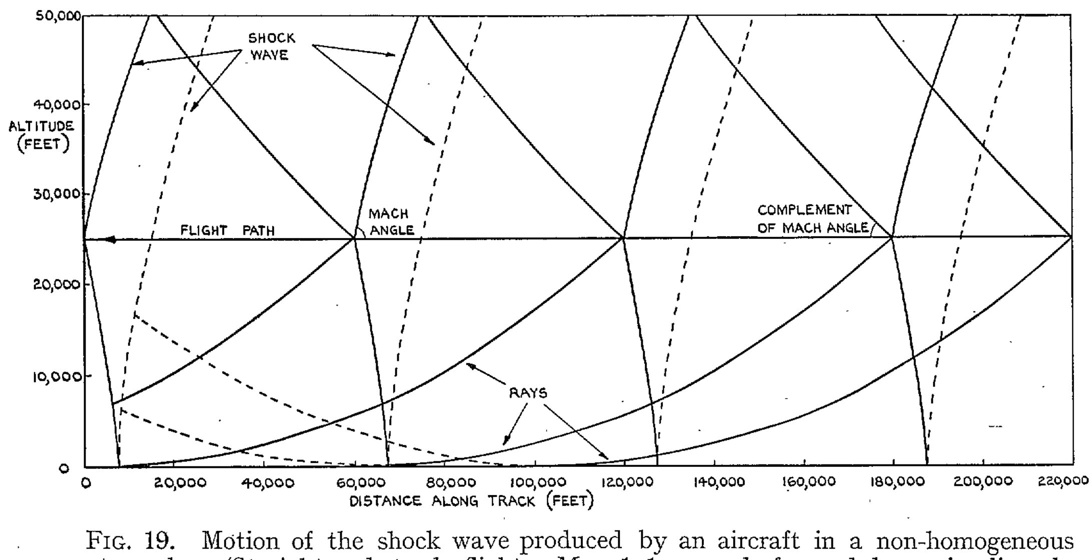

FIG. 19. Motion of the shock wave produced by an aircraft in a non-homogeneous atmosphere (Straight and steady flight ; $M = 1 \cdot 1$; speed of sound decreasing linearly with altitude).

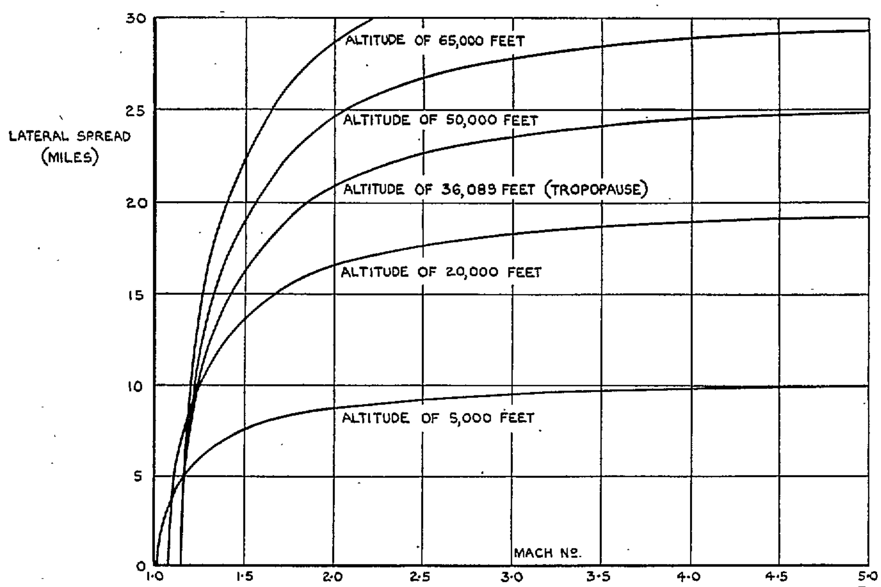

FIG. 20. Lateral spread of sonic bangs produced by an aircraft in steady level flight (Measured from flight track).

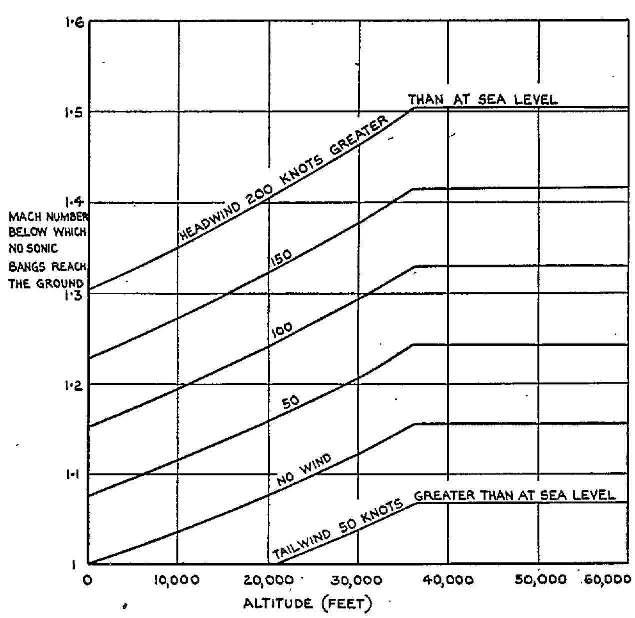

FIG. 21. Effects of refraction and winds on the sonic bangs from an aircraft flying in the I.C.A.O. standard atmosphere.

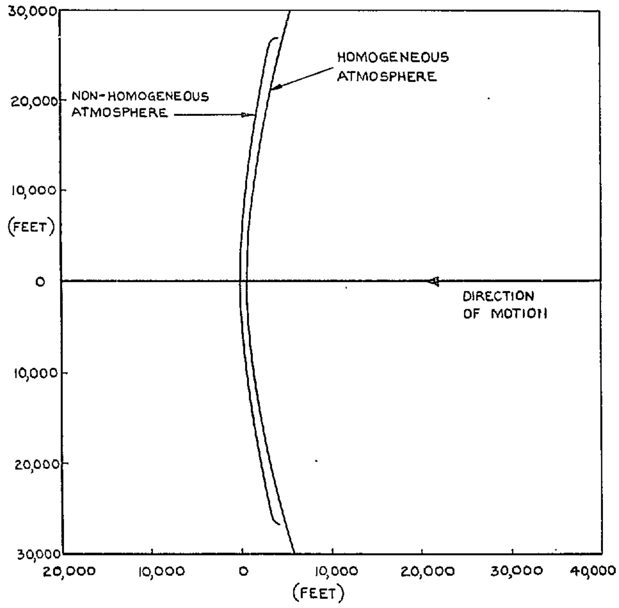

FIG. 22. The effect of refraction on the shock wave at the ground (Aircraft in steady level flight at an altitude of 30,000 ft and at a Mach number of 1·15).

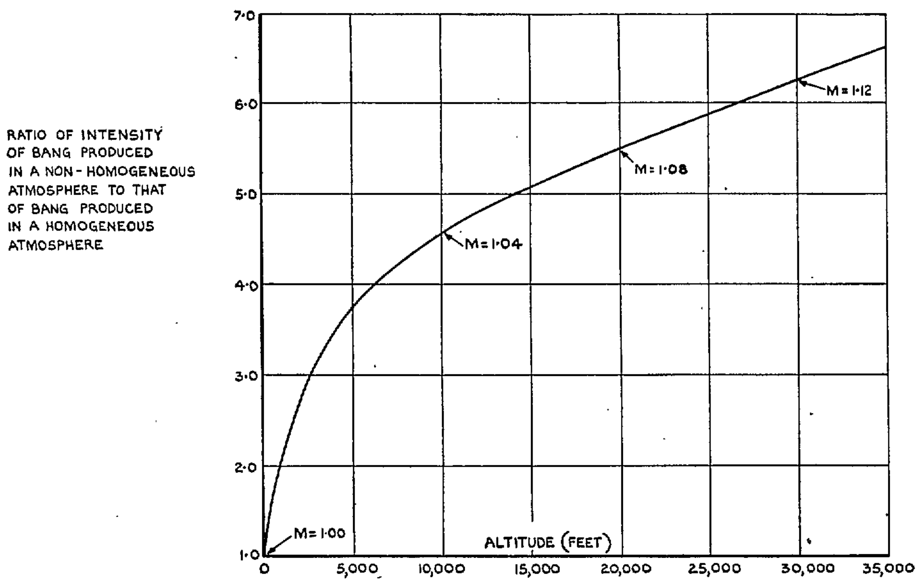

FIG. 23. Increase, due to refraction, in intensity of bangs along the track produced during steady level flight (The bangs are assumed just to reach the ground).

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