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Methods for Determining the Wave Drag of Non-Lifting Wing-Body Combinations

By

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Summary.—The area-rule, moment of area-rule and transfer-rule methods for estimating the wave drag of wing-body combinations are discussed. It is pointed out that the moment of area rule and the transfer rule are different forms of the area rule, and that the transfer rule expresses the interference wave drag in a simple form. The existing methods of wave-drag estimation are restricted to combinations with bodies having continuous surface slope and here an extension to combinations with bodies having discontinuous surface slope is given. This paper is concerned with the theoretical methods and their associated numerical techniques and no numerical results or particular applications are presented.

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1. *Introduction.*—Important problems in the design of transonic and supersonic aircraft are the calculation of the wave drag of the configuration and the design of the fuselage so that the overall wave drag is a minimum for certain specified conditions. Solutions of these two problems can now be found by using the area rule. The justification of the area rule at transonic speeds depends solely upon experiment.

Originally the area rule was applicable at sonic speed only and, in this form, is due to Oswatitsch, Whitcomb[1] and Lord.[2] Whitcomb[1] based his formulation of the sonic area rule upon an experimentally verified equivalence between wings and bodies at sonic speed; it was shown that there does exist a marked similarity between the shock wave patterns around a wing-body combination and that body of revolution which has the same cross-sectional area distribution. This result is clearly of a non-linearised nature since shock waves are considered. A different form of the sonic area rule is due to Lord[2] who used the supersonic linearised theory to derive an expression for the wave drag; thus quantitative theoretical comparisons can be made between the sonic wave drags of different configurations. The design of optimum wing-body combinations with minimum sonic wave drag has been examined by Lord[3, 4] while the calculation of the sonic wave drag has been simplified remarkably by Eminton and Lord[4].

The extension of the area rule to supersonic speeds is due to Jones[5] and Whitcomb, who have made the implicit assumption that the effect of the interference velocity potential on the wave drag is negligible. Several aspects of this extension have been examined by Lord[6] and Warren[7]. An important presentation of the mathematical basis of the area rule has been given by Ward[8, 9, 10], who also suggested an alternative form, which is described as the transfer rule. A special case of the supersonic area rule, called the moment-of-area rule, was given by Baldwin and Dickey[11], whose result has been obtained independently in a different form. It is interesting to note that the supersonic area rule could be deduced directly from a wing drag result obtained by Hayes[13] in 1947.

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A comprehensive summary of American work on the wave drag of wing-body combinations has been given recently by Lomax and Heaslet[13]. This paper[13] and the references cited therein contain important theoretical analyses of the wave drag of configurations under conditions for which the area rule is not applicable. The area rule applies to combinations of non-lifting thin wings and slender bodies, which can be represented solely by surface source distributions with the source strengths proportional to the local surface slopes in the streamwise direction. In certain circumstances these source distributions may be discontinuous and it appears that, in so far as the use of the area rule for the estimation of wave drag is concerned, discontinuities in source distributions give rise to difficulties only when they occur along straight lines inclined such that the component of free-stream velocity normal to the line is supersonic (or sonic), in the case of a wing, or when they occur around the circumference of a cross-section of a body. Such discontinuous source distributions are not discussed by Lomax and Heaslet[13] and so some consideration is given to them in this report. The cases of lifting configurations, which require the introduction of surface doublet distributions, and combinations with non-slender bodies, which can be treated by using axial distributions of multipoles as well as simple sources, are discussed in detail by Lomax and Heaslet[13] and are not considered here. Further information on configurations incorporating non-slender bodies is given in Ref. 14.

This present report is concerned only with theoretical methods and their associated numerical techniques and no numerical results or particular applications are given. In Section 2 a summary is presented of the theoretical methods for estimating the wave drag of configurations to which the area rule does apply, namely combinations of wings and bodies with continuous source representations, and the problem of body design for low total wave drag is also discussed; the sonic area rule, supersonic area rule, moment of area rule and transfer rule are treated separately. In Section 3 extensions are given of the area and transfer rules to configurations incorporating bodies represented by discontinuous source distributions. The appropriate numerical techniques are described in Section 4. Since this report is concerned with idealised combinations of wings and bodies, Section 5 contains some remarks of a more practical nature and it is indicated how the methods developed for wing-body combinations may be applied to aircraft. Finally, the main conclusions of the report are given in Section 6.

2. *Combinations with Bodies Represented by Continuous Source Distributions.*—2.1. *Sonic Area Rule.*—Before preceeding with a description of the area rule and its alternative forms a statement of the sonic area rule will be given, since this is a simple special case of the general area rule and thus provides a useful introduction to it.

2.1.1. *Estimation of wave drag.*—Consider a non-lifting configuration of unit length and let $S(x)$, $0 \leqslant x \leqslant 1$, be the axial distribution of cross-sectional area. The area distribution $S(x)$ must be such that $S'(x)$ $(= dS/dx)$ is continuous and $S'(0) = 0 = S'(1)$. Then the sonic wave drag D is given by

$$\frac{D}{q} = \frac{1}{2\pi}\int_0^1\int_0^1 S''(x_1)S''(x_2)\log\left(\frac{1}{|x_1 - x_2|}\right)dx_1\,dx_2\,, \qquad (1)$$

where q is the kinetic pressure and $S''(x)$ denotes d^2S/dx^2. An alternative expression for the sonic wave drag when $S''(x)$ is continuous is given by Legendre[15] in the form

$$\frac{D}{q} = \frac{1}{2\pi}\int_0^1 \frac{[S'(x_1)]^2}{x_1(1 - x_1)}\,dx_1 + \frac{1}{4\pi}\int_0^1\int_0^1\left[\frac{S'(x_1) - S'(x_2)}{x_1 - x_2}\right]^2 dx_1\,dx_2\,.$$

This form has the advantage that it is non-singular, introduces only $S'(x)$ and shows that the drag is positive. A more general form of this result when $S''(x)$ is not continuous is more complex, although equation (1) remains valid as it stands.

The development of the area rule began after the work of Whitcomb[1] who obtained experimental verification of the importance of the cross-sectional area distribution at transonic speeds. However, the physical interpretation of the sonic wave drag given by equation (1) requires further elucidation and must depend upon experiment unless an adequate non-linear theory, which is applicable to the transonic range, can be found. Following the suggestions of Lord[3], the sonic wave drag of a wing-body combination may be interpreted physically as a measure of the sudden transonic drag rise of the configuration, when such a drag rise can be defined.

It is worthy of note that the usefulness of equation (1) is greatly enhanced by the fact that the linearised theory predicts zero lift-dependent wave drag at $M = 1$. Thus equation (1) is not restricted to non-lifting configurations since it gives the total sonic wave drag of a lifting configuration (for a discussion of lift-dependent wave drag, *see* for example, Lomax and Heaslet[13]).

2.12. *Design for low wave drag.*—The design of a wing-body combination with low sonic wave drag requires the determination of optimum area distributions whose associated wave drag, given by equation (1), is a minimum. Lord and Eminton have investigated this problem and presented optimum area distributions for a number of specified conditions. Thus in order to obtain low sonic wave drag it is necessary to alter the body, and possibly the wing also, so that the cross-sectional area distribution becomes an optimum, or near optimum. A simple example of an optimum area distribution is that of a so-called Sears-Haack body, which is pointed at both ends and has minimum wave drag for given length and volume.

2.2. *Area Rule.*—2.2.1. *Estimation of wave drag.*—The area-rule method of estimating wave drag is due to Jones[5] and Whitcomb. However, the derivation of the area rule requires the use of an assumption even though the wing-body combination is represented by source distributions only. The assumption is that the effect of the interference velocity potential on the wave drag is negligible, *i.e.*, the perturbation velocity potential of a wing-body combination is assumed to be equal to the perturbation velocity potential of the isolated exposed wing together with the perturbation velocity potential of the isolated body.

It is convenient to define an elemental area distribution. Consider the family of parallel Mach planes inclined at the Mach angle to the body axis and such that the orthogonal plane containing the body axis is inclined at the angle θ to the ' plane of the wing '. A given family of Mach planes cuts the wing-body combination obliquely and thus defines an oblique sectional area distribution $s(x, \theta, M)$, where M is the Mach number under consideration. The cross-sectional area distribution $S(x, \theta, M) = s(x, \theta, M) \sin \mu$, where μ is the Mach angle, is defined to be an elemental area distribution. Its associated wave drag is given by equation (1) as

$$\frac{D(\theta, M)}{q} = \frac{1}{2\pi} \int\int S''(x_1, \theta, M)\, S''(x_2, \theta, M) \log \left(\frac{1}{|x_1 - x_2|}\right) dx_1\, dx_2 \,; \qquad (2)$$

where $S''(x, \theta, M)$ denotes $\partial^2 S(x, \theta, M)/\partial x^2$ and it is assumed that $S'(x, \theta, M)$ is continuous everywhere. When $M = 1$ it is seen that all the elemental area distributions are the same, being equal to the cross-sectional area distribution.

Hayes[12] and Heaslet, Lomax and Spreiter[16] obtained a result which applies to wings only but its application to wing-body combinations follows immediately from the assumption that the effect of the interference velocity potential on the wave drag is negligible*. The wave drag of the configuration therefore can be expressed in the form

$$D = \frac{1}{2\pi} \int_0^{2\pi} D(\theta, M)\, d\theta \,, \qquad (3)$$

i.e., the wave drag of a wing-body combination at a Mach number M is the mean of the wave drags associated with all the elemental area distributions. This is a complete statement of the

* In a private communication L. E. Fraenkel has shown that this assumption is valid for small values of the ratio of body radius to wing chord. This condition may be derived by considering a configuration with a circular cylindrical body and then observing that the important parameter of an interference flow field is BR_t/c where $B = \sqrt{(M^2 - 1)}$, R_t is a typical body radius and c is the wing-root chord. Thus the interference velocity potential can be neglected for small values of BR_t/c because its effect vanishes when $R_t = 0$ and when $B = 0$ (i.e., $M = 1$). When the above condition is not satisfied the error introduced by the neglect of the interference velocity potential is not known. However, it is possible that the error will not be large because the interference velocity potential can be neglected also when the ratio BR_t/c is large, any interference effects vanishing when $c = 0$ or $B = \infty$.

area rule originally given by Jones[5]. Ward[8] has derived the same result by using operational methods, while Graham, Beane and Licher[17] have also given this result. When $M = 1$, equations (2) and (3) reduce to the sonic area-rule result, equation (1).

It is important to note that the derivation of the area rule (equation (3)), required all the elemental area distributions to be such that $S'(x, \theta, M)$ is continuous everywhere. Nevertheless, provided that almost all the elemental area distributions satisfy this requirement the wave drag may still be given by equation (3). This condition is much less restrictive than the original one*.

An elemental area distribution of a thin-wing–slender-body combination can be written in the approximate form $S(x, \theta, M) = S(x) + S_W(x, \theta, M)$, where $S(x)$ is the cross-sectional area distribution of the body and $S_W(x, \theta, M)$ is the contribution from the exposed wing. The wing is taken to lie in the plane $z = 0$ (*see* Fig. 1) and its thickness at the point (x, y) is denoted by $T(x, y)$.

Consider any Mach plane cutting the axis at the point $x = x_1$. Then this Mach plane cuts the wing obliquely along an area which, when projected on to the plane $x = x_1$, defines the area $S_W(x, \theta, M)$. This projection may be regarded as a projection on to a plane normal to the wing followed by a projection on to the plane $x = x_1$. The former projection defines an area equal to the wing thickness integrated along the line in which the Mach plane cuts the wing, *i.e.*, $x - x_1 = y \tan \nu$ where $\tan \nu = \cot \mu \cos \theta$. It follows that

$$S_W(x, \theta, M) = \int_{T \neq 0} T(x + y_1 \tan \nu, y_1) \, dy_1 \, .$$

Heaslet, Lomax and Spreiter[16] obtained this result in an equivalent form.

The above result applies to a wing lying in the plane $z = 0$. However, its extension to a more general wing is not difficult and follows immediately from the definition of an elemental area distribution. The general result, which includes the previous result as a special case, can be written in the form

$$S_W(x, \theta, M) = \int_{T \neq 0} T(\bar{\mathbf{R}}) \, d\mathbf{r} \, ,$$

where an element of the wing plan-form is denoted by $dx \, d\mathbf{r}$, a general point in the co-ordinate system of Fig. 1 is denoted by $\mathbf{R} = (x, \mathbf{r})$, $\bar{\mathbf{R}} = (x + B\mathbf{r}_0 \, \mathbf{r}, \mathbf{r})$, where $B = \cot \mu$, $\mathbf{r}_0$ is the unit vector $(\cos \theta, \sin \theta)$ and θ is defined by $y = r \cos \theta$, $z = r \sin \theta$.

2.2.2. *Design for low wave drag.*—The problem of reducing the supersonic wave drag of a wing-body combination with a given wing was examined first by Jones[5]. At sonic speed, a body may be modified so that the equivalent body, which has the same cross-sectional area distribution as the wing-body combination, is an optimum with the same total volume, *i.e.*, the modified body is formed by shaping the optimum equivalent body in the region of the wing. At higher speeds the wing may be influenced by, and may itself influence, a larger part of the body. Thus the corresponding optimum equivalent body shaping extends outside the region of the wing and may be shown to be over that part of the body within the Mach diamond of the wing. Details of this drag-reduction procedure are given in section 2.4.2. where the transfer rule is discussed.

Another problem of interest is that of reducing the wave drag of a general configuration and this has been discussed by Jones[5] in a qualitative form. All the elemental area distributions should be such that their associated wave drags are as low as possible. The design of combinations

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* Since this report was completed, Lock[18] has shown that the wave drag of a rectangular wing can be found using the area rule (equations (2), (3)). Thus the smoothness condition that $S'(x, \theta, M)$ be continuous everywhere appears to be unnecessary for the wing of a configuration.

satisfying this requirement appears to be very difficult except in certain special cases; for example, at low supersonic speeds only a small number of elemental area distributions are required and so an iteration method can be used to obtain a modified configuration with low wave drag.

In the case of an isolated wing it is interesting to use the area rule to determine the optimum thickness distribution for minimum wave drag, the plan-form and volume of the wing being given. A particular case of this problem was first examined by Jones[19] who showed that if the plan-form is elliptic then the wing section must be parabolic-arc biconvex with thickness/chord ratio proportional to local chord. However, Graham, Beane and Licher[17] have used the area rule to obtain the same result and have shown that all the elemental area distributions represent Sears-Haack bodies (*i.e.*, bodies of minimum wave drag for given length and volume.) The result for the wave-drag coefficient of the optimum wing of elliptic plan-form is

$$C_D = 2\pi A\tau_0^2 \left(1 + \frac{\pi^2 A^2 B^2}{32}\right) \Big/ \left(1 + \frac{\pi^2 A^2 B^2}{16}\right)^{3/2}$$

where C_D is based upon the wing plan-form area

A is the aspect ratio of the wing

τ_0 is the thickness/chord ratio at the wing root.

Also $\tau_0 = (32V)/(\pi^2 Ac^3)$, where V is the volume of the wing and c is the wing-root chord.

2.3. *Moment of Area Rule*.—2.3.1. *Estimation of wave drag*.—The moment of area-rule method is due to Baldwin and Dickey[11], who expanded the wave drag in a series of powers of $B^2 = (M^2 - 1)$. The results presented later in this section are similar and were obtained independently of Baldwin and Dickey[11] whose results will be described first.

The first step towards a series solution in ascending powers of B^2 was made by Adams and Sears[20] who developed a 'not-so-slender' wing theory by obtaining an expansion up to the first power of B^2, the term independent of B^2 being the slender-wing solution. These results emphasise the fact that the series solution is actually an expansion in ascending powers of B^2k^2 where k is the slenderness parameter introduced by Adams and Sears[20]. For example, in some wing problems k may be defined as the aspect ratio.

Before proceeding with a description of the moment of area rule developed by Baldwin and Dickey[11], it is necessary to obtain an alternative form of equation (1). Using the substitution $x = \frac{1}{2}(1 - \cos\varphi)$, $0 \leqslant x \leqslant 1$, $S'(x)$ is expressed as a Fourier sine series in the new variable φ; namely

$$S'(x) = \sum_{n=1}^{\infty} A_n \sin n\varphi, \qquad 0 \leqslant \varphi \leqslant \pi .$$

Substituting this series in equation (1) shows that the wave drag is given by

$$\frac{D}{q} = \frac{\pi}{4} \sum_{n=1}^{\infty} nA_n^2 .$$

Therefore, using similar substitutions equation (3) can be written as

$$\frac{D}{q} = \frac{1}{8} \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} nA_n^2(\theta, M) \right\} d\theta ,$$

where $A_n(\theta. M)$ is the Fourier coefficient appropriate to the elemental area distribution $S(x, \theta, M)$. Baldwin and Dickey[11] expanded the Fourier coefficient $A_n(\theta, M)$ in a series of powers of

$B^2 = (M^2 - 1)$ and it is found that, within the usual order of approximation, the wave drag of a configuration with the wing in the plane $z = 0$ (*see* Fig. 1) depends solely upon the moments of area defined by

$$M_0(x) = S(x) + \int_{T \neq 0} T(x, y)\, dy$$

and

$$M_i(x) = \int_{T \neq 0} T(x, y)\, y^i\, dy, \qquad i \text{ even}, i \geqslant 2\,,$$

where $S(x)$ is the cross-sectional area distribution of the body and, in the notation of Fig. 1, $T(x, y)$ is the wing thickness at the point (x, y). Clearly, $M_0(x)$ is equal to the cross-sectional area distribution of the wing-body combination and $M_2(x)$ is equal to the distribution of the cross-sectional polar moment of inertia of the wing.

The final result for the wave drag may be written in the form $D/q = a_0 + a_2 B^2 + 0(B^4)$, where a_0 depends upon $M_0(x)$ and a_2 depends upon both $M_0(x)$ and $M_2(x)$. The higher terms in this series are more complex and so, as Baldwin and Dickey[11] have stated, the method is simple to apply only at low supersonic speeds. Nevertheless, within this speed range it does offer a simple qualitative method for reducing wave drag; a useful requirement is that the moment of area distributions be as smooth and 'slender' as possible. It is suggested that, in general, it will be sufficient to consider only $M_0(x)$ and $M_2(x)$.

Beginning from equatons (2) and (3) it has been shown that, for a configuration of unit length, the wave drag is given by

$$\begin{aligned}\frac{D}{q} = &-\frac{1}{2\pi}\int_0^1\int_0^1 S''(x_1)S''(x_2)\log|x_1 - x_2|\, dx_1\, dx_2 \\ &-\frac{B^2}{2\pi}\int_0^1\int_0^1\Big[\tfrac{1}{2}S''(x_1)\frac{\partial^4}{\partial x_2{}^4}\{\mathscr{C}_{2,0}(x_2)\} + \tfrac{1}{2}\frac{\partial^3}{\partial x_1{}^3}\{\mathscr{C}_{1,1}(x_1)\}\frac{\partial^3}{\partial x_2{}^3}\{\mathscr{C}_{1,1}(x_2)\} \\ &+ \tfrac{1}{2}\frac{\partial^3}{\partial x_1{}^3}\{\mathscr{S}_{1,1}(x_1)\}\frac{\partial^3}{\partial x_2{}^3}\{\mathscr{S}_{1,1}(x_2)\}\Big]\log|x_1 - x_2|\, dx_1\, dx_2 + O(B^4)\,, \quad \ldots \quad \ldots \quad (4)\end{aligned}$$

where $\mathscr{C}_{i,\lambda}(x)$, $\mathscr{S}_{i,\lambda}(x)$ are generalised moments of area defined by

$$\mathscr{C}_{i,\lambda}(x) = \frac{1}{i+2}\int_0^{2\pi} R^{i+2}(x, \theta)\cos\lambda\theta\, d\theta\,,$$

$$\mathscr{S}_{i,\lambda}(x) = \frac{1}{i+2}\int_0^{2\pi} R^{i+2}(x, \theta)\sin\lambda\theta\, d\theta\,,$$

where $r = R(x, \theta)$ is the equation of the cross-section at the station x. It is assumed that the series in equation (4) is convergent and that all necessary continuity and differentiability conditions are satisfied. Furthermore, since the body is slender it is important to note that the expressions for the generalised moments of area can be simplified. Within the usual order of approximation, the contribution of the body can be neglected for all $\mathscr{C}_{i,\lambda}$, $\mathscr{S}_{i,\lambda}(i \neq 0)$. Thus in equation (4) the body contributes to the wave drag only in so far as does $S(x)$, the cross-sectional area distribution of the wing-body combination. For a wing symmetrical about the planes $\theta = 0, \pi/2$ (*see* Fig. 1) the generalised moments of area can be written in the simpler forms

$$\mathscr{C}_{i,\lambda}(x) = 0\,, \qquad \lambda \text{ odd},$$

$$\mathscr{S}_{i,\lambda}(x) = 0\,, \qquad \text{all } \lambda,$$

and since $\mathscr{C}_{i,0}(x)$ is the ith moment of area, it follows that $\mathscr{C}_{i,0}(x) = M_i(x)$.

Thus equation (4) becomes

$$D = D\{S\} + \frac{B^2}{4}[D\{S + M_2''\} - D\{S\} - D\{M_2''\}] + O(B^4) , \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where $D\{S\}$, $D\{M_2''\}$, $D\{S + M_2''\}$ denote the wave drag of 'bodies' with the cross-sectional 'area' distributions $S(x)$, $M_2''(x)$, $S(x) + M_2''(x)$ respectively and it has been assumed that $M_2'''(x)$ is continuous everywhere. At first sight this condition seems to limit very seriously the applicability of equation (5) but it is not unduly restrictive since, for example, it is satisfied by an unswept, tapered wing with straight edges and biconvex section of constant thickness ratio. Moreover, the condition is satisfied by the optimum second-moment distribution given in Section 2.3.2, although it is not satisfied by the Jones elliptic wing discussed in Section 2.2.2. When the wing is defined numerically, equation (5) has the serious disadvantage that the wave drag cannot be determined very accurately since it is necessary to carry out two successive numerical differentiations in order to calculate $M_2''(x)$.

The importance of equation (5) lies in the fact that it reduces the calculation of the wave drag of a 'not-so-slender' configuration to the calculation of the wave drag of three 'bodies'. Thus numerical evaluation of the wave drag is considerably simplified and can be carried out using the methods of Section 4.2. However, it should be emphasised that both $S'(x)$ and $M_2'''(x)$ must be continuous everywhere before equation (5) can be used. Therefore the Fourier-series method described earlier in this Section may be used, the substitutions being

$$x = \tfrac{1}{2}(1 - \cos\varphi) , \qquad 0 \leqslant x \leqslant 1 ,$$

$$S'(x) = \sum_{n=1}^{\infty} A_n \sin n\varphi$$

and

$$M_2'''(x) = \sum_{n=1}^{\infty} B_n \sin n\varphi .$$

Equation (5) now becomes:

$$\frac{D}{q} = \frac{4}{\pi}\sum_{n=1}^{\infty} nA_n^2 + \frac{2B^2}{\pi}\sum_{n=1}^{\infty} nA_nB_n + O(B^4) .$$

It will be noticed that only the coefficients B_n for which the corresponding coefficient A_n is non-zero appear in the wave drag. Thus this form of the wave drag is useful when most of the A_n vanish. It is especially useful when the area distribution $S(x)$ is an optimum distribution with a finite Fourier series. For example, the Sears-Haack body has $A_n = 0$, $n \neq 2$.

2.3.2. *Design for low wave drag.*—It has already been emphasised that the main usefulness of the moment of area rule is as a qualitative design procedure for low supersonic speeds. Baldwin and Dickey[11] have examined the use of the method in designing configurations with minimum wave drag under specified conditions. The procedure used is a step-by-step optimisation technique whereby successive terms in the series solution are optimised one after the other. Thus a unique set of moment distributions is found.

The optimum set of moment distributions for minimum wave drag, the lengths and the 'volumes' being fixed, is given by Baldwin and Dickey[11]. In this context the 'volume" is the pth moment 'volume'* of the configuration defined by

$$V_p = \int_0^{l_p} M_p(x)\, dx ,$$

* The pth moment 'volume' does not have any physical significance unless $p = 0$ when it is the actual volume of the configuration.

where l_p is the length of the pth moment distribution, i.e., $l_0 = l$, the length of the combination and, for $p \geqslant 2$, $l_p = l_2$, the length of the projection of the wing on the body axis. The result is that the optimum moment distributions which vanish at $x = 0$, l_p, are

$$M_p(x) = M_p(l_p/2)\left[1 - \left(\frac{x - l_p/2}{l_p/2}\right)^2\right]^{p+3/2} \qquad p \text{ even, } 0 \leqslant x \leqslant l_p$$

and that the wave drag of the optimum configuration is given by

$$\frac{D}{q} = \frac{\pi}{4}\left\{18\left[\frac{M_0(l/2)}{l}\right]^2 + 16537 \cdot 5\left[\frac{M_2(l_2/2)}{l_2^3}\right]^2 B^4\right\} + O(B^8) \,.$$

It will be noticed that no term in B^2 appears in this result.

Since all the terms in the above expression for the wave drag are positive, it has been pointed out by Baldwin and Dickey[11] that the wave drag of a configuration, which is optimum for their conditions, must increase monotonically with increasing Mach number and be very large when the second term is very much greater than the first term, i.e.,

$$B^2 \to O\left(\frac{M_0(l/2)l_2^3}{M_2(l_2/2)l}\right) .$$

Thus the Mach number indicated by this relation can be used to find an obvious upper limit to the range of applicability of the optimum technique. This emphasises once again the restriction of the method to low supersonic speeds and therefore that it is necessary to consider only the first few terms of the series solution. Baldwin and Dickey[11] have pointed out that the step-by-step optimisation technique fails at higher supersonic speeds because it eliminates all except positive definite terms in the expression for the wave drag. It appears that a more realistic optimisation procedure, based on equation (5), will be obtained if the term in B^2 is negative instead of zero as in the above procedure. The fact that the term in B^2 can be negative is shown readily by considering an unswept wing, such as the Jones elliptic wing discussed in Section 2.2.2.

Experimental confirmation of the usefulness of the moment of area rule for wing-body combinations has been given by Baldwin and Dickey[11]. Firstly, the optimum area distribution was found and then an optimum second moment distribution was obtained by adding bodies of revolution near the wing tips while modifying the body to retain the optimum area distribution. The tests involved an unmodified configuration, one with the optimum area distribution as well as one with both the optimum area distributions and, it will be noted, an optimum, lengthened second moment distribution. By considering the difference between the wave drag of the wing–body combination and the wave drag of the body alone the agreement between theory and experiment was found to be good over the Mach-number range $M = 1 \cdot 0$ to $M = 1 \cdot 4$, for an elliptic wing of aspect ratio 2. At $M = 1 \cdot 4$ the moment of area-rule model had a slightly lower wave drag than the unmodified configuration. Thus the sonic area rule is extended to low supersonic speeds by using the moment of area rule. However, because l_2 has been increased without changing $M_2(0)$, these experimental results do not justify anything more than the use of smooth, ‘ slender ’ second moment distributions.

It will be noted that the above technique requires the addition of bodies to the wing. This is in contrast to the method described in Section 2.2.2, whereby the main body only was shaped for minimum wave drag. Thus it would be desirable to examine combinations with subsidiary bodies near the wing tips by using the methods applicable at any supersonic speed, namely, the area rule and the transfer rule. The problem of shaping such subsidiary bodies is examined in Section 5.2.2.

2.4. *Transfer Rule*.—2.4.1. *Estimation of the wave drag*.—The transfer rule method is due to Ward[8, 9, 10], who has based his considerations upon the drag of source distributions. The result obtained by Ward[10] is that

$$\frac{D}{q} = -\frac{1}{2\pi}\iint T''(\mathbf{R}_1)T''(\mathbf{R}_2)\cosh^{-1}\left(\frac{|x_1 - x_2|}{B|\mathbf{r}_1 - \mathbf{r}_2|}\right) d\sigma_1\, d\sigma_2$$
$$-\frac{1}{\pi}\iint T''(\mathbf{R}_1)S''(x_2)\cosh^{-1}\left(\frac{x_1 - x_2}{B|r_1|}\right) d\sigma_1\, dx_2$$
$$-\frac{1}{2\pi}\iint S''(x_1)S''(x_2)\log(|x_1 - x_2|)\, dx_1\, dx_2\,, \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

where $d\sigma_1$, $d\sigma_2$ denote elements of area of the wing surface

$\mathbf{R}_1$, $\mathbf{R}_2$ denote points on the wing surface (*i.e.*, using Fig. 1, $\mathbf{R}$ is the vector (x, y, z))

$\mathbf{r}_1$, $\mathbf{r}_2$ denote the vector distance of the elements $d\sigma_1$, $d\sigma_2$ from the body axis (*i.e.*, using Fig. 1, $\mathbf{r}$ is the vector (y, z))

T'' denotes $\partial^2 T/\partial x^2$, T being the wing thickness,

and the integrations are to be taken over real values of the integrands. Ward[10] has not placed any restrictions on the validity of equation (6) but it is valid only if the effect of the interference velocity potential on the wave drag is negligible.

It is of interest to note that if the three terms of equation (6) are denoted by D_W, D_{WB} and D_B respectively, then equation (6) can be written in the form $D = D_W + D_{WB} + D_B$, where D_W is the isolated exposed wing wave drag, D_{WB} is the interference wave drag and D_B is the isolated body wave drag. Following Ward[10] the interference wave drag D_{WB} can be expressed in a simpler form and a useful alternative to equation (6) derived. Put

$$A(x) = \frac{1}{\pi}\int \frac{dV_1}{\sqrt{\{B^2 r_1^2 - (x - x_1)^2\}}}\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where $dV_1 = T(\mathbf{R}_1)\, d\sigma_1$ is an element of volume of the wing and the integration is over that part of the wing for which $|x - x_1| \leqslant B|\mathbf{r}_1|$. This shows that $A(x)$ vanishes outside the Mach diamond which encloses the wing. Furthermore, Ward[8] shows that equation (7) can be used for wings carrying slender bodies (*e.g.*, tip tanks).

Using equation (7) and assuming that $\partial A/\partial x$ is continuous for all values of x, equation (6) can be written as

$$\frac{D}{q} = -\frac{1}{2\pi}\iint T''(\mathbf{R}_1)T''(\mathbf{R}_2)\cosh^{-1}\left(\frac{|x_1 - x_2|}{B|\mathbf{r}_1 - \mathbf{r}_2|}\right) d\sigma_1\, d\sigma_2$$
$$+\frac{1}{2\pi}\iint A''(x_1)A''(x_2)\log(|x_1 - x_2|)\, dx_1\, dx_2$$
$$-\frac{1}{2\pi}\iint [S''(x_1) + A''(x_1)][S''(x_2) + A''(x_2)]\log(|x_1 - x_2|)\, dx_1\, dx_2\,, \qquad (8)$$

where $A''(x)$ denotes $\partial^2 A/\partial x^2$. This result enables the interference wave drag to be expressed in the simpler form

$$\frac{D_{WB}}{q} = -\frac{1}{\pi}\iint S''(x_1)A''(x_2)\log(|x_1 - x_2|)\, dx_1\, dx_2\,. \qquad \ldots \quad \ldots \quad \ldots \quad (9)$$

Thus D_{WB} depends upon Mach number since $A(x)$ does.

The transfer rule is the name suggested by Ward[9] for the wave drag result given by equation (8). Therefore $A(x)$ will be referred to as the transferred area distribution of the wing. It follows from equation (7) that the transferred area distribution can be determined by spreading each element of volume of the wing over the x axis according to the law $dV_1/\pi\sqrt{\{B^2r_1^2 - (x - x_1)^2\}}$. Hence a volume element at the point $(x_1, \mathbf{r}_1)$ contributes to the transferred area distribution only within the range $x_1 - B|\mathbf{r}_1| < x < x_1 + B|\mathbf{r}_1|$. In addition, the volume of a body with the cross-sectional area distribution $A(x)$ is equal to the volume of the wing.

The work of Jones[5] suggests an alternative method for calculating $A(x)$. Attention is concentrated upon a point on the x-axis rather than a point on the wing. Then it is implicit in the work of Jones[5] that $A(x)$ must be equal to the mean wing elemental area distribution. This result can be verified for a wing lying the the (x, y) plane bu putting $dV_1 = T(\mathbf{R}_1)\, dx_1\, dy_1$ and changing the variable of integration x_1 in equation (7) from x_1 to $\theta = \cos^{-1}\{(x_1 - x)/By_1\}$, $0 \leqslant \theta \leqslant \pi$. Thus the double integration in equation (7) is replaced by an integration with respect to y_1 followed by an integration with respect to θ and, using Section 2.2.1, it can be shown that

$$A(x) = \frac{1}{2\pi}\int_0^{2\pi} S_W(x, \theta, M)\, d\theta\,,$$

where $S_W(x, \theta, M)$ defines a wing elemental area distribution.

Now, returning to a consideration of the interference wave drag D_{WB}, the wave drag, given by equation (8), is written in the form

$$D = D_W + D\{S + A\} - D\{A\}\,, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

where $D\{S + A\}$, $D\{A\}$ denote the wave drag, given by equation (1), associated with the area distributions $S(x) + A(x)$, $A(x)$, and D_W is the wave drag of the exposed wing. Equation (10) shows that the interference wave drag can be expressed as $D_{WB} = D\{S + A\} - D\{S\} - D\{A\}$. For a slender wing, $D_W = D\{A\}$ to the order of approximation considered and so $D = D\{S + A\}$, which is the standard result of slender-body theory.

However, equation (10) is not in a form suitable for computation since D_W is the wave drag of the exposed wing, not the wave drag of some suitably chosen gross wing. Denote a suitably chosen gross wing by W_0 and the portion blanketed by the body by $\varDelta W = W_0 - W$. Furthermore, denote the body source representation by Q_B, the wing source representation by Q_W and so on. Then, the fundamental assumption that the interference velocity potential can be neglected shows that the source distribution representing the wing-body combination is $Q = Q_W + Q_B$. Therefore $Q = Q_{W0} + Q_B - Q_{\varDelta W}$, where $Q_{\varDelta W}$ is the source distribution representing the portion of the gross wing blanketed by the body. Thus $Q_{\varDelta W}$ does not necessarily represent a slender body although $\varDelta W$ is a low-aspect-ratio wing. But, $Q_{\varDelta W}$ can be incorporated with the slender-body source distribution Q_B, i.e., for the purposes of wave drag evaluation $Q_{\varDelta W}$ may be replaced by the equivalent slender-body source distribution $Q_{\varDelta A}$, where $\varDelta A$ is the (transferred) area distribution of the blanketed wing $\varDelta W$. The wave drag of the configuration can now be simplified and equation (10) becomes[5]

$$D = D_{W0} + D\{S + A\} - D\{A + \varDelta A\}\,, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

it being assumed that $\varDelta A'(x)$ is continuous everywhere. This result is advantageous when D_{W0} is known from existing results in supersonic wing theory.

An alternative method of changing equation (10) into a form suitable for computation is to replace D_W by D_{WN}, the wave drag of the net wing. To construct the net wing, the segments of the exposed wing are placed together to form a continuous wing. This procedure is not unreasonable because $D_W = D_{WN}$ at $M = 1, \infty$. The fact that $D_W \neq D_{WN}$ at other Mach numbers seriously limits the accuracy of any results obtained by setting $D_W = D_{WN}$ in equation (10).

It has been pointed out that the moment-of-area rule is applicable at low supersonic speeds and, since this method is useful, it is desirable to investigate the transfer rule at such speeds. From the results given in the previous discussion it will be seen that a simpler expression for the transferred area distribution of the wing system is required. This is readily found at $M = 1$ since the transferred area distribution is then equal to the cross-sectional area distribution of the wing. At low supersonic speeds the transferred area distribution $A(x)$ can be replaced by a distribution derived as follows.

Consider a narrow streamwise strip of the wing at a distance R from the axis and with the cross-sectional area distribution $\Psi(x)$. It has been shown that $A(x)$ is equal to the mean wing elemental area distribution and so, denoting the contribution of the streamwise strip to $A(x)$ by $\delta A(x)$, it follows that

$$\delta A(x) = \frac{1}{\pi}\int_0^\pi \Psi(x - BR\cos\theta)\, d\theta$$

or

$$\delta A(x) = \frac{2}{\pi}\int_0^{\pi/2} \tfrac{1}{2}\{\Psi(x - BR\cos\theta) + \Psi(x + BR\cos\theta)\}\, d\theta\ .$$

Thus, when $BR = 0$, $\delta A(x) = \Psi(x)$ and $A(x)$ is equal to the cross-sectional area distribution of the wing. When BR is small the integrand is almost constant and may be replaced by $\frac{1}{2}\{\Psi(x - BR\cos\pi/4) + \Psi(x + BR\cos\pi/4)\}$, since $\theta = \pi/4$ is a mean value of θ in the range of integration, i.e.,

$$\delta A(x) \simeq \tfrac{1}{2}\left\{\Psi\left(x - \frac{BR}{\sqrt{2}}\right) + \Psi\left(x + \frac{BR}{\sqrt{2}}\right)\right\}$$

and, if $\Psi''(x)$ is continuous everywhere,

$$\delta A(x) \simeq \Psi(x) + \tfrac{1}{4}B^2R^2\Psi''(x)$$

(this last result may be derived by expanding $A(x)$ in ascending powers of B^2R^2).

Therefore, at low supersonic speeds where BR is small, the simplified procedure for calculating $A(x)$ is to transfer an element of wing cross-sectional area $d\Psi$, situated at the point $(x_1, \mathbf{r}_1)$ by placing elements $d\Psi_1/2$ at the points $|x - x_1| = B|\mathbf{r}_1|\sqrt{2}$. This procedure can be applied easily and quickly.

2.4.2. *Design for low wave drag.*—The problem of determining the optimum fuselage when the wing is given has been examined by Ward[9,10]. From equation (10), it follows that the wave drag, at a given Mach number, will be a minimum when the wave drag, $D\{S + A\}$, of the 'body' with the cross-sectional area distribution $S(x) + A(x)$ is a minimum. Thus the problem has been reduced to an equivalent slender-body problem which can be treated using the methods given by Lord and Eminton. For a given wing and Mach number, the optimum body can be found but it will be noticed that the new exposed wing usually differs slightly from the given wing. Following Ward[10] this difference is taken into account by combining it with the body. However, since the effect on the body shape is small, this additional refinement can be neglected if it is desired to simplify the analysis.

The application of this procedure at low supersonic speeds can be compared with the moment-of-area rule method, which is described in Section 2.3.2. The area-rule and transfer-rule methods consider configurations with fixed wings and derive a body shape which, it will be noted, varies with Mach number. On the other hand, the moment-of-area rule considers variations in wing geometry, such as the addition of bodies near the wing tips, and derives one body shape for the entire low supersonic Mach-number range.

A more general minimum drag problem arises if, for example, only the wing plan area, span and volume are fixed as well as the body length and volume. Such problems have not been examined in the literature although Ward[9] has made some relevant comments, when the Mach diamond enclosing the wing lies within the body length. He shows that the wave drag of a wing-body combination subject to these restrictions cannot be reduced to that of the Sears-Haack body with the same total volume and length. This result does not necessarily imply that the wave drag cannot be reduced to a low value.

It is suggested by Ward[9] that the fuselage should always be designed sufficiently long for the wing system to be included within its Mach diamond, i.e., $S(x)$ is chosen so that it is non-zero wherever $A(x)$ is non-zero.

An important class of optimisation problems has been examined by Lomax and Heaslet[13], who have investigated the optimum fuselage design for a range of Mach numbers, the wing being fixed (it has been shown earlier in this Section that $D\{S + A\}$ is then the only term depending on the fuselage). Lomax and Heaslet[13] consider in effect the minimisation of the integral

$$\int_{M_1}^{M_2} f(M) \frac{D}{q} \{S + A\} \, dM ,$$

where $M_1 \leqslant M \leqslant M_2$ is the range of Mach number being examined and $f(M)$ is a weighting function appropriate to the particular design problem. For example, if transonic acceleration is important then $f(M)$ can be chosen to emphasise the transonic contribution to the above integral.

Let $\overline{\frac{D}{q} \{S + A\}}$ denote the 'mean' value defined by

$$\overline{\frac{D}{q}\{S + A\}} = \int_{M_1}^{M_2} f(M) \frac{D}{q}\{S + A\} \, dM \Bigg/ \int_{M_1}^{M_2} f(M) \, dM .$$

Then, using the double integral of equation (1) it can be shown that

$$\overline{\frac{D}{q}\{S + A\}} - \overline{\frac{D}{q}\{A\}} = \frac{D}{q}\{S + \bar{A}\} - \frac{D}{q}\{\bar{A}\} .$$

Therefore, for a fixed wing (i.e., $A(x)$ fixed), minimising $\overline{(D/q)\{S + A\}}$ is equivalent to minimising $(D/q)\{S + \bar{A}\}$. This latter problem is of the same type as that already considered earlier in this section and so the minimisation can be carried out directly once the 'mean' $\bar{A}(x)$ has been calculated. The simplicity of this procedure is noteworthy.

Lomax and Heaslet[13] have solved only the case for which the optimum area distribution is a cylinder, i.e., $S(x) + \bar{A}(x) = \text{constant}$. The present analysis has generalised this particular result to arbitrary optimum area distributions, which include the constant (or cylindrical) area distribution as a special case. Furthermore, as pointed out by Lomax and Heaslet[13], the present analysis can be applied to minimising expressions of the form

$$\sum_{i=1}^{n} f(M_i) \frac{D}{q}\{S + A_{M\,i}\} ,$$

which consider the wave drag at a discrete number of Mach numbers only. Here $A_{M\,i}$ is the transferred area distribution appropriate to the Mach number M_i. The definitions of 'mean' values follow exactly as in the above analysis.

The result for $\overline{\frac{D}{q}\{S + A\}}$ obtained above, namely

$$\overline{\frac{D}{q}\{S + A\}} - \overline{\frac{D}{q}\{A\}} = \frac{D}{q}\{S + \bar{A}\} - \frac{D}{q}\{\bar{A}\} ,$$

can be used to derive an expression for $\bar{C}_D$, the mean value of the wave-drag coefficient of a wing-body combination throughout a Mach-number range. From equation (10) it follows that $\bar{C}_D = \bar{C}_{D\,W} + C_{D\{S+\bar{A}\}} - C_{D\{\bar{A}\}}$, where all the wave-drag coefficients are based upon the same reference area.

3. *Configurations Represented by Discontinuous Source Distributions.*—3.1. *Isolated Wings.*—Ward[10] has shown that the wave drag associated with a discontinuous surface source distribution is

$$D = -\frac{\rho_0}{4\pi}\int_1\int_2 \cosh^{-1}\left(\frac{|x_1 - x_2|}{B|\mathbf{r}_1 - \mathbf{r}_2|}\right) dS_1\, dQ(\mathbf{R}_1)\, dS_2\, dQ(\mathbf{R}_2)\,, \qquad (12)$$

where ρ_0 is the free-stream density

$\rho_0 Q(\mathbf{R})$ is the source density at the point $\mathbf{R} = (x, y, z)$

$\mathbf{r}$ is the position vector in a plane $x =$ constant, i.e., $\mathbf{r} = (y, z)$

$dS\, dx$ denotes a general volume element,

the co-ordinates (x, y, z) are chosen as in Fig. 1 and the integration is over real values of the integrand. Furthermore, since $Q(\mathbf{R})$ is a discontinuous function of x the above integral is a double Stieltjes integral which must be written in this form. It is possible to replace $dQ(\mathbf{R})$ by $Q'(\mathbf{R})\, dx$ only when $Q(\mathbf{R})$ is a continuous function of x.

It follows directly from equation (12) that the wave drag of a wing is given by

$$D = -\frac{\rho_0 U^2}{4\pi}\int_1\int_2 \cosh^{-1}\left(\frac{|x_1 - x_2|}{B|\mathbf{r}_1 - \mathbf{r}_2|}\right) dT'(\mathbf{R}_1)\, d\mathbf{r}_1\, dT'(\mathbf{R}_2)\, d\mathbf{r}_2\,, \qquad (13)$$

where $dx\, d\mathbf{r}$ denotes an element of area of the wing, $T(\mathbf{R})$ denotes the wing thickness at the point $\mathbf{R}$, and the integration is over those parts of the wing for which the integrand is real. This result has been given by Ward[10]. When the wing has rounded subsonic leading edges it is modified by the addition of a leading-edge drag force. Such forces are examined in Section 5.

It is of interest to note that Ward[10] has suggested that equations (12) and (13) should be modified when the total source strength is not zero, i.e., the body is not closed. Such a modification is needed to allow for the base drag term. Equation (13) gives the external wave drag of a wing, except for the base drag, whether the total source strength is zero or not. This result can be proved most easily if the sources are included in the equation of continuity and the aerodynamic forces are calculated by the method of Ward[21].

A particular form of the result, equation (13), has been given by Lighthill[22] for a plane wing with polygonal streamwise sections. The wave drag of a wing lying in the plane $z = 0$ (*see* Fig. 1) is found to be

$$D = -\frac{2\rho_0}{\pi}\sum_r\sum_n\int\int a_n(y_1)a_r(y_2)\cosh^{-1}\left(\frac{c_n(y_1) - c_r(y_2)}{B|y_1 - y_2|}\right) dy_1\, dy_2\,,$$

where the discontinuities in T' occur along the lines $x = c_n(y)$ and are of magnitude $2/U a_n(y)$ $(n = 1, 2, 3, \ldots)$.

Ward[8] has observed that equation (13) is exactly equivalent to the wave-drag result obtained by integrating the pressure distribution over the wing plan-form. Furthermore, in view of the usefulness of the sonic area rule, the result as $M \to 1$ in equation (13) is of interest. In many cases the presence of discontinuities in T' implies that the linearised theory yields infinite wave drag at $M = 1$ but there are cases for which it yields finite wave drag at $M = 1$. This point has been discussed by Heaslet, Lomax and Spreiter[16] and by Lord, Ross and Eminton[23]. It would appear that the wave drag is finite at $M = 1$ when the rate of change of the cross-sectional area distribution is continuous everywhere.

3.2. *Isolated Bodies.*—The wave drag of a slender body which can be represented by a discontinuous surface source distribution has been given by Fraenkel and Portnoy[24]. The surface slope η is discontinuous on a finite number of contours $C_i (i = 1, 2, \ldots, n)$ lying in planes $x = x_i$ and separated by distances of magnitude $O(1)$, the body length being $O(1)$. Then, within the usual order of approximation, the wave drag of this body is given by

$$\frac{D}{q} = \frac{1}{2\pi} \int_1^{*} \int_2 \log \left(\frac{1}{|x_1 - x_2|} \right) dS'(x_1)\, dS'(x_2)$$
$$+ \sum_{i=1}^{n} \frac{1}{2\pi} (\Delta S_i')^2 \log \frac{2}{B}$$
$$+ \sum_{i=1}^{n} \oint_{C_i} [(\phi_h)_{x=x_i-}\, \Delta\eta_i - (\Delta\phi_h)_i\, \eta_{x=x_i+}]\, d\tau \,, \qquad \ldots \qquad (14)$$

where ϕ_h is that portion of the slender body perturbation velocity potential which can be determined from the two-dimensional incompressible cross-flow

$(\Delta\phi_h)_i$ is the increase in ϕ_h at the point of discontinuity $x = x_i$

$\Delta\eta_i$ denotes the discontinuity in the surface slope at the point $x = x_i$

$\Delta S_i'$ denotes the discontinuity in $S'(x)$ at the point $x = x_i$

$*$ denotes the finite part of a double Stieltjes integral taken over all values of x_1 and x_2 from just upstream of the body nose to just downstream of the body base

and the integration in the third term is over the contour C_i.

Fraenkel and Portnoy[24] have shown that the wave drag, given by equation (14), remains unchanged if the stream direction is reversed. Furthermore, the variation of the wave drag with Mach number depends solely on the discontinuities in $S'(x)$ and does not depend upon the cross-sectional shape. The case when the body profile is discontinuous has been examined by Fraenkel and Portnoy[24] but no applications of the result to engine intakes have been given. Finally, it should be mentioned that the result obtained by Ward[10] differs from equation (14) and is not correct for a slender body of arbitrary cross-sectional shape.

The effect of cross-sectional shape on the wave drag of a slender body can be investigated using equation (14). The presence of the third term in equation (14) shows that the cross-sectional shape at and within the neighbourhood of a discontinuity in surface slope is important and that therefore the wave drag of slender bodies does not depend solely on the cross-sectional area distribution. It follows that the wave drag is independent of the cross-sectional shape away from the discontinuities. Ward[10] has pointed out that the wave drag is a maximum for the body of revolution. Bodies of elliptic cross-section have been examined by Fraenkel[25] who has shown that the wave drag of the equivalent body of revolution is greater by an amount

$$\frac{1}{2\pi} \sum_{i=1}^{n} (\Delta S_i')^2 \log \tfrac{1}{2}[(1 - e^2)^{1/4} + (1 - e^2)^{-1/4}] \,,$$

where e is the eccentricity of the elliptic cross-section. It was pointed out by Fraenkel[25] that if the cross-section is 'nearly' circular, in the sense that $e = O(t^{1/4})$, where t is the thickness ratio of the body, then the wave drag, which is $O(t^4 \log t)$, will not change to this order. Hence it may be assumed that most slender bodies of practical importance have a wave drag independent of cross-sectional shape.

The form of equation (14) for a body of revolution was given first by Lighthill[26] and, for a body with radius R_i at the point x_i, is found to be

$$\frac{D}{q} = \frac{1}{2\pi} \int\int \log\left(\frac{1}{|x_1 - x_2|}\right) S''(x_1)S''(x_2)\, dx_1\, dx_2$$
$$+ 2 \sum_{i=1}^{n} R_i \Delta\eta_i \int \log\left(\frac{1}{|x - x_i|}\right) S''(x)\, dx$$
$$+ 4\pi \sum_{i>}^{n} \sum_{j=1}^{n} R_i R_j \Delta\eta_i \Delta\eta_j \log\left(\frac{1}{|x_i - x_j|}\right)$$
$$+ 2\pi \sum_{i=1}^{n} R_i^2 (\Delta\eta_i)^2 \log\frac{2}{BR_i}, \quad \ldots \quad (15)$$

where the integrations are Riemann integrations ignoring the points at which $S''(x)$ is undefined and the first three terms are the expanded form of the finite part of the double Stieltjes integral given in equation (14). The last term in equation (15) gives the result that the wave drag of a slender body with discontinuities in $S'(x)$ is infinite at $M = 1$ according to this theory. Also, as already noted, it follows from equation (14) that equation (15) gives the wave drag of a slender body which has circular sections at and within the neighbourhood of the discontinuity contours C_i.

3.3. *Wing-Body Combinations.*—An extension of the area rule and the transfer rule to configurations incorporating bodies with discontinuities in surface slope is given below. The cross-sectional shape of the body is arbitrary.

Consider the flow field on a circular cylindrical control surface at a large distance from the combination. Then the wave drag is to be found by calculating the rate of flux of momentum across this surface. Heaslet, Lomax and Spreiter[16], for example, have shown that the perturbation velocity potential due to the wing alone, at a given angular position θ on the control surface, is the same as that associated with the appropriate wing elemental area distribution. Let the effect of any interference velocity potential on the wave drag be neglected†. The flow field on the control surface is then taken as the sum of the perturbation velocity potentials due to the body alone and to an appropriate wing elemental area distribution so that, at a given angular position on the control surface, the source distributions representing the body and the appropriate elemental area distribution are superimposed. The wave drag may be written as

$$D = \frac{1}{2\pi} \int_0^{2\pi} D\{S + S_W(x, \theta, M)\}\, d\theta, \quad \ldots \quad (16)$$

where $D\{S + S_W(x, \theta, M)\}$ is interpreted by examining the flow field near the axis, i.e., $D\{S + S_W(x, \theta, M)\}$ denotes the wave drag of a 'combined' slender body. This 'combined' body is a stream surface of the discontinuous surface source distribution Q_B, representing the isolated body, and the continuous source distribution defined by $S_W'(x, \theta, M)$. Since almost all the $S_W'(x, \theta, M)$ are continuous everywhere those discontinuous source distributions associated with the discontinuous $S_W'(x, \theta, M)$ are omitted from the integral in equation (16).

Using equation (12) the wave drag of the 'combined' body can be expressed as

$$D\{S + S_W(x, \theta, M)\} = D\{S\} + D\}S_W(x, \theta, M)\}$$
$$- \frac{\rho_0 U}{2\pi} \int_1 \int \cosh^{-1}\left(\frac{|x_1 - x_2|}{B|\mathbf{r}_1|}\right) dS_1\, dQ_B(\mathbf{R}_1)\, S_W''(x_2, \theta, M)\, dx_2,$$

where $D\{S\}$, the isolated body wave drag, is given by equation (14) and $D\{S_W(x, \theta, M)\}$, the wave drag associated with $S_W(x, \theta, M)$, is given by equation (1), within the usual order of approximation.

† This assumption is not valid for slender wing-body combinations when the source distribution representing the body (and/or the wing cross-sectional area distribution) is discontinuous within the region of the wing because equation (14), derived by Fraenkel and Portnoy[24], cannot be obtained by neglecting the interference velocity potential. The assumption is valid for any other type of slender wing-body combination.

Following Ward's analysis[10] for slender bodies of revolution the inverse cosh in this result is expanded in terms of logarithms. The unknown body source density $Q_B(\mathbf{R})$ is eliminated by noting that the total source strength in a plane $x =$ constant is equal to $US'(x)$, where $S(x)$ is the cross-sectional area distribution of the body[10]. Since $S_W'(x, \theta, M)$ vanishes at its end points, it can be shown quite simply that the wave drag of the ' combined ' body becomes, within the usual order of approximation,

$$D\{S + S_W(x, \theta, M)\} = D\{S\} + D\{S_W(x, \theta, M)\} - \frac{\rho_0 U^2}{2\pi} \int_1 \int \log |x_1 - x_2| \, dS'(x_1) \, S_W''(x_2, \theta, M) \, dx_2 \quad \ldots \quad (17)$$

Equations (16) and (17) provide a generalisation of the area rule, equations (2) and (3), to combinations incorporating bodies with discontinuities in surface slope.

The transferred wing area distribution $A(x)$ is equal to the mean wing elemental area distribution, i.e.,

$$A(x) = \frac{1}{2\pi} \int_0^{2\pi} S_W(x, \theta, M) \, d\theta \, .$$

Thus, using equation (17), equation (16) becomes

$$D = D\{S\} + D_W - \frac{\rho_0 U^2}{2\pi} \int_1 \int \log |x_1 - x_2| \, dS'(x_1) \, A''(x_2) \, dx_2 \, , \quad \ldots \quad (18)$$

where $D\{S\}$ is given by equation (14) and D_W by equation (13), for example. This result is the extended form of the transfer rule, equations (8) and (10).

Equation (18) implies that the interference wave drag D_{WB} is given by

$$D_{WB} = -\frac{q}{\pi} \int_1 \int \log |x_1 - x_2| \, dS'(x_1) \, A''(x_2) \, dx_2 \, ,$$

which is the generalisation of equation (9). It should be noted that the interference wave drag D_{WB} is independent of the cross-sectional shape of the body. The result for the interference wave drag can be derived directly from equation (12) with $Q(\mathbf{R}) = Q_B(\mathbf{R}) + Q_W(\mathbf{R})$; $Q_W(\mathbf{R}) = UT'(\mathbf{R})$. It is necessary to use Ward's integral equation[10] for the transferred area distribution $A(x)$.

In Section 2.4.1 it was shown that the wing-body combination can be replaced by an equivalent combination made up of a suitably chosen gross wing W_0 and a modified body which is denoted by $B - \Delta A$, where ΔA denotes the (transferred) area distribution of the portion of the gross wing blanketed by the body. It is assumed that $S'(x)$ is continuous where $\Delta A(x)$ is defined and that $\Delta A'(x)$ is continuous everywhere as otherwise the source distribution representing the modified body is not equal (in the wave-drag sense) to that representing the body less blanketed wing, i.e., $Q_{(B-\Delta A)} \not\equiv Q_{B-\Delta W}$. The generalised form of equation (11) follows from equation (18) and can be shown to be

$$D \simeq D_{W0} + D\{S + A\} - D\{A + \Delta A\} \, , \quad \ldots \quad (19)$$

where $D\{S + A\}$ denotes the wave drag of a ' combined ' body, i.e.,

$$D\{S + A\} = D\{S\} + D\{A\} - \frac{\rho_0 U^2}{2\pi} \int_1 \int \log |x_1 - x_2| \, dS'(x_1) \, S_W''(x_2, \theta, M) \, dx_2 \, .$$

4. *Numerical Evaluation of Wave Drag.*—In the previous sections attention has been concentrated upon giving the theoretical methods rather then the practical application of these methods. The present section is intended to give a brief description of the numerical methods that have been developed for evaluating wave drag. It will be seen that these methods have been developed for isolated wings or isolated bodies but their application to wing-body combinations is not difficult because the results obtained for the wave drag of wing-body combinations require only the evaluation of wing and ' body ' wave drags.

4.1. *Isolated Wings*.—There has been a large number of investigations in the field of supersonic wing theory and no attempt will be made to summarise them here. However, it is sufficient to notice that the results for wave drag have been found by integrating pressure distributions. Also, the wing plan-forms are straight-edged and the wing sections simple shapes. A summary of the results obtained has been given by Bishop and Cane[27]. Some results for sonic speed have been obtained by Lord, Ross and Eminton[23] using the sonic area rule.

From the results given in Section 3.1 it is suggested that equation (13) be used to calculate the wave drag provided that the pressure distribution is not required. Otherwise it would probably be better to determine the pressure distribution first and then the wave drag. If the pressure distribution is required it can be found using the method of Thomson[28].

Nevertheless, it is apparent that the area rule (equations (2) and (3)) offers an alternative method for calculating the wave drag of an isolated wing. However, a difficulty arises if any straight lines, along which the wing planar source distribution is discontinuous, are inclined so that the component of free-stream velocity normal to the line is supersonic (for example, along sharp edges the source distribution is discontinuous). But, since almost all the wing elemental area distributions have source representations continuous everywhere, the wave drag may be calculated using these elemental area distributions only and it may not be necessary to use the results of Section 3.2 for bodies with discontinuous source representations. Lord[29] has pointed out that a numerical application of this procedure to wings with supersonic edges is not satisfactory.

The area-rule method is particularly useful for wings with curved plan-forms and has been used by Lord and Bennett. Details may be found in Refs. 30 and 31.

4.2. *Isolated Bodies*.—Recently a number of authors have investigated numerical methods for evaluating the double integral of equation (1), *i.e.*, methods have been developed for calculating the wave drag of slender bodies without discontinuities in surface slope. The more general case when the source representation is not continuous will be examined later in the present section.

There are two main methods for evaluating $D\{S\}$, equation (1), namely, the Fourier-series method and the optimum method of Eminton[31]. Considering a body of unit length, the Fourier-series method relies upon expressing $S'(x)$ as a sine series in the new variable $\varphi = \cos^{-1}(1 - 2x)$. The disadvantage of this method lies in the determination of $S'(x)$, which requires a numerical differentiation of the cross-sectional area distribution $S(x)$. Furthermore, it is not clear how many terms in the Fourier series are required to give reasonable accuracy in the answer for the wave drag.

The difficulties of the Fourier-series method can be overcome most satisfactorily by using the method of Eminton[4,31] for evaluating the double integral of equation (1). This method depends upon some results in the theory of optimum area distributions and details can be found in the papers cited[4,31]. The given area distribution is replaced by the optimum area distribution with minimum wave drag for a large number of fixed areas at equally spaced points along the body. Thus the evaluation of the wave drag, given by the double integral of equation (1), requires only a knowledge of the areas at a large number of points, usually nineteen, along the body. The derivative $S'(x)$ is not required for this method.

Slender bodies with discontinuous source representations cannot be treated in general but can be treated provided that the cross-section is circular at and within the neighbourhood of a discontinuity in surface slope, *i.e.*, the wave drag is given by equation (15) and depends upon the cross-sectional area distribution only.

A numerical procedure was developed first by Dickson and Jones[32], who utilised a transformation of equation (15) whereby Fourier series could be used directly to evaluate the integrals involved. This approach would be expected to have a counterpart in the theory of optimum area distributions if such optima could be found for this more general case. Eminton is investigating an extension of the optimum method to the present problem, thus determining the optimum area distribution with minimum wave drag (equation (15)) for prescribed values of

$\Delta S_i'$ at the points $x = x_i$ where the cross-sectional area is $S(x_i)$ ($i = 1, 2, \ldots n$). It appears that a simple and illuminating extension may be possible and that the optimum method will be extended to the evaluation of equation (15). However, since the method of Dickson and Jones[32] seems to be the only published method and since their report[32] may not be generally available, their method will be summarised here.

Consider a body of unit length and introduce the function $P(x)$ defined by

$$P'(x) = S'(x) - \sum_{i=1}^{j} \Delta S_i' + x \sum_{i=1}^{n} \Delta S_i' , \qquad x_j \leqslant x \leqslant x_{j+1} .$$

The function $P(x)$ is chosen by Dickson and Jones[32] because $P'(x)$ is continuous everywhere. Now introduce the Fourier-series variable $\phi = \cos^{-1}(1 - 2x)$ and put

$$P'(x) = \sum_{m=1}^{\infty} A_m \sin m\varphi ,$$

so that equation (15) becomes

$$\begin{aligned} \frac{D}{q} = & -\frac{1}{4\pi}\left(\sum_{i=1}^{n} \Delta S_i'\right)^2 \\ & + \frac{1}{2\pi}\sum_{i=1}^{n} (\Delta S_i')^2 \log \frac{2}{BR_i} \\ & + \frac{\pi}{4}\sum_{m=1}^{\infty} mA_m^{\,2} + \sum_{i=1}^{n} \Delta S_i' \sum_{m=1}^{\infty} \frac{A_{2m}}{[(2m)^2 - 1]} \\ & + \sum_{i=1}^{n} \Delta S_i' \left(\sum_{m=1}^{\infty} A_m \cos m\varphi_i\right) \\ & + \frac{1}{\pi}\sum_{i=1}^{n} \Delta S_i' \sum_{j=1}^{n} \Delta S_j'\{(1 - x_j)\log(1 - x_j) + x_j \log x_j\} \\ & + \frac{1}{\pi}\sum_{i>}^{n}\sum_{j=1}^{n} \Delta S_i' \, \Delta S_j' \log\left(\frac{1}{|x_i - x_j|}\right) , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned} \tag{20}$$

where $x_i = \frac{1}{2}(1 - \cos\varphi_i)$ and $\pi R_i^{\,2} = S(x_i)$. This result is in a somewhat different form from that given by Dickson and Jones[32].

4.3. *Wing-Body Combinations.*—In Sections 2 and 3 it has been shown that the calculation of the wave drag of a wing-body combination can be reduced either to a wing problem together with two 'body' problems by using the transfer rule or to a number of 'body' problems by using the area rule. Hence the numerical methods for wings and bodies given in Sections 4.1 and 4.2 enable the wave drag of a wing-body combination to be found. Procedures for calculating the transferred area distribution of the wing are described in the Appendix.

5. *Some Miscellaneous Remarks.*—5.1. *Configurations with Round-Nose Wing Sections.*—Ward[10] has pointed out that the results of linearised theory are incorrect if the wing of the wing-body combination has rounded edges where $T'(\mathbf{R})$ is infinite. Nevertheless, the linearised-theory result for wings with rounded subsonic leading edges can be obtained by allowing for the effect of the leading-edge singularity in the way suggested by Jones[33]. The result obtained is that to the existing result (equation (6), for example), there must be added a leading-edge drag F per unit length given by

$$\frac{F}{q} = \frac{\pi r \cos^3 \Lambda}{[1 - (M \cos \Lambda)^2]^{1/2}} ,$$

where q is the kinetic pressure, Λ is the local sweepback angle of the leading edge, and r is the local nose radius.

It should be noted that this result is restricted to wings for which there are no abrupt changes in the curvature of the leading edge. Moreover, the result shows that the linearised-theory wave drag of a wing with a rounded leading edge is infinite if the leading edge is sonic (i.e., $M \cos \Lambda - 1$). This infinite result for a wing with a sonic leading edge is obviously incorrect and emphasises the fact that the linearised theory is not satisfactory for wings with rounded edges. Thus, in the absence of non-linear theory, experimental results are required for determining the wave drag of such wings. However, since the transfer rule separates the wing wave drag from the total wave drag of a wing-body combination, it can be used to estimate the interference wave drag.

5.2. *Systems of Wings and Bodies*.—The discussion so far has been restricted to wing-body combinations, although it could also have been given for wing-system–body-system combinations. The wing system includes wing-like elements of the configuration as well as any subsidiary bodies situated on these elements while the body system is essentially axial and may include fins and tailplanes. Since the area rule, moment of area rule and transfer rule are all equally applicable to systems of wings and bodies as to wing-body combinations, no essentially new problem is introduced.

5.2.1. *Arrangements of bodies*.—Arrangements of bodies have been examined by Friedman and Cohen[34] and Rennemann[35]. In both cases the analysis does not use any of the methods given in Section 2 but such an analysis could be carried out using either the area rule or the transfer rule. Rennemann[35] has examined the shape of a body of revolution in the pressure field of a larger body and has shown that the optimum shape for minimum wave drag can be calculated sufficiently exactly by ignoring the larger body, *i.e.*, the Sears-Haack body based upon the given length and volume is the best body shape to use for the smaller body.

Friedman and Cohen[34] have used this result to examine the positioning of auxiliary bodies alongside a main body. For two-body or three-body systems the most favourable location is where the maximum cross-section of the auxiliary body is just upstream of the Mach cone from the tail of the main body while the least favourable location is within the Mach diamond enclosing the main body. These conclusions are not absolutely definite and so a further analysis of this problem may be needed.

A suggested application of an analysis of the above type is as follows: Firstly, consider the main and auxiliary bodies of the given system and position these for minimum wave drag at the Mach number under consideration. Secondly, by assuming that the wave drag of these bodies is considerably greater than the wave drag of a wing system, it follows that the next step is to introduce a wing system containing these bodies and shape all the bodies according to the nature of the wing system. This latter procedure can be carried out using the methods discussed in Section 2. Details are given in Section 5.2.2 below.

5.2.2. *Combinations with low wave drag*.—Consider, for example, the problem of shaping and positioning a number of subsidiary bodies such as external stores on the wing of a configuration so that the overall wave drag is a minimum. To a first approximation the most important effect of a change in the wing system will be in the wave drag of the auxiliary bodies alone and the interference wave drag between them and the wing (this result assumes that the subsidiary bodies are not close to the body and to each other). By considering separately each subsidiary body and the wing, the interference problem can be treated by using equation (10) and so, for a given wing, it follows that the variable wave drag term will be $D\{A_1 + \Psi\}$ where $A_1(x)$ is the transferred area distribution of the wing system, less the appropriate subsidiary body, on to the auxiliary body axis and $\Psi(x)$ is the cross-sectional area distribution of the subsidiary body. Thus the optimisation procedure for each subsidiary body has been reduced to minimising $D\{A_1 + \Psi\}$ subject to certain prescribed conditions. Finally the optimisation procedure is completed by examining the main body, which is shaped using the transfer-rule procedure given in Section 2.4.2.

In many applications it may be necessary to repeat this optimisation procedure because any variation in $A_1(x)$, due to subsequent modifications of other subsidiary bodies, is not considered when minimising $D\{A_1 + \Psi\}$. Thus repetition may be essential in those cases where the application of a symmetry condition implies that there will be corresponding changes in $A_1(x)$ for any alteration to $\Psi(x)$. However, no repetition at all will be necessary if each subsidiary body does not influence, or is not influenced by, any other subsidiary body.

It will be noted that this optimisation procedure examines firstly the subsidiary bodies and then the main body. But, since the subsidiary bodies may require a repetition of the procedure it is recommended that the whole wing system be considered before any shaping of the main body is investigated.

The most important example of wing-system–body-interference occurs when there are two subsidiary bodies only. In this case the assumption that the bodies are not ' close ' to each other is unnecessary since the wave drag can be found by breaking down the wing system into a number of simple wing–body problems. It can be shown that the interference wave drag for the isolated wing system consists of three terms: the interference wave drag between each subsidiary body and the wing (two terms) together with that between the two subsidiary bodies. The last term is evaluated by selecting one subsidiary body as a ' main ' body and then applying the transfer rule, equation (10). It is this body-body interference term which would be neglected, sometimes without justification, by the simplified optimisation procedure presented for a general wing system. However, although the inclusion of this term does make any optimisation procedure more complex, it does enable, also, estimates of the wave drag of the complete configuration to be made.

5.3. *Problems Associated with Intakes and Exhausts.*—The presence of engine intakes and exhausts within an aircraft has not been considered in the previous discussions which are applicable only to ideal configurations with no intakes and exhausts. It will be shown that the use of two simplifying assumptions enables the total external wave drag to be estimated. As well as the wave drag arising from the pressure forces acting upon the external surface of the aircraft it is necessary now to consider the external wave drag due to the pressure forces acting upon the pre-entry and post-exit stream tubes. The pre-entry stream tubes extend from infinity upstream to each engine intake and separate the air flowing into the intakes from that flowing around them while the post-exit stream tubes extend from each engine exhaust to infinity downstream and separate the high-speed gases flowing out of the exhausts from the air which passes over the external surfaces of the configuration, *i.e.*, the pre-entry and post-exit stream tubes separate the external flow from the internal engine flow.

Thus, in principle, an effective ' solid ' combination, including all the pre-entry and post-exit stream tubes can be found. The methods of wave drag estimation presented in Sections 2 and 3 are not necessarily strictly applicable to this ' solid ' configuration because large perturbation velocities would be associated with any regions of large ' surface ' slopes on either the intake cowls or the pre-entry and post-exit stream tubes. Nevertheless, part of the following discussion is based upon the linearised theory and an approximate method for treating engine intakes and exhausts is described.

The usual method of calculating the total external wave drag requires its division into three components: the external wave drag proper, which is associated with the actual external surfaces of the aircraft; the pre-entry wave drag, which is associated with the pre-entry stream tubes; the post-exit wave drag, which is associated with the post-exit stream tubes. The external wave drag proper cannot be evaluated directly and therefore it is convenient to use the linearised theory methods of Sections 2 and 3 in order to calculate it for an internal flow in which all the pre-entry and post-exit stream tubes are of constant cross-sectional area. But, since the pre-entry and post-exit stream tubes are not of constant cross-sectional area, in general, there are a number of terms to be added before the actual external wave drag proper is found. The largest of these additional terms is likely to be that associated with the flow over the intake cowl and can be estimated using existing techniques for isolated intakes. The remaining additional terms are neglected since they are assumed to be small.

It is suggested that calculations of external wave drag be carried out using this assumption together with the additional assumption that the pre-entry and post-exit wave drags can be estimated by examining each intake-engine-exhaust system in isolation. There is, however, one case for which these assumptions may not be required. For slender bodies, the linearised theory techniques of Fraenkel and Portnoy[24] and Ward[21] would enable the intake problem to be solved exactly, provided that the intake has sharp edges and there is no spillage around these edges. The treatment of an exhaust upstream of the tail is difficult because of the high-speed exhaust gases. For arbitrary configurations the assumptions are necessary and, since they may give rise to incorrect wave-drag estimates, an investigation of the problems involved in treating engine intakes and exhausts may be desirable.

5.4. *Non-linear Effects.*—Since the present report deals with the linearised theory, non-linear effects are not examined. Confining attention to the transfer rule, it is seen that only the interference wave drag need be considered, since the linearised-theory results for the isolated wing and body wave drags can be modified by using correlations between experiment and linearised theory as a guide to the magnitude of any non-linear effects. Furthermore, from the form of equation (10) it appears that satisfactory modifications may be able to be carried out by modifying only the transferred area distribution $A(x)$.

However, at the present time, the estimation of non-linear effects must be based upon known effects on isolated wings and isolated bodies. The above suggestion is purely tentative and is intended to emphasise the need for some satisfactory method of estimating non-linear effects on the wave drag of wing-body combinations.

In the transonic range, non-linear effects on wave drag have been examined by Owatitsch and Keune[36] for low-aspect-ratio wings. In the supersonic speed range the effects of strong shock and expansion waves have not been considered except in certain particular cases such as two-dimensional wings.

5.5. *Boundary-Layer Corrections.*—An important aspect of wave-drag calculations for a specific configuration is the determination of corrections that allow for the boundary-layer displacement thickness. Consider a slender body whose wave drag is given by the double integral of equation (1). In equation (1), $S''(x_1)$ may be regarded as a surface slope term and $S''(x_2)$ as a local pressure term. Warren has pointed out that the nature of the boundary-layer correction to the wave drag $D\{S\}$ requires careful consideration because only the local pressure term is altered. Let $\mathscr{S}(x)$ be the increment in $S(x)$ due to the boundary-layer displacement thickness. Then the wave drag is given by equation (1) with $S''(x_1)$ unaltered and $S''(x_2)$ replaced by $S''(x_2) + \mathscr{S}''(x_2)$. But, since the boundary layer correction is small, it can be shown that the wave drag is given approximately by $D\{S + \frac{1}{2}\mathscr{S}\}$. Similar considerations apply to equation (13) which gives the wave drag of a wing. Thus, in order to obtain an effective solid body, half the boundary-layer displacement thickness is added everywhere to the profile of a wing-body combination.

6. *Conclusion.*—The wave drag of a combination of a thin wing and a smooth slender body can be estimated subject to the usual limitations of linearised theory and, it should be noted, subject to the validity of the assumption that the interference velocity potential is negligible. For smooth combinations, which are aerodynamically slender, the wave drag is given by the sonic area rule, which states that the wave drag depends only on the axial distribution of cross-sectional area. For smooth combinations which are 'not-so-slender', it is necessary to use the moment of area rule, which states that the wave drag depends upon the axial distribution of the second moment of area as well as upon the cross-sectional area distribution. For arbitrary combinations the wave drag is given by the supersonic area rule, which expresses the wave drag as the mean of the wave drags associated with the so-called elemental area distributions. If the wave drag of the isolated wing of an arbitrary configuration is known, it is convenient to use the transfer rule, which expresses the sum of the body wave drag and the interference wave drag in a simple form.

The methods of wave-drag estimation can be interpreted to yield practical methods of designing for low wave drag under prescribed conditions. For a given wing, the general problem of designing a body so that the combination has low wave drag throughout a range of Mach number can be solved. The method of designing the body is simple at all Mach numbers, but the possible benefits of an optimum design decrease quite rapidly with increase of Mach number. The achievement of low wave drag is facilitated by allowing the wing to be altered, an example of which is the design for low wave drag according to the moment of area rule.

Previous discussions of the methods of estimation of the wave drag of wing-body combinations have been confined to combinations with smooth slender bodies. Here an extension of these methods is given for the cases when the combinations incorporate bodies with discontinuities in surface slope.

All the theoretical methods presented here require suitable numerical techniques for their practical application and it is found that the existing techniques are very satisfactory.

No numerical results or particular applications are described but since most of the methods discussed in this report are somewhat idealised some miscellaneous remarks are made to help in their application to practical aircraft. In particular, it is pointed out that the present method of allowing for the air which passes through the engines of an aircraft is far from satisfactory and a more adequate procedure is essential.

NOTATION

a_0, a_2		Coefficients of B^0, B^2 in series expansion of D
$a_n(y)$		$U/2$ multiplied by discontinuity in T' along the line $x = c_n(y)$
A		Aspect ratio
A_{Mi}		Transferred area distribution appropriate to the Mach number M_i
$A(x)$		Transferred area distribution (equation (7))
$A_1(x)$		Transferred area distribution of wing system on to axis of subsidiary body
$\delta A(x)$		Contribution of streamwise strip of wing to $A(x)$
$\Delta A(x)$		(Transferred) area distribution of portion of gross wing system blanketed by the body
A_n		Fourier coefficient in expansion of $S'(x)$ using the variable φ
$A_n(\theta, M)$		Fourier coefficient in expansion of $S'(x, \theta, M)$ using the variable φ
B	=	$\{M^2 - 1\}^{1/2}$
B_n		Fourier coefficient in expansion of $M_2'''(x)$ using the variable φ
c		Wing-root chord
$c_n(y)$		Defines line along which discontinuities in T' occur
$\mathscr{C}_{i,\lambda}(x)$		Generalised moment of area (*see* Section 2.3.1)
C_i		Contour lying in the plane $x = x_i$ and along which the body surface slope η is discontinuous
C_D		Wave-drag coefficient D/qS, S being an appropriate area
D		Wave drag

$D\{S\}$		Wave drag of 'body' with the cross-sectional area distribution $S(x)$ and the prescribed shape
$D(\theta, M)$		Wave drag associated with an elemental area distribution
D_{WB}		Interference wave drag
$D_{B-\Delta A}$		Wave drag of body less portion of gross wing blanketed by the body
e		Eccentricity of elliptic cross-section
$f(M)$		Mach-number function used in Section 2.4.2
F		Leading-edge drag force per unit length due to curvature of wing section
I_n		Integral defined in Appendix
k		Slenderness parameter
K	=	$(2N + 1)$
l		Length of wing-body combination
l_p		Length of pth moment of area distribution
M		Mach number
$M_n(x)$		nth moment of area distribution
n		Number of discontinuities in $S'(x)$ (also a subscript)
N		Number of elements into which streamwise strip of wing is divided
$P(x)$		Modified form of $S(x)$ (*see* Section 4.2)
q		Kinetic pressure $\frac{1}{2}\rho_0 U^2$
Q		Source distribution
$Q_{\Delta A}$		Source distribution representing $\Delta A(x)$
$Q(\mathbf{R})$		$1/\rho_0$ multiplied by the source density at the point $\mathbf{R}$
r		Radial co-ordinate (Fig. 1) or leading-edge nose radius
$\mathbf{r}$		Vector (y, z)
$\mathbf{r}_0$		Unit vector $(\sin\theta, \cos\theta)$
R		Distance of body from axis
R_i, R_j		Radius of body of revolution at the points of discontinuity $x = x_i, x_j$
R_t		Typical body radius
$\mathbf{R}$		Position vector (x, y, z) or $(x, \mathbf{r})$
$\bar{\mathbf{R}}$		Position vector $(x + B\mathbf{r}_0\,\mathbf{r}, \mathbf{r})$
$R(x, \theta)$		Defines cross-section of configuration at the station x on axis
$\mathscr{S}(x)$		Increment in $S(x)$ due to boundary-layer displacement thickness
$s(x, \theta, M)$	=	$\operatorname{cosec}\mu\, S(x, \theta, M)$
$S(x)$		Cross-sectional area distribution of body (and also of wing-body combination)
S		Constant value of S_n
S_n		Constant element of $\Psi(x)$ in the range $X_n \leqslant x \leqslant X_{K-n}$
$S(x, \theta, M)$		Elemental area distribution
$\mathscr{S}_{i,\lambda}(x)$		Generalised moment of area (*see* Section 2.3.1)

dS		Element of area used in sense that $dS\,dx$ is an element of volume
$\Delta S_i'$		Increase in $S'(x)$ at a point of discontinuity $x = x_i$
t		Body thickness ratio
$T(x, y)$, $T(\mathbf{R})$		Wing thickness at the point (x, y), $\mathbf{R}$
U		Free-stream velocity
V		Wing volume
V_p		pth moment volume of the configuration (*see* Section 2.3.2)
dV		Element of volume
W		Wing
W_0		Suitably chosen gross wing
ΔW	$\equiv$	$W_0 - W$
(x, y, z)		Rectangular Cartesian co-ordinates (*see* Fig. 1)
x_i, x_j		Points at which body surface slope is discontinuous
X_n		One of $2N$ division points for the area distribution $\Psi(x)$, $X_1 \leqslant x \leqslant X_{2N}$
η		Body surface slope relative to free-stream direction
$\Delta\eta_i$		Increase in η at the point of discontinuity $x = x_i$
θ	$=$	$\tan^{-1}(z/y)$ (*see* Fig. 1)
Λ		Angle of leading-edge sweepback
μ		Mach angle ($= \sin^{-1}(1/M)$)
ν	$=$	$\tan^{-1}(B\cos\theta)$
ρ_0		Free-stream density
$d\sigma$		Element of area of the wing surface
τ_0		Wing-root thickness ratio
$d\tau$		Element of arc length on the contour C_i
φ	$=$	$\cos^{-1}(1 - 2x)$
φ_i	$=$	$\cos^{-1}(1 - 2x_i)$
ϕ_h		That portion of the slender-body perturbation velocity potential which can be determined from the two-dimensional incompressible cross-flow
$(\Delta\phi_h)_i$		Increase in ϕ_h at the point of discontinuity $x = x_i$
$\Psi(x)$		Cross-sectional area distribution of a body, or a streamwise wing strip, at a distance R from axis
$d\Psi$		Element used in the sense that $d\Psi\,dx$ is an element of wing volume
$i, j, n, m, p, r, \lambda$		Subscripts denoting different members of a family, e.g., $M_i(x)$ is the ith moment of area
$A, B, W, W0, WN$		Subscripts denoting transferred area distribution, body, wing, gross wing, net wing respectively
$1, 2$		Subscripts usually applied to variables of integration
dashes		Denote derivatives with respect to x, e.g., $S'(x) = \partial S/\partial x$
bar		Denotes a 'mean' value, e.g., $\bar{A}(x)$

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APPENDIX

Numerical Procedures for Calculating the Mean or Transferred Area Distribution of a Wing

Since the transfer rule has been recommended for wave-drag calculations it is advisable to develop a simple numerical procedure for calculating the mean or transferredare a distribution $A(x)$ (*see* equation (7)). The elemental area distributions (Section 2.2.1) could be calculated and then the mean found. This method has the advantage that all the elemental area distributions calculated for a given Mach number can be incorporated in calculations for higher Mach numbers; however, it is likely to be useful only when a small number of elemental area distributions are required to represent the wing. A numerical procedure dealing with only the mean area distribution has been given by Faget[37]; a simple derivation of this method is presented below. Sufficient details are given for the method to be used without reference to Faget's paper[37], which may not be readily available.

Firstly, the wing is divided into a number of streamwise strips so that the cross-sectional area distribution of each strip defines a body which is to be placed along the centre-line of the strip. Take one of these bodies and let its distance from the axis be R and its cross-sectional area distribution be $\Psi(x)$. Then $\Psi(x)$ is replaced by a number, N, of finite overlapping elements, each of constant cross-sectional area. Let the nth element, having the cross-sectional area S_n, extend from $x = X_n$ to $x = X_{K-n}$ ($n = 1, 2, \ldots, N$; $K = 2N + 1$). Thus $\Psi(x)$ vanishes outside the range $X_1 \leqslant x \leqslant X_{2N}$ and

$$\Psi(x) \equiv \sum_{n=1}^{N} S_n ; \qquad S_n = S \text{ for } X_n \leqslant x \leqslant X_{K-n} \quad (K = 2N + 1) .$$

Faget[37] chooses the areas S_n so that they are equal to each other. Now, equation (7) can be written as

$$\delta A(x) = \frac{1}{\pi} \int_{X_1}^{X_{2N}} \frac{\Psi(x_1)\, dx_1}{\sqrt{\{B^2R^2 - (x - x_1)^2\}}} ,$$

where $\delta A(x)$ is the contribution of a streamwise strip to the transferred area distribution $A(x)$. Therefore

$$\delta A(x) = \frac{S}{\pi} \sum_{n=1}^{N} I_n$$

where, for $X_{K-n} - BR \leqslant X_n + BR$

$$I_n = \begin{cases} \dfrac{\pi}{2} + \sin^{-1}\left(\dfrac{x - X_n}{BR}\right), & X_n - BR \leqslant x \leqslant X_{K-n} - BR \\ \sin^{-1}\left(\dfrac{X_{K-n} - x}{BR}\right) + \sin^{-1}\left(\dfrac{x - X_n}{BR}\right), & X_{K-n} - BR \leqslant x \leqslant X_n + BR \\ \sin^{-1}\left(\dfrac{X_{K-n} - x}{BR}\right) + \dfrac{\pi}{2}, & X_n + BR \leqslant x \leqslant X_{K-n} + BR \end{cases}$$

and, for $X_{K-n} - BR \geqslant X_n + BR$

$$I_n = \begin{cases} \dfrac{\pi}{2} + \sin^{-1}\left(\dfrac{x - X_n}{BR}\right), & X_n - BR \leqslant x \leqslant X_n + BR \\ \pi , & X_n + BR \leqslant x \leqslant X_{K-n} - BR \\ \sin^{-1}\left(\dfrac{X_{K-n} - x}{BR}\right) + \dfrac{\pi}{2}, & X_{K-n} - BR \leqslant x \leqslant X_{K-n} + BR \end{cases}$$

(Note that I_n is defined for $X_n - BR \leqslant x \leqslant X_{K-n} + BR$ and that $I_n = 0$ when $x = X_n - BR$, $X_{K-n} + BR$).

The function I_n/π given here can be expressed in terms of inverse cosines; then it is identical with one of Faget's functions[37]. Further details of the numerical procedure for evaluating $A(x)$ may be found in Ref. 37. The fact that I_n is symmetrical about $x = \frac{1}{2}(X_n + X_{K-n})$ may enable the calculations to be simplified.

In any application of Faget's procedure[37] it is not recommended that N be large or that a large number of streamwise strips be taken. For example, the width of the streamwise strips may conveniently be chosen to be of the same order as the body diameter while the value of N used will depend upon the wing section shape. Furthermore, the approximation to $A(x)$ obtained by this procedure can always be improved since the end points of the transferred area distribution are defined exactly by the Mach diamond enclosing the wing system. The volume condition

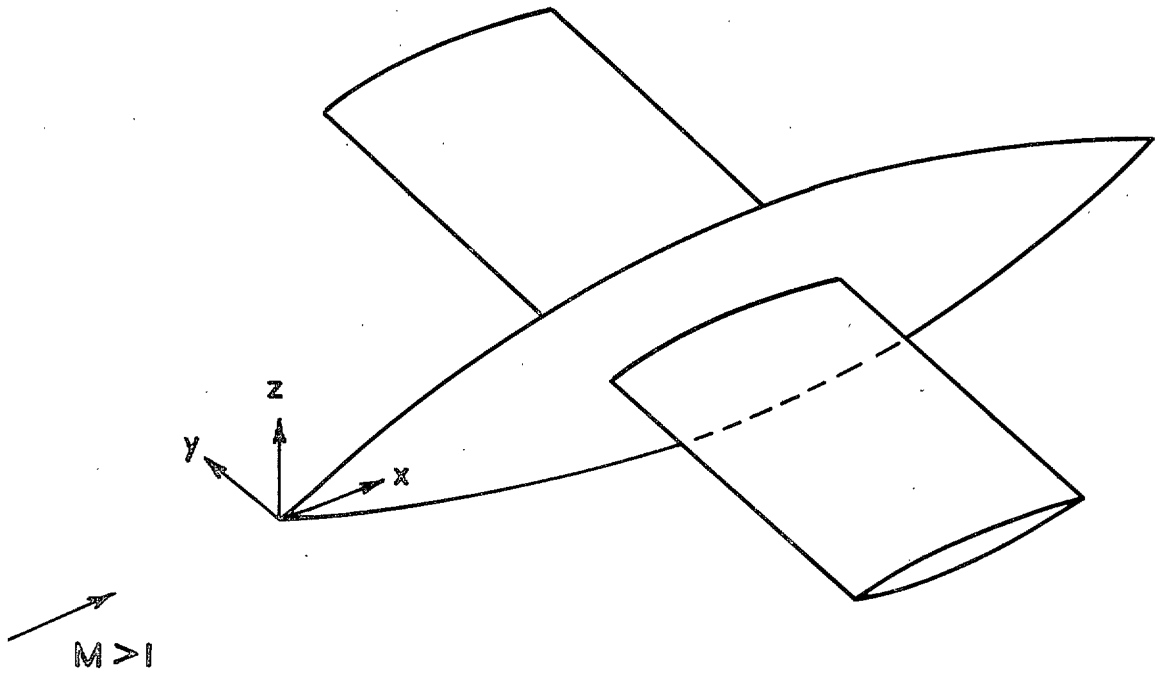
$$\int \delta A(x)\, dx = \int \Psi(x)\, dx$$

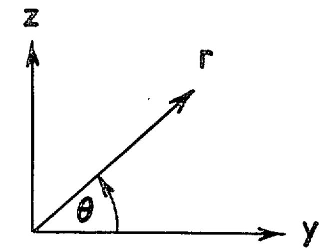
is also useful.

At low supersonic speeds the procedure can be simplified and, from Section 2.4.1, it follows that

$$\delta A(x) \simeq \tfrac{1}{2} \left\{ \Psi\left(x - \frac{BR}{\sqrt{2}}\right) + \Psi\left(x + \frac{BR}{\sqrt{2}}\right) \right\} .$$

This result is valid when BR is 'not-so-small'. When BR is small it becomes $\delta A(x) \simeq \Psi(x)$, which is the slender-body result for $\delta A(x)$.





PLANE x=CONSTANT

FIG. 1. System of co-ordinates for wing-body combination.

(71666) Wt. 52/8943 K.7 4/58 Hw.

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