R. & M. No. 3020
(17,787)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

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LONDON: HER MAJESTY'S STATIONERY OFFICE
1957
PRICE 5s. 6d. NET

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Communicated by the Director-General of Scientific Research (Air), Ministry of Supply

Reports and Memoranda No. 3020*

December, 1954

Summary.—Real flow patterns are produced by formally placing a pair of conjugate complex sources at conjugate complex points on the axis of symmetry. These complex singularities are shown to be equivalent to a non-uniform distribution of real doublets on a real disc. Reciprocal relationships are formulated between these new singularities and the well-known simple source ring and vortex ring. While the latter are simpler physically, the new type of singularity is easier to handle in mathematical analysis, involving only square roots instead of elliptic integrals.

Sufficient conditions are determined under which an axisymmetric body may be generated by a real distribution of sources and sinks along the axis of symmetry, and the formula for the source intensity is given when these conditions are satisfied. An example deals with the flow about an oblate spheroid.

1. *Introduction.*—Rankine's source-sink method (Ref. 1) consists of disposing a distribution of sources and sinks along a line and superimposing a uniform stream parallel to this line. Provided that the algebraic sum of the strengths of the sources and sinks is zero, then a closed stream surface is generated, and the interior of this surface may be replaced by a rigid body. The resulting flow pattern exterior to the surface is easily calculated and represents the flow pattern produced when a uniform stream of infinite extent flows past the rigid body.

A serious drawback of Rankine's method is that only a certain class of bodies can be generated in this manner. For example, although any prolate spheroid can be so generated, no oblate spheroid can. Moreover, it is clear that the surface of a rigid body generated by such a method can never have a discontinuity of any type (for example, an abrupt change of curvature or slope) at any point which is not located on the axis of symmetry.

To meet this objection, the method has recently been extended by Weinstein (Ref. 2) and van Tuyl (Ref. 3) to employ sources or doublets distributed uniformly round a circle, or uniformly over a disc, whose plane is normal to the direction of the stream. The formulae for the potential and stream functions of such simple source or doublet rings or discs, involve complete elliptic integrals, and have been recorded in convenient form by Sadowsky and Sternberg (Ref. 4). In computing flow patterns regarded as generated by the superposition of a uniform stream on a continuous distribution of these axisymmetric singularities, however, the occurrence of elliptic integrals in the very kernel function, as it were, of the computation, is an inconvenience.

The singularities which form the subject of the present paper are regarded from a mathematical point of view as complex sources located at complex points on the axis of symmetry. The analytical formulae for the resulting potential and stream functions are accordingly no more complicated than those of Rankine's original method, except in that they contain complex

* A.R.E. Report 59/54, received 4th August, 1955.

numbers. In terms of practical computing, this means merely that the operation of taking square roots has to be performed more frequently. Moreover, integration over simple surface shapes can be performed analytically.

From a hydrodynamical point of view, the new singularities will be shown to consist essentially of a circle of branch points introduced into the flow. This implies the existence of a shell of doublets, the shell being of arbitrary shape provided that its rim coincides with the circle of branch points.

The fundamental notion of the equivalence of continuous distributions of singularities at imaginary points on the axis of symmetry, and continuous distributions of singularities at real points with axial symmetry about this axis, was first put forward by Bateman (Ref. 5), although the work reported in the present paper was performed in ignorance of this earlier work. Bateman studied the equivalence in a rather more sophisticated way, from the standpoint of mathematical potential theory. He apparently did not study the physical nature of the individual point complex sources which are the main concern of this paper. Bateman's work was, in one respect, more general, as it considered similar equivalences for sets of singularities not possessing axial symmetry.

The method will here be used to consider the question whether the flow past a given axisymmetric body can or cannot be regarded as generated by a set of real sources and sinks distributed along the axis of symmetry. Particular examples considered include certain spindles, and spheroids (the latter having been also treated by Bateman).

2. *Complex Source-Sink Pairs Generating Real Flow Patterns.*—Let (x, y) be rectangular co-ordinates in a meridian plane of an axisymmetric field, the x-axis being the axis of symmetry. Let us consider the flow pattern generated by a complex source of strength $p + iq$ located at $(i\eta, 0)$, together with a complex source of strength $p - iq$ located at $(-i\eta, 0)$, (p, q, η are all real). Deriving the potential function ϕ_{pq} of this source pair formally by the standard method, we obtain:

$$4\pi\phi_{pq} = \frac{p + iq}{\{y^2 + (x - i\eta)^2\}^{1/2}} + \frac{p - iq}{\{y^2 + (x + i\eta)^2\}^{1/2}}, \qquad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

the convention being that the velocity is the negative gradient of the potential function.

Let

$$R^4 = (y^2 + x^2 - \eta^2)^2 + 4x^2\eta^2 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

and

$$\left.\begin{aligned} \tan 2\omega &= 2x\eta/(y^2 + x^2 - \eta^2)\,, \\ \sin 2\omega &= 2x\eta/R^2\,, \quad \cos 2\omega = (y^2 + x^2 - \eta^2)/R^2 \end{aligned}\right\}, \qquad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

so that (2.1) becomes:

$$\begin{aligned} 4\pi\phi_{pq} &= \frac{p + iq}{Re^{-i\omega}} + \frac{p - iq}{Re^{i\omega}} \\ &= 2R^{-1}(p \cos \omega - q \sin \omega)\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4) \end{aligned}$$

This potential function is real since all the symbols contained in it represent real quantities.

In order to deduce the stream function ψ_{pq} and velocity components (u_{pq}, v_{pq}) of the flow pattern, we notice first that:

$$\begin{aligned} \frac{\partial}{\partial x}\{y^2 + (x - i\eta)^2\}^{-1/2} &= -\frac{x - i\eta}{\{y^2 + (x - i\eta)^2\}^{3/2}} \\ &= -\eta^{-1}R^{-3}(\tfrac{1}{2}R^2 \sin 2\omega - i\eta^2)(\cos 3\omega + i \sin 3\omega)\,, \end{aligned}$$

and

$$\frac{\partial}{\partial y}\{y^2 + (x - i\eta)^2\}^{-1/2} = -\frac{y}{\{y^2 + (x - i\eta)^2\}^{3/2}}$$
$$= -R^{-3}y(\cos 3\omega + i \sin 3\omega) .$$

Now,

$$u_{pq} = -\frac{\partial}{\partial x}\phi_{pq} = -\frac{1}{y}\frac{\partial}{\partial y}\psi_{pq} ,$$

so that

$$4\pi u_{pq} = \frac{(p + iq)(x - i\eta)}{\{y^2 + (x - i\eta)^2\}^{3/2}} + \frac{(p - iq)(x + i\eta)}{\{y^2 + (x + i\eta)^2\}^{3/2}} \quad (2.5)$$
$$= \eta^{-1}R^{-3}\{p(2\eta^2 \sin 3\omega + R^2 \sin 2\omega \cos 3\omega) + q(2\eta^2 \cos 3\omega - R^2 \sin 2\omega \sin 3\omega)\} ;$$

and

$$v_{pq} = -\frac{\partial}{\partial y}\phi_{pq} = \frac{1}{y}\frac{\partial}{\partial x}\psi_{pq} ,$$

so that

$$4\pi v_{pq} = \frac{(p + iq)y}{\{y^2 + (x - i\eta)^2\}^{3/2}} + \frac{(p - iq)y}{\{y^2 + (x + i\eta)^2\}^{3/2}} \quad (2.6)$$
$$= 2yR^{-3}\{p \cos 3\omega - q \sin 3\omega\} ;$$

and

$$4\pi \psi_{pq} = \frac{(p + iq)(x - i\eta)}{\{y^2 + (x - i\eta)^2\}^{1/2}} + \frac{(p - iq)(x + i\eta)}{\{y^2 + (x + i\eta)^2\}^{1/2}} \quad (2.7)$$
$$= 2\eta^{-1}R^{-1}\{p \sin \omega(\eta^2 + R^2 \cos^2 \omega) + q \cos \omega(\eta^2 - R^2 \sin^2 \omega)\} .$$

The physical significance of the quantities R and ω may be deduced by observing that the definitions (2.2) and (2.3) are equivalent to the following relations:

$$R^4 = \{(\eta - y)^2 + x^2\}\{(\eta + y)^2 + x^2\} , \quad (2.8)$$

and

$$\tan 2\omega = \frac{(\eta - y)/x + (\eta + y)/x}{1 - (\eta - y)/x \, . \, (\eta + y)/x} . \quad (2.9)$$

Thus, R is the geometric mean of the lengths of the lines joining the field-point (x, y) to the points $(0, \pm \eta)$, whilst 2ω is the angle between these lines, measured in a counter-clockwise sense (*see* Fig. 1).

If the field-point (x, y) follows a closed contour in a clockwise sense encircling the point $(0, \eta)$, but not the point $(0, -\eta)$, it is clear that the angle 2ω increases by 2π, so that all the functions ϕ_{pq}, u_{pq}, v_{pq}, and ψ_{pq} simply change sign. An additional circuit of the contour causes the functions to resume their original values. The four flow functions are, in fact, two-valued, with branch-points at $(0, \eta)$ and $(0, -\eta)$. In order to render the functions single-valued it is sufficient to join these two points by a 'cut' of arbitrary shape, stipulating that the field-point must never cross this cut.

In three-dimensional space the cut represents a surface having a circular rim. The four flow functions all have discontinuities at this surface equal in magnitude to twice the magnitude of the function on either side. It is convenient to regard the cut as a surface distribution of singularities, whose nature and density are determined by the discontinuities across it.

It is convenient in the present instance to consider the cut surface as forming a disc $x = 0$, $|y| \leqslant \eta$. The angle 2ω may then be defined to lie between the values $\pm \pi$. On the two sides of this disc, according to whether $x = \pm 0$, we find that (*see* Figs. 2 and 3):

$$R^2 = \eta^2 - y^2 \quad , \quad \omega = \pm \tfrac{1}{2}\pi$$

and hence

$$\left.\begin{aligned} 4\pi\phi_{pq} &= \mp \frac{2q}{(\eta^2 - y^2)^{1/2}} \\ 4\pi u_{pq} &= \mp \frac{2p\eta}{(\eta^2 - y^2)^{3/2}} \\ 4\pi v_{pq} &= \pm \frac{2qy}{(\eta^2 - \eta^2)^{3/2}} \\ 4\pi \psi_{pq} &= \pm \frac{2p\eta}{(\eta^2 - y^2)^{1/2}} \end{aligned}\right\} \begin{matrix} (x = \pm 0) \\ (0 < y < \eta) \end{matrix} \qquad \ldots \quad (2.10)$$

From the above examination of the discontinuities we shall next determine the precise nature of the real singularities on the disc which produce the same flow pattern as that due to the complex source-pair on the x-axis. There are two equivalent ways of carrying out this analysis. First, a discontinuity in potential function or stream function may be regarded as indicating the strength of a doublet distribution orientated perpendicular to or parallel to the disc, respectively. Second, a discontinuity in normal or tangential velocity components may be regarded as indicating the strength of a source or vortex distribution, respectively.

The discontinuity in the potential function ϕ_{pq} indicates a doublet layer perpendicular to the disc ($x = 0$, $|y| \leqslant \eta$) of density $-q(\eta^2 - y^2)^{-1/2}/\pi$. But this is known to be equivalent to a vortex distribution with a strength equal to the negative space rate of change of the normal doublet distribution, namely, $qy(\eta^2 - y^2)^{-3/2}/\pi$. The latter is, of course, in agreement with the vortex strength deduced from the discontinuity in the tangential velocity v_{pq}. By integration over the disc, the total doublet strength is found to be:

$$-2q\int_0^\eta (\eta^2 - y^2)^{-1/2} y \, dy = -2q\eta \, . \qquad \ldots \quad (2.11)$$

For the p-component of the flow it is simpler to start from the discontinuity in u_{pq}, which indicates a source strength of density $-p\eta(\eta^2 - y^2)^{-3/2}/\pi$. Thus, in any ring element of area $2\pi y \, dy$ there is a total source strength of $-2p\eta y(\eta^2 - y^2)^{-3/2} \, dy$. By simple integration with respect to y from 0 to η it then appears that the total negative source strength would be infinite. This is at variance with the original complex source distribution, which obviously has a total strength of $2p$.

The source distribution, however, may be regarded as due to a doublet distribution orientated radially outwards, the total doublet strength in any ring element $2\pi y \, dy$ being given by integration of the source strength as $2p\eta(\eta^2 - y^2)^{-1/2} \, dy$. This doublet disc is precisely equivalent to the source disc except at its extremes, namely, at the edge ($y = \eta$) and at the centre ($y = 0$).

At the centre, the pure doublet distribution would imply a terminal point sink of strength $-2p$. Since, however, the expression for u_{pq} in equation (2.10) has a finite limit as $y \to 0$, it is clear that this point sink does not exist. Hence, we have to add to our doublet disc a point source at the centre with strength $+2p$. This is of exactly the right strength to agree with the original complex source-pair, the net source strength of the doublet disc itself being zero.

At the edge of the disc ($y = \eta$), the doublet distribution implies a peripheral source ring of infinite strength. This does in fact exist, and cancels the previously obtained infinite negative integral of the source disc.

To sum up, the formal placing of purely imaginary sources $\pm iq$ at the purely imaginary points $(\pm i\eta, 0)$, respectively, is equivalent to a surface doublet distribution of strength $-q(\eta^2 - y^2)^{-1/2}/\pi$ per unit area, orientated parallel to the x-axis, over the disc $x = 0$, $|y| \leqslant \eta$. The total doublet strength, obtainable either by integration or directly from the moments of the complex sources about the origin, is $-2q\eta$, whilst the source strength is zero. Again, the formal placing of a real source p at each of the purely imaginary points $(\pm i\eta, 0)$ is equivalent to a point source $2p$ at the point $(0, 0)$, together with a surface doublet distribution of strength $p\eta(\eta^2 - y^2)^{-1/2}/(\pi y)$ per unit area, orientated radially outwards, over the disc $x = 0$, $|y| \leqslant \eta$. (The doublet strength of this layer could, in fact, be directly deduced from the discontinuity in ψ_{pq}.) The total source strength is $2p$, whilst the total doublet strength is zero.

It is of interest to write down the nature of the flow pattern in certain other special parts of the field. For example, at a great distance from the origin:

$$R^2 \sim x^2 + y^2 , \qquad \omega \sim x\eta/(x^2 + y^2) ,$$

so that

$$4\pi\phi_{pq} \sim \frac{2p}{(x^2 + y^2)^{1/2}} - \frac{2q\eta x}{(x^2 + y^2)^{3/2}} , \qquad (x^2 + y^2 \to \infty) . \qquad \ldots \quad (2.12)$$

This formula is in agreement with a total source strength of $2p$ and a total doublet strength of $-2q\eta$.

Again, along the x-axis ($x >$ or < 0, $y = 0$) we find that $R^2 = \eta^2 + x^2$, $\tan \omega = \eta/x$, $\sin \omega = \pm \eta/(\eta^2 + x^2)^{1/2}$, $\cos \omega = \pm x/(\eta^2 + x^2)^{1/2}$, so that

$$\left.\begin{aligned} 4\pi\phi_{pq} &= \pm 2\frac{px - q\eta}{\eta^2 + x^2} \\ 4\pi u_{pq} &= \pm 2\frac{p(x^2 - \eta^2) - 2q\eta x}{(\eta^2 + x^2)^2} \\ 4\pi v_{pq} &\sim \pm 2y\frac{p(x^3 - 3x\eta^2) - q\eta(3x^2 - \eta^2)}{(\eta^2 + x^2)^3} \\ 4\pi\psi_{pq} \mp 2p &\sim \pm y^2\frac{p(\eta^2 - x^2) + 2q\eta x}{(\eta^2 + x^2)^2} \end{aligned}\right\} \begin{matrix} (x > \text{or} < 0) \\ (y = 0) \end{matrix} \qquad \ldots \quad (2.13)$$

Again, along the equatorial plane outside the branch points ($x = 0$, $|y| > \eta$), we find that $R^2 = y^2 - \eta^2$, $\omega = 0$, so that

$$\left.\begin{aligned} 4\pi\phi_{pq} &= \frac{2p}{(y^2 - \eta^2)^{1/2}} \\ 4\pi u_{pq} &= \frac{2q\eta}{(y^2 - \eta^2)^{3/2}} \\ 4\pi v_{pq} &= \frac{2py}{(y^2 - \eta^2)^{3/2}} \\ 4\pi\psi_{pq} &= \frac{2q\eta}{(y^2 - \eta^2)^{1/2}} \end{aligned}\right\} \begin{matrix} (x = 0) \\ (|y| > \eta) \end{matrix} \qquad \ldots \quad (2.14)$$

It will be noticed that the p-component of velocity is directed away from the branch point, in accordance with the existence of a positive source ring at the rim of the negative source disc.

Finally, in the neighbourhood of $(0, \eta)$, we may write (*see* Fig. 1), $x = \varepsilon \sin 2\bar{\omega}$, $y = \eta + \varepsilon \cos 2\bar{\omega}$, where ε is small. Then $R^2 \sim 2\varepsilon\eta$, $\omega \sim \bar{\omega}$, and hence

$$\left.\begin{aligned} 4\pi\phi_{pq} &\sim 2(2\varepsilon\eta)^{-1/2}(p \cos \bar{\omega} - q \sin \bar{\omega}) \\ 4\pi u_{pq} &\sim 2\eta(2\varepsilon\eta)^{-3/2}(p \sin 3\bar{\omega} + q \cos 3\bar{\omega}) \\ 4\pi v_{pq} &\sim 2\eta(2\varepsilon\eta)^{-3/2}(p \cos 3\bar{\omega} - q \sin 3\bar{\omega}) \\ 4\pi \psi_{pq} &\sim 2\eta(2\varepsilon\eta)^{-1/2}(p \sin \bar{\omega} + q \cos \bar{\omega}) \end{aligned}\right\} (\varepsilon \text{ small}) . \quad \ldots \quad \ldots \quad (2.15)$$

For the sake of simplicity, we have assumed in this discussion that the real parts of the abscissae of the complex sources are zero. If the sources were to be moved from the points $(\pm i\eta, 0)$ to the points $(\xi \pm i\eta, 0)$ where ξ is real, then the only change necessary in all the formulae would be to replace x by $(x - \xi)$.

3. *Reciprocal Relations between Complex Sources and Ring Singularities.*—Let us now examine the real singularity distribution which corresponds to a complex line source of strength $2(pb + iq\eta)(b^2 - \eta^2)^{-1/2}$ per unit of η, placed along the line $x = i\eta$, where $-b \leqslant \eta \leqslant b$.

From the q-component we have a surface distribution of doublets, orientated parallel to the x-axis, distributed over the disc $0 \leqslant \eta \leqslant b$. The area density of the distribution is found to be Q, where:

$$Q = -\frac{q}{\pi}\int_y^b \frac{2\eta\, d\eta}{(b^2 - \eta^2)^{1/2}(\eta^2 - y^2)^{1/2}}$$

$$= -\frac{q}{\pi}\int_{y^2}^{b^2} \frac{d(\eta^2)}{[\{\frac{1}{2}(b^2 - y^2)\}^2 - \{\eta^2 - \frac{1}{2}(b^2 + y^2)\}^2]^{1/2}} .$$

Setting $\eta^2 - \frac{1}{2}(b^2 + y^2) = \frac{1}{2}(b^2 - y^2) \sin t$, this integral reduces to:

$$Q = -\frac{q}{\pi}\int_{\pi/2}^{\pi/2} dt = -q . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

Since the doublet density of the disc is uniform, it may be replaced by a simple vortex ring at its edge.

From the p-component we have a surface distribution of doublets, orientated radially outwards, distributed over the disc $0 \leqslant \eta \leqslant b$. The strength per unit area of the distribution is found to be:

$$P = \frac{pb}{\pi y}\int_y^b \frac{2\eta\, d\eta}{(b^2 - \eta^2)^{1/2}(\eta^2 - y^2)^{1/2}} = \frac{pb}{y} . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

A ring element accordingly has an outward strength of radial density $2\pi pb$. Since this is a constant, the only effect of the doublet disc is to produce a ring source of strength $2\pi pb$ on the ring $(0, b)$, together with a point sink of strength $2\pi pb$ at the origin. But our complex distribution of sources also yields a point source at the origin of total strength:

$$4pb \int_0^b (b^2 - \eta^2)^{-1/2}\, d\eta = 2\pi pb , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

which just cancels the terminal sink due to the radial doublet distribution.

To sum up, the distribution of complex sources and sinks is equivalent to a simple source ring of total strength $2\pi pb$ at $(0, b)$, together with a simple vortex ring at $(0, b)$ with a total moment parallel to the x-axis of $-\pi q b^2$.

The relationship between the complex axial singularities and the real off-axis singularities may be expressed in a form having some degree of symmetry. For this purpose, a point source p at $x = i\eta$ and a point source $-p$ at $x = 0$ [the pair having a ' moment ' about the origin of $(p) \times (i\eta) + (-p) \times (0) = ip\eta$] may be replaced by a line distribution of doublets with uniform density ip extending from $x = 0$ to $x = i\eta$. Similarly, a point source iq at $x = i\eta$ and a point source $-iq$ at $x = i\eta$ [the pair having a moment about the origin of $(iq) \times (i\eta) + (-iq) \times (-i\eta) = (-q) \times (2\eta)$] may be replaced by a line distribution of doublets with uniform density $-q$ extending from $x = -i\eta$ to $x = i\eta$. It should be noticed that the total strength of such a uniform line distribution of doublets is obtained simply by multiplying the line density by the length of the line, without regard to the direction of the line in the complex plane.

Using the symbol j to denote a positive rotation of $\frac{1}{2}\pi$ in the (x, y)-plane, the relation obtained in section 2 may then be expressed as follows:

Complex x-plane		Real (x, y)-plane	
Doublet distribution with line density $+ip - q$ from $x = 0$ to $x = i\eta$; $-ip - q$ from $x = 0$ to $x = -i\eta$	$\equiv$	Doublet distribution with radial density $\dfrac{2(jp\eta - qy)}{(\eta^2 - y^2)^{1/2}}$ at $(0, y)$ from $y = 0$ to $y = \eta$	(3.4)

On the other hand, the relation which has just been obtained in the present section may be expressed in the following form:

Complex x-plane		Real (x, y)-plane	
Source distribution with line density $\dfrac{2(pb + iq\eta)}{(b^2 - \eta^2)^{1/2}}$ at $x = i\eta$ from $\eta = -b$ to $\eta = b$	$\equiv$	Source ring at $(0, b)$ with arc length density p + vortex ring at $(0, b)$ with circulation $(-q)$	(3.5)

In the theory of the two-dimensional complex potential function, a line vortex with circulation $-q$ is precisely equivalent to a line source with strength jq per unit length. If the vortex ring of equation (3.5) is interpreted in this light, it becomes apparent that the relations (3.4) and (3.5) exhibit a certain degree of symmetry.

Since it can very easily be shown that the flow pattern past any axisymmetric body can be generated by a suitable vortex sheet covering its surface, it follows from equation (3.5) that the flow pattern past any axisymmetric body whatever can be generated by a suitable distribution of complex sources at complex points on the axis of symmetry. This point will be elaborated in section 4.

It is easily verified from equation (3.5) that the flow functions at a general field point (x, y) produced by a source ring or a vortex ring at $(0, b)$ are given by complete elliptic integrals, as obtained by Sadowsky and Sternberg (Ref. 2).

From the relation (3.4) it would be possible, using Sadowsky and Sternberg's basic formulae, to derive a number of academically interesting identities. On the left-hand side would appear the flow function for a complex conjugate source-pair, involving only square roots. On the right-hand side would appear a definite integral, the integrand of which would contain complete elliptic integrals.

For the practical problem of generating a closed stream surface of a specified shape it is usually necessary to distribute singularities with respect to the x-variable and then integrate to find the total effect. If the elemental singularity is regarded as the source ring or vortex ring, the distribution is over an axially symmetric surface in the (x, y)-plane, and since the integrand contains complete elliptic integrals, analytic integration is very difficult, if not impossible. If, on the other hand, the elemental singularity is regarded as the complex source-pair, the distribution is over a curve in the complex plane of x. Since the integrand now involves only square roots, it is possible to effect the integration analytically for a number of simple density distributions. The shape of the path of integration is, under certain conditions, immaterial, only the terminal points being significant.

In fairness it must, however, be admitted that, if the problem is to be solved by purely numerical iterative solution of an integral equation, then it may be simpler to use the vortex ring as the elemental singularity, since this yields a very simple integral equation, albeit with a difficult kernel (Ref. 6).

4. *Determination of Sufficient Conditions under which a Body may be Generated by a Distribution of Real Sources and Sinks along the Axis.*—Let us assume that the potential flow past a certain axisymmetric rigid body is known. We shall further suppose that the profile of the body is expressed by a single-valued differentiable function $y = y^*(x)$, $(x_1 \leqslant x \leqslant x_2)$ and that the tangential velocity of the fluid along this profile is $w(x)$, the velocity at infinity being taken as unity, directed parallel to the positive x-axis. The rigid body may now be replaced by an identical region of fluid at rest, and this will imply a discontinuity of tangential velocity, but not of normal velocity, across the interface. The only singularity in the resulting flow pattern, apart from the uniform stream at infinity, is a vortex sheet with circulation $w(x)$ per unit arc length of meridian along the surface of the original body.

Let the stream function induced at a field point (x, y) by a vortex ring with unit circulation at (x^*, y^*) be denoted by the function $\Psi\ (x, y\,; x^*, y^*)$. Then the stream function of the flow pattern about the body, for any external point (x, y), is given by:

$$\psi(x, y) = -\tfrac{1}{2}y^2 + \int_{x_1}^{x_2} w(x^*)\Psi(x, y\,; x^*, y^*)\{1 + (dy^*/dx^*)^2\}^{1/2}\, dx^* . \quad \ldots \quad \ldots \quad (4.1)$$

But we have shown that, so far as the effect at external points is concerned, a vortex ring with circulation $w(x)$ may be replaced by a certain distribution of complex sources in the complex x-plane. Thus, according to equations (3.5) and (2.7), we may replace equation (4.1) by the formula:

$$\psi(x, y) = -\tfrac{1}{2}y^2 + \frac{1}{4\pi}\int_{x_1}^{x_2} w(x^*)\{1 + (dy^*/dx^*)^2\}^{1/2} \times \left[\int_{-y^*}^{y^*} \frac{-2i\eta}{(y^{*2} - \eta^2)^{1/2}} \frac{x - x^* - i\eta}{\{y^2 + (x - x^* - i\eta)^2\}^{1/2}}\, d\eta\right] dx^* . \quad \ldots \quad (4.2)$$

Writing γ for η/y^*, this last equation becomes:

$$\psi(x, y) = -\tfrac{1}{2}y^2 + \frac{1}{4\pi}\int_{x_1}^{x_2} w(x^*)\{1 + (dy^*/dx^*)^2\}^{1/2} \times \left[\int_{-1}^{+1} \frac{-2i\gamma}{(1 - \gamma^2)^{1/2}} \frac{x - x^* - i\gamma y^*}{\{y^2 + (x - x^* - i\gamma y^*)^2\}^{1/2}}\, y^*\, d\gamma\right] dx^* . \quad \ldots \quad (4.3)$$

In order to understand clearly the meaning of this repeated integral it is convenient to construct a three-dimensional picture (Fig. 4) with co-ordinate axes (x^*, η, γ). The repeated integral is then obviously equal to the double integral over a ruled surface whose equation is given by $\eta = \gamma y^*(x^*)$.

As expressed in equation (4.3), the double integral is calculated by first integrating along the generators of the surface, *i.e.*, with respect to γ (x^* constant), and subsequently with respect to x^*. It is permissible, however, to calculate the double integral in the reverse way, namely, by integrating first along horizontal curves, *i.e.*, with respect to x^* (γ constant), and subsequently with respect to γ, as follows:

$$\psi(x, y) = -\tfrac{1}{2}y^2 + \frac{1}{4\pi}\int_{-1}^{+1} \frac{-2i\gamma}{(1-\gamma^2)^{1/2}} \times \left[\int_{x_1}^{x_2} w(x^*) \frac{\{1 + (dy^*/dx^*)^2\}^{1/2}(x - x^* - i\gamma y^*)}{\{y^2 + (x - x^* - i\gamma y^*)^2\}^{1/2}} y^* \, dx^*\right] d\gamma \, .$$

In the inside integral let us now make the substitution:

$$x' = x^* + i\gamma y^*(x^*)$$
$$dx' = dx^* \, (1 + i\gamma \, dy^*/dx^*) \, . \qquad \qquad (4.4)$$

We assume that the first of these relations, giving x' as an analytic function of x^*, can be inverted to give x^* as an analytic function of x', and, by the second of these relations, dx'/dx^* as an analytic function of x'. It then follows that:

$$\frac{y^* w(x^*)\{1 + (dy^*/dx^*)^2\}^{1/2}}{1 + i\gamma \, dy^*/dx^*} = g(x', \gamma) \, , \qquad \qquad (4.5)$$

g being a known analytic function of x', with γ as a parameter. Thus $\psi(x, y)$ may be written as:

$$\psi(x, y) = -\tfrac{1}{2}y^2 + \frac{1}{4\pi}\int_{-1}^{+1} \frac{-2i\gamma}{(1-\gamma^2)^{1/2}} \left[\int_{x_1}^{x_2} \frac{g(x', \gamma)(x - x')}{\{y^2 + (x - x')^2\}^{1/2}} dx'\right] d\gamma \, , \qquad \qquad (4.6)$$

using the fact that $y^*(x_1) = y^*(x_2) = 0$. Now x' is a complex variable whose Argand diagram is depicted by horizontal planes in Fig. 4. The path of integration in the horizontal plane is determined by equation (4.4) for each separate value of γ. This path of integration, however, may be deformed without affecting the value of the integral, provided that the integrand, regarded as a function of x', has no singularities between the original and the deformed path. In practice, since (x, y) is a field point either outside or on the contour of the body, the only possible singularities in the integrand must arise from the function $g(x', \gamma)$.

Provided, then, that $g(x', \gamma)$ has no singularities between the real axis of x' and the curved path defined by equation (4.4), the paths of integration may all be deformed to lie along the real axis of x'. The path of integration is then independent of γ, so that the double integration may be performed first with respect to γ (x' constant) and subsequently with respect to x', thus:

$$\psi(x, y) = -\tfrac{1}{2}y^2 + \frac{1}{4\pi}\int_{x_1}^{x_2} \frac{x - x'}{\{y^2 + (x - x')^2\}^{1/2}} f(x') \, dx' \, , \qquad \qquad (4.7)$$

where

$$f(x') = \int_{-1}^{+1} \frac{-2i\gamma}{(1-\gamma^2)^{1/2}} g(x', \gamma) \, d\gamma \, . \qquad \qquad (4.8)$$

The variable x' is now real, and the physical significance of equation (4.7) is clearly that the body may be generated by superimposing a uniform stream on a real distribution of sources with density $f(x)$ located on the x-axis.

We conclude that a sufficient condition that an axisymmetric body should be capable of being generated by an axial distribution of sources and sinks is simply that $g(x', \gamma)$ should have no singularities between the curve $x' = x^* + i\gamma y^*(x^*)$ and the real axis, for any value of the parameter γ between ± 1.

In the case where singularities of g occur actually on the real axis of x', the paths of integration have to be indented by small semi-circles in the usual way. These semi-circles lie on the positive side of the real axis when $\gamma > 0$, and on the negative side when $\gamma < 0$, so that the above inversion of the order of integration is no longer strictly valid. It might be possible, however, still to justify it in the case where integration along the indentations themselves makes a contribution which vanishes in the limit as the radius tends to zero, as in the case of the prolate spheroid in the example considered below.

In practice, the form of $y^*(x)$ is part of the data of the problem, whilst $w(x)$ is not. According to equation (4.5), singularities of $g(x', \gamma)$ might arise from singularities either of $y^*(x)$ or of $w(x)$ when each is expressed as a function of x'. From a knowledge of the profile $y^*(x)$ only, it will sometimes be possible to state that our sufficient condition is *not* fulfilled. A knowledge of the velocity distribution also, however, would be necessary in order to be able to assert that our sufficient condition *is* fulfilled. Although the condition has not been shown to be necessary as well as sufficient, nevertheless I suspect that it is so, since failure to fulfill it, at least in certain cases, coincides with the impossibility of generating the flow pattern by real axial sources and sinks.

For example, in the case of an elliptic profile with semi-axes l_x and l_y parallel to the x-axis and y-axis respectively, we have $(x^{*2}/l_x^2) + (y^{*2}/l_y^2) = 1$. Substituting this relation in equation (4.4) and solving the resulting quadratic equation for either x^* or y^*, we obtain expressions containing the surd $h(x')$, where

$$h(x') = (x'^2 - l_x^2 + \gamma^2 l_y^2)^{1/2} . \qquad (4.9)$$

This surd clearly has a branch point given by $x'^2 = l_x^2 - \gamma^2 l_y^2$. For a prolate spheroid $(l_x > l_y)$ the branch points all lie on the real axis, and the paths of integration may be deformed to obtain equation (4.7). For an oblate spheroid $(l_x < l_y)$, on the other hand, there are branch points extending along the imaginary axis between the foci of the elliptic profile, and deformation of the path of integration is not permissible. It is known that, while a prolate spheroid can be generated by a real distribution of axial sources and sinks, an oblate spheroid cannot.

It is interesting to use the above theoretical considerations to hazard a prediction in a case which has not yet been solved, namely the parabolic arc spindle. The profile of this spindle has the equation:

$$\nu x^{*2} = 1 - y^* , \qquad (4.10)$$

so that $x' = x^* + i\gamma(1 - \nu x^{*2})$ and $2\nu\gamma x^* = -i \pm \{-1 + 4\nu\gamma(\gamma + ix')\}^{1/2}$. The function $x^*(x', \gamma)$ accordingly has a singularity at $x' = x_0'(\gamma)$, where $x_0' = i\{\gamma - 1/(4\nu\gamma)\}$. Singularities occur above the real axis if and only if $\nu > \frac{1}{4}$, which leads us to formulate the following conjecture:

The spindle whose profile has the equation (4.10) may be generated by a doublet distribution extending along its entire axis of symmetry, provided that $\nu < \frac{1}{4}$. On the other hand, if $\nu > \frac{1}{4}$, the spindle may be generated by a doublet distribution extending along its entire axis of symmetry, together with a ring doublet distribution over the disc $|y| \leqslant 1 - 1/(4\nu)$.

5. *Example : The Oblate Spheroid.*—It is a well known fact that a closed stream surface having the shape of a prolate spheroid may be generated in a uniform stream by placing a source distribution of density $4\pi mx$ (m being a real constant) along the x axis from $x = -1$ to $x = 1$.

We shall now investigate the effect of placing in a uniform stream a purely imaginary source distribution, of density $-4\pi im\eta$, at the imaginary point $(i\eta, 0)$ on the x axis from $\eta = -1$ to $\eta = 1$. The stream function ψ_m due to the source distribution is then obtained from equation (2.7) as:

$$\psi_m = -im\int_{-1}^{+1} \frac{(x - i\eta)\,\eta\,d\eta}{\{y^2 + (x - i\eta)^2\}^{1/2}} . \qquad (5.1)$$

Hence, writing $x - i\eta = y \sinh\vartheta$, we obtain

$$\psi_m = -im\int_{\vartheta_1}^{\vartheta_2} y \sinh\vartheta (x - y\sinh\vartheta)\,d\vartheta$$

$$= -im\Big\{xy\cosh\vartheta - \tfrac{1}{2}y^2(\tfrac{1}{2}\sinh 2\vartheta - \vartheta)\Big\}_{\vartheta_1}^{\vartheta_2} \qquad (5.2)$$

where

$$y\sinh\vartheta_1 = x + i , \qquad y\sinh\vartheta_2 = x - i . \qquad (5.3)$$

Since ϑ_1 and ϑ_2 are obviously conjugate complex imaginary quantities, it is convenient to write:

$$\vartheta_1 + \vartheta_2 = 2\lambda\,, \qquad \vartheta_1 - \vartheta_2 = 2i\mu\,, \quad \ldots \qquad (5.4)$$

where λ and μ are real. Equation (5.2) may then be rewritten as:

$$\begin{aligned}\psi_m &= -\,im[2xy \sinh \lambda \sinh(-\,i\mu) - \tfrac{1}{2}y^2\{\cosh 2\lambda \sinh(-\,2i\mu) + 2i\mu\}] \\ &= -\,my^2(\mu - \tfrac{1}{2}\sin 2\mu)\,, \quad \ldots \qquad (5.5)\end{aligned}$$

by virtue of the fact that, from (5.3),

$$2x = 2y \sinh \lambda \cosh(i\mu) \ldots \qquad (5.6)$$

Accordingly, if this source distribution is superimposed on a uniform stream of unit velocity parallel to the negative x-axis, the combined stream function ψ is given by:

$$\psi = \tfrac{1}{2}y^2\{1 - m(2\mu - \sin 2\mu)\} \ldots \qquad (5.7)$$

The stream-surface $\psi = 0$ consists of the x-axis, together with the closed surface $\mu = \mu_0$, where:

$$2\mu_0 - \sin 2\mu_0 = 1/m\,. \qquad (5.8)$$

The nature of the surfaces $\mu =$ constant is deduced by noticing that, from (5.3),

$$2i = 2y \cosh \lambda \sinh(i\mu)\,, \ldots \qquad (5.9)$$

which, together with equation (5.6), yields:

$$-\frac{1}{\sinh^2(i\mu)} - \frac{x^2}{\cosh^2(i\mu)} = y^2\,.$$

Hence

$$x^2 \tan^2 \mu + y^2 \sin^2 \mu = 1\,, \qquad (5.10)$$

which is the equation of an ellipse with foci at $(0, \pm 1)$, eccentricity $\sin \mu$, and with major and minor semi-axes equal to $\operatorname{cosec} \mu$ and $\cot \mu$ respectively. The surface $\mu =$ constant is therefore an oblate spheroid.

The real distribution of singularities required to generate this closed surface is seen to be a surface distribution of doublets, orientated parallel to the x-axis, over the disc $x = 0$, $|y| \leqslant 1$. The surface density is clearly given by:

$$4\pi m \int_y^1 \eta(\eta^2 - y^2)^{-1/2}\, d\eta/\pi = 4m(1 - y^2)^{1/2}\,, \ldots \qquad (5.11)$$

and the total strength by:

$$8\pi m \int_0^1 (1 - y^2)^{1/2} y\, dy = 8\pi m/3 = M, \text{ say}, \ldots \qquad (5.12)$$

which is the same result as would be obtained directly by taking moments about the origin of the complex source distribution, thus:

$$4\pi m \int_{-1}^{+1} \eta^2\, d\eta = 8\pi m/3 \ldots \qquad (5.13)$$

The corresponding analysis in the case of a prolate spheroid is formally almost identical with that given above, but omitting the factor i. In that case it may be shown that a closed stream surface is generated whose equation is $\mu = \mu_0$, where:

$$\sinh 2\mu_0 - 2\mu_0 = 1/m\,, \qquad (5.14)$$

and

$$x^2 \tanh^2 \mu + y^2 \sinh^2 \mu = 1\,. \qquad (5.15)$$

The meridional section of the surface $\mu =$ constant is therefore an ellipse with foci at $(\pm 1, 0)$, eccentricity $\tanh \mu$, and major and minor semi-axes equal to $\coth \mu$ and $\operatorname{cosech} \mu$ respectively.

The singularity distribution in the case of the prolate spheroid may be regarded as a source distribution along the x-axis of density $4\pi mx$ $(-1 \leqslant x \leqslant 1)$, or alternatively by the equivalent doublet distribution of $2\pi m(1 - x^2)$, from $x = -1$ to $x = +1$. Once again, the total doublet strength is given by:

$$2\pi m \int_{-1}^{+1} (1 - x^2)\, dx = 8\pi m/3 = M \,, \quad \ldots \quad (5.16)$$

in close analogy with equation (5.12).

If V is the volume of the spheroid, and V' the associated volume of liquid for translatory motion along the axis of symmetry, then Lamb (Ref. 1) shows that V' may be deduced from the total doublet strength of the included singularity distribution. Thus $V' + V = M$, so that

$$V'/V = M/V - 1 \,. \quad \ldots \quad (5.17)$$

Now

$$V = \frac{4\pi}{3} \begin{matrix}\operatorname{cosec}^2 \\ \operatorname{cosech}^2\end{matrix} \mu \begin{matrix}\cot \\ \coth\end{matrix} \mu \quad \ldots \quad (5.18)$$

in the case of the oblate and prolate spheroid respectively. Hence, replacing $M = 8\pi m/3$ by the expressions given in equations (5.8 and 14) respectively, we obtain, after a little reduction,

$$V'/V = \frac{\begin{matrix}\tan \\ \tanh\end{matrix} \mu - \mu}{\mu - \frac{1}{2} \begin{matrix}\sin \\ \sinh\end{matrix} (2\mu)} \,. \quad \ldots \quad (5.19)$$

These two formulae are in agreement with those given by Lamb (Ref. 1), apart from a printer's error in Lamb (p. 701) in the case of the oblate spheroid.

It is customary to define an associated volume coefficient (V'/V_s), where V_s is the volume of the sphere tangent to the spheroid at the equator. This ratio, which is obviously given by multiplying V'/V by $\cos\mu$ or $\cosh\mu$, respectively, has the formula:

$$V'/V_s = \frac{\begin{matrix}\sin \\ \sinh\end{matrix} \mu - \mu \begin{matrix}\cos \\ \cosh\end{matrix} \mu}{\mu - \frac{1}{2} \begin{matrix}\sin \\ \sinh\end{matrix} (2\mu)} \,. \quad \ldots \quad (5.20)$$

It has the advantage of remaining finite throughout the entire range of spheroids. The associated volume coefficients V'/V and V'/V_s are tabulated in Table 1 and plotted graphically in Fig. 5.

6. *Conclusion.*—It has been shown in this paper that Rankine's source-sink method may be generalized by employing a pair of sources, the strength of each member being the complex conjugate of the other, situated at conjugate imaginary points on the axis of symmetry. The flow function so generated is entirely real, and corresponds to a doublet disc of non-uniform density. Reciprocal relations have been formulated between this new type of singularity and the well-known simple source ring and vortex ring.

For generating bodies by the inverse method, the new type of singularity is considerably easier to manipulate analytically and numerically than the source ring or vortex ring, since it involves only square roots, as opposed to complete elliptic integrals. On the other hand, for the solution of the direct problem of flow about a given body, using integral equation techniques, it is probably necessary to use the vortex ring as the basic singularity.

Assuming the flow pattern about a given body is known, a partial answer has been given to the question: Could this body be generated by a real axial source-sink distribution? In cases when it can, the formula for the source distribution density has been given.

The new type of singularity has been used to determine, in a particularly simple way, the flow past an oblate spheroid.

NOTATION

b	Radius of ring singularity
e	Eccentricity of an ellipse
f	Source strength function, defined in equation (4.8)
g	A function defined in equation (4.5)
h	An auxiliary function defined in equation (4.9)
i	Positive rotation of $\frac{1}{2}\pi$ in the complex x-plane
j	Positive rotation of $\frac{1}{2}\pi$ in the (x, y)-plane
l	Semi-axis of an ellipse
m	A parameter defining a linear source distribution in section 5
M	Total doublet strength inside an ellipsoid of revolution
p	Source strength (real part)
P	Radial doublet strength per unit area
q	Source strength (imaginary part)
Q	Axial doublet strength per unit area
r	Distance from the origin
R	Geometric mean of the distances of (x, y) from $(0, \pm \eta)$
t	An auxiliary variable in equation (3.1)
u	x-component of fluid velocity
v	y-component of fluid velocity
V	Volume of ellipsoid of revolution
V'	Associated volume of ellipsoid of revolution
V_s	Volume of sphere with same equatorial radius
w	Tangential velocity of slip
x	Co-ordinate parallel to the axis of symmetry
x'	Defined in equation (4.4)
y	Distance from the axis of symmetry
γ	The ratio η/y^*
ε	Distance from the branch point
η	The imaginary part of x, considered as a complex variable
ϑ	An auxiliary variable of integration in section 5
λ, μ	Elliptic co-ordinates in section 5
ν	Parameter defining a parabolic arc spindle
$\bar{\omega}$	Angular co-ordinate near the branch point
ξ	The real part of x, considered as a complex variable
ϕ	Velocity potential function
ψ	Stokes's stream function
Ψ	Stream function of a vortex ring
ω	A co-ordinate angle defined in equation (2.3)

Affixes:

Subscript $_{pq}$	Pertaining to the source distribution $p + iq$
$*$	Pertaining to the profile of a body

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TABLE 1

	Prolate spheroid		Oblate spheroid	
Axis ratio	V'/V	V'/V_s	V'/V	V'/V_s
0	0	0	∞	0·637
0·1	0·021	0·207	6·184	0·618
0·2	0·059	0·296	3·008	0·602
0·3	0·105	0·351	1·953	0·586
0·4	0·156	0·391	1·428	0·571
0·5	0·210	0·420	1·115	0·558
0·6	0·266	0·443	0·908	0·545
0·7	0·323	0·461	0·761	0·533
0·8	0·381	0·477	0·651	0·521
0·9	0·440	0·489	0·567	0·510
1·0	0·500	0·5	0·500	0·500

Acknowledgement.—The author is happy to acknowledge the assistance of his colleagues, Dr. A. Coombs and Mr. A. D. Cox, in helping to prepare this paper for publication. He is also indebted to Dr. M. van Dyke for the reference to Bateman's earlier work on this subject.

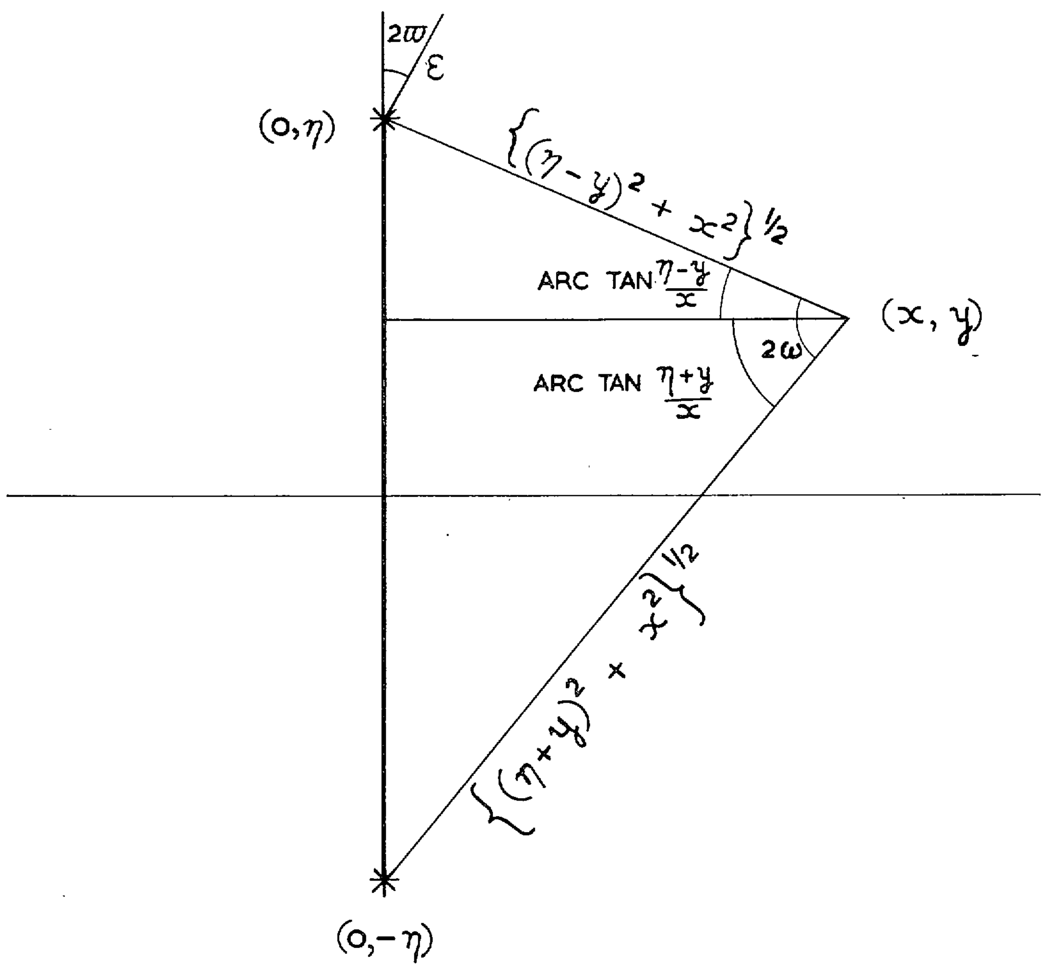

Fig. 1. Geometrical Relations.

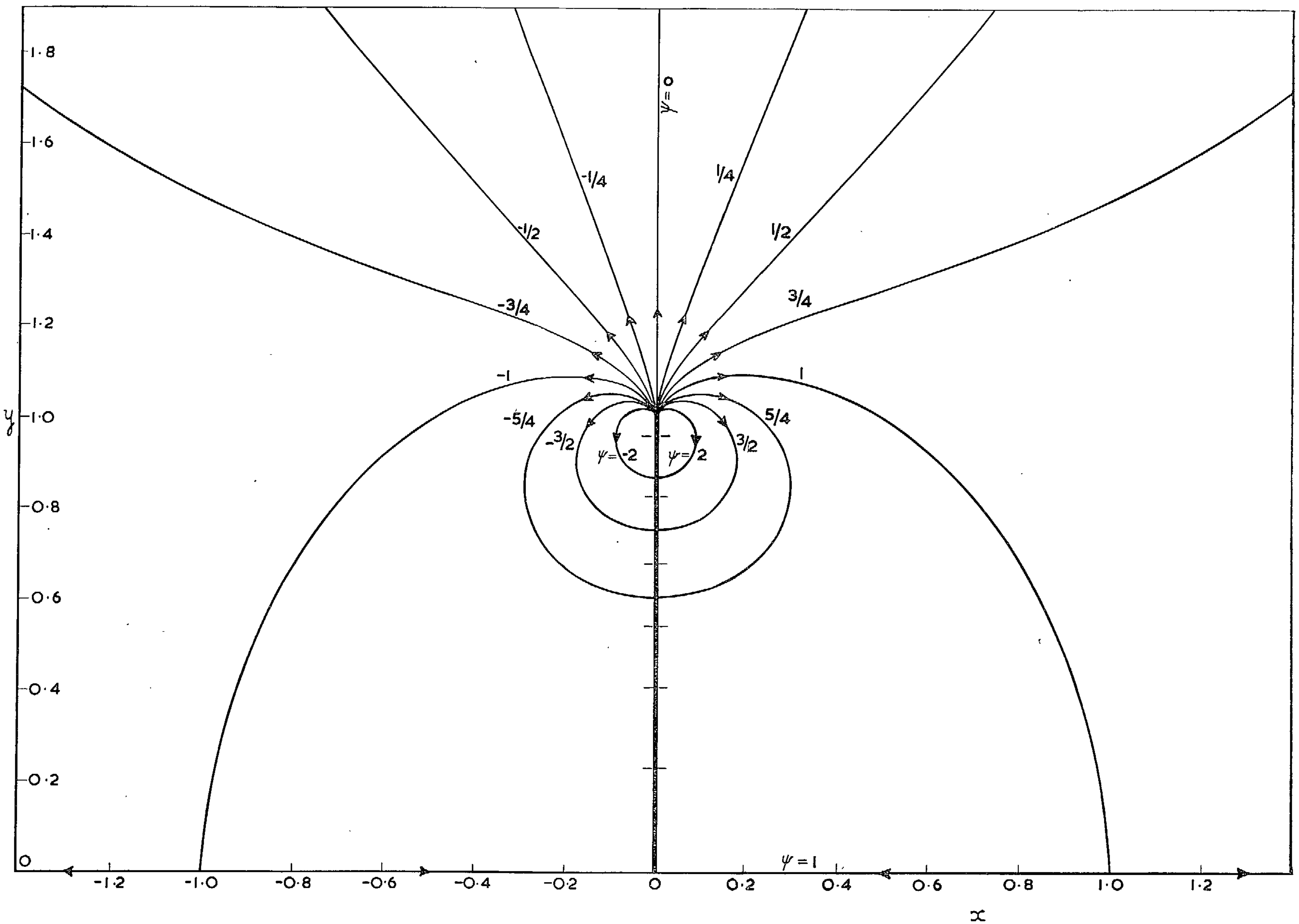

Fig. 2. Streamlines for: Real source at $(x = i, y = 0)$ and real source at $(x = -i, y = 0)$.

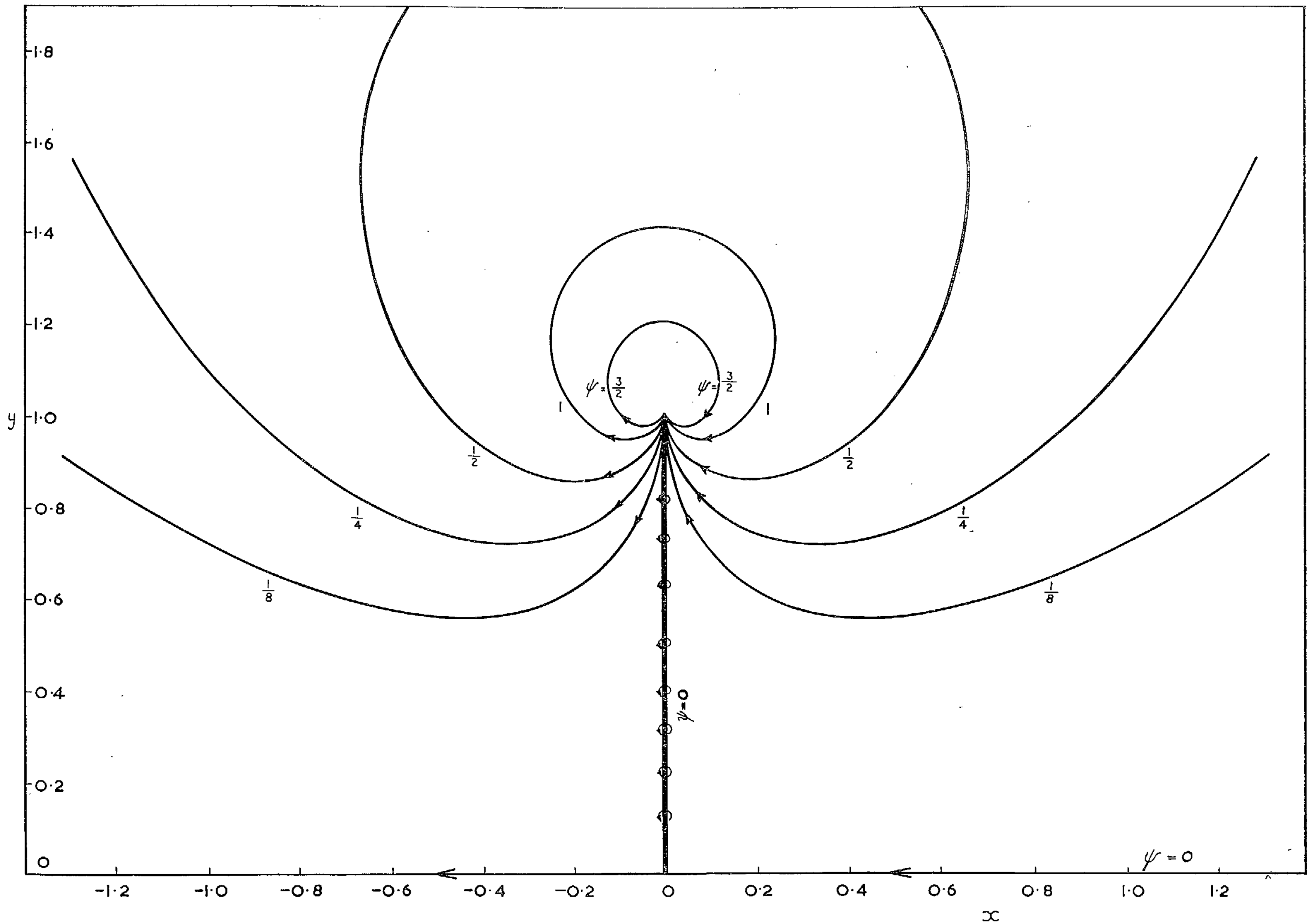

Fig. 3. Streamlines for imaginary source at ($x = i$, $y = 0$) and imaginary sink at ($x = -i$, $y = 0$).

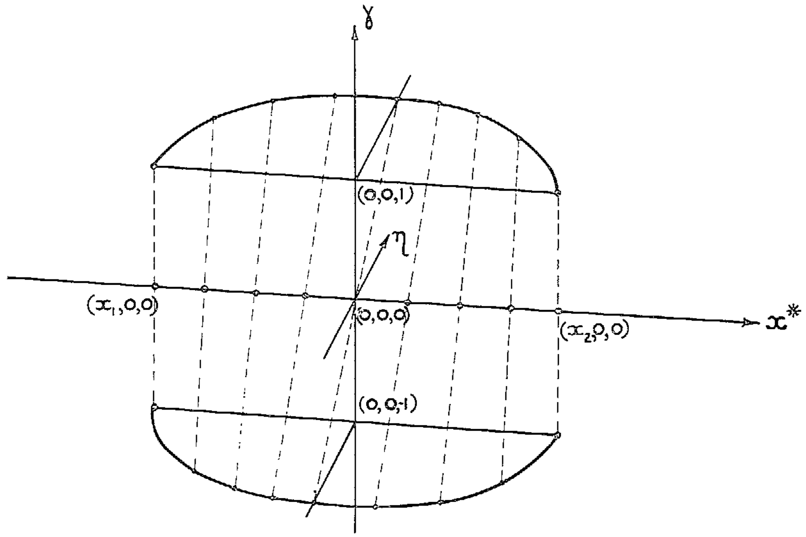

Fig. 4. Ruled surface of integration.

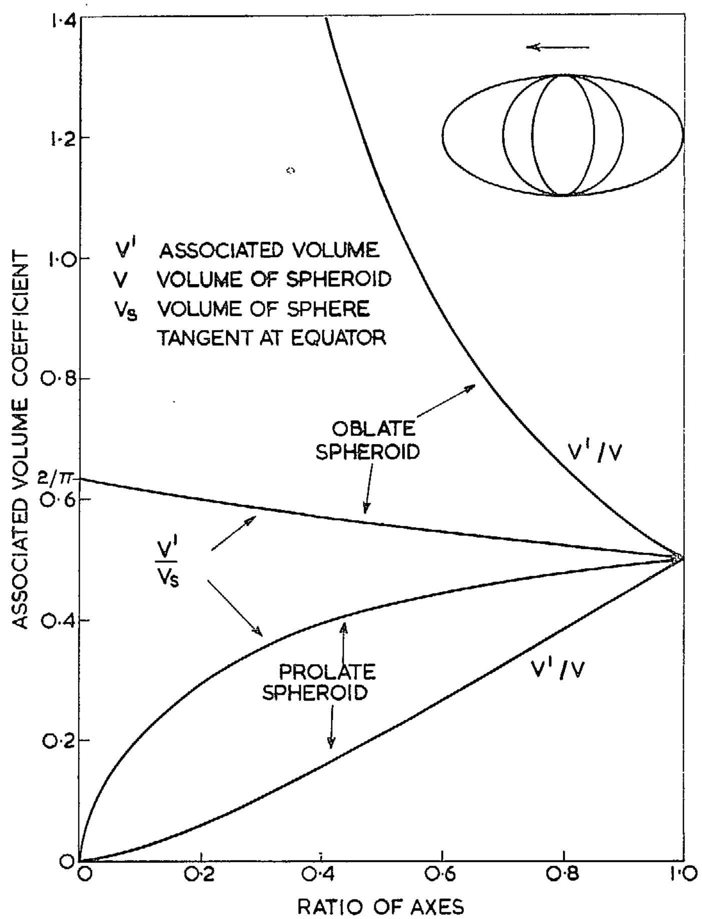

Fig. 5.

(68306) Wt. 20/9036 K7 11/57 Hw.

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