R. & M. No. 3073
(15,610)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

On the Post-Buckling Behaviour of Stiffened Plane Sheet Under Shear

By
E. H. MANSFIELD, M.A.

LONDON: HER MAJESTY'S STATIONERY OFFICE
1958
PRICE 7*s.* 6*d.* NET

On the Post-Buckling Behaviour of Stiffened Plane Sheet Under Shear

By

E. H. MANSFIELD, M.A.

COMMUNICATED BY THE DIRECTOR-GENERAL OF SCIENTIFIC RESEARCH (AIR),
MINISTRY OF SUPPLY

Reports and Memoranda No. 3073*
November, 1952

Summary.—The post-buckling behaviour of a flat plate reinforced by stringers and frames is considered theoretically, attention being concentrated on the case of pure shear loading. Formulae and graphs are presented for the rapid determination of the shear stiffness, shear strain and induced compressive stresses in stringers and frames.

The analysis is an extension of work by Kromm and Marguerre.

1. *Introduction.*—It is well known that a reinforced sheet can carry shear loads considerably in excess of the initial buckling load. The behaviour of the sheet under very high loads has been fully explained by the tension-field theory of Wagner[1]. The transition from initial buckling to a pure tension field is of particular importance in the aeronautical field and it has been considered theoretically by Kromm and Marguerre[2], Kuhn[3] and Leggett[4] and experimentally by Crowther[5], Van der Neut[6] and others.

The equations derived by Kromm and Marguerre are complicated, and for pure shear loading the solutions were confined to the four limiting combinations of zero and infinite direct stiffness of the stringers and frames. In the present report these equations are modified so that the post-buckling behaviour may be predicted quickly with moderate accuracy for any combination of stringer and frame stiffness. Formulae and graphs are presented for the post-buckled state to determine the overall shear stiffness of the sheet, the overall shear strain of the sheet, the stringer stress, and the frame stress.

1.1. *Assumptions.*—The following assumptions are made regarding the structure (*see* Fig. 1):

(*a*) The material is elastic

(*b*) The frames are not attached to the skin

(*c*) The stringers have negligible torsional rigidity

(*d*) The frames and stringers do not bend.

Assumptions made in the analytical solution are given in the text and in Appendix I.

* R.A.E. Report Struct. 136, received 5th February, 1953.

2. *Statement of the Problem.*—The structure of the reinforced plate is specified, apart from overall scale factors, by the relative direct stiffnesses of the stringers (S) and the frames (F). The loading on the reinforced plate is specified by the shear stress τ and the nominal direct stress σ. The six unknown quantities are:

γ	the overall shear strain
σ_S	the average stringer compressive stress
σ_F	the frame compressive stress
λ	the buckle wave-number
α	the buckle wave-angle.
Δ	the maximum plate deflection.

2.1. *Use of Non-dimensional Symbols.*—Considerable simplification in the analysis and in the presentation of results is effected by using non-dimensional symbols. The symbols mentioned above are non-dimensional because they are expressed in terms of their initial buckling values. Precise definitions are given in the List of Symbols. It will be noted that at the onset of buckling under pure shear

$$\left.\begin{aligned} \tau = \gamma = \lambda = \alpha = 1 \\ \sigma = \sigma_S = \sigma_F = \Delta = 0 \end{aligned}\right\}, \qquad (1)$$

while at the onset of buckling under pure compression in the direction of the stringers

$$\left.\begin{aligned} \sigma = \sigma_S = \lambda = 1 \\ \tau = \gamma = \alpha = \Delta = 0 \end{aligned}\right\}, \qquad (2)$$

provided F is zero.

The amounts of material in the stringers, frames and plate are in the ratio

$$S : F : 1 .$$

A value of Δ equal to unity corresponds to a deflection of about $1 \cdot 7$ times the skin thickness.

2.2. *Equations for Solution.*—The six equations containing the six unknowns γ, σ_S, σ_F, λ, α, Δ are, essentially, those obtained by Kromm and Marguerre[1] who used a strain-energy method of solution. Brief details of the analysis leading up to the equations are given in Appendix I. The equations are:

$$(1 + \nu)(\gamma - \tau) = \frac{\alpha\lambda^2\Delta^2}{2 + \alpha^2}, \qquad (3)$$

$$(1 + F)\sigma_F - \nu\{S\sigma_S - \sigma(1 + S)\} = \left(1 + \frac{\lambda^2\alpha^2}{2 + \alpha^2}\right)\Delta^2, \qquad (4)$$

$$\sigma_S - \nu F\sigma_F + \{S\sigma_S - \sigma(1 + S)\} = \frac{2\lambda^2\Delta^2}{2 + \alpha^2}, \qquad (5)$$

$$F\sigma_F = \frac{\lambda^4 - 1}{4} - \frac{2\Delta^2}{(2 + \alpha^2)^2}, \qquad (6)$$

$$\frac{2\tau}{\alpha} - F\sigma_F = \frac{\lambda^2 + 3}{2} - \frac{2\Delta^2}{\lambda^2(2 + \alpha^2)^2}, \qquad (7)$$

$$\alpha\tau - \{S\sigma_S - \sigma(1 + S)\} = \frac{\lambda^2 + 1}{2} + \frac{\Delta^2}{(2 + \alpha^2)}\left\{\lambda^2 + \frac{\alpha^2}{\lambda^2(2 + \alpha^2)}\right\}. \qquad (8)$$

Equation (3) comes from a consideration of the average shear strain in the plate. Equations (4) and (5) come from a consideration of equilibrium of direct loads in stringers, frames and plate and from compatibility of direct strains. Equations (6) to (8) are derived from minimising the total strain energy with respect to λ, α and Δ.

In solving equations (3) to (8), Kromm and Marguerre considered only the special cases in which F was zero or infinite. In the present report no such restriction is made but attention is concentrated on the case of pure shear loading, *i.e.*, we take $\sigma = 0$. The solutions so obtained may be applied indirectly to all cases in which σ is not zero, because S occurs only in the combination $\{S\sigma_S - \sigma(1 + S)\}$. A solution obtained for

$$\left.\begin{aligned} S &= S_1 \\ \sigma &= 0 \end{aligned}\right\} \qquad (9)$$

will be a solution for

$$\left.\begin{aligned} S &= S_2 \\ \sigma &= \left(\frac{S_2 - S_1}{1 + S_2}\right)\sigma_S \end{aligned}\right\} \qquad (10)$$

2.3. *Solution of the Equations with $\sigma = 0$.*—In any particular problem S and F are known and we require the behaviour of γ, $d\tau/d\gamma$ (the shear stiffness), σ_S and σ_F as τ increases. Two distinct methods of solution are presented here. The first solution is by elimination and is obtained by regarding λ as the independent variable instead of τ. This involves considerable computation and is unsatisfactory for finding $d\tau/d\gamma$. The method is summarised in section 3. The second method of solution is indirect and is based on the exact behaviour of $d\tau/d\gamma$, γ, σ_S, σ_F immediately after buckling ($\tau = 1$) and under very high shear loads ($\tau \to \infty$), but makes use of the known general behaviour of the functions for predicting the behaviour in the range $1 < \tau < \infty$. The method, though approximate, admits of considerable generality and can be relied upon to be accurate to within 2 to 3 per cent. The method is discussed in detail in section 4.

3. *Direct Parametric Solution.*—By regarding λ as the independent variable, equations (3) to (8) can be manipulated to give:

$$\left.\begin{aligned} &[\alpha^2(1 + S)(\lambda^4 + 2\lambda^2 + 5) - 2(\lambda^2 + 1)\{2(1 + S) + \nu S(\lambda^2 - 1)\}] \times \\ &\times [F\alpha^4\lambda^2(\lambda^2 + 1) + \alpha^2 F\{2\lambda^2(\lambda^2 + 2) - \nu(\lambda^4 + \lambda^2 + 2)\} + 2\lambda^2(1 + 3F - \nu\lambda^2 F)] \\ &= [2(\lambda^2 + 1)\{(1 + F)(\lambda^2 - 1) + 2\nu F\} - \nu\alpha^2 F(\lambda^4 + 2\lambda^2 + 5)] \times \\ &\times [\alpha^2\{\lambda^4 + \lambda^2 + 2 + S(3\lambda^4 + \lambda^2 + 2)\} + 2\lambda^4 + 2S\lambda^2(3\lambda^2 - \nu)] \end{aligned}\right\} \qquad (11)$$

as an equation for determining α, and

$$8\Delta^2 = \frac{\lambda^2(2 + \alpha^2)^2[\alpha^2(1 + S)(\lambda^4 + 2\lambda^2 + 5) - 2(\lambda^2 + 1)\{2(1 + S) + \nu S(\lambda^2 - 1)\}]}{\alpha^2\{\lambda^4 + \lambda^2 + 2 + S(3\lambda^4 + \lambda^2 + 2)\} + 2\lambda^4 + 2\lambda^2 S(3\lambda^2 - \nu)} \qquad (12)$$

$$\tau = \frac{\alpha}{8}(\lambda^4 + 2\lambda^2 + 5) - \frac{\alpha\Delta^2(\lambda^2 + 1)}{\lambda^2(2 + \alpha^2)^2}. \qquad (13)$$

Once λ, α, Δ, τ are known the functions γ, σ_S and σ_F may be found directly from equations (3), (8) and (6) respectively. For values of τ lying in the range $1 < \tau < 5$, the approximate range for λ is $1 < \lambda < 1{\cdot}5$. A numerical example based on these equations is given in Appendix II. The corresponding equations in which σ is non-zero are given in Appendix III.

4. *Indirect Solution.*—Before this method can be applied expressions are required for

$$\left(\frac{d\tau}{d\gamma}\right)_1, \quad \left[\frac{d}{d\tau}\left(\frac{d\tau}{d\gamma}\right)\right]_1, \quad \left(\frac{d\sigma_S}{d\tau}\right)_1, \quad \left(\frac{d^2\sigma_S}{d\tau^2}\right)_1, \quad \left(\frac{d\sigma_F}{d\tau}\right)_1, \quad \left(\frac{d^2\sigma_F}{d\tau^2}\right)_1, \quad \left(\frac{d\tau}{d\gamma}\right)_\infty, \quad \left(\frac{d\sigma_S}{d\tau}\right)_\infty, \quad \left(\frac{d\sigma_F}{d\tau}\right)_\infty$$

and these are derived in Appendix IV.

The application of the method to the particular case of the shear stiffness will now be considered in detail. The other cases considered (shear strain, stringer stress and frame stress) are similar and the results only will be given.

4.1. *Post-Buckling Behaviour of the Shear Stiffness*.—Prior to buckling the shear stiffness $d\tau/d\gamma$ is unity. Immediately after buckling the shear stiffness drops to a value given by

$$\left(\frac{d\tau}{d\gamma}\right)_1 = \left[1 + \frac{1 + S + F + SF(1 - \nu^2)}{(1 + \nu)\{1 + 2S + 5F + SF(6 + 4\nu - \nu^2)\}}\right]^{-1}, \quad \ldots \quad \ldots \quad (14)$$

which has been plotted in Fig. 2.

Similarly, the rate of change of the shear stiffness with shear load $\left[\frac{d}{d\tau}\left(\frac{d\tau}{d\gamma}\right)\right]_1$ is a known function. The shear stiffness for large values of τ, plotted in Fig. 3, is given by

$$\left(\frac{d\tau}{d\gamma}\right)_\infty = \frac{(1 + \nu)(1 + 3S)}{1 + \nu + S(3 + \nu) + (1 + S)\alpha_\infty^{\,2}}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

where

$$\alpha_\infty^{\,2} = \sqrt{\left\{\frac{2(1 + 3S)(1 + F) - 2SF\nu^2}{F(1 + S)}\right\}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

Up to the present the analysis is exact in so far as it represents the solution of equations (3) to (8). The stiffness for values of τ in the range $1 < \tau < \infty$ is represented approximately by an equation of the form

$$\frac{d\tau}{d\gamma} = \left(\frac{d\tau}{d\gamma}\right)_\infty + \left\{\left(\frac{d\tau}{d\gamma}\right)_1 - \left(\frac{d\tau}{d\gamma}\right)_\infty\right\}\tau^{-K}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

which is correct at $\tau = 1$ and ∞ and should give a good representation of the known general behaviour at intermediate values. The constant K is chosen so that $\left[\frac{d}{d\tau}\left(\frac{d\tau}{d\gamma}\right)\right]_1$ will be correct, whence

$$K = \frac{-\left[\frac{d}{d\tau}\left(\frac{d\tau}{d\gamma}\right)\right]_1}{\left\{\left(\frac{d\tau}{d\gamma}\right)_1 - \left(\frac{d\tau}{d\gamma}\right)_\infty\right\}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

and is plotted in Fig. 4. Figs. 5 and 6 have been prepared from equation (17) and demonstrate the change in shear stiffness due to changes in S and F. A detailed example is given in section 5.

4.1.1. *Optimum value of S for maximum post-buckled shear stiffness*.—For a given amount of stiffening material $W(= S + F)$ there is an optimum value of S for which the stiffness immediately after buckling is a maximum. This value, obtained by differentiating equation (14), reduces to

$$S = \frac{W(1 + 2\nu) - 1}{3(1 + \nu)}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

Equation (19) will not, or course, be valid unless it yields a value of S sufficiently large to warrant the original assumption of negligible stringer bending.

4.2. *Post-Buckling Behaviour of the Shear Strain*.—The shear strain could be obtained by integrating equation (17) but this would have to be done graphically and it is simpler to obtain γ from the approximate relation

$$\frac{\tau - 1}{\gamma - 1} = \left(\frac{d\tau}{d\gamma}\right)_\infty + \left\{\left(\frac{d\tau}{d\gamma}\right)_1 - \left(\frac{d\tau}{d\gamma}\right)_\infty\right\}\tau^{-K/2}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

4.3. *Post-Buckling Behaviour of the Stringer Stress.*—At the onset of buckling the stringer compressive stress is determined by

$$\left(\frac{d\sigma_S}{d\tau}\right)_1 = \frac{2 + F(2 + 4\nu)}{1 + 2S + 5F + SF(6 + 4\nu - \nu^2)}, \quad \ldots \quad (21)$$

while for very high values of the shear load

$$\left(\frac{d\sigma_S}{d\tau}\right)_\infty = \frac{2(\alpha_\infty^2 + \nu)}{(1 + 3S)\alpha_\infty} \quad \ldots \quad (22)$$

and the approximate relation for determining the stringer stress is

$$\frac{\sigma_S}{\tau - 1} = \left(\frac{d\sigma_S}{d\tau}\right)_\infty - \left\{\left(\frac{d\sigma_S}{d\tau}\right)_\infty - \left(\frac{d\sigma_S}{d\tau}\right)_1\right\}\tau^{-K'} \ldots \quad (23)$$

The functions $(d\sigma_S/d\tau)_1$, $(dS\sigma_S/d\tau)_1$, $(d\sigma_S/d\tau)_\infty$, κ' are plotted in Figs. 7, 8, 9 and 10.

4.4. *Post-Buckling Behaviour of the Frame Stress.*—At the onset of buckling the frame compressive stress is determined by

$$\left(\frac{d\sigma_F}{d\tau}\right)_1 = \frac{4 + S(4 + 2\nu)}{1 + 2S + 5F + SF(6 + 4\nu - \nu^2)}, \quad \ldots \quad (24)$$

while for very high values of the shear load

$$\left(\frac{d\sigma_F}{d\tau}\right)_\infty = \frac{2}{F\alpha_\infty} \quad \ldots \quad (25)$$

and the approximate relation for determining the frame stress is

$$\frac{\sigma_F}{\tau - 1} = \left(\frac{d\sigma_F}{d\tau}\right)_\infty - \left\{\left(\frac{d\sigma_F}{d\tau}\right)_\infty - \left(\frac{d\sigma_F}{d\tau}\right)_1\right\}\tau^{-K''}, \quad \ldots \quad (26)$$

The functions $(d\sigma_F/d\tau)_1$, $(dF\sigma_F/d\tau)_1$, $(d\sigma_F/d\tau)_\infty$, κ'' are plotted in Figs. 11, 12, 13, and 14.

5. *Numerical Example.*—Consider a structure in which

$$t = 0\cdot048 \text{ in.}$$
$$b = 4\cdot8 \text{ in.}$$
$$a = 20 \text{ in.}$$
$$\text{Section area of stringer} = 0\cdot115 \text{ sq in.}$$
$$\text{Section area of frame} = 0\cdot192 \text{ sq in.}$$
$$E = 10 \times 10^6 \text{ lb/sq in.}$$
$$\nu = 0\cdot3 .$$

We require the shear stiffness, shear strain, stringer stress and frame stress at shear stresses of 5,000, 10,000 and 15,000 lb/sq in.

With these dimensions

$$S = \frac{0\cdot115}{4\cdot8 \times 0\cdot048} = 0\cdot5$$

$$F = \frac{0\cdot192}{20 \times 0\cdot048} = 0\cdot2$$

$$\sigma^* = \frac{\pi^2 E t^2}{3(1 - \nu^2)b^2} = 3{,}600 \text{ lb/sq in.}$$

$\tau = 1$ corresponds to a buckling stress of

$$\sqrt{(2)}\sigma^* = 5{,}100 \text{ lb/sq in.}$$

This value may be compared with the known exact value, for simply supported edges, of 4,850 lb/sq in. The conditions along the edges are those of 'continuity' rather than simple support so that the error suggested by these figures (6 per cent) will not in fact be so great (the error arises, of course, from the assumed mode of buckling). For convenience we can assume initial buckling stress of 5,000 lb/sq in.

At 5,000 lb/sq in. ($\tau = 1$) the relative shear stiffness drops to a value given by equation (14) or Fig. 2:

$$\left(\frac{d\tau}{d\gamma}\right)_1 = 0\cdot73$$

From Figs. 3, 4, 7, 9, 10, 11, 13 and 14 we have

$$\left(\frac{d\tau}{d\gamma}\right)_\infty = 0\cdot34, \quad \kappa = 1\cdot66, \quad \left(\frac{d\sigma_S}{d\tau}\right)_1 = 0\cdot71, \quad \left(\frac{d\sigma_S}{d\tau}\right)_\infty = 1\cdot80, \quad \kappa' = 0\cdot62,$$

$$\left(\frac{d\sigma_F}{d\tau}\right)_1 = 1\cdot70, \quad \left(\frac{d\sigma_F}{d\tau}\right)_\infty = 4\cdot73, \quad \kappa'' = 0\cdot37.$$

We are now in a position to consider the state of the structure at 10,000 lb/sq in. ($\tau = 2$) and at 15,000 lb/sq in. ($\tau = 3$). Thus, from equation (17)

$$\frac{d\tau}{d\gamma} = 0\cdot34 + 0\cdot39/\tau^{1\cdot66},$$

whence

$$\left(\frac{d\tau}{d\gamma}\right)_{\tau=2} = 0\cdot463,$$

$$\left(\frac{d\tau}{d\gamma}\right)_{\tau=3} = 0\cdot403.$$

From equation (20)

$$\frac{\tau - 1}{\gamma - 1} = 0\cdot34 + 0\cdot39/\tau^{0\cdot83},$$

whence

$$(\gamma)_{\tau=2} = 2\cdot79,$$

$$(\gamma)_{\tau=3} = 5\cdot02.$$

From equation (23)

$$\frac{\sigma_S}{\tau - 1} = 1\cdot80 - 1\cdot09/\tau^{0\cdot62}$$

whence

$$(\sigma_S)_{\tau=2} = 1\cdot09,$$

corresponding to an actual stress of $1\cdot09\sigma^* = 3{,}920$ lb/sq in. Similarly,

$$(\sigma_S)_{\tau=3} = 2\cdot49, \quad (\sigma_F)_{\tau=2} = 2\cdot39, \quad (\sigma_F)_{\tau=3} = 5\cdot42.$$

These results have been plotted in Fig. 15 and compared with values obtained by the lengthier direct solution considered numerically in Appendix II.

6. *Conclusions*.—The post-buckling behaviour under shear of plane sheet reinforced by stringers and frames has been considered theoretically. An approximate method of analysis, based on work by Kromm and Marguerre[2], has been developed. Formulae and graphs have been presented from which it is possible to determine the shear stiffness, shear strain, stringer stress and frame stress in the post-buckled state.

LIST OF SYMBOLS

t		Skin thickness
b		Stringer pitch
a		Frame pitch
S		Section area of stringer divided by bt
F		Section area of frame divided by at
E		Young's modulus
G		Shear modulus
ν		Poisson's ratio
σ^*		Buckling stress of plate under pure compression
	$=$	$\dfrac{\pi^2 E t^2}{3(1-\nu^2)b^2}$
σ_S		Compressive stress in stringer divided by σ^*
σ_F		Compressive stress in frame divided by σ^*
σ		Nominal applied compressive stress divided by σ^* (load applied to one stringer and one panel $= (1 + S)bt\,\sigma\sigma^*$)
τ		Nominal shear stress divided by $\sqrt{(2)}\sigma^*$, so that $\tau = 1$ at initial buckling under pure shear
$w_{\max}$		Maximum plate deflection $= \frac{1}{2}$(crest to trough deflection)
Δ	$=$	$\dfrac{\sqrt{\{6(1-\nu^2)\}}}{4}\dfrac{w_{\max}}{t}$
ψ		Angle between the nodal lines and stringers (*see* Fig. 1)
α	$=$	$\sqrt{2}\cot\psi$, so that $\alpha = 1$ at onset of buckling under pure shear
λ	$=$	b divided by distance between nodal lines (*see* Fig. 1)
γ	$=$	G multiplied by shear strain divided by $\sqrt{(2)}\sigma^*$, so that prior to buckling $\gamma = \tau$.
W	$=$	$S + F$
$\kappa, \kappa', \kappa''$		Decay factors

Suffices $_1$ and $_\infty$ refer to conditions when $\tau = 1$ and as $\tau \to \infty$.

REFERENCES

No.	Author	Title, etc.
1	H. Wagner	Ebene Blechwandträger mit sehr dünnen Stegblechen. *Z. Flugtech. Motorluftsch.* Vol. 20. p. 200. 1929.
2	A. Kromm and K. Marguerre ..	Verhalten eines von Schub und Druckkraften beanspruchten Plattenstreifens oberhalb der Beulgrenze. *L.F.F.* Vol. 14. p. 627. 1937.
3	Paul Kuhn	Investigations on the incompletely developed plane diagonal-tension field. N.A.C.A. Report 697. 1940.
4	D. M. A. Leggett..	The stresses in a flat panel under shear when the buckling load has been exceeded. R. & M. 2430. September, 1940.
5	F. Crowther and N. Sanderson..	Experimental investigation into plate web spars under shear. R.A.E. Reports Struct. 7 and 8. A.R.C. 11,570. February, 1948, and A.R.C. 11,573. March, 1948.
6	A. Van der Neut and W. K. G. Floor	Experimental investigation of the post-buckling behaviour of flat plates loaded in shear and compression. N.L.L. Report S.341. 1949.
7	E. H. Mansfield	Some identities on structural flexibility after buckling. R.A.E. Report Struct. 137. 1952.

APPENDIX I

Derivation of the Equations Obtained by Kromm and Marguerre

Additional Notation

Ox, Oy	Cartesian co-ordinates, Ox parallel to stringers (*see* Fig. 1)
ϕ	Airy's stress function
w	Deflection normal to plane of plate
l	Wavelength of buckles
G	Shear modulus
D	Flexural rigidity of plate $= Et^3/12(1 - \nu^2)$
p_x, p_y	Average values of the compressive stresses in the plate
q	Average shear stress in the plate
ε_{xx}, ε_{yy}	Average compressive strains in the stringers and frames
ε_{xy}	Average shear strain in the plate
V	Strain energy

Suffix $_x$ or $_y$ after ϕ or w indicates differentiation.

The equations governing the post-buckling behaviour of the plate are

$$\nabla^4\phi = E\{(w_{xy})^2 - w_{xx}w_{yy}\}\,, \qquad (27)$$

$$D\nabla^4 w = t\{\phi_{yy}w_{xx} - 2\phi_{xy}w_{xy} + \phi_{xx}w_{yy}\}\,. \qquad (28)$$

Owing to the extreme difficulty of obtaining an exact solution of these equations, approximate methods must be employed. A form for w is chosen which includes various parameters and which gives a good representation of the type of distortion observed in practice. An approximate solution of equation (27) is then possible by simple integration. The various parameters are found from a minimum-energy theorem which is used in preference to equation (28).

The assumed form for the deflection is

$$\left.\begin{aligned} w &= w_{\max} \cos \frac{\pi y}{b} \cos \frac{\pi}{l}(x - y \cot \psi) \\ &= w_{\max} \cos \frac{\pi y}{b} \cos \left\{ \left(\frac{\pi \lambda \sin \psi}{b} \right) (x - y \cot \psi) \right\} \end{aligned}\right\} \quad \ldots \quad (29)$$

Substituting this expression in equation (27) and integrating gives, for the particular integral,

$$\phi^{(p)} = - \frac{E w_{\max}^2}{16(2 + \alpha^2)} \left\{ \frac{1}{\lambda^2} \cos \frac{2\pi}{l}(x - y \cot \psi) + \lambda^2 \cos \frac{2\pi y}{b} \right\} . \quad \ldots \quad (30)$$

For the complementary function only terms which give rise to average values of the stresses are considered;

$$\phi^{(c)} = - \tfrac{1}{2} p_y x^2 - q x y - \tfrac{1}{2} p_x y^2 , \quad \ldots \quad (31)$$

and we have

$$\phi = \phi^{(p)} + \phi^{(c)} . \quad \ldots \quad (32)$$

Restricting ϕ in this way means that is is not possible to satisfy the boundary conditions exactly, but average values for the overall shear and compressive strains may be found:

$$\varepsilon_{xy} = \frac{q}{G} + \frac{\pi^2 w_{\max}^2 \cot \psi}{4 l^2} , \quad \ldots \quad (33)$$

$$\varepsilon_{yy} = \frac{p_y - \nu p_x}{E} + \frac{\pi^2 w_{\max}^2}{8 l^2} \left\{ \left(\frac{l}{b} \right)^2 + \cot^2 \psi \right\} , \quad \ldots \quad (34)$$

$$\varepsilon_{xx} = \frac{p_x - \nu p_y}{E} + \frac{\pi^2 w_{\max}^2}{8 l^2} . \quad \ldots \quad (35)$$

Equations (33), (34), (35) correspond to equations (3), (4), (5) respectively of the main text.

It is convenient to regard the strains ε_{xx}, ε_{yy} and ε_{xy} as fixed and then express the fact that the strain energy of the plate is a minimum with respect to l, ψ, $w_{\max}$ (the static analogue of Kelvin's minimum energy theorem).

From well-established theory the strain energy in a strip of plate of length l is

$$\begin{aligned} V = \tfrac{1}{2} \int_{-l/2}^{l/2} \int_{-b/2}^{b/2} \Big[& \frac{t}{E} \{ (\phi_{xx} + \phi_{yy})^2 - 2(1 + \nu)(\phi_{xx}\phi_{yy} - \phi_{xy}^2) \} \\ & + D\{ (w_{xx} + w_{yy})^2 - 2(1 - \nu)(w_{xx} w_{yy} - w_{xy}^2) \} \Big] \, dx \, dy \quad \ldots \quad (36) \end{aligned}$$

and if the values of ϕ and w, given by equations (32) and (29) are substituted we find

$$\begin{aligned} V = Etbl \Big[& \frac{\pi^4 w_{\max}^4 (1 + \lambda^4) \sin^4 \psi}{256 b^4} + \frac{\pi^4 t^2 w_{\max}^2}{96 b^4 (1 - \nu^2)} \{ (1 + \lambda^2)^2 + 4\lambda^2 \cos^2 \psi \} \\ & + \frac{1}{E^2} \{ \tfrac{1}{2} p_x^2 + \tfrac{1}{2} p_y^2 - \nu p_x p_y + (1 + \nu) q^2 \} \Big] \quad \ldots \quad (37) \end{aligned}$$

and the average strain energy per unit length is

$$\bar{V} = V / l . \quad \ldots \quad (38)$$

The conditions of minimum strain energy are now

$$\frac{\partial \bar{V}}{\partial l} \text{ or } \frac{\partial \bar{V}}{\partial \lambda} = 0\,, \quad \ldots \quad (39)$$

$$\frac{\partial \bar{V}}{\partial \psi} \text{ or } \frac{\partial \bar{V}}{\partial \alpha} = 0\,, \quad \ldots \quad (40)$$

$$\frac{\partial \bar{V}}{\partial w_{\max}} \text{ or } \frac{\partial \bar{V}}{\partial \Delta} = 0\,, \quad \ldots \quad (41)$$

subject to equations (33) to (35) being satisfied.

On introducing the non-dimensional parameters used in the main text, and equating ε_{xx}, ε_{yy} to the strains in the stringers and frames, equations (39), (40), (41) may be written, respectively:

$$\frac{2\lambda^2\Delta^2}{2+\alpha^2} + 2\{S\sigma_S - \sigma(1+S)\} + \alpha^2(F\sigma_F + 1) - 4\alpha\tau + \tfrac{1}{2}(1+\lambda^2)(2+\alpha^2) = 0\,, \quad \ldots \quad (42)$$

$$\frac{(1+\lambda^4)\Delta^2}{(2+\alpha^2)\,\lambda^2} + \{S\sigma_S - \sigma(1+S)\} - F\sigma_F + (2-\alpha^2)\tau/\alpha - 1 = 0, \quad \ldots \quad (43)$$

$$\frac{(1+\lambda^4)\Delta^2}{(2+\alpha^2)\,\lambda^2} + \{S\sigma_S - \sigma(1+S)\} + F\sigma_F\left(\frac{2+\alpha^2}{2\lambda^2} + \frac{\alpha^2}{2}\right) - $$
$$- 2\alpha\tau + \tfrac{1}{2}\alpha^2 + \frac{(1+\lambda^2)^2(2+\alpha^2)}{8\lambda^2} = 0\,. \quad \ldots \quad (44)$$

Equations (6) to (8) of the main text are derived by suitable combination of equations (42) to (44).

APPENDIX II

Example Based on Direct Parametric Solution

The example considered here is the same as that considered in section 5, for which $S = 0{\cdot}5$, $F = 0{\cdot}2$, and $\nu = 0{\cdot}3$.

We now choose an arbitrary value for λ and substitute in equation (11). Let us take, for convenience, $\lambda = \sqrt{2}$ so that $\lambda^2 = 2$. Equation (11) is now

$$(19{\cdot}5\alpha^2 - 18{\cdot}90)(1{\cdot}2\alpha^4 + 2{\cdot}72\alpha^2 + 5{\cdot}92) - (7{\cdot}92 - 0{\cdot}78\alpha^2)(16\alpha^2 + 19{\cdot}4) \equiv Z = 0\,. \quad (45)$$

This equation may be solved by trial and error.

With $\alpha^2 = 2$, $Z = -2{\cdot}09$; with $\alpha^2 = 2{\cdot}1$, $Z = +40{\cdot}2$, so that, on interpolating, we can take $\alpha^2 = 2{\cdot}005$.

From equation (12) we now find $\Delta^2 = 1{\cdot}573$ and from equation (13) $\tau = 2{\cdot}10$, whence, from equations (3), (8) and (6), $\gamma = 2{\cdot}95$, $\sigma_S = 1{\cdot}18$ and $\sigma_F = 2{\cdot}77$.

Values of λ^2 equal to $1{\cdot}1$ and $2{\cdot}5$ have also been taken. The numerical results of all the calculations are presented in tabular form below.

λ^2	α^2	α	Δ^2	τ	γ	σ_S	σ_F
1·0	1·00	1·00	0	1·00	1·00	0	0
1·1	1·102	1·050	0·1156	1·08	1·11	0·06	0·14
2·0	2·005	1·416	1·573	2·10	2·95	1·16	2·77
2·5	2·377	1·542	2·530	2·85	4·56	2·14	5·25

APPENDIX III

For the case of combined shear and compression it is convenient to obtain the solution for the case in which the ratio of applied compressive load/applied shear load remains constant during loading. If we denote this ratio by β we have

$$\beta = \sigma(1 + S)/\tau \quad \ldots \quad (46)$$

and the equations corresponding to those of para. 3 are

$$\begin{aligned}
&[\{\alpha^2(1 + S) + \alpha\beta\}(\lambda^4 + 2\lambda^2 + 5) - 2(\lambda^2 + 1)\{2(1 + S) + \nu S(\lambda^2 - 1)\}] \times \\
&\quad \times [\alpha^4 F\lambda^2(\lambda^2 + 1) + \alpha^2 F\{2\lambda^2(\lambda^2 + 2) - \nu(\lambda^4 + \lambda^2 + 2)\} + 2\lambda^2(1 + 3F - \nu F\lambda^2)] \\
&\quad = [2(\lambda^2 + 1)\{(1 + F)(\lambda^2 - 1) + 2\nu F\} - \nu F\alpha^2(\lambda^4 + 2\lambda^2 + 5)] \times \\
&\quad \times [\alpha^2\{\lambda^4 + \lambda^2 + 2 + S(3\lambda^4 + \lambda^2 + 2)\} + \alpha\beta(\lambda^2 + 1) + 2\lambda^4 + 2S\lambda^2(3\lambda^2 - \nu)] , \quad \ldots \quad (47)
\end{aligned}$$

$$8\Delta^2 = \frac{\lambda^2(2 + \alpha^2)^2[\{\alpha^2(1 + S) + \alpha\beta\}(\lambda^4 + 2\lambda^2 + 5) - 2(\lambda^2 + 1)\{2(1 + S) + \nu S(\lambda^2 - 1)\}]}{\alpha^2\{\lambda^4 + \lambda^2 + 2 + S(3\lambda^4 + \lambda^2 + 2)\} + \alpha\beta(\lambda^2 + 1) + 2\lambda^4 + 2\lambda^2 S(3\lambda^2 - \nu)} , \quad (48)$$

$$\tau = \frac{\alpha}{8}(\lambda^4 + 2\lambda^2 + 5) - \frac{\alpha\Delta^2(\lambda^2 + 1)}{\lambda^2(2 + \alpha^2)^2} , \quad \ldots \quad (49)$$

$$\sigma(1 + S) = \beta\tau . \quad \ldots \quad (50)$$

Knowing λ, α, Δ, τ, σ the functions γ, σ_S and σ_F may be found directly from equations (3), (8) and (6) respectively.

APPENDIX IV

Derivation of the Differentials $(d\tau/d\gamma)_1$, $\left[\frac{d}{d\tau}\left(\frac{d\tau}{d\gamma}\right)\right]_1$, *etc.*

In determining the differentials $(d\tau/d\gamma)_1$, $\left[\frac{d}{d\tau}\left(\frac{d\tau}{d\gamma}\right)\right]_1$, etc., use will be made of some identities[7] in which the flexibility of a structure is related to the flexibility of a structure constrained to buckle in certain fixed modes. Identities (*C*) and (*G*) of Ref. 7 state that the flexibility $d\gamma/d\tau$ and cross-flexibilities, e.g., $d\sigma_S/d\tau$, at the onset of buckling do not depend on the rate of change of mode shape. Thus we may take $\lambda = \alpha = 1$ in equations (3), (4), (5) and (44), *i.e.*, (41), which on writing Γ for $(1/3)\Delta^2$ reduce simply to

$$\left.\begin{aligned}
(1 + \nu)(\gamma - \tau) &= \Gamma \\
(1 + F)\sigma_F - \nu S\sigma_S &= 4\Gamma \\
(1 + S)\sigma_S - \nu F\sigma_F &= 2\Gamma \\
\tau - 1 - F\sigma_F - \tfrac{1}{2}S\sigma_S &= \Gamma
\end{aligned}\right\} . \quad \ldots \quad (51)$$

The flexibilities on this basis do not vary with τ so that by simple elimination we can deduce equations (14), (21), (24) of the main text.

To calculate $\left[\frac{d}{d\tau}\left(\frac{d\tau}{d\gamma}\right)\right]_1$ we observe that

$$\frac{d}{d\tau}\left(\frac{d\tau}{d\gamma}\right) \equiv -\frac{d^2\gamma}{d\tau^2} \div \left(\frac{d\gamma}{d\tau}\right)^2$$

and we shall calculate $(d^2\gamma/d\tau^2)_1$ from identity (*E*) of Ref. 7.

$$\left(\frac{d^2\sigma_S}{d\tau^2}\right)_1 \text{ and } \left(\frac{d^2\sigma_F}{d\tau^2}\right)_1$$

will be calculated from identity (H).

Consider now that λ is the independent variable and consider the state of the structure when λ has increased to $(1 + d\lambda)$. From equation (11) by substitution and neglecting second order quantities it will be found that α has increased to $(1 + \alpha' d\lambda)$, where

$$\alpha' = \frac{3}{2}\left(\frac{1 + 2S + F + SF(2 + 2\nu - \nu^2)}{1 + S + 7F + SF(7 + 3\nu - \nu^2)}\right). \qquad (52)$$

Furthermore, from equation (12):

$$\left(\frac{d\Delta^2}{d\lambda}\right)_1 = \frac{9}{2}\left(\frac{1 + S + F + SF(1 - \nu^2)}{1 + S + 7F + SF(7 + 3\nu - \nu^2)}\right), \qquad (53)$$

so that from equation (13), $\tau = 1 + \mu\, d\lambda$, where

$$\mu = \frac{3}{2}\left(\frac{1 + 2S + 5F + SF(6 + 4\nu - \nu^2)}{1 + S + 7F + SF(7 + 3\nu - \nu^2)}\right). \qquad (54)$$

Now let the structure be constrained to buckle in this mode alone ($\lambda = 1 + d\lambda$, $\alpha = 1 + \alpha' d\lambda$). The stiffness of such a constrained structure may be determined from a set of equations similar to equation (51), namely,

$$\left.\begin{aligned}(1 + \nu)(\gamma - \tau) &= \Gamma[1 + \{2 + (1/3)\alpha'\}\, d\lambda] \\ (1 + F)\sigma_F - \nu S\sigma_S &= 4\Gamma[1 + \{\tfrac{1}{2} + (1/3)\alpha'\}\, d\lambda] \\ (1 + S)\sigma_S - \nu F\sigma_F &= 2\Gamma[1 + \{2 - (2/3)\alpha'\}\, d\lambda] \\ \tau - 1 - F\sigma_F\{1 - (3/2)\, d\lambda\} - \tfrac{1}{2}S\sigma_S\, (1 - \alpha'\, d\lambda) &= \Gamma\{1 - (5/3)\alpha'\, d\lambda\}\end{aligned}\right\} \qquad (55)$$

If these equations are solved by simple elimination they give expressions for the flexibilities in the form

$$\left.\begin{aligned}\left(\frac{d\gamma}{d\tau}\right)_m &= \left(\frac{d\gamma}{d\tau}\right)_1 + A_1\, d\lambda \\ &= \left(\frac{d\gamma}{d\tau}\right)_1 + \frac{A_1}{\mu}\, d\tau\end{aligned}\right\} \qquad (56)$$

from equation (54).

And similarly

$$\left.\begin{aligned}\left(\frac{d\sigma_S}{d\tau}\right)_m &= \left(\frac{d\sigma_S}{d\tau}\right)_1 + \frac{A_2}{\mu}\, d\tau \\ \left(\frac{d\sigma_F}{d\tau}\right)_m &= \left(\frac{d\sigma_F}{d\tau}\right)_1 + \frac{A_3}{\mu}\, d\tau\end{aligned}\right\} \qquad (57)$$

Now since

$$\left(\frac{d\gamma}{d\tau}\right)_m \equiv \left(\frac{d\gamma}{d\tau}\right)_1 + \left[\frac{d}{d\tau}\left(\frac{d\gamma}{d\tau}\right)_m\right]_1 d\tau + \text{etc.}, \qquad (58)$$

we have by equating coefficients in equations (56) and (58)

$$\left[\frac{d}{d\tau}\left(\frac{d\gamma}{d\tau}\right)_m\right]_1 = \frac{A_1}{\mu}$$

whence, from identity (E) of Ref. 7:

$$\left(\frac{d^2\gamma}{d\tau^2}\right)_1 = \frac{3A_1}{2\mu}. \qquad (59)$$

Similarly we have from identity (H):

$$\left.\begin{aligned}\left(\frac{d^2\sigma_S}{d\tau^2}\right)_1 &= \frac{2A_2}{\mu} - \frac{A_1\left(\dfrac{d\sigma_S}{d\tau}\right)_1}{2\mu\left\{\left(\dfrac{d\gamma}{d\tau}\right)_1 - 1\right\}} \\ \left(\frac{d^2\sigma_F}{d\tau^2}\right)_1 &= \frac{2A_3}{\mu} - \frac{A_1\left(\dfrac{d\sigma_F}{d\tau}\right)_1}{2\mu\left\{\left(\dfrac{d\gamma}{d\tau}\right)_1 - 1\right\}}\end{aligned}\right\} \quad \ldots \quad (60)$$

$(d\tau/d\gamma)_\infty$, etc., are found by standard methods if we observe that $\lambda \to \infty$ as $\tau \to \infty$.

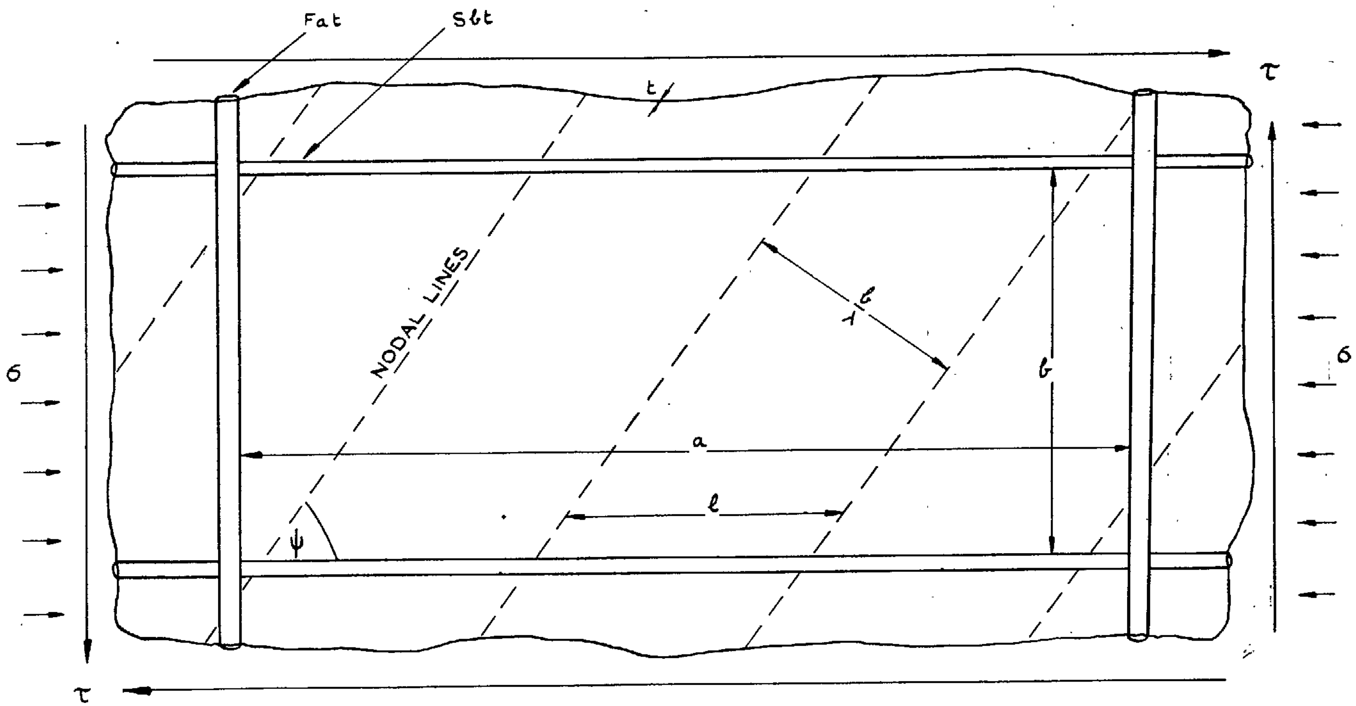

FIG. 1. Figure showing notation.

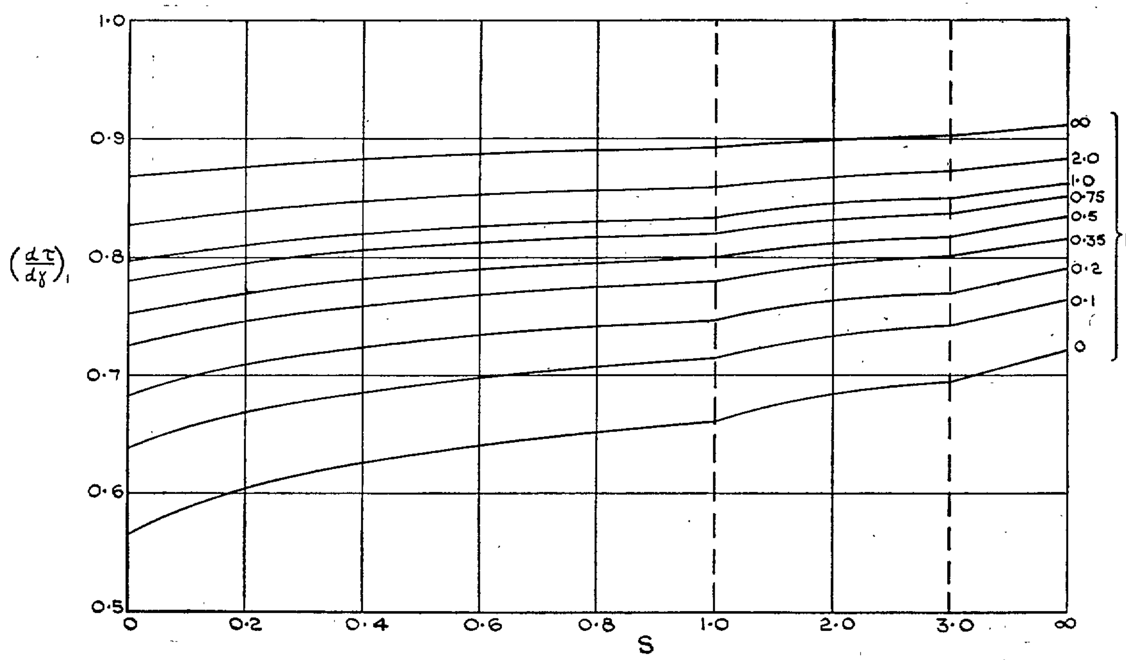

FIG. 2. Shear stiffness at onset of buckling.

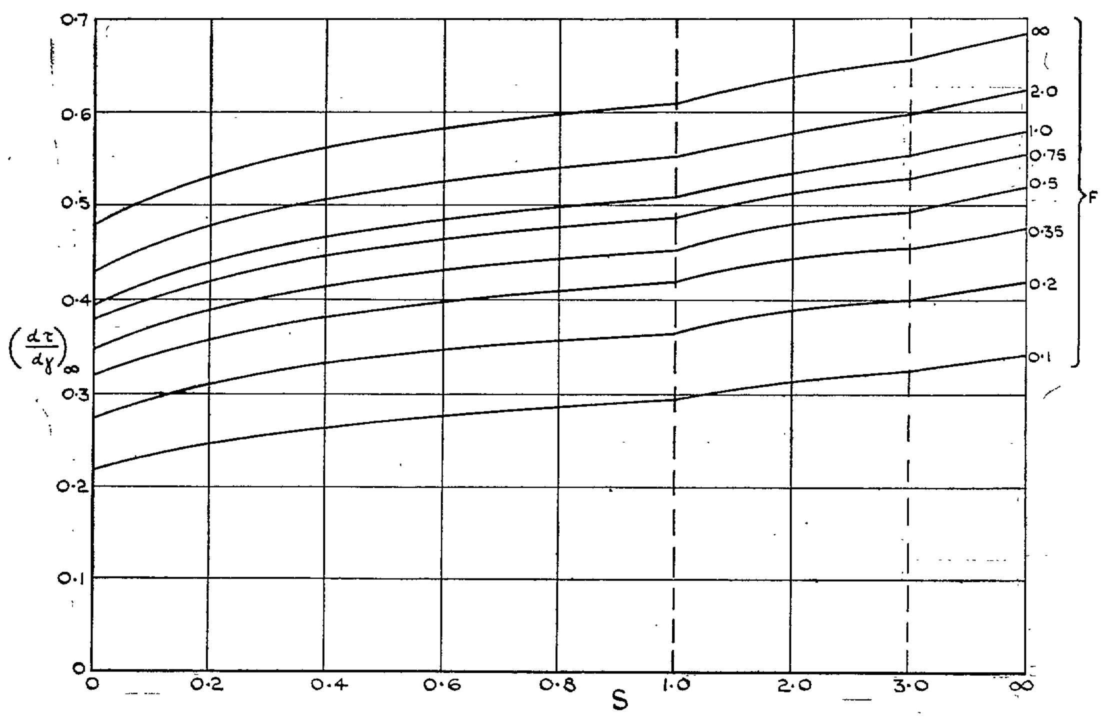

FIG. 3. Shear stiffness under high shear loads.

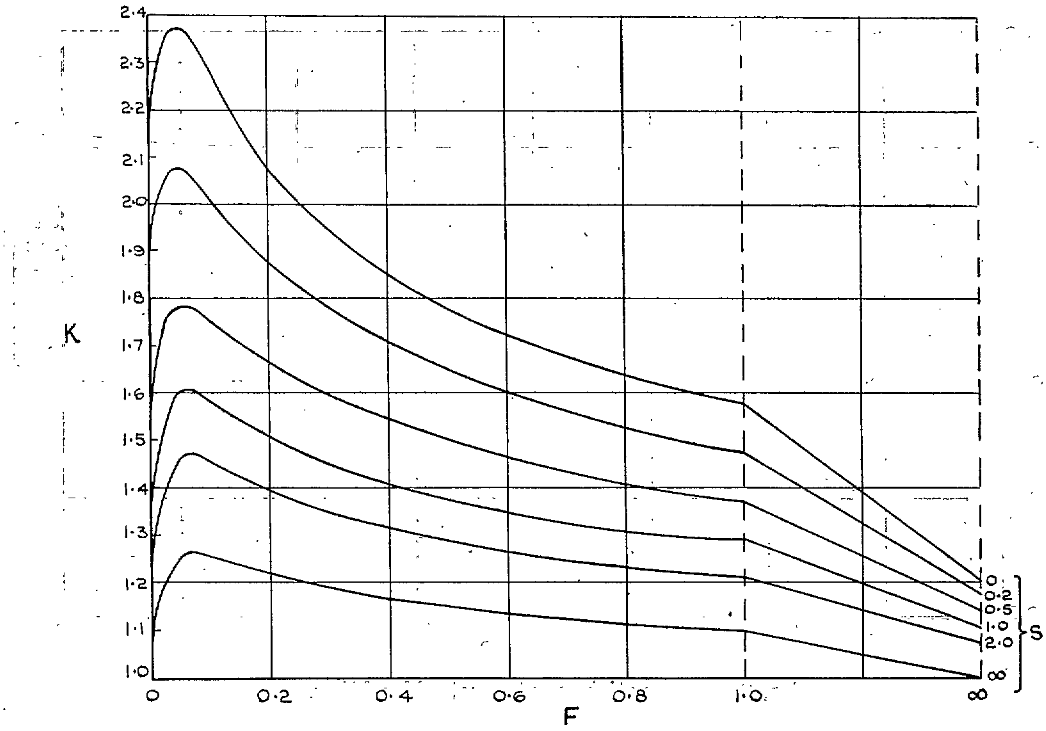

FIG. 4. The decay factor κ.

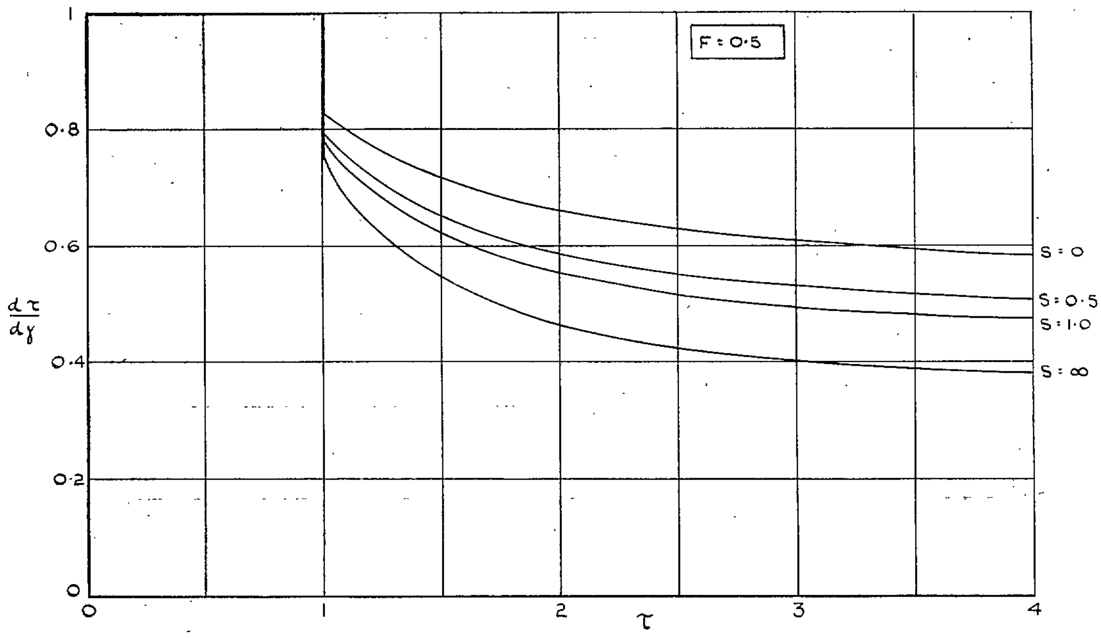

FIG. 5. Variation of stiffness with τ and S.

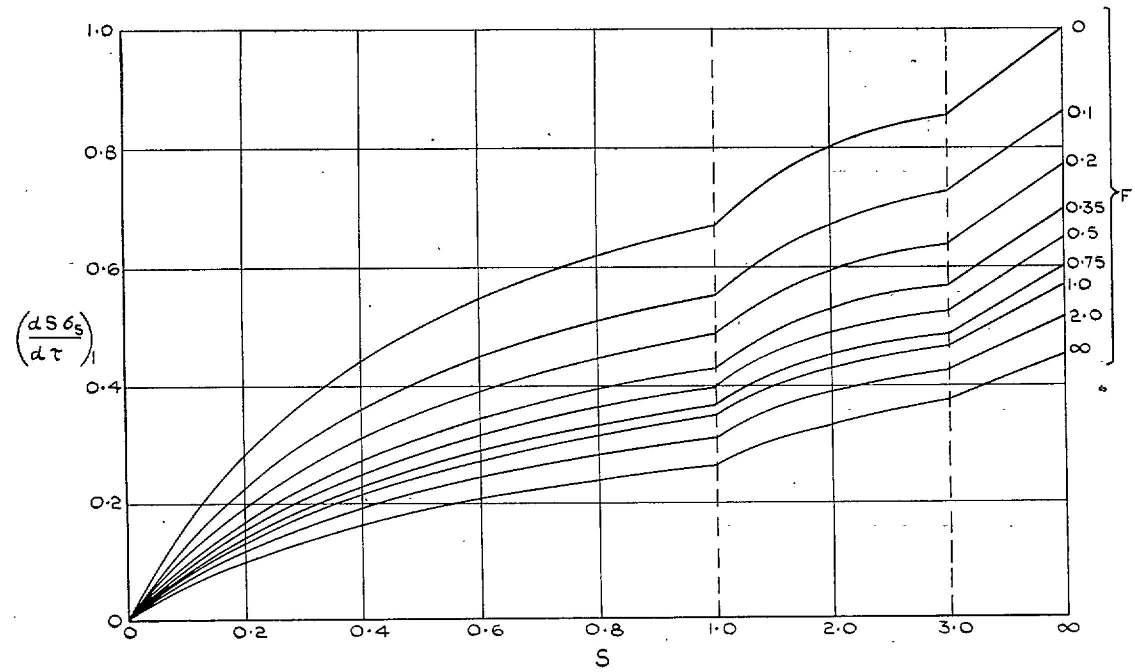

FIG. 8. Rate of increase of stringer load at onset of buckling.

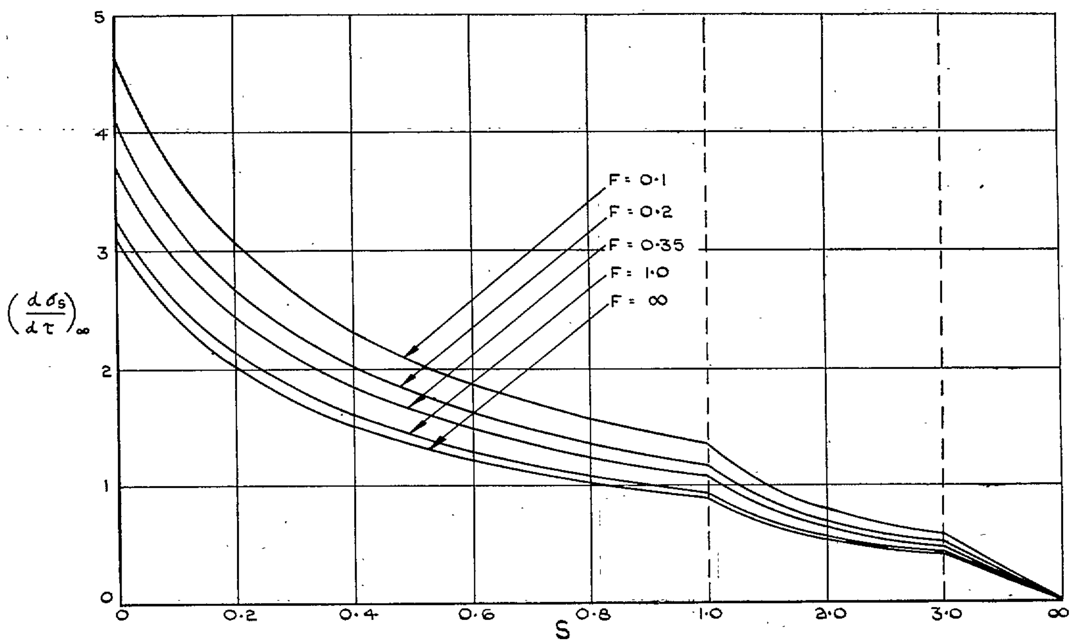

FIG. 9. Rate of increase of stringer stress under high shear loads.

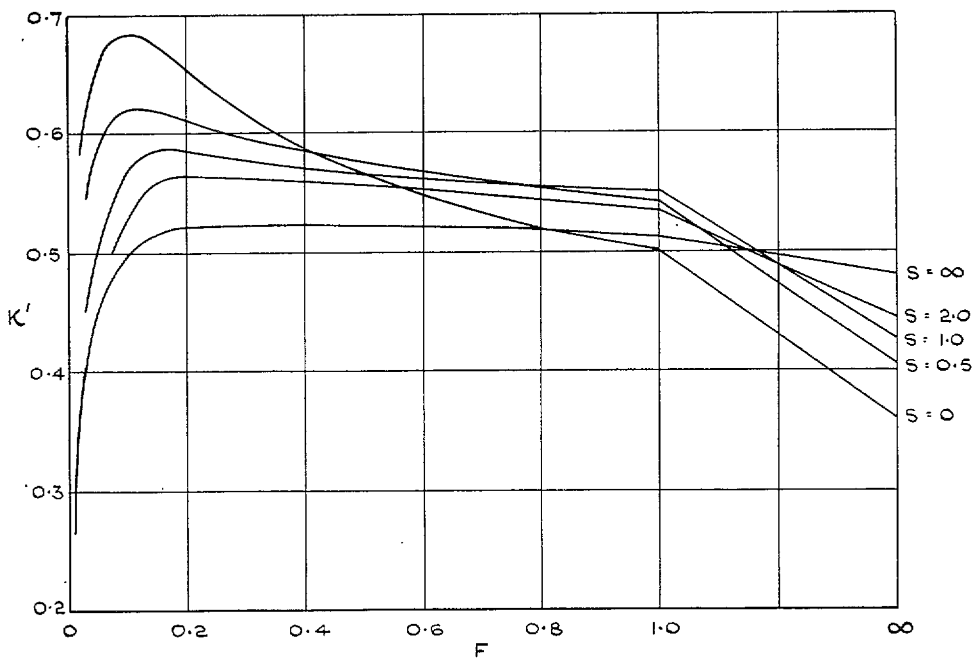

FIG. 10. The decay factor κ'.

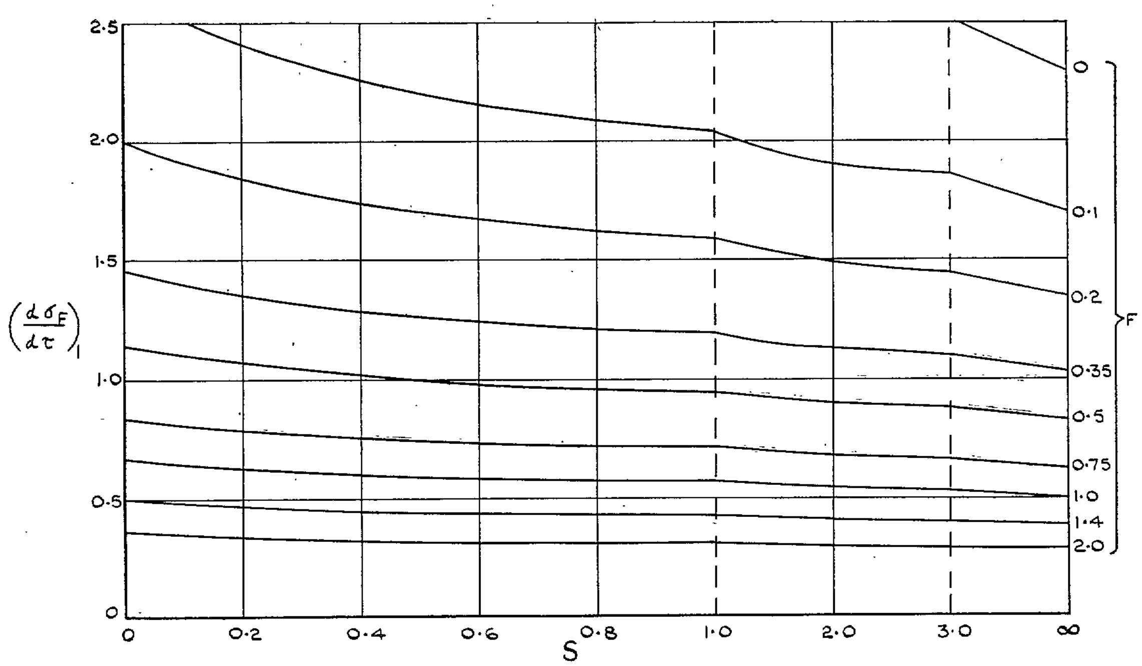

FIG. 11. Rate of increase of frame stress at onset of buckling.

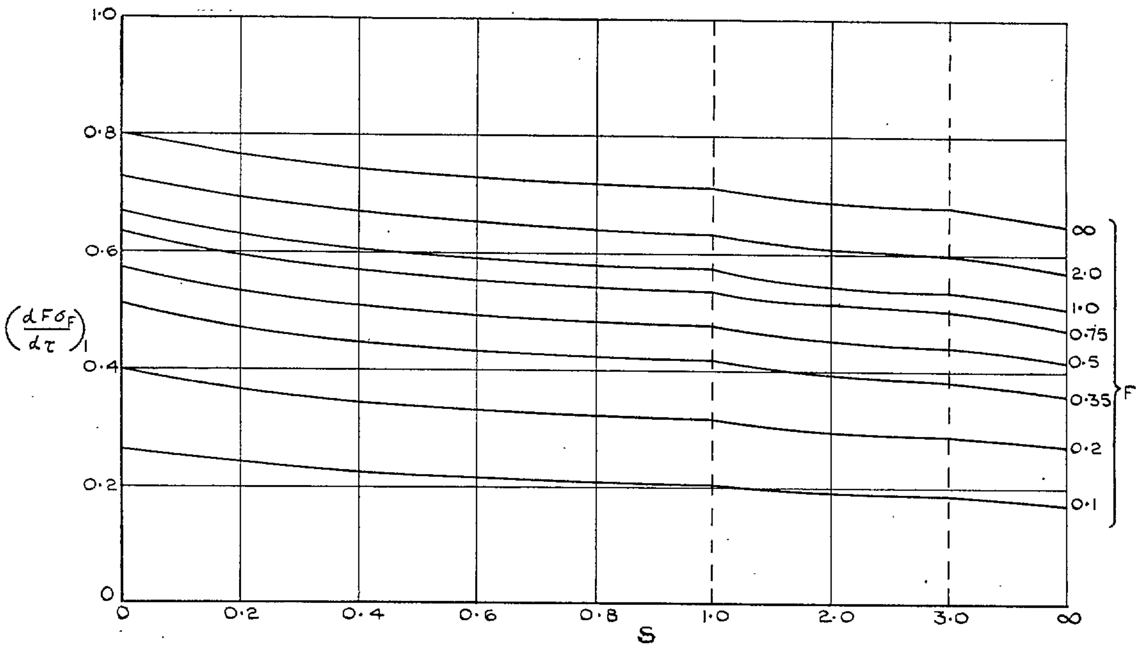

FIG. 12. Rate of increase of frame load at onset of buckling.

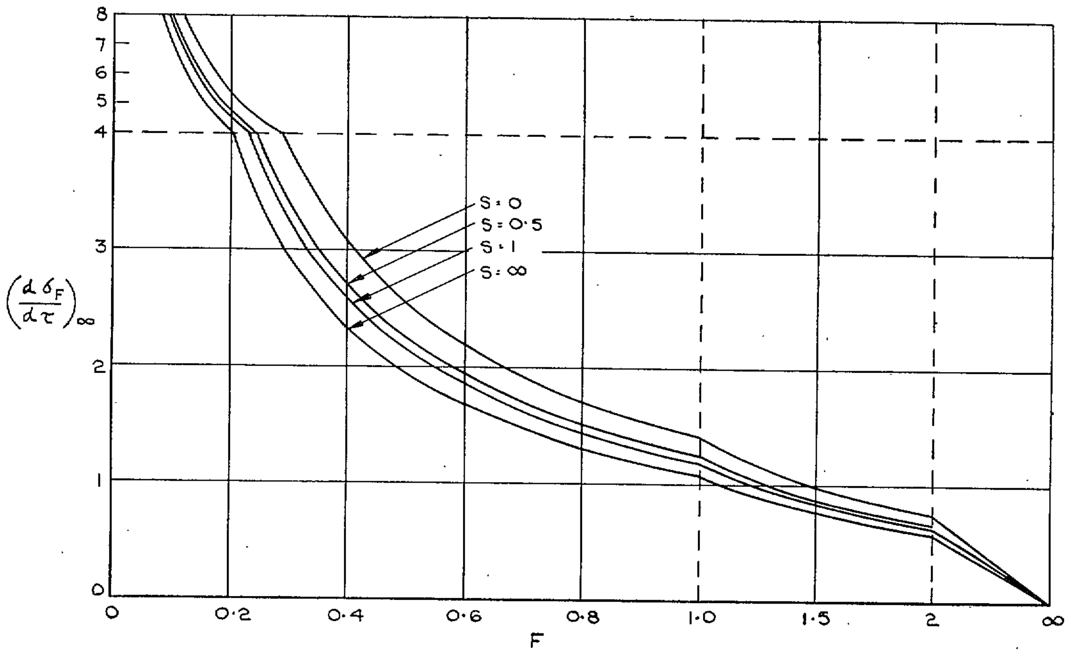

FIG. 13. Rate of increase of frame stress under high shear loads.

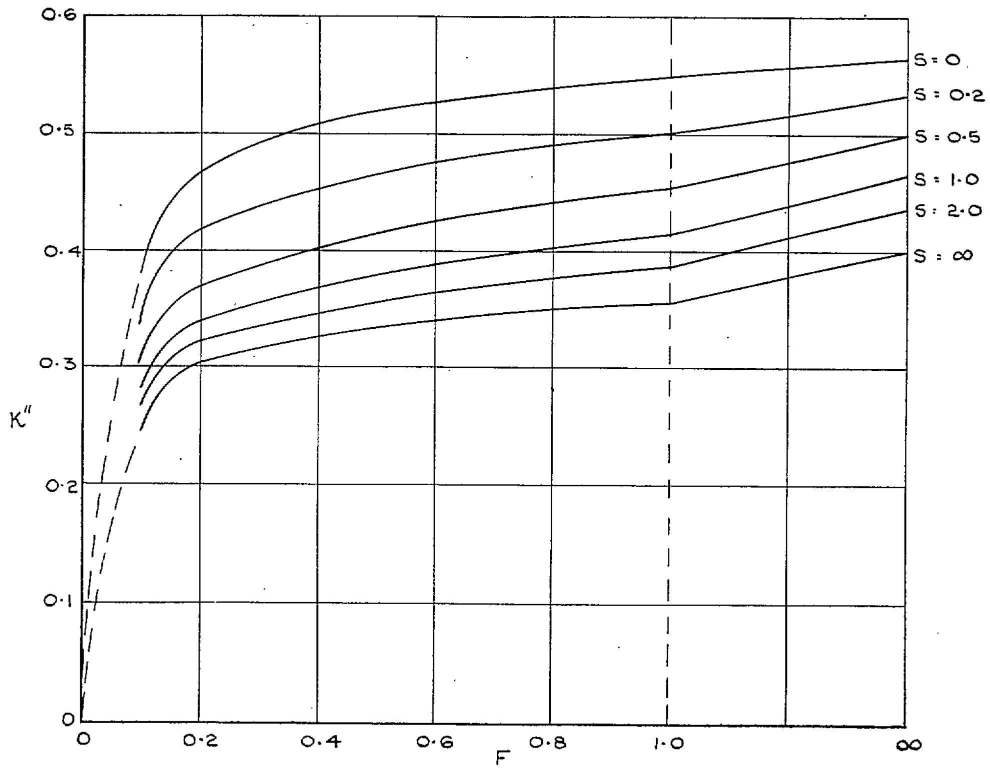

FIG. 14. The decay factor κ''.

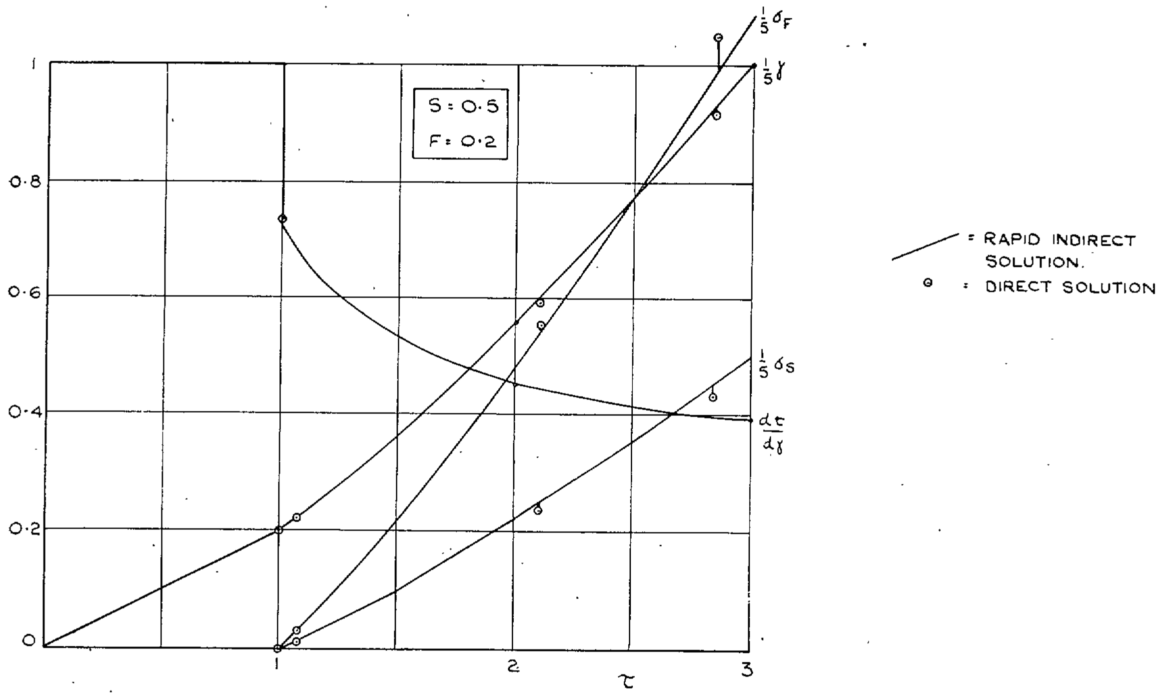

FIG. 15. Example considered in section 5 and Appendix II.

(71667) Wt. 52/8943 K7 5/58 Hw.

PRINTED IN GREAT BRITAIN

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)

1939 Vol. I. Aerodynamics General, Performance, Airscrews, Engines. 50s. (52s.).
Vol. II. Stability and Control, Flutter and Vibration, Instruments, Structures, Seaplanes, etc. 63s. (65s.)

1940 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Icing, Stability and Control, Structures, and a miscellaneous section. 50s. (52s.)

1941 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Stability and Control, Structures. 63s. (65s.)

1942 Vol. I. Aero and Hydrodynamics, Aerofoils, Airscrews, Engines. 75s. (77s.)
Vol. II. Noise, Parachutes, Stability and Control, Structures, Vibration, Wind Tunnels. 47s. 6d. (49s. 6d.)

1943 Vol. I. Aerodynamics, Aerofoils, Airscrews. 80s. (82s.)
Vol. II. Engines, Flutter, Materials, Parachutes, Performance, Stability and Control, Structures. 90s. (92s. 9d.)

1944 Vol. I. Aero and Hydrodynamics, Aerofoils, Aircraft, Airscrews, Controls. 84s. (86s. 6d.)
Vol. II. Flutter and Vibration, Materials, Miscellaneous, Navigation, Parachutes, Performance, Plates and Panels, Stability, Structures, Test Equipment, Wind Tunnels. 84s. (86s. 6d.)

1945 Vol. I. Aero and Hydrodynamics, Aerofoils. 130s. (132s. 9d.)
Vol. II. Aircraft, Airscrews, Controls. 130s. (132s. 9d.)
Vol. III. Flutter and Vibration, Instruments, Miscellaneous, Parachutes, Plates and Panels, Propulsion. 130s. (132s. 6d.)
Vol. IV. Stability, Structures, Wind Tunnels, Wind Tunnel Technique. 130s. (132s. 6d.)

Annual Reports of the Aeronautical Research Council—

1937 2s. (2s. 2d.) 1938 1s. 6d. (1s. 8d.) 1939–48 3s. (3s. 5d.)

Index to all Reports and Memoranda published in the Annual Technical Reports, and separately—

April, 1950 - - - - - R. & M. 2600 2s. 6d. (2s. 10d.)

Author Index to all Reports and Memoranda of the Aeronautical Research Council—

1909—January, 1954 R. & M. No. 2570 15s. (15s. 8d.)

Indexes to the Technical Reports of the Aeronautical Research Council—

December 1, 1936—June 30, 1939	R. & M. No. 1850	1s. 3d. (1s. 5d.)
July 1, 1939—June 30, 1945	R. & M. No. 1950	1s. (1s. 2d.)
July 1, 1945—June 30, 1946	R. & M. No. 2050	1s. (1s. 2d.)
July 1, 1946—December 31, 1946	R. & M. No. 2150	1s. 3d. (1s. 5d.)
January 1, 1947—June 30, 1947	R. & M. No. 2250	1s. 3d. (1s. 5d.)

Published Reports and Memoranda of the Aeronautical Research Council—

Between Nos. 2251–2349	R. & M. No. 2350	1s. 9d. (1s. 11d.)
Between Nos. 2351–2449	R. & M. No. 2450	2s. (2s. 2d.)
Between Nos. 2451–2549	R. & M. No. 2550	2s. 6d. (2s. 10d.)
Between Nos. 2551–2649	R. & M. No. 2650	2s. 6d. (2s. 10d.)
Between Nos. 2651–2749	R. & M. No. 2750	2s. 6d. (2s. 10d.)

Prices in brackets include postage

HER MAJESTY'S STATIONERY OFFICE

York House, Kingsway, London W.C.2; 423 Oxford Street, London W.1; 13a Castle Street, Edinburgh 2; 39 King Street, Manchester 2; 2 Edmund Street, Birmingham 3; 109 St. Mary Street, Cardiff; Tower Lane, Bristol 1; 80 Chichester Street, Belfast, *or through any bookseller.*

S.O. Code No. 23-3073