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A Subsonic Lifting-Surface Theory for Low-Aspect-Ratio Wings

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Summary.—A method is developed for determining the loading on low-aspect-ratio wings. By allowing for down stream effects on the flow at a station on the wing and for the trailing-edge condition, the method improves on R. T. Jones's theory for wings of very small aspect ratio. Calculations have been made for thirteen different wings and a comparison with other methods of solution is given for some cases.

1. *Introduction.*—In recent years several methods for obtaining numerical solutions of the lifting surface problem have been proposed, notably those of W. P. Jones[1] (1943), Falkner[2] (1943), Multhopp[3] (1950) and Küchemann[4] (1952). Although these have been successfully applied to wings of medium aspect ratio, they are essentially high-aspect-ratio methods since they determine the spanwise variation of the loading more accurately than the chordwise variation. Their application to low-aspect-ratio swept wings is also restricted by their use of either Birnbaum expansions for the chordwise loading[1,2,3] or the concept of effective incidence[4].

For wings of very small aspect ratio the theory of R. T. Jones provides a useful first approximation. This solution is not wholly satisfactory since (*a*) it predicts zero load on any part of the wing downstream of the section having maximum span and (*b*) it does not always satisfy the Kutta-Joukowski condition of zero load at the trailing edge. Several theories based on slender-body concepts overcome these objections. The first was that of Wieghardt[6] (1939), who solved the problem for the particular case of the low-aspect-ratio rectangular wing. His results were later improved in accuracy by Lomax and Sluder[7] (1951), who also gave solutions for slender delta wings. A theory applicable to low-aspect-ratio wings of almost arbitrary plan-form has been given by Lawrence[8] (1951). His method does not allow the complete load distribution to be determined, however, since it gives only the chordwise variation of total lift. Recently Legras[9] (1954) has obtained a second approximation to the loading on slender pointed wings which takes partial account of the trailing-edge condition.

The method proposed in this paper utilises certain slender-wing concepts but provides an improvement on R. T. Jones' theory. For some plan-forms it bridges the gap between slender-wing theory and the methods of Refs. 1 to 4.

2. *Statement of the Problem.*—Consider the incompressible flow* past a finite thin wing at a small angle of incidence α. Let the origin of the rectangular co-ordinates X, Y, Z be at the vertex (or mid-point of the leading edge) of the wing and let the scale and orientation of the co-ordinate system be adjusted so that the trailing edge passes through the point (1,0,0). Denote by S_W and S_T the projections on the plane $Z = 0$ of the wing and wake respectively (*see* Fig. 1).

* The linearized subsonic flow past a given wing is related to the incompressible flow past a deformed wing by the Göthert rule [10].

The following restrictions are imposed on the wing geometry:

(*a*) The wing must be symmetrical with respect to the plane $Y = 0$ and have a straight trailing edge.

(*b*) The local span must be a non-decreasing function of X.

(*c*) The equation of the mean surface may be written $Z = g(X)$.

The perturbation velocity potential due to the wing is equivalent to that of a doublet distribution over the wing and wake of strength $\gamma(X,Y)$ defined by:

$$\gamma(X, Y) = \frac{1}{4\pi}\left\{\phi(X,Y, +0) - \phi(X,Y, -0)\right\}$$

$$= \frac{1}{4\pi}\int^{X}\left\{u(X,Y, +0) - u(X,Y, -0)\right\} dX$$

$$= -\frac{1}{8\pi}\int^{X}\left\{C_p(X,Y, +0) - C_p(X,Y, -0)\right\} dX,$$

where $U_0 \cos \alpha\phi$ is the perturbation potential. It follows that the doublet strength at any point is a measure of the load carried by the chordwise section of the wing ahead of this point and also that the doublet strength is independent of X over the wake.

The problem is now to determine the doublet distribution on $S_W + S_T$ which satisfies the boundary conditions:

(i) tangential flow over the mean surface

(ii) smooth flow at the trailing edge

(iii) continuous pressure across the wake.

If U_0 is the speed of the undisturbed stream and $(uU_0 \cos \alpha,\ vU_0 \cos \alpha,\ wU_0 \cos \alpha)$ is the perturbation velocity due to the wing, the appropriate approximation to (i) becomes:

$$W(X,Y,0) = \frac{dg}{dX} - \alpha \qquad \text{on } S_W. \qquad \ldots \qquad (1)$$

Also

$$C_p = \frac{2(p - p_0)}{\rho U_0^2} = -2\,(u + \alpha w) - (v^2 + w^2), \qquad \ldots \qquad (2)$$

so that (iii) implies: $u = 0$ on S_T.

If $dg/dX = 0(\alpha)$, equation 1 is correct to $0(\alpha^2)$ as required in slender-wing theory.

3. *Fundamental Assumptions*.—The analysis of the next section is based on the hypothesis that for a large class of wings of small aspect ratio (conforming to the restrictions of section 2), the doublet distribution on $S_W + S_T$ has an approximately elliptic spanwise variation. This has been confirmed in several particular cases:

(*a*) An elliptic variation of the doublet strength is given by the slender-wing theory of R. T. Jones[5] and also by the extension to this theory by Adams and Sears[11]

(*b*) Both Wieghardt[6] and Scholz[12] have shown that the loading on low-aspect-ratio rectangular wings is very nearly elliptic

(*c*) An elliptic variation of the doublet strength is given by the author's exact solution for the incompressible flow past infinite wings of parabolic plan-form.

(*d*) Garner's[13] solution of the lifting-surface problem for a particular delta wing has shown that the hypothesis may be true for some flat wings of aspect ratios as high as three.

In all the above cases the approximately elliptic doublet distributions give rise to a constant value of the upwash across the span of the wing (i.e., $\partial/\partial Y w(X, Y, 0) = 0$ for $-s(X) < Y < s(X)$). It may be assumed that this is true for all wings for which the hypothesis is valid. Thus the required simplification of the lifting-surface integral equation can be achieved by assuming an elliptic spanwise variation of the doublet strength and by satisfying the tangency condition on the centre-line of S_W.

4. *Derivation of the Integral Equation.*—The relationship between the perturbation velocity potential $U_0 \cos \alpha\, \phi$ and its discontinuity on $S_W + S_T$ is[14]:

$$4\pi\phi(X,Y,Z) = \iint_{S_W+S_T} \left\{\phi(X,Y,+0) - \phi(X,Y,-0)\right\} \frac{\partial}{\partial z}\left(\frac{1}{r}\right) dy\, dx$$

where $$r^2 = (X-x)^2 + (Y-y)^2 + Z^2.$$

The upwash is then given by

$$w(X,Y,0) = \lim_{Z\to 0} \frac{\partial \phi}{\partial Z}.$$

This rather cumbersome limiting process can be avoided by using Hancock's formula[15]:

$$2\pi w(X,Y,0) = -\iint_{S_W+S_T} \frac{(X-x)\, u(x,y+0) + (Y-y)\, v(x,y+0)}{r^3}\, dy\, dx \quad .. \quad (3)$$

where $r^2 = (X-x)^2 + (Y-y)^2$, and the principal value of the y integration is to be taken before that of the x integration.

In accordance with the remarks of the previous section the doublet strength is taken to be:

$$\gamma(X,Y) = \frac{1}{2\pi} f(X) \sqrt{\{s^2(X) - Y^2\}},$$

where $s(X)$ is the local semi-span and $f(X)$ is a weighting function whose variation is to be determined.

Hence
$$u(X,Y+0) = 2\pi \frac{\partial \gamma}{\partial X} = f'(X)\sqrt{(s^2 - Y^2)} + f \frac{ss'(X)}{\sqrt{(s^2 - Y^2)}}$$

and
$$v(X,Y+0) = 2\pi \frac{\partial \gamma}{\partial Y} = -f \frac{Y}{\sqrt{(s^2 - Y^2)}}.$$

For $Y = 0$, the y integration in (3) may be achieved using the substitution $y = s\, cn(t,k)$, where $cn(t,k)$ is the Jacobian elliptic function of modulus $k(X,x) = s(x)/\sqrt{\{(X-x)^2 + s^2(x)\}}$.

Thus
$$\int_{-s}^{s} \frac{\sqrt{(s^2 - y^2)}}{r^3}\, dy = \frac{2k^3}{s}\int_0^K \frac{sn^2(t,k)}{dn^2(t,k)}\, dt = -\frac{k^3}{k'^2 s}\int_0^K sn(t,k) \frac{d}{dt}\frac{cn(t,k)}{dt(t,k)}\, dt$$

$$= \frac{2k^3}{k'^2 s}\int_0^K cn^2(t,k)\, dt = \frac{2k}{k'^2 s}\left\{E(k) - k'^2 K(k).\right\}$$

$$\int_{-s}^{s} \frac{1}{\sqrt{(s^2 - y^2)}} \frac{dy}{r^3} = \frac{2k^3}{s^3} \int_0^K \frac{dt}{dn^2(t,k)}$$

$$= \frac{2k^3}{s^3} \int_0^K dt + \frac{2k^5}{s^3} \int_0^K \frac{sn^2(t,k)}{dn^2(t,k)} dt = \frac{2k^3}{k'^2 s^3} E(k) .$$

$$\int_{-s}^{s} \frac{y^2}{\sqrt{(s^2 - y^2)}} \frac{dy}{r^3} = \frac{2k^3}{s} \int_0^K \frac{cn^2(t,k)}{dn^2(t,k)} dt$$

$$= \frac{2k^3}{s} \int_0^K \frac{dt}{dn^2(t,k)} - \frac{2k^3}{s} \int_0^K \frac{sn^2(t,k)}{dn^2(t,k)} dt = \frac{2k}{s} \{K(k) - E(k)\} ,$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kinds respectively.

Now on S_T (i.e., $x \geqslant 1$) : $s(x) = b/2$ and by condition (iii), $f(x) = f(1)$, thus using $dk/dx = (2/b)k'k^2$* the contribution from the wake may be obtained:

$$\int_1^{\infty} k(K - E)\, dx = -\tfrac{1}{2}b \int_0^{k(X,1)} \frac{K - E}{k'k} \mathrm{dk} = \tfrac{1}{2}b \Big[k'K \Big]_0^{k(X,1)}$$

$$= \tfrac{1}{4}\pi b + \tfrac{1}{2}b\, k'(X,1)K.$$

The upwash equation may now be written:

$$-\pi w(X,0,0) = \int_0^1 \left(\frac{E - k'^2 K}{k'} f' + \frac{s'}{s} \frac{k^2 E}{k'} f + \frac{k}{s} (K - E) f \right) dx$$

$$+ f(1)\, (\tfrac{1}{2}\pi + k'(X,1)K) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4)$$

and since

$$\frac{d}{dx}(k'K) = -\frac{K - E}{k'k} \left(\frac{s'}{s} kk'^2 + k' \frac{k^2}{s} \right) ,$$

an integration by parts yields the more compact form:

$$-\pi w(X,0,0) = \int_0^1 \frac{E}{k'} f'\, dx + \int_0^1 \frac{s'}{s} \frac{E - k'^2 K}{k'} f\, dx + \tfrac{1}{2}\pi\, \{f(0) + f(1)\}. \quad \ldots \qquad (5)$$

For a rectangular wing $s(X) = b/2$ and $f(0) = 0$, and it is found that (5), regarded as an equation in $f'(x)$, is identical to that used by Wieghardt[6] and Lomax and Sluder[7] for this plan-form.

The integral equations derived from (4) and (5) are not amenable to numerical solution and it was found that a more suitable form of the upwash integral can be obtained by an integration with respect to X.

Thus, since $\dfrac{\partial}{\partial X} = \dfrac{k'k^2}{s} \dfrac{d}{dk}$, (4) may be written:

$$-\pi w(X,0,0) = \frac{\partial}{\partial X} \int_0^1 \left\{ \frac{s}{k} (K - E) f' + s'kKf - k'Kf \right\} dx + f(1)\, \{\tfrac{1}{2}\pi + k'(X,1)K\}$$

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* In this report k' does not denote $\sqrt{(1 - k^2)}$, for convenience it is defined by

$$k' = (X - x)/\sqrt{\{(X - x)^2 + s^2(x)\}} = \pm \sqrt{(1 - k^2)}.$$

and integrating by parts:

$$- \pi w(X,0,0) = \frac{\partial}{\partial X}\int_0^1 \frac{E}{k'} f\, dx - \frac{\partial}{\partial X}\left\{\frac{K-E}{k(X,1)}\right\} f(1)\, \tfrac{1}{2}b + f(1)\,\{\tfrac{1}{2}\pi + k'(X,1)K\}$$

$$= \frac{\partial}{\partial X}\int_0^1 \frac{E}{k'} f\, dx + f(1)\left\{\tfrac{1}{2}\pi + \frac{E}{k'(X,1)}\right\} . \qquad \ldots \qquad (6)$$

On applying boundary condition (i) (equation 1), the required integral equation may be written:

$$g(X_1) - g(X_2) + \alpha\,(X_2 - X_1) = \int_0^1 G(X_2,x)\, f(x)\, dx - \int_0^1 G(X_1,x)\, f(x)\, dx$$

$$+ f(1)\int_{X_1}^{X_2} \{G(X,1) + \tfrac{1}{2}\}\, dX\,, \qquad \ldots \qquad (7)$$

$$0 < X_1 < X_2 < 1\,,$$

where

$$G(X,x) = \frac{E(k)}{\pi k'}\,, \qquad k' = \frac{(X-x)}{\sqrt{\{(X-x)^2 + s^2(x)\}}}\,.$$

Under the slender-wing assumption $s^2 \ll (X-x)^2$,

$$\frac{E}{k'} = \frac{\pi}{2}\frac{(X-x)}{|X-x|}$$

and equation (6) becomes:

$$- w(X, 0, 0) = f(X)\,, \quad 0 \leqslant X \leqslant 1$$

$$= f(1)\,, \quad 1 \leqslant X\,.$$

Thus in the limit $A \to 0$, equations (6) and (7) yield the slender-body solution.

Similarly in the limit $A \to \infty$,

$$\frac{E}{k'} \to \frac{s}{(X-x)}\,,$$

so that (6) reduces to the integral equation of two-dimensional thin-aerofoil theory:

$$- w(X,0,0) = \frac{1}{\pi}\int_0^1 \frac{u(x,0,+0)}{(X-x)}\, dx.$$

5. *Method of Solution.*—Approximate solutions of the integral equation can be obtained by finite-difference methods. Application of the trapezium-rule approximation to the integrals on the right hand side of (7) reduces the integral equation to a system of linear simultaneous algebraic equations.

To obtain an N-point solution, divide the interval (0, 1) into N sub-intervals; the resulting equations become well-conditioned by taking:

$$X_1 = \frac{2m-1}{2N}\,, X_2 = \frac{2m+1}{2N}\,, \quad m = 1, 2 \ldots . N\,.$$

This results in $N - 1$ equations in the unknowns:

$$f_N\left(\frac{n}{N}\right), \quad n = 1, 2, \ldots . N,$$

(the coefficient of $f_N(0)$ is zero.)

The Nth equation is obtained from the Kutta condition; for if the flow leaves the trailing edge smoothly it may be assumed that:

$$w(X,0,0) = \left.\frac{dg}{dX}\right|_{X=1} - \alpha \quad \text{for } 1 \leqslant X \leqslant 1 + \tfrac{1}{2}N,$$

or

$$\int_{1-\frac{1}{2}N}^{1+\frac{1}{2}N} w(X,0,0)\, dX = g\left(\frac{2N+1}{2N}\right) - g\left(\frac{2N-1}{2N}\right) - \frac{\alpha}{N} \quad \ldots \quad \ldots \quad (8)$$

if

$$g\left(\frac{2N+1}{2N}\right) = g(1) + \frac{1}{2N}\left.\frac{dg}{dX}\right|_{X=1}. \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

When the Kutta condition is applied in this way the equations become:

$$\sum_{n=1}^{N-1}\left\{G\left(\frac{2m+1}{2N}, \frac{n}{N}\right) - G\left(\frac{2m-1}{2N}, \frac{n}{N}\right)\right\} f_N\left(\frac{n}{N}\right) + \left\{G\left(\frac{2m+1}{2N}, 1\right) + \tfrac{1}{2}\right\} f_N(1)$$
$$= g\left(\frac{2m-1}{2N}\right) - g\left(\frac{2m+1}{2N}\right) + \alpha \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

for $m = 1, 2, \ldots, N$ and $g\{(2N+1)/2N\}$ defined by (9).

Values of $|G|$ as a function of k'^2 are given for $k'^2 \geqslant 0 \cdot 14$ in Tables 1 and 2. For $k'^2 \leqslant 0 \cdot 14$ it is recommended that G be calculated from the alternative form $G(X, x) = \{s/(X - x)\}(E/k)$. Values of E/k are given in Table 3.

The numerical work may sometimes be simplified by assuming the wing to be of length $2N$ so that $X - x$ takes integral values only; k'^2 is best calculated from:

$$k'^2 = \frac{(X-x)^2}{s^2}\left\{1 + \frac{(X-x)^2}{s^2}\right\}^{-1}.$$

The form of the equations makes them suitable for solution by relaxation, the starting values being deduced from lower order solutions. In using the relaxation method the following rule was found to assist convergence: 'over-relax if residue has same sign as adjacent unrelaxed residues, under-relax if residue is of opposite sign to adjacent unrelaxed residues'.

6. *Evaluation of the Aerodynamic Forces.*—The aerodynamic coefficients can be obtained from the values of $f_N(X)$ by suitable finite difference methods.

The loading is given by:

$$\Delta C_p = C_p(X,Y,-0) - C_p(X,Y,+0) = 4u(X,Y+0), \quad \ldots \quad \ldots \quad (11)$$

the most convenient expression for u being:

$$u(X,Y+0) = \sqrt{\left(1 - \frac{Y^2}{s^2}\right)}\frac{d}{dX}(fs) + f\frac{s'}{s^2}\frac{Y^2}{\sqrt{\left(1 - \frac{Y^2}{s^2}\right)}} \quad \ldots \quad \ldots \quad (12)$$

$$= \sin\theta\,\frac{d}{dX}(fs) + \cos\theta\cot\theta\, f\, s' \text{ if } Y = s\cos\theta\,.$$

This is because the function fs is more suitable for numerical differentiation (at half-way points) than f since $f(0)\ s(0) = 0$ and $fs = \frac{1}{2}f(1)b$ for $X \geqslant 1$.

The chordwise variation of the lift force is given by:

$$\frac{\partial L}{\partial X} = \pi\rho U_0^2 \frac{d}{dX}(s^2 f), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

so that

$$C_L = \tfrac{1}{2}\pi \frac{b^2}{S} f(1) = \tfrac{1}{2}\pi A f(1). \qquad \ldots \quad \ldots \quad \ldots \quad (14)$$

The pitching-moment coefficient (about $X = 0$) is:

$$-C_m = C_L - \frac{2\pi}{S}\int_0^1 s^2 f\, dX \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

and

$$h = -\frac{\partial C_m}{\partial C_L} = 1 - \frac{4}{b^2 f(1)}\int_0^1 s^2 f\, dX. \qquad \ldots \quad \ldots \quad \ldots \quad (16)$$

The spanwise variation of total lift is elliptic; the variation of local aerodynamic centres is given by:

$$h(Y) = 1 - \frac{4}{f(1)\sqrt{(b^2 - 4Y^2)}}\int_{X_0}^1 f\sqrt{(s^2 - Y^2)}\, dX, \qquad \ldots \quad \ldots \quad (17)$$

where the lower limit of integration is defined by $s^2(X_0) = Y^2$.

Sufficient accurate estimations of the integrals in (15), (16) and (17) may be obtained by judicious use of the Gregory* formula, except for rectangular wings where Bickley's* formula should be used near $X = 0$.

In subsequent sections of this report the symbols C_{LN}, h_N, etc., will be used to denote the values of these coefficients based on N-point solutions of the integral equation.

7. *Convergence and Asymptotic Values.*—No attempt will be made to establish formally the convergence of the solutions of this paper. However, some conclusions as to their behaviour for large N may be obtained by a technique similar to Richardson's 'deferred approach to the limit '[17, 18].

The method is essentially a statistical one based on the 32 solutions of equation (10) which have so far been obtained. These solutions are listed in Table 4. A detailed discussion of the results is given in section 8. The basis of the method is illustrated in Figs. 10 and 11, which show that there is a linear correlation between $\{f_N(1)\}/\alpha$ and $k'^2\{(2N+1)/2N, 1\} = 1/(1 + b^2N^2)$ and also between h_N and $1/(1 + b^2N^2)$. Since values of $\{f_N(1)\}/\alpha$ and h_N are determined by purely algebraic processes which are invariant for all finite N, an extrapolation to $1/(1 + b^2N^2) = 0$ will yield asymptotic values of these derivatives. Thus it appears that the solutions of this paper converge like $1/N^2$ but it is not apparent that their limiting values are equal to the exact values.

These remarks do not apply to the results obtained for wing H. In this case the variation of $\{f_N(1)\}/\alpha$ with N was negligible, the variation of h_N appeared to take the form $h_N = a + (b/N)$.

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* These formulae are given in Ref. 17.

8. *Discussion of the Results.*—The thirteen flat $\{g(X) \equiv 0\}$ wings listed in Table 4 and sketched in Fig. 2 were chosen to test the validity of the method over a wide range of plan-forms. No undesirable feature of the method was revealed by this investigation.

Some typical solutions for $f_9(X)$ are shown in Fig. 3; Figs. 4 to 9 show the chordwise variation of the loading at the centre-line for all thirteen wings. Fig. 6 illustrates the effect of cropping a 15-deg (apex semi-angle) delta wing.

Several comparisons with existing theoretical results are possible. In Fig. 9 the results obtained by Lomax and Sluder for a square plan-form are plotted alongside those of this report. Both curves represent 9-point solutions. Values of $\partial C_L/\partial \alpha$ and h given by various authors are:

Lomax and Sluder[7]:	$\partial C_L/\partial\alpha = 1\cdot465$	$h = 0\cdot168$
Falkner[19]:	$1\cdot49$	$0\cdot148$
Scholz[12]:	$1\cdot45$	$0\cdot179$
Present method:	$1\cdot44$	$0\cdot192$.

In Fig. 10 a comparison is shown with the results of Garner[13] and Hancock[16] for wing G_1 (aspect ratio three). The agreement between the values for $N = 7$ and those of Garner, near the centre of the wing, is surprisingly good since the present method is applicable primarily to wings of lower aspect ratio. The less satisfactory agreement near the tips is probably due to the divergence of the true circulation distribution from the assumed elliptic form. Values of $\partial C_L/\partial\alpha$ and h obtained by this and other methods are:

Garner[13]:	$\partial C_L/\partial\alpha = 3\cdot038$	$\overset{h}{\cancel{H}} = 0\cdot533$
Multhopp[3]:	$3\cdot057$	$0\cdot535$
Falkner[19]:	$3\cdot192$	$0\cdot531$
Present method:	$3\cdot284$	$0\cdot534$.

Figs. 13 and 14 show the variation with aspect ratio of the asymptotic values of $\{f(1)\}/\alpha = (2/\pi A)\,(\partial C_L/\partial\alpha)$ and h for slender parabolic and triangular wings. It is thought that the values for very low-aspect-ratio triangular wings given by the present method are more accurate than those given by Lomax and Sluder since the method of obtaining asymptotic values reduces the error due to approximations. Lomax and Sluder's curve appears to have been based on results for aspect ratios $0\cdot90$, $1\cdot26$ and $1\cdot79$, in which case slight changes resulting from higher order solutions might seriously affect the shape of the extrapolated curve.

Calculated values of $\partial C_L/\partial\alpha$ and h for all the wings investigated are listed in Table 5.

9. *Conclusions.*—The proposed method of obtaining solutions of the lifting-surface problem has been found satisfactory for a large variety of low-aspect-ratio plan-forms, and it seems satisfactory for delta wings for aspect ratios up to three.

10. *Acknowledgments.*—This investigation was made under the supervision of Professor H. B. Squire of Imperial College. The author is also indebted to Professor W. G. Bickley for valuable discussion and to Mrs. P. Armitage for assistance with the numerical work.

NOTATION

X, Y, Z		Rectangular cartesian co-ordinates
x,y		Variables of integration
u,v,w		The X, Y, Z components of the perturbation velocity
α		Angle of incidence
$g(X)$		Function defining the mean surface
U_0		Speed of main stream
S_W,S_T		Projections on $Z = 0$ of the wing and wake respectively
S		Area of S_W
$s(X)$		Local semi-span
b		Maximum span $= 2s(1)$
$K(k),E(k)$		Complete elliptic integrals of the first and second kinds
$k(X,x)$		Modulus of the elliptic integrals
$G(X,x)$		Kernel of the integral equation
$f(X)$		Chordwise weighting factor
N		Order of the approximate solutions
$f_N(X)$		Approximate solution of order N
C_p	=	$(p - p_0)/\frac{1}{2}\rho U_0^2$.
ΔC_p	=	$C_{p\ \text{(lower surface)}} - C_{p\ \text{(upper surface)}}$.
C_L		Lift coefficient
C_M		Moment coefficient about $X = 0$.
h		Position of aerodynamic centre.
A		Aspect ratio

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15	G. J. Hancock	The design of thin finite wings in incompressible flow. A.R.C. 15,185. September, 1952. Also *Aero. Quart.* V, p. 119. July, 1954.
16	G. J. Hancock	Method for the determination of the pressure distribution over a finite thin wing at a steady low speed. C.P. 128. September, 1952.
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19	V. M. Falkner	Calculated loadings due to incidence on a number of straight and swept-back wings. R. & M. 2596. June, 1948.

TABLE 1

$$|G| = \frac{E(k)}{\pi|k'|}$$

Table 1 is for linear interpolation, δ'' being less than 5.

k'^2	$\|G\|$	$\delta'_{1/2}$
·75	·53937	
		— 197
·76	·53740	
		— 193
·77	·53547	
		— 189
·78	·53358	
		— 185
·79	·53173	
		— 181
·80	·52992	
		— 178
·81	·52814	
		— 174
·82	·52640	
		— 171
·83	·52469	
		— 167
·84	·52302	
		— 164
·85	·52138	
		— 161
·86	·51977	
		— 158
·87	·51819	
		— 156
·88	·51663	
		— 152
·89	·51511	
		— 150
·90	·51361	
		— 147
·91	·51214	
		— 144
·92	·51070	
		— 142
·93	·50928	
		— 139
·94	·50789	
		— 137
·95	·50652	
		— 135
·96	·50517	
		— 133
·97	·50384	
		— 130
·98	·50254	
		— 128
·99	·50126	
		— 126
1·00	·50000	

TABLE 2

$$|G| = \frac{E(k)}{\pi |k'|}$$

Table 2 has $\delta''' < 60$ and $\delta'''' < 20$ and is intended for use with Bessel's formula to second differences: $G_n = G_0 + n\delta'_{1/2} + B''(\delta''_0 + \delta''_1)$.

k'^2	$\lvert G \rvert$	$\delta'_{1/2}$	$\delta''_0 + \delta''_1$
·14	·96642		
		— 2669	— 574
·15	·93973		
		— 2407	— 486
·16	·91566		
		— 2183	— 417
·17	·89383		
		— 1992	— 356
·18	·87391		
		— 1827	— 309
·19	·85564		
		— 1683	— 270
·20	·83881		
		— 1557	— 238
·21	·82324		
		— 1445	— 211
·22	·80879		
		— 1346	— 188
·23	·79533		
		— 1257	— 168
·24	·78276		
		— 2284	— 579
·26	·75992		
		— 2024	— 476
·28	·73968		
		— 1808	— 396
·30	·72160		
		— 1628	— 333
·32	·70532		
		— 1475	— 283
·34	·69057		
		— 1345	— 244
·36	·67712		
		— 1231	— 213
·38	·66481		
		— 1132	— 185
·40	·65349		
		— 1046	— 162
·42	·64303		
		— 970	— 143
·44	·63333		
		— 903	— 128
·46	·62430		
		— 842	— 115
·48	·61588		
		— 788	— 103
·50	·60800		
		— 739	— 93
·52	·60061		
		— 695	— 84
·54	·59366		
		— 655	— 76
·56	·58711		
		— 619	— 69
·58	·58092		
		— 586	— 64
·60	·57506		
		— 1336	— 347
·65	·56170		
		— 1181	— 284
·70	·54989		
		— 1052	— 236
·75	·53937		

TABLE 3

Table 3 is for linear interpolation.

k'^2	E/k	$\delta'_{1/2}$
0	1·0000	
		2111
·01	1·02111	
		1794
·02	1·03905	
		1686
·03	1·05591	
		1625
·04	1·07216	
		1586
·05	1·08802	
		1559
·06	1·10361	
		1539
·07	1·11900	
		1525
·08	1·13425	
		1517
·09	1·14942	
		1512
·10	1·16454	
		1509
·11	1·17963	
		1509
·12	1·19472	
		1511
·13	1·20983	
		1515
·14	1·22498	

TABLE 4

Wings Investigated.

Wing	Plan-form	A	t	Solutions obtained for:
A	Parabolic	0·5	0	$N =$ 3, 6, 9
B	Parabolic	1·0	0	3, 6, 9
C	Delta	0·35	0	3, 6, 9, 12, 15
D	Delta	0·705	0	5, 10
E	Delta	1·072	0	3, 6, 9
E_1	Cropped delta	0·858	1/9	9
F_1	Cropped delta	2·0	1/7	7
F_2	Cropped delta	1·6	1/4	4, 8
F_3	Cropped delta	1·33	1/3	3, 6, 9
G_1	Cropped delta	3·0	1/7	7
G_2	Cropped delta	2·4	1/4	4, 8
G_3	Cropped delta	2·0	1/3	3, 6, 9
H	Square	1·0	1·0	3, 6, 9

Notes:

1. t = (tip chord) $\div$ (root chord).
2. The plan-forms are sketched in Fig. 2.
3. Wings C, D, & E have semi apex angles 5 deg, 10 deg and 15 deg respectively.
4. Wing E_1 is derived by cropping wing E.
5. Wings F_1, F_2, F_3, are obtained by cropping a delta wing of aspect ratio 8/3; wings G_1, G_2, G_3 are similarly obtained from a delta of aspect ratio 4.

TABLE 5

Results of Calculations.

Wing	Plan-form	$\partial C_L / \partial \alpha$	h	Remarks
A	Parabolic	·736	·480	Asymptotic values
B	Parabolic	1·389	·470	Asymptotic values
C	Delta	·519	·650	Asymptotic values
D	Delta	·994	·639	Asymptotic values
E	Delta	1·445	·628	Asymptotic values
E_1	Cropped delta	1·232	·578	$N = 9$ values
F_1	Cropped delta	2·474	·543	$N = 7$ values
F_2	Cropped delta	2·097	·494	Asymptotic values
F_3	Cropped delta	1·833	·454	Asymptotic values
G_1	Cropped delta	3·284	·534	$N = 7$ values
G_2	Cropped delta	2·818	·487	Asymptotic values
G_3	Cropped delta	2·488	·450	Asymptotic values
H	Square	1·441	·192	Asymptotic values

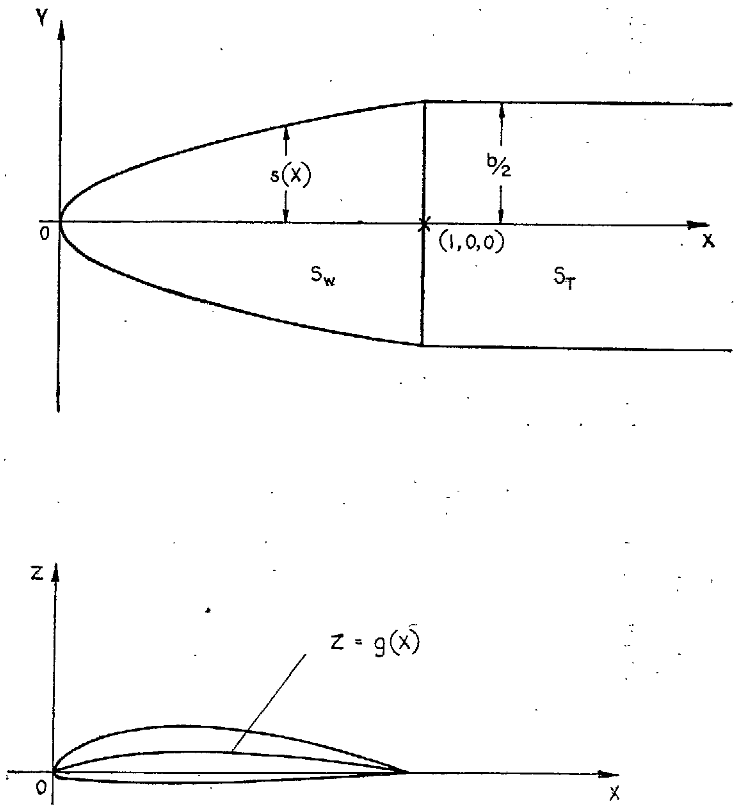

FIG. 1. Plan and elevation of wing.

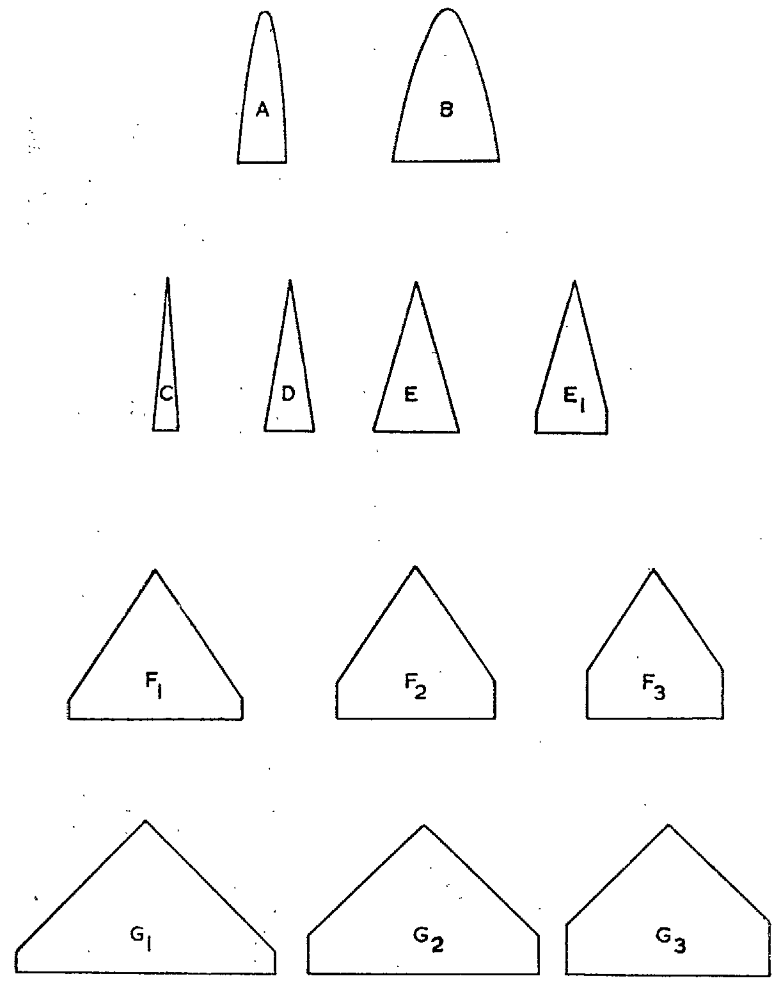

FIG. 2. Plan views of wings A to G_3.

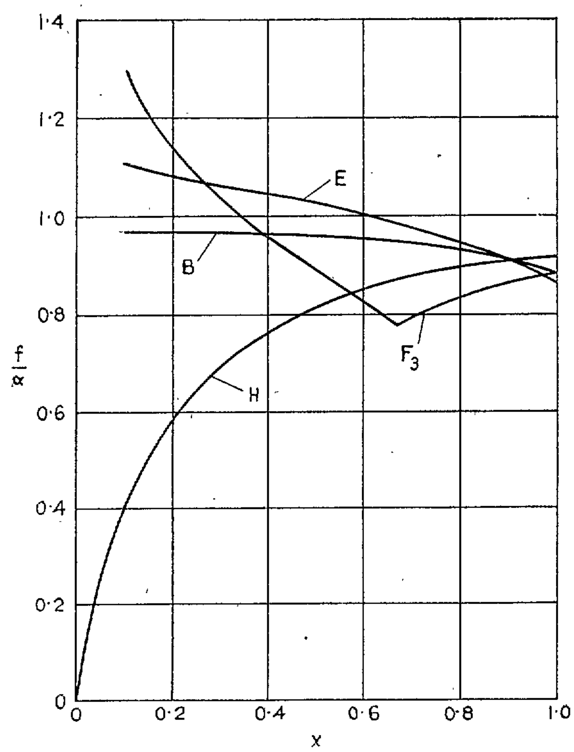

FIG. 3. Some typical solutions for the weighting factor.

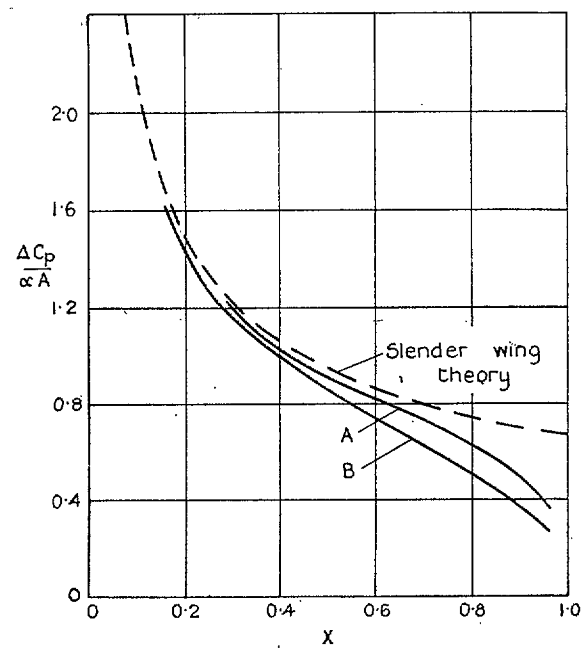

FIG. 4. Variation of loading along centre-lines of parabolic wings.

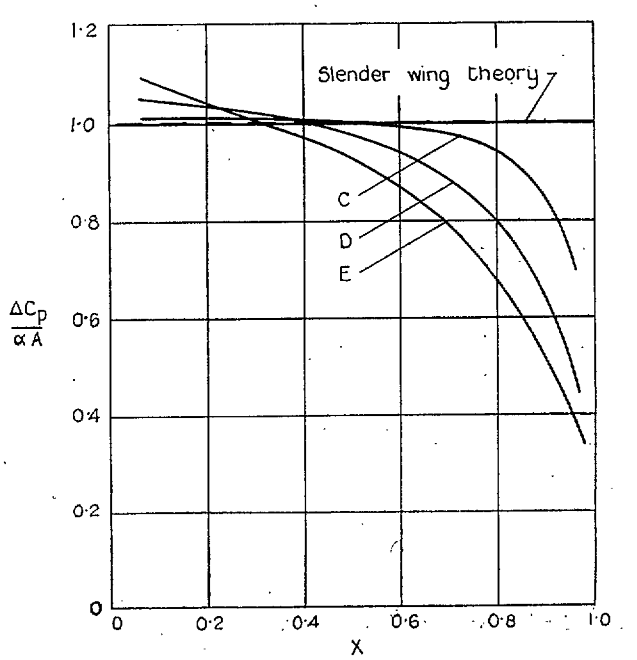

FIG. 5. Variation of loading along the centre-lines of triangular wings.

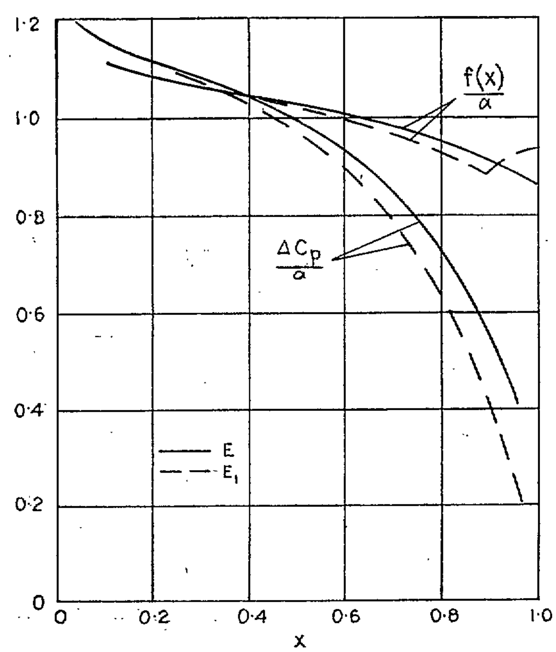

FIG. 6. Variation of f/α and $\Delta C_p/\alpha$ along centre-lines of wings E_1 and E_2.

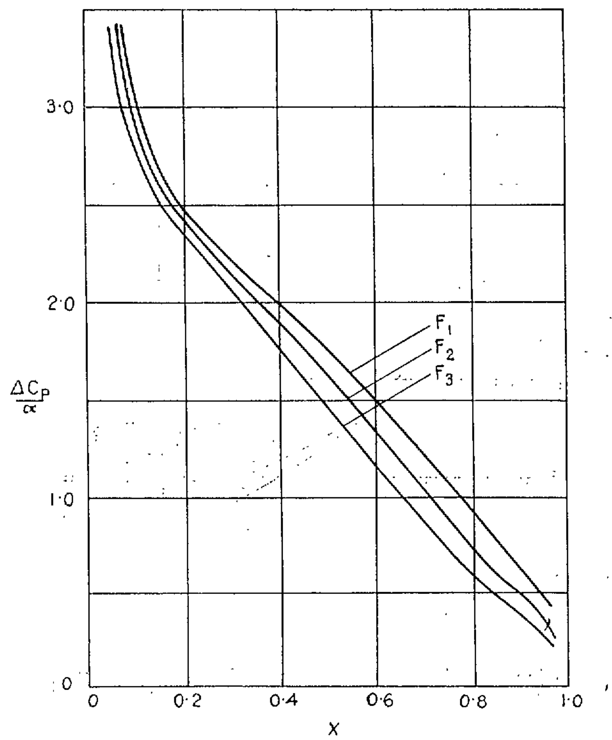

FIG. 7. Variation of loading on centre-lines of wings F_1, F_2 and F_3.

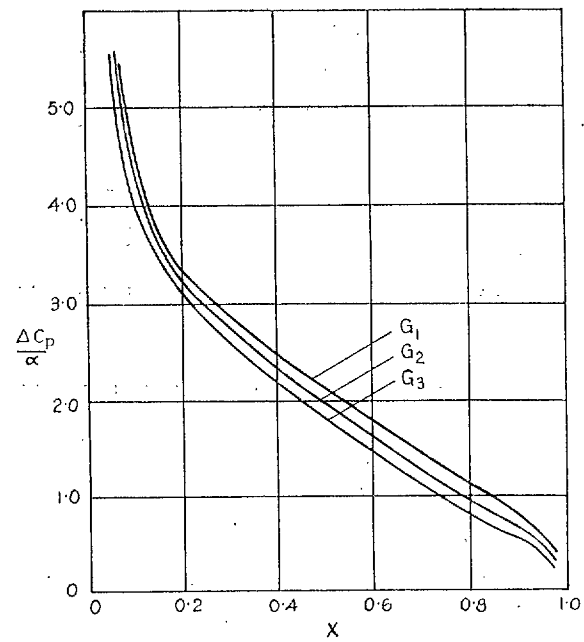

FIG. 8. Variation of loading on centre-lines of wings G_1, G_2 and G_3.

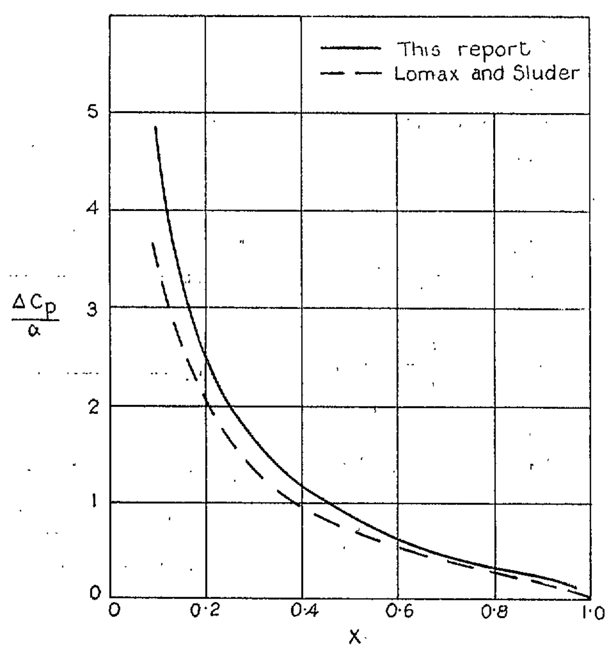

FIG. 9. Variation of loading along the centre-line of a square wing.

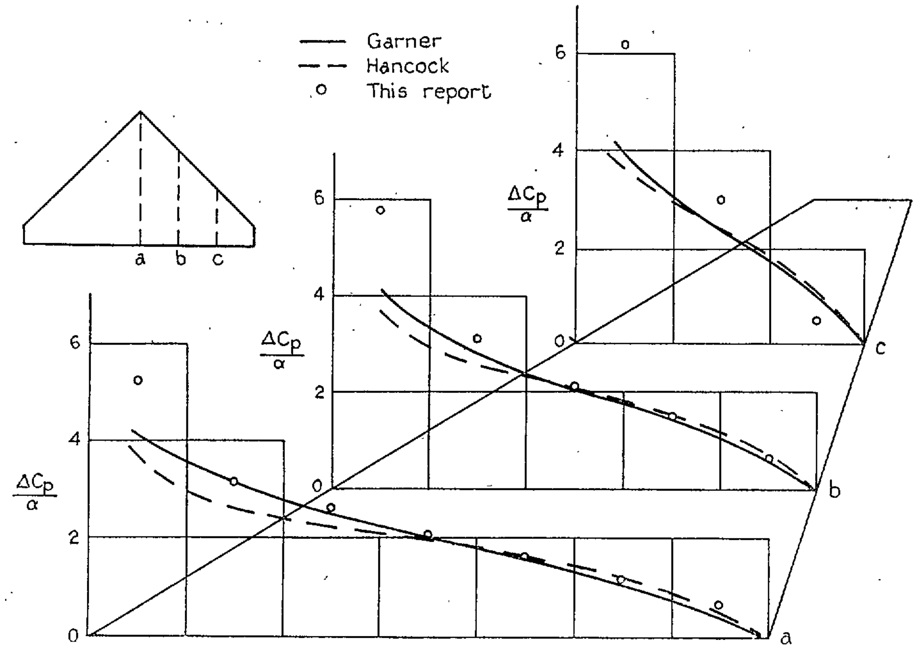

FIG. 10. Load distribution on wing G_1.

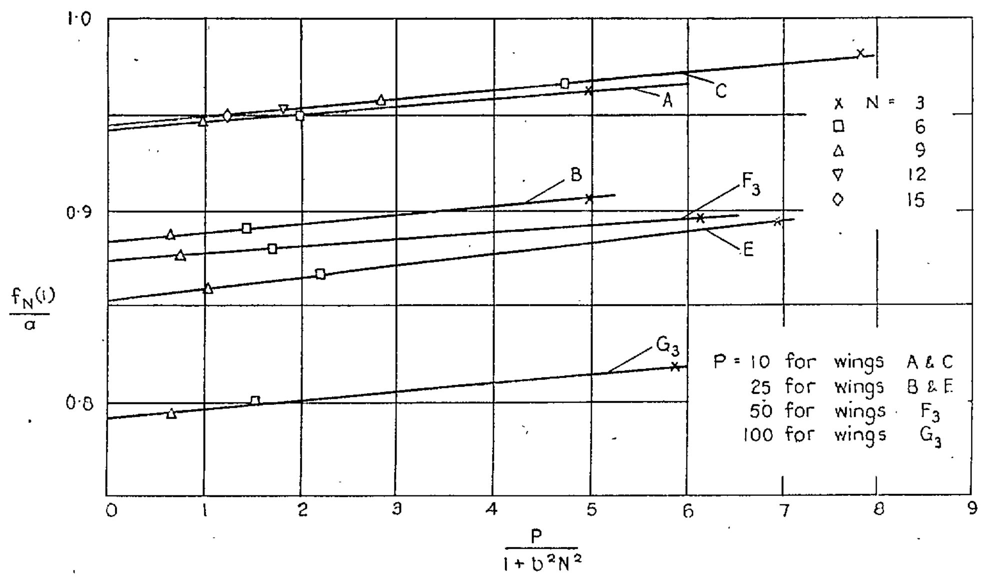

FIG. 11. Variation of $\{f_N(1)\}/\alpha$ with N.

(40854) Wt. 52 K7 7/57 F. M. & S.

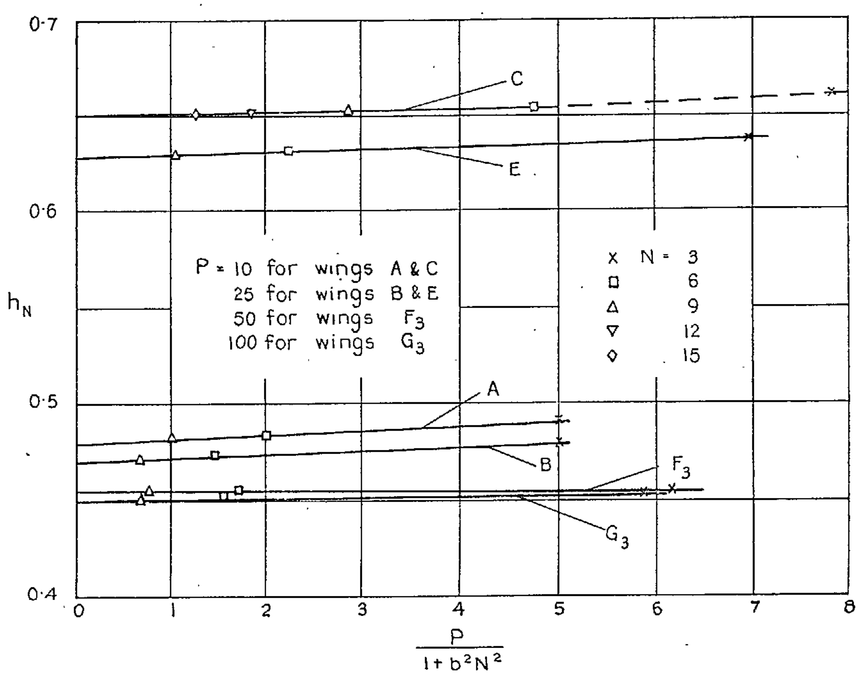

FIG. 12. Variation of h_N with N.

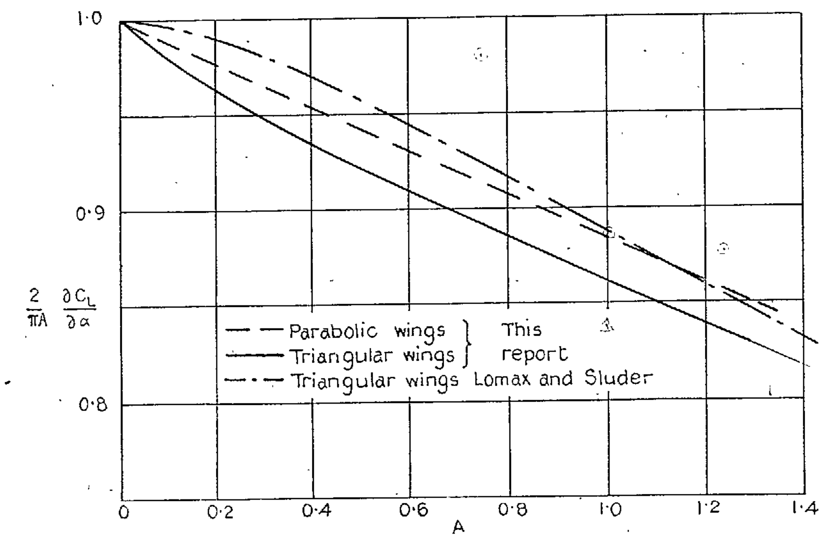

FIG. 13. Variation of $\partial C_L/\partial \alpha$ with aspect ratio.

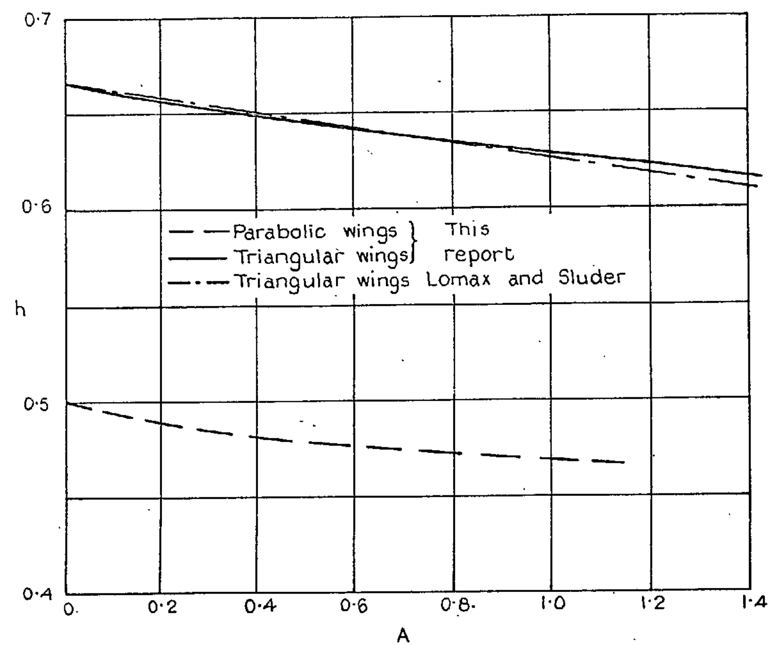

FIG. 14. Variation of h with aspect ratio.

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