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Summary.—The laminar boundary-layer equation, for a linearly retarded velocity in the main stream, $U = 1 - \frac{1}{8}x$ in reduced variables, has been solved numerically by working in finite intervals in x, with a correction for the finite length of x-interval. The method was first tried out on the region near the forward stagnation point, where the results could be checked from tables given by Howarth, and proved very satisfactory. The separation point has been determined by two independent methods to be close to $x = 0{\cdot}959$, in excellent agreement with Howarth's value. The nature of the singularity at the separation point is discussed.

1. *Introduction.*—The equations of the laminar boundary layer, in their usual form, are partial differential equations in two variables, and though in a few special cases the variables can be separated, this is not possible in general. A number of approximate methods, of various kinds, have been developed for obtaining approximate solutions in more general cases. A survey and critical discussion of methods then available was given by Howarth[7] in 1934, and other methods have since been developed by Kármán and Millikan[9], Howarth[8] and others.

More recently, a rather general method for the numerical or mechancial solution of partial differential equations with suitable forms of boundary conditions has been proposed and investigated by Hartree and Womersley[5], and a test of this method on a simple form of the equation of heat conduction was entirely satisfactory and showed that the method was manageable in practice, and, in that case, would give results of quite good accuracy (five figures) without undue labour in numerical work. The method, which is outlined in section 2 of the present report, is also very suitable for the use of mechanical methods of integration such as the differential analyser of Dr. Bush[1 6], if not such a high accuracy in the solution is required, and it has been applied successfully to the solution of the equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + \beta e^{\theta} ,$$

which arises in the theory of the thermal breakdown of dielectrics in alternating fields[2], using the differential analyser at Manchester University.

These successful trials of the method led to the hope that it could also be applied successfully to the equations of the laminar boundary layer. This application is certainly a more ambitious one than the previous ones attempted, as the equation is of a higher order and more elaborate, and the range of integration is formally infinite in a direction normal to the boundary, whereas, apart from a few experiments, previous applications had been concerned with a finite range of integration in the corresponding variable. But the boundary conditions are of the form which allows the application of the method, and the effective range of integration in practice did not seem likely to be too large to be convenient.

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The solution of the boundary-layer equations depends on the pressure distribution over the solid boundary, or the equivalent velocity distribution† in the main stream just outside the boundary layer. It was proposed in the first instance to attempt the solution of the boundary-layer equations by Hartree and Womersley's method for two cases, namely for Schubauer's experimental pressure distribution[13] for an ellipse of axial ratio 3 : 1, and for a linearly retarded velocity in the main stream. For the former case it was proposed to use the differential analyser for carrying out mechanically the integrations involved; for the latter case it seemed desirable to work to a greater nominal accuracy than that obtainable from the differential analyser, and it was proposed to do the integration numerically. The present report is concerned primarily with the latter work.

The case of a linearly-retarded velocity in the main stream has been examined by Kármán and Millikan[9] and by Howarth[8]. Kármán and Millikan's method gives separation at $x^* = 0 \cdot 102$ (in Howarth's notation) whereas Pohlhausen's method[10] gives separation at $x^* = 0 \cdot 156$. Howarth estimates the position of the separation point as between $x^* = 0 \cdot 119$ and $0 \cdot 129$ (Ref. 8, p. 555), and probably close to $x^* = 0 \cdot 120$ (p. 564). It may be said at once that the results of the present calculations entirely confirm this last result of Howarth, and in fact show that his result is correct nearly to one further decimal; they indicate that the separation point is close to $8x^* = 0 \cdot 959$.

The fact that for this case the pressure distribution is given by a formula, and so can be evaluated and interpolated with certainty to any accuracy required, makes this case a satisfactory one for comparative trials of different methods of obtaining approximate solutions of the boundary-layer equations. An experimentally observed pressure distribution, such as Schubauer's, has the disadvantage for this purpose that the observational material can be analysed by different workers in slightly different ways in deducing, for example, the pressure-gradient distribution or its derivative from the observed pressure. If the flow is very sensitive to the pressure distribution, as it is in that case, it is very difficult to make sure to what extent differences between the results of different approximations are real and how much they depend on slightly different interpretations of the observational material.

This is why it seemed worth while carrying out the present calculations to a rather high nominal accuracy, whereas a lower accuracy was regarded as adequate for the solution of the equations with Schubauer's pressure distribution.

Nov., 1948. The work covered in this and the succeeding Reports was carried out at the University of Manchester, and reported to the Aeronautical Research Committee before the war. It was approved for publication, but revision for publication was interrupted by the outbreak of war. The methods used and the results obtained, however, still seem of sufficient interest to put on record. Further, two other investigations (Refs. 14 and 15) in the theory of the laminar boundary layer in the immediate neighbourhood of the separation point were suggested directly by the results of this work, and other references have also been made to it[16].

2. *Outline of the Method of Integration.*—For equations in two independent variables, the essential idea of Hartree and Womersley's method is the replacement of the partial differential equation by an approximately equivalent ordinary differential equation, by replacing derivatives with respect to *one* of the variables by corresponding finite difference ratios, retaining the derivatives with respect to the other variable to be integrated either mechanically or by some standard process for the numerical integration of ordinary differential equations. In the present case, as will be seen, derivatives parallel to the boundary are replaced by finite differences, and integration is carried out along successive normals to the boundary at finite intervals, so that from the distribution of velocity across one section of the boundary layer, the distribution of velocity across another section at an interval downstream is calculated.

† On the approximations of the boundary-layer theory, the pressure is uniform throughout any one section of the boundary layer. The pressure is the quantity which explicitly appears in the equation of motion, but it is often convenient to express it in terms of an "equivalent" velocity in the main stream.

Further, Richardson's process of "h^2-extrapolation"[12] can be used to estimate and correct for the leading terms in the error made by working with finite intervals in one of the variables. The use of this process involves covering the same range by two independent integrations, one with intervals of half the length of those used in the other. It can be shown that under certain conditions, satisfied in the case of the boundary-layer equation, the aggregate error is proportional to the square of the interval length, so that from the difference between the results of these two integrations, the error in each can be estimated and an appropriate correction applied. This process of correction is of course not exact, and only deals with the leading terms in the error involved by the use of finite intervals, but the residual error can usually be kept small by the use of sufficiently small intervals.

It is of interest to compare the kind of approximation made in the present method with those made in some other methods for obtaining approximate solutions of the boundary-layer equations.

Pohlhausen's[10] method and some others involve the assumption that the velocity distribution in the boundary layer at any one section is given to an adequate approximation by a member of a one-parameter set of functions. In Pohlhausen's method the appropriate member of the set, at each section, is selected by use of the momentum-integral equation; the separate solutions represent the velocity distribution in the form of a quartic in the distance normal to the boundary, which does not seem a satisfactory form for a distribution which must tend asymptotically to that in the main stream.

Howarth's method for a general pressure distribution (Ref. 8, Part II), also involves the assumption that the velocity profile at each section can be matched exactly by a member of a one-parameter set of functions, but in this case these functions represent the set of velocity distributions through the boundary layer for some standard pressure distribution, namely that corresponding to a linearly-retarded velocity distribution in the main stream.

Some of these approximations appear rather artificial and formal, and in many cases it is difficult to assess the errors they are likely to introduce. In the present method the only approximation is the replacement of a derivative by a finite difference; this seems a straightforward approximation, and is one whose effect it is possible to assess quantitatively, and, ideally, the errors it introduces can be made as small as required by taking small enough intervals.

3. *The Boundary-layer Equation.*—Since the work is done in terms of non-dimensional reduced variables throughout, it is convenient to start from the boundary-layer equation in terms of these variables.

The following notation will be used:—

U_0	a representative velocity } of the system considered
l	a representative length } of the system considered
ν	kinematic viscosity of fluid
R	$U_0 l/\nu$
lx	distance measured along boundary ($x = 0$ at forward stagnation point)
$(l/R^{1/2})y$	distance measured normal to boundary ($y = 0$ at boundary)
$U_0 u$	tangential component of velocity
$(U_0/R^{1/2})v$	normal component of velocity
$U_0 U(x)$	velocity in main stream at distance x downstream
$\rho U_0^2 p$	pressure $[p(x) = p_0 - \frac{1}{2}U^2(x)]$*
$(l\nu U_0)^{1/2}\Psi$	stream function†.

* This relation between p and U depends on the approximations of the boundary-layer theory; it is sometimes convenient to express a pressure distribution in terms of the "equivalent" velocity distribution given by this relation.

† It is convenient to distinguish between the solution of equation (1) which will be regarded as the "exact" boundary layer equation (in the sense that it is the equation for which a solution is required, not that it is an exact expression of the physical situation) and that of the approximate equation by which it will later be replaced. Ψ will be used for the former, and ψ for the latter.

In the reduced variables x, y, p, U, Ψ so defined, the boundary layer equation is

$$\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} = -\frac{dp}{dx} + \frac{\partial^3 \Psi}{\partial y^3} \quad \ldots \quad (1)$$

$$\doteq U\frac{dU}{dx} + \frac{\partial^3 \Psi}{\partial y^3}, \quad \ldots \quad (2)$$

and the reduced stream function Ψ is related to the reduced velocity components u, v by

$$u = \frac{\partial \Psi}{\partial y}, \qquad v = -\frac{\partial \Psi}{\partial x}. \quad \ldots \quad (3)$$

The boundary conditions satisfied by Ψ are

$$\Psi = 0, \qquad \frac{\partial \Psi}{\partial y} = 0 \text{ at } y = 0, \quad \ldots \quad (4)$$

$$\frac{\partial \Psi}{\partial y} \to U(x) \text{ as } y \to \infty. \quad \ldots \quad (5)$$

Since in the approximate treatment of this equation to be considered in this report, the x derivatives are replaced by finite differences, while the y derivatives are retained, it is convenient to use dashes to indicate y derivatives. Hence we write (1) in the form

$$\Psi''' = \Psi'\frac{\partial \Psi'}{\partial x} - \Psi''\frac{\partial \Psi}{dx} + \frac{dp}{dx}. \quad \ldots \quad (6)$$

For reference later, we require the values of various orders of y derivatives of Ψ at $y = 0$ in the absence of a singularity at the boundary. These will be indicated by a suffix 0, and can most easily be derived by putting $\Psi = \Psi' = 0$ in the results of successive differentiation of (6) with respect to y. The first few are found to be as follows:—

$$\left.\begin{aligned}
\Psi_0''' &= \frac{dp}{dx} && \text{(a)} \\
\Psi_0^{(iv)} &= 0 && \text{(b)} \\
\Psi_0^{(v)} &= \Psi_0''\frac{d\Psi_0''}{dx} = u_0'\frac{du_0'}{dx} && \text{(c)} \\
\Psi_0^{(vi)} &= 2\frac{d^2p}{dx^2}\Psi_0'' = 2\frac{d^2p}{dx^2}u_0' && \text{(d)} \\
\Psi_0^{(vii)} &= 2\frac{dp}{dx}\frac{d^2p}{dx^2} && \text{(e)} \\
\Psi_0^{(viii)} &= 4\Psi_0''\frac{d\Psi_0^{(v)}}{dx} - 5\Psi_0^{(v)}\frac{d\Psi_0''}{dx} = 4u_0'^2\frac{\partial^2 u_0'}{\partial x^2} - u_0\left(\frac{\partial u_0'}{\partial x}\right)^2 && \text{(f)} \\
\Psi_0^{(ix)} &= 10(u_0')^2\frac{d^3p}{dx^3} - 13u_0'\frac{du_0'}{dx}\frac{d^2p}{dx^2} + 9\frac{d}{dx}\left(u_0'\frac{du_0'}{dx}\right)\frac{dp}{dx} && \text{(g)} \\
\Psi_0^{(x)} &= 8u_0'\left[5\frac{dp}{dx}\frac{d^3p}{dx^3} - 2\left(\frac{d^2p}{dx^2}\right)^2\right]. && \text{(h)}
\end{aligned}\right\} \quad \ldots \quad (7)$$

Of these, (7) (a), (b), (d), (e) have been given by Goldstein*, and (c), (f) are implied by formulae in the same paper†.

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* Ref. 3, formulae (3).

† Ref. 3, formula (5), *see also* Howarth, Ref. 8, formula (27), Prandtl, Ref. 11; the denominator of the last term given by Howarth should read 8 $(\partial u/\partial y)^3$.

The relations (7) (d), (e), (h) suggest that it would be an advantage to take the case of a uniform *pressure* gradient, rather than a uniform *velocity* gradient in the main stream, as the standard case for which to attempt to work out a rather accurate numerical solution. This procedure was contemplated for other reasons before the work was started, but Howarth's work had already been done for a uniform velocity gradient, and in order that results of the present work should be strictly comparable with those of Howarth's, it was judged better to carry out the work for the same case. It might have been still better to repeat Howarth's work for a uniform retarding pressure gradient*.

For convenience of comparison with Howarth's results, we suppose the representative velocity and length chosen so that, in the reduced variables,

$$U = 1 - \tfrac{1}{8}x \,. \qquad \ldots \qquad (8)$$

For a flat plate at zero incidence, this means that U_0 is taken as the velocity in the main stream at the leading edge, and the typical length is the length in which the velocity in the main stream decreases to $\frac{7}{8}U_0$. Then our reduced variable x is the quantity which Howarth writes $8x^*$, in terms of which his expansion of the stream function is carried out. Howarth's results give separation at $8x^* = 0 \cdot 96$, so that the range of x over which integration is to be carried out is of the order of unity, which is convenient on numerical grounds.

4. *Approximate Form of the Boundary-layer Equation.*—As already outlined in section 2, the process used in this work for obtaining an approximation to the solution of a partial differential equation in two variables depends essentially on replacing the derivatives with respect to one of the variables by finite differences.

Since the boundary conditions to be satisfied are at $x = 0$ and at both ends of the range of integration in y, the x derivative is the one which it is appropriate to replace by a finite difference coefficient, while leaving the y derivatives as such, to be integrated by numerical or mechanical means.

We will indicate by a small letter ψ the solution of the approximate equation by which the exact equation (6) is replaced. For the x interval from x_1 to x_2, values of ψ etc. at x_1 and x_2 will be indicated by suffices 1 and 2 respectively. The equation satisfied by ψ is obtained by replacing x derivatives by corresponding ratios of finite differences, e.g., $\partial\Psi/\partial x$ by $(\psi_2 - \psi_1)/(x_2 - x_1)$, and by replacing other quantities by the arithmetic mean of their values at the beginning and end of the interval. Thus (6) is replaced by

$$\tfrac{1}{2}(\psi_2''' + \psi_1''') = \tfrac{1}{2}(\psi_2' + \psi_1')\frac{\psi_2' - \psi_1'}{x_2 - x_1} - \tfrac{1}{2}(\psi_2'' + \psi_1'')\frac{\psi_2 - \psi_1}{x_2 - x_1} + P \,, \qquad \ldots \qquad (9)$$

where P takes the place of dp/dx.

The appropriate expression to take for P, in general, gave rise to an unexpected difficulty. At first sight the obvious replacement appears to be

$$P = \frac{p_2 - p_1}{x_2 - x_1} = -\tfrac{1}{2}\frac{U_2^2 - U_1^2}{x_2 - x_1} \,; \qquad \ldots \qquad (10)$$

but, since p is a given function of x, so that p_2 does not involve the unknown ψ_2, there would be no difficulty in using the alternative

$$P = \tfrac{1}{2}\left[\left(\frac{dp}{dx}\right)_2 + \left(\frac{dp}{dx}\right)_1\right] ; \qquad \ldots \qquad (11)$$

the order of the error is the same as for (10).

† For $U = b_0\surd(1 - 2x^*)$ (in Howarth's notation) which gives the same pressure and pressure gradient at $x = 0$ as in the case considered by Howarth, the differences from Howarth's work are that the term $-\frac{1}{8}$ in equation (7) of Howarth's paper is omitted, and the conditions (14) on $f_2', f_3' \ldots$ at infinity are replaced by $f_2' = -1/2^6$, $f_3' = 1/2^9$, $f_4' = -5/2^{14}, \ldots$.

When the pressure is a polynomial of degree not higher than the second in x, and only then, the two approximations for P are identical. Thus, for example, no discrimination between these two expressions for P is necessary for the calculations with which this report is primarily concerned, in which U is linear, and so p quadratic, in x; but the point may, and did, arise in a rather acute form when calculations are carried out for empirical pressure distributions such, for example, as Schubauer's*.

It would seem desirable to ensure, if possible, that the approximate solution should have the right behaviour at both ends of the range of y integration; by this means it might be hoped that the correction for finite x-interval length would be kept nearly as small as possible.

For $y = 0$, (9) becomes

$$\tfrac{1}{2}\,(\psi_2''' + \psi_1''') = P\,,$$

and if ψ''' is to satisfy the same condition as Ψ''', namely $\psi_0''' = dp/dx$ (cf. (7a)), clearly the expression (11) is the appropriate one to use for P. On the other hand for $y \to \infty$, we would require ψ''' and ψ'' to tend to 0, when (9) becomes

$$0 = \tfrac{1}{2}\,[(\psi_2')^2 - (\psi_1')^2]/(x_2 - x_1) + P\,,$$

and then expression (10) is the appropriate one to use.

This establishes definitely the appropriate expression to be taken at each end of the range of y integration, and since in general these will give different values of P, it is necessary to have some rule for interpolating between them for intermediate values of y. It would seem that the tangential velocity, or its square, might be a suitable variable in terms of which to interpolate between the extreme values of P, since it provides a measure of the extent to which the flow is different from that at the boundary or at infinity. Interpolation between the extreme values of P, linearly in ψ', would make $\partial P/\partial y \neq 0$ at $y = 0$, whereas on differentiating (9) and putting $\psi = \psi' = 0$ at $y = 0$,

$$(\psi_2^{(iv)} + \psi_1^{(iv)})_0 = \left(\frac{\partial P}{\partial y}\right)_0 .$$

Hence, in order that ψ should be as good an approximation as possible to Ψ, we require the left-hand side to be zero (cf. (7b)). This condition is satisfied by interpolating between the extreme values of P, linearly in $\psi_1'^2$,† so that

$$P = P_0 + (\psi_1'/U_1)^2\,\Delta P\,, \qquad \qquad (12)$$

where P_0 is given by (11) and ΔP is the difference between the values of P given by (10) and (11).

The value of ΔP is usually small, and the exact way in which the interpolation between the extreme values of P is carried out is rather a refinement. The above argument is not conclusive, but it shows what to avoid, and on the basis of it the substitution (11) has been used when ΔP was appreciable. A test on a rather large x-interval, for which ΔP was nearly **15** per cent. of P_0, for Schubauer's pressure distribution, gave very satisfactory results (see R. and M. 2427).

It is interesting to put $y = 0$ in the results of differentiating (9) successively, and to compare the results with the relations (7) to be satisfied with the derivatives of the exact solution Ψ at

* The work leading to the analytical results of this section was stimulated largely by difficulties encountered in the work on Schubauer's observed pressure distribution, but it is convenient to summarise it here, as some of the results are relevant in connection with the present work.

† Possibly linear interpolation in $(\psi_1' + \psi_2')^2$ would be better, but ψ_1' is known before the integration is started while ψ_2' is not. Since the variation of P is usually small, this method of interpolation is probably adequate.

the origin. It will be assumed that P is given by (12), and also that ψ_1 satisfies the conditions (7) so that, for example, from $(\psi_2^{(iv)} + \psi_1^{(iv)})_0 = 0$ it follows that also $(\psi_2^{(iv)} - \psi_1^{(iv)})_0 = 0$. Then

$$\left.\begin{aligned}
(\psi_2''' + \psi_1''')_0 &= 2P_0 && \text{(a)}\\
(\psi_2^{(iv)} + \psi_1^{(iv)})_0 &= 0 && \text{(b)}\\
(\psi_2^{(v)} + \psi_1^{(v)})_0 &= \frac{[(\psi_2'')^2 - (\psi_1'')^2]_0}{x_2 - x_1} + 2\left(\frac{\partial^2 P}{\partial y^2}\right)_0 && \text{(c)}\\
(\psi_2^{(vi)} + \psi_1^{(vi)})_0 &= 2(\psi_2'' + \psi_1'')_0 \frac{(\psi_2''' - \psi_1''')_0}{x_2 - x_1} + 2\left(\frac{\partial^3 P}{\partial y^3}\right)_0 && \text{(d)}\\
(\psi_2^{(vii)} + \psi_1^{(vii)})_0 &= \frac{2[(\psi_2''')^2 - (\psi_1''')^2]_0}{x_2 - x_1} + 2\left(\frac{\partial^4 P}{\partial y^4}\right)_0 && \text{(e)}\\
(\psi_2^{(viii)} + \psi_1^{(viii)})_0 &= 4(\psi_2'' + \psi_1'')_0 \frac{(\psi_2^{(v)} - \psi_1^{(v)})_0}{x_2 - x_1}\\
&\quad - 5\,(\psi_2^{(v)} + \psi_1^{(v)})_0 \frac{(\psi_2'' - \psi_1'')_0}{x_2 - x_1} - 2\left(\frac{\partial^5 P}{\partial y^5}\right)_0 && \text{(f)}
\end{aligned}\right\} \quad \ldots \quad (13)$$

These relations simplify considerably when the pressure gradient is linear in x, for then, first, the two expressions (10) and (11) for P are the same, so that P given by (12) is independent of y, and the various derivatives of P on the right-hand side of (13) then vanish. And secondly, we then have

$$(\psi_2''' - \psi_1''')_0 = \left(\frac{dp}{dx}\right)_2 - \left(\frac{dp}{dx}\right)_1 = (x_2 - x_1)\,\frac{d^2p}{dx^2} \quad \ldots \quad (14)$$

exactly, so that 13 (d) and (e) become

$$\left.\begin{aligned}
(\psi_2^{(vi)} + \psi_1^{(vi)})_0 &= 2\,(\psi_2'' + \psi_1'')_0 \frac{d^2p}{dx^2}\\
(\psi_2^{(vii)} + \psi_1^{(vii)})_0 &= 2\left[\left(\frac{dp}{dx}\right)_2 + \left(\frac{dp}{dx}\right)_1\right]\frac{d^2p}{dx^2}\,;
\end{aligned}\right\} \quad \ldots \quad (15)$$

hence if ψ_1 satisfies (7) (d) and (e), then so does ψ_2.

Thus, for the case with which this report is primarily concerned, it appears that the method of solution adopted necessarily imposes the conditions (7) (a), (b), (d), (e) on the approximate solution obtained. This may be significant in discussing how much information about the flow in the immediate neighbourhood of the separation point can be deduced from the results of these calculations (*see* section 10).

On the other hand (13) (c) and (f) are not exactly equivalent to (7) (c) and (f) respectively.

5. *Form of Equation Suitable for Integration for Small x.*—The form (6) of the boundary-layer equation (or the form (9) by which it is replaced) is not suitable for integration in the neighbourhood of the leading edge of a plate, on account of the presence of a singularity there. The nature of this singularity and the fact that when the pressure gradient is zero, Ψ is a function of $y/x^{1/2}$ only, suggests the use of the variables

$$\xi = x, \qquad \eta = y/2x^{1/2}\,, \quad \ldots \quad (16)$$

in this neighbourhood.

Since for zero pressure gradient the flow is a function of η only, it is probable that for other pressure distributions the " thickness " of the boundary layer, and hence the range of integration normal to the boundary to be covered before the flow becomes sensibly that in the main stream, will be more nearly the same at different sections when expressed in terms of η rather than in terms of y. Also the velocity distributions through the boundary layer at different sections are likely to be more nearly the same when expressed as functions of η rather than of y, and particularly so near the leading edge. For both reasons, the approximations made in replacing the partial differential equation by an ordinary equation are likely to be smaller than when the equation in (ξ, η) is used rather than that in (x, y), at any rate for small x.

In terms of (ξ, η), equation (6) becomes

$$\frac{\partial^3 \Phi}{\partial \eta^3} = \xi \frac{\partial}{\partial \xi}\left[\left(\frac{\partial \Phi}{\partial \eta}\right)^2 - 4U^2\right] - \left(\Phi + 2\xi \frac{\partial \Phi}{\partial \xi}\right)\frac{\partial^2 \Phi}{\partial \eta^2}, \quad \ldots \quad (17)$$

where

$$\Phi = \Psi / x^{1/2}, \quad \ldots \quad (18)$$

so that

$$u = \tfrac{1}{2}\partial \Phi / \partial \eta . \quad \ldots \quad (19)$$

The function Φ defined by (18) is the function $\psi/(b_0 x \nu)^{1/2}$ of Howarth's paper*. Also for the velocity distribution in the main stream (8), our x is Howarth's $8x$*, as already mentioned, hence

$$\Phi = f_0(\eta) - x f_1(\eta) + x^2 f_2(\eta) - x^3 f_3(\eta) \ldots, \quad \ldots \quad (20)$$

where the f's are the functions tabulated by Howarth.

Using Howarth's tables, it is easy to evaluate the velocity distribution through the boundary layer, and other data required for starting the integration, at a value of x away from the singularity at the leading edge, but still near enough for the values derived from Howarth's series, taken as far as his tables go (up to f_6), to be quite reliable to the last figure required. The starting point chosen was $x = 0{\cdot}4$, but in addition, as a test of the method of integration, and of the accuracy to be expected of the method of correction for the finite size of x-interval, it was decided to carry out an integration from $x = 0$ to $0{\cdot}4$ also. In terms of η, there is no singularity at $x = 0$, but a " velocity distribution " there is needed to start the integration, since the boundary layer has a non-zero thickness in η at $x = 0$; from (20) the required information at $x = 0$ is given by the function f_0 and its derivatives.

6. *Approximate Form of the Boundary-layer Equation in* (ξ, η).—The replacement of the boundary-layer equation in the form (17) by an equation for integration through a finite ξ-interval raised two points in connection with the appropriate replacement of derivatives by finite differences.

The first is concerned with the appropriate replacement for $\xi \partial \Phi / \partial \xi$ and similar terms. Replacement of ξ by its mean value and $\partial \Phi / \partial \xi$ by the appropriate finite difference ratio gives

$$\tfrac{1}{2}(\xi_2 + \xi_1)\frac{\phi_2 - \phi_1}{\xi_2 - \xi_1}, \quad \ldots \quad (21)$$

(where ϕ has been written for the function satisfied by the approximate equation), and the same expression is obtained if we use the identity

$$\xi \frac{\partial \phi}{\partial \xi} = \frac{\partial}{\partial \xi}(\xi \Phi) - \Phi ,$$

and carry out the appropriate replacements.

* Ref. 8, formula (1).

Thus (17) is replaced by

$$\frac{\partial^3}{\partial\eta^3}(\phi_2+\phi_1)=\frac{\xi_2+\xi_1}{\xi_2-\xi_1}\left[\left(\frac{\partial\phi_2}{\partial\eta}\right)^2-\left(\frac{\partial\phi_1}{\partial\eta}\right)^2\right]$$
$$-\tfrac{1}{2}\left[(\phi_2+\phi_1)+\frac{2(\xi_2+\xi_1)}{\xi_2-\xi_1}(\phi_2-\phi_1)\right]\times\frac{\partial^2}{\partial\eta^2}(\phi_2+\phi_1)-2Q\,, \quad \ldots \quad (22)$$

where Q is the quantity used to replace the term $4\xi\partial(U^2)/\partial\xi$ in (17). The second point is the appropriate replacement for this term.

As in the case of the term in equation (6) not involving ψ, this term could be replaced, to equal accuracy, in two ways, namely by the arithmetic mean of its initial and final values in the x-interval.

$$Q=2\left[\xi_2\left(\frac{\partial(U^2)}{\partial\xi}\right)_2+\xi_1\left(\frac{\partial(U^2)}{\partial\xi}\right)_1\right], \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

or by using the form of replacement (21)

$$Q=2(\xi_2+\xi_1)\frac{U_2^2-U_1^2}{\varepsilon_2-\xi_1}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

With $U=1-\tfrac{1}{8}\xi$, these give respectively

$$\left.\begin{aligned} Q&=-\tfrac{1}{2}[(\xi_2+\xi_1)-\tfrac{1}{8}(\xi_2^2+\xi_1^2)] \qquad &\text{(a)}\\ Q&=-\tfrac{1}{2}[(\xi_2+\xi_1)-\tfrac{1}{16}(\xi_2+\xi_1)^2]\,, \qquad &\text{(b)}\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

and these are not the same, even for a linear pressure gradient. The former appears the more natural if the term $4\xi\partial(U^2)/\partial\xi$ is written in (17) in the explicit form $-\xi(1-\tfrac{1}{8}\xi)$ which it takes with $U=1-\tfrac{1}{8}\xi$, the latter appears the more natural if one starts from the general form of (17).

On further investigation (25) (b) appears likely to be the better, since on substituting (24) in (22) it will be seen that the boundary condition

$$\frac{\partial\Phi}{\partial\eta}=2U$$

at infinity can then be satisfied exactly by the approximate solutions ϕ.

7. *Process of Solution.*—The process of solution of the ordinary equation (9) or (22) was the same in principle in all cases. An outline of the process for equation (9) will be given, followed by notes of the main points at which the integration of (22) differs from it.

The quantities required from the solution are the velocity profile, that is to say the distribution of tangential velocity ψ' through the boundary layer, and the normal gradient of the tangential velocity at the boundary, ψ_0'', which gives the skin friction and is needed to determine the separation point. The stream function ψ itself is of no particular interest; from the point of view of the integration it is merely an auxiliary variable introduced as the most convenient way of ensuring that the equation of continuity is satisfied.

For any one x-interval, say from $x=x_1$ to $x=x_2$, the flow at the initial section $x=x_1$ is known, and as we will see, for the equation in the form (9) the only data necessary is ψ_1' as a function of y; correspondingly ψ_2' as a function of y is required to form the data for the next interval $x=x_2$ to $x=x_3$.

The equation to be solved, for each x-interval, is third-order non-linear, with two boundary conditions specified at one end of the range of integration ($\psi' = \psi = 0$ at $y = 0$) and one at the other ($\psi' \to U$ as $y \to \infty$). To obtain a solution satisfying them, it is necessary to use some form of trial-and-error method, and the form of the boundary conditions suggests integrating from $y = 0$ with different trial values of $(\psi_2'')_0$, and adjusting the value of ψ_0'' so that the solution satisfies the condition at $y = \infty$.

The process of integration for any one trial value of $(\psi_2'')_0$ is straightforward. The integration formula

$$\delta(2f) = \int_{y_0}^{y_0+\delta y} 2f'dy = \delta y\,[f_0' + f_1' - \tfrac{1}{12}(\delta^2 f_0' + \delta^2 f_1')] \qquad \ldots \quad \ldots \quad \ldots \quad (26)$$

(here suffixes 0 and 1 indicate values at the ends of the interval δy of the y integration) was used for each integration; the interval of integration taken throughout was $\delta y = 0{\cdot}1$, and the second difference terms in (26) were usually small and often negligible. As suggested by (26), the quantities actually calculated are:—

$$\frac{\partial^3}{\partial y^3}(\psi_2 + \psi_1),\quad 2\frac{\partial^2}{\partial y^2}(\psi_2 + \psi_1),\quad 4\frac{\partial}{\partial y}(\psi_2 \pm \psi_1),\quad 8(\psi_2 - \psi_1),$$

the powers of 2 in the coefficients being chosen to avoid a large number of divisions by 2*.

For convenience in using these quantities, the equation is taken in the form

$$\frac{\partial^3}{\partial y^3}(\psi_2 + \psi_1) = \frac{1}{16(x_2 - x_1)}\left[\left\{4\frac{\partial}{\partial y}(\psi_2 + \psi_1)\right\}\left\{4\frac{\partial}{\partial y}(\psi_2 - \psi_1)\right\}\right.$$
$$\left. - \left\{8(\psi_2 - \psi_1)\right\}\left\{2\frac{\partial^2}{\partial y^2}(\psi_2 + \psi_1)\right\} + 32P(x_2 - x_1)\right]. \qquad \ldots \quad \ldots \quad (27)$$

For $U = 1 - \frac{1}{8}x$, the term $32P(x_2 - x_1)$ which is equal to $-16(U_2^2 - U_1^2)$ (*see* (10)) becomes $\frac{1}{4}[16 - (x_2 + x_1)](x_2 - x_1)$.

The quantities obtained by integration are $2\frac{\partial^2}{\partial y^2}(\psi_2 + \psi_1)$, $4\frac{\partial}{\partial y}(\psi_2 + \psi_1)$, $8(\psi_2 - \psi_1)$; $4\frac{\partial}{\partial y}(\psi_2 - \psi_1)$ is obtained by subtracting $8\,\partial\psi_1/\partial y$ from $4\frac{\partial}{\partial y}(\psi_2 + \psi_1)$, and this is the only point in the integration at which information concerning the velocity profile at x_1 is required.

The solutions with different starting values of $[\partial^2(\psi_2 + \psi_1)/\partial y^2]_0$ diverge rather rapidly from one another† beyond about $y = 2$, especially for the smaller values of $x_2 - x_1$, whereas the integration has to be taken to about $y = 6$ or $y = 8$ before the flow in the boundary layer has become substantially that in the mainstream. It is therefore usually necessary to interpolate, at some intermediate y, between the solutions calculated with different trial starting values of ψ_0'', and to start a new integration from the interpolated values. This interpolation process may have to be repeated many times (usually 10 to 20 times in the present work) before trial solutions are obtained over the whole range, and near enough together for the solution satisfying the conditions at infinity to be interpolated between them.

* I am indebted to Dr. L. J. Comrie for suggesting this arrangement, which has been very successful.

† The difference between two such solutions may increase by a factor 2,000 or more in a range of 1 in y.

Before starting a solution from interpolated values of $2\,(\psi_2'' + \psi_1'')$ etc., it is necessary to verify that the use of linear interpolation is justified. This can be done by interpolating $2\,\frac{\partial^2}{\partial y^2}(\psi_2 + \psi_1)$, $4\,\frac{\partial}{\partial y}(\psi_2 \pm \psi_1)$, $8\,(\psi_2 - \psi_1)$, and evaluating $\frac{\partial^3}{\partial y^3}(\psi_2 + \psi_1)$ from them using (27); if the result agrees with the interpolated value of $\frac{\partial^3}{\partial y^3}(\psi_2 + \psi_1)$, then linear interpolation is justified*.

Approaching the separation point, the solutions for different trial values of $(\psi_2'')_0$ run nearly parallel to one another for a long way (out to about $y = 2$) and then separate rapidly. Study of the numerical details of the integration shows that $\partial^3(\psi_2 + \psi_1)/\partial y^3$ is remarkably insensitive to the value taken for $[\partial^2(\psi_2 + \psi_1)/\partial y^2]_0$, although the separate terms in (27) may individually have considerably different values in the two integrations. This means that the integration of a trial solution has to be continued over quite a considerable range from $y = 0$ before the behaviour of ψ shows in which way the solution is going to diverge. The result is that, unless care is taken, the accumulated effects of rounding-off errors may mask the real effects of a change of trial value of $[\partial^2(\psi_2 + \psi_1)/\partial y^2]$. Thus in this region $(\psi_2'')_0$ is not very well defined by the condition at $y = \infty$. and special precautions (such as the retention of an extra place of decimals to keep down the effect of rounding-off errors) are necessary to obtain a good approximation to the position of the separation point.

This behaviour is probably not accidental. Rather similar behaviour was found in the case of the solutions of a related equation arising in Falkner and Skan's treatment of the boundary-layer equation†, and this behaviour is probably related also to the difficulty that was met in trying to find any approximation to a solution downstream from the separation point.

In the integration for a single x-interval it is only necessary to work with the functions $\psi_2 \pm \psi_1$ and their derivatives. Once a "final" solution, namely one satisfying the boundary condition at $y = \infty$, has been obtained, $8\,(\psi_2' - \psi_1')$ is calculated for it and added to the given $8\psi_1'$ to give $8\psi_2'$, which is them smoothed before being used as the given function $8\psi_1'$ for the next x-interval.

This smoothing of the values of ψ_1' is important, as the process used to correct for the finite length of x-interval tends to exaggerate small irregularities in the data or integration as will be seen shortly. Also the quantity actually calculated in the integration is $4\,(\psi_2' + \psi_1')$, and any irregularities (due to rounding-off errors) in it are, in effect, doubled in calculating $8\psi_2'$, and these irregularities might exaggerate the effects of rounding-off errors in $8\psi_1'$.

Similarly it is advisable to ensure that the numerical values of $8\psi'$ for small y are in fact consistent with the conditions (7) (a), (b), (d), (e), which we have seen the approximate solution should satisfy exactly (apart from rounding-off errors) when p is quadratic in x. With $U = 1 - \frac{1}{8}x$, these conditions become

$$\psi_0''' = \tfrac{1}{8}\,(1 - \tfrac{1}{8}x)\,,\ \psi_0^{(iv)} = 0\,,\ \psi_0^{(vi)} = -\tfrac{1}{32}\,\psi_0''\ \psi_0^{(vii)} = -\tfrac{1}{256}\,(1 - \tfrac{1}{8}x)\,,$$

so that

$$8\psi' = 8\psi_0''\,y + \tfrac{1}{2}(1 - \tfrac{1}{8}x)\,y^2 + \frac{8\psi_0^{(v)}}{4!}\,y^4 - \frac{\psi_0''}{480}\,y^5 - \frac{1 - \frac{1}{8}x}{32\times 720}\,y^6 + \frac{8\psi_0^{(viii)}}{7!}\,y^7\,. \qquad (28)$$

Thus we should have

$$\left[\frac{8\psi'}{y} - \left\{8\psi_0'' + \tfrac{1}{2}(1 - \tfrac{1}{8}x)\,y - \frac{\psi_0'}{480}\,y^4 - \frac{1 - \frac{1}{8}x}{32\times 720}\,y^5\right\}\right]\frac{1}{y^3} = \frac{8\psi_0^{(v)}}{4!} + \frac{8\psi_0^{(viii)}}{7!}\,y^3 + \ldots\,.$$

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* This criterion is not a general one, but applies in the present case because departures from linearity appear first in the highest derivative and increase rapidly with y.

† Ref. 4. The behaviour referred to is the insensitiveness of y', at large x, to $y''(0)$ for small $\beta - \beta_0$, so that $y''(0)$ is not well determined by the condition on y' at large x (*see* section 3 of the paper quoted). $\beta = \beta_0$ corresponds, in a sense, to separation.

The left-hand side is evaluated from the values of $8\psi'$ and plotted against y^3, usually for values of y up to $1 \cdot 5$. For small y (less than about $y = 0 \cdot 5$) the plotted points are usually irregular on account of the considerable contribution from a unit in the last decimal when divided by y^4, but this can be indicated in plotting. Allowing for this, a smooth curve (usually a straight line) is drawn " through " the points, the values of the right-hand side read off, and values of $8\psi'$ reconstructed from them*.

In the smoothing and adjustment of the values of $8\psi'$, it was very seldom necessary to alter any value by more than a unit in the last decimal, and only about one in ten of the values were altered by a unit.

The process of correction for finite length of x-interval was as follows. The interval x_1 to $x_3 = x_1 + 2X$ was covered by two independent integrations, one with a single interval of length $2X$ and the other with two intervals of length X each. If θ_I and θ_{II} are the values of any quantity θ, at $x = x_3$ and the same value of y, calculated from the 1-interval and 2-interval integrations, respectively, then Richardson's process of h^2-extrapolation[12] gives

$$\theta_{II} + \tfrac{1}{3}(\theta_{II} - \theta_I) \qquad \qquad (29)$$

as the value corrected for finite length of x-interval. The correction is not, of course, exact, but the two leading terms in the error are eliminated by this process.

As already mentioned, this process tends to exaggerate small irregularities in the initial data. This can be seen as follows. Suppose that at $x = x_1$, a single one of the values of ψ_1' is in error by ε so that the value used in the integration is $\psi_1' + \varepsilon$. This single irregularity would not greatly affect the integration, and in any case, since $(\psi_2' + \psi_1')$ is obtained by integration of $(\psi_2'' + \psi_1'')$, $\psi_2' + \psi_1'$ is smooth (apart from rounding-off errors) so that ψ_2', deduced from the integrated $(\psi_2' + \psi_1')$ and the given $(\psi_1' + \varepsilon)$, is in error by $- \varepsilon$. Similarly the result ψ_3' for the second interval ψ_3' would be in error by $+ \varepsilon$. But ψ_3' obtained in 1-interval would be in error by $- \varepsilon$, and using (29) it will be seen that the "h^2-extrapolated" value would be in error by $5\varepsilon/3$. This shows how irregularities may build up, and why it is desirable to smooth the results at each stage to prevent this occurring.†

For the equation in the form (22), the data required at the initial section consists of both $\partial\phi_1/\partial\eta$ and ϕ_1; the latter is best obtained by integrating $\partial\phi_1/\partial\eta$. The equation can be put in a slightly more convenient form for numerical work, but at best it is appreciably more complicated than (27) on account of the appearance of factors involving ξ explicitly, and of $(\phi_2 + \phi_1)$ as well as $\phi_2 - \phi_1$.

8. *Trial Solution, $\xi = 0$ to $0 \cdot 4$.*—As already mentioned, it was decided first to carry out a trial solution of (17) from $\xi = 0$ to $0 \cdot 4$, as a general test of the method.

In planning the details of the arrangement of the first experimental work, it happened that the equation was written with the term $- 4\xi\partial(U^2)/\partial\xi$ in the explicit form $\xi(1 - \frac{1}{8}\xi)$ which it takes with $U = 1 - \frac{1}{8}x$, and consequently the substitution (25a) for Q was used at first without enquiry as to whether a more suitable substitution could be found. The first integrations, in which three decimals were retained in $\partial^2\phi/\partial\eta^2$ and $\partial\phi/\partial\eta$, were hardly precise enough to bring into prominence the difficulty of satisfying the boundary conditions with this substitution for Q; but the results were so encouraging that it seemed worth repeating the integrations with an extra decimal throughout, and then the difficulty was plain. This led to a further examination of the approximate equation from the point of view of the terminal conditions to be satisfied, and this showed the formal advantage of the substitution (25b).

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* For this method of adjusting ψ' to the series, I am indebted to Mr. W. Hartree.

In the work as actually carried out, the coefficient of y^6 in (28) was taken, in error, to be $(1 - \frac{1}{8}x)/16 \times 720$; this error was not noticed till long after the work was completed, but it is not likely to have effected the results seriously. I am indebted to Dr. C. W. Jones for the correction.

† Some part of the accumulation of errors in this way can be avoided by taking *four* short intervals and *two* long ones before carrying out the process of h^2-extrapolation.

To see whether this formally more satisfactory substitution gave quantitatively better results in practice, integrations were carried out with it as well. In all integrations the interval $\xi = 0$ to $0 \cdot 4$ was covered by two independent calculations, one with one interval $\xi = 0$ to $0 \cdot 4$, and the other with two intervals $\xi = 0$ to $0 \cdot 2$, $\xi = 0 \cdot 2$ to $0 \cdot 4$. The results are summarised in Table 1. This shows the advantage of the substitution (25b) over the substitution (25a) on all counts; the errors due to finite x-interval are smaller, and the h^2-extrapolated values are more accurate.

The full results with substitution (25b) are shown in Table 2, and compared with results of calculations from Howarth's tables. The results of the integration seem highly satisfactory. The maximum error in the values of $2\partial\phi/\partial\eta$ is $0 \cdot 0004$ out of a maximum of $3 \cdot 8$, or 1 in 10,000 of the maximum velocity. As the fourth decimal was the last retained in the integration, which covered a range of 4 in η, this error is hardly greater than the possible cumulative rounding-off error. The value of $(\partial^2\phi/\partial\eta^2)_{\eta=0}$ at $\xi = 0 \cdot 4$ agrees almost exactly with the value calculated from Howarth's series, but the extreme closeness of the agreement is probably a numerical accident.

These results were regarded as good enough to justify the retention of a fourth decimal in the further work. The accuracy and range of Howarth's tables are only just enough to guarantee a fourth decimal in $2\partial\phi/\partial\eta$ at $x = 0 \cdot 4$*, and in going further we soon get beyond the range where the results can be tested by comparison with his tables. But the satisfactory results of this trial suggest that some confidence can be placed in the results of the continuation of the integration by the same method.

9. *Continuation of the Solution.*—Starting from the values of $2\partial\phi/\partial\eta$ at $\xi = 0 \cdot 4$, calculated from Howarth's tables and tabulated in Table 2, the range $\xi = 0 \cdot 4$ to $0 \cdot 8$ was covered in one and two steps. The results of these two separate integrations, and the results obtained from them by h^2-extrapolation and smoothed, are given in Table 3. The maximum difference between the results of the one-step and two-step calculations is about $1 \cdot 7$ times as large as for the interval $\xi = 0$ to $0 \cdot 4$, and occurs at about the same place ($\eta = 1 \cdot 5$).

The calculation was first carried out to three decimals in $2\partial\phi/\partial\eta$, and then repeated to four decimals; this or a similar process was used throughout a large part of the work; the three-decimal solution forms a valuable guide in the four-decimal work, in providing approximate values for the variation of the solution with variation of the trial starting value of $[\partial^2(\phi_2+\phi_1)/\partial\eta^2]_0$, and in suggesting the appropriate interpolation between two trial solutions.

It is perhaps worth noting that the final results of the three-decimal calculations nowhere differ by more than 2_5 in the third decimal from the results of the four-decimal calculations. In view of the large number of possibilities of accumulation of rounding-up errors in the various integrations and interpolations, this degree of accuracy in the last decimal retained seems rather surprising. The fact that the solution has to fit boundary conditions at both ends of the range is not in itself an adequate explanation, for it might be possible for the solution to be appreciably in error for finite y, and still fit the boundary condition at $y = \infty$. This accuracy in the last decimal has been noticed in other calculations of this kind (for example in the solution of the equation $\partial\theta/\partial t = \partial^2\theta/\partial x^2 + 1$ used by Hartree and Womersley as a test of the method of integration), and appears to be characteristic of them. It suggests that also the fourth decimal in the four-decimal calculations may be given some significance, although it was really only retained to avoid the accumulation of rounding-off errors in the third decimal.

By the time $x = 0 \cdot 8$ had been reached, the awkward form of equation (21) for practical work had become very clear to all who had taken part in the numerical work. Since the integration had by then been taken well away from the singularity at the leading edge, so that the argument in favour of (22) rather than (9) had no longer much weight, and it was anticipated that with the best conditions the approach to the separation point might be rather troublesome, it was decided to change over to the use of the simpler equation (9), with y as the co-ordinate normal to the boundary.

* The contributions from the functions $f'_7, f'_8 \ldots$, which have not been calculated by Howarth, may amount to 1 in the fourth decimal at $\xi = 0 \cdot 4$, but they can probably be estimated to adequate accuracy. Five decimals in $\partial\phi/\partial\eta$ were kept in the calculations from Howarth's tables, to avoid the accumulation of rounding-off errors.

At $x = 0\cdot8$, $\eta = y/2.(0\cdot8)^{1/2} = 0\cdot559017y$, and the solution was available at exact values of η (multiples of $0\cdot1$), as tabulated in Table 3. Values of $2\partial\phi/\partial\eta$ were interpolated at multiples of $0\cdot1$ in y, and converted into values of $8\partial\psi/\partial y$ for use in equation (9) in form (27); these values, smoothed and adjusted by (28), are given in the second column in Table 4, and formed the starting point for the further integration. It would have been possible to continue using values of y corresponding to the exact values of η at $x = 0\cdot8$, but it was judged that the time and trouble of interpolating to exact values of y would be repaid in quicker and easier working later, and this was undoubtedly the case.

The aim of the subsequent work was twofold. First, it was required to determine the separation point as accurately as possible, and, if possible, the flow in the immediate neighbourhood of the separation point and the nature of the singularity at that point. Secondly, it was required to calculate the flow through the whole boundary layer, for evaluation of the integrals $\int_0^\infty (1 - u/U)\, dy$ for the "displacement thickness" and $\int_0^\infty (u/U)\,(1 - u/U)\, dy$ for the "momentum thickness".

For the determination of the separation point and the flow in its neighbourhood it was not necessary to integrate the equations through the whole thickness of the boundary layer, as in the trial-and-error determination of the starting value of $(\psi_2'' + \psi_1'')$, in order to obtain a solution with the right behaviour at ∞, solutions with different trial starting values diverge so rapidly beyond about $y = 2\cdot5$ that the behaviour of two trial solutions at about $y = 3$ is already enough to show how the required solution must be interpolated between them, accurately enough to establish the fourth decimal in this solution out to about $y = 2\cdot5$.

Thus for this purpose it is only necessary to take successive integrations out progressively less and less far from the boundary. On the other hand, as already explained, it is desirable to carry out these integrations to the highest degree of numerical accuracy which the data and method will provide. Four decimals were retained through this part of the work, the four-decimal calculations being usually preceded by three-decimal calculations to provide a guide as already explained.

After some experiments with longer x steps, the following were adopted:

Stage (A). $x = 0\cdot8$ to $0\cdot88$ in one and two steps; correction at $x = 0\cdot88$ for finite size of x steps.

Stage (B). Using the final results of stage (A), $x = 0\cdot88$ to $0\cdot94$ in one and two steps; correction at $x = 0\cdot94$ for finite size of x steps.

Stage (C). Using the final results of stage (B), $x = 0\cdot94$ to $0\cdot956$ and $0\cdot94$ to $0\cdot958$, each in one and two steps, and correction for finite size of x steps.

The final results of each stage were smoothed and adjusted to the series (28) as already explained before being used as the starting values for the next stage.

For stage (B), an attempt was first made to go from $x = 0\cdot88$ to $0\cdot96$ in one and two steps, but the difference between the results at $x = 0\cdot96$ in one and two steps was too large to give confidence in the results as part of the data for determining the separation point. The corrections for the finite length of x-interval increase rapidly on close approach to the separation point, and it seemed advisable to approach it by intervals as small as were practicable.

For stage (C), it was at first intended to go from $x = 0\cdot94$ to $0\cdot96$ in one and two steps, but no solution could be found for the second step, and it was later found that $x = 0\cdot96$ is just beyond the separation point.

The results at $x = 0\cdot956$ and $0\cdot958$ were both calculated by one and two steps from $x = 0\cdot94$; the correction for the finite size of x-interval at $x = 0\cdot958$ was about three times that at $x = 0\cdot956$; this rapid increase is presumably due to the close approach to the singularity at the separation point. Despite this, an attempt was made to reach the separation point itself, and appeared to be successful; this will be discussed in section 10.

For the calculation of the flow well away from the boundary, it seemed adequate to dispense with the fourth decimal, and three decimals in $8\partial\psi/\partial y$ were kept throughout; also it seemed adequate to work with larger x-intervals beyond $x = 0\cdot88$. A unit in the third decimal represents about 1 part in 8,000 of the velocity in the main stream, and the retention of this decimal is probably enough to guard against the accumulation of rounding-off errors to any extent which would be appreciable in the use of the results. The greatest correction for finite interval length at $x = 0\cdot88$ and $x = 0\cdot94$ was only 1_5 in the third decimal, so the results at $x = 0\cdot84$ and $0\cdot92$ should not be in error by more than 1 in the last decimal, as far as errors due to the finite length of x-interval are concerned.

A complete table of the results is given in Table 4. For $x = 0\cdot84$, $0\cdot91$, $0\cdot92$, $0\cdot948$, $0\cdot949$ only results calculated in one step are available, and these are tabulated. For $x = 0\cdot88$, $0\cdot94$, $0\cdot956$, $0\cdot958$ results calculated by one step and two steps are given to show the magnitude of the difference, and also " final " results, namely, results corrected for finite size of x-interval and, for $x = 0\cdot88$ and $0\cdot94$ for which the results formed the starting point for further integrations, smoothed and ajusted by the series (28).

Values of the displacement thickness $\int_0^\infty (1 - u/U)\, dy$, the momentum thickness $\int_0^\infty (u/U)\,(1 - u/U)\, dy$, and $2\,(\partial u/\partial y)_0$ are given in Table 5.

The " final " values of $2\,(\partial u/\partial y)_0$ which are not enclosed in brackets have been obtained by Richardson's process of h^2-extrapolation from the results of calculation by one and two steps, and subsequent adjustment to fit the series (28). This adjustment is responsible for the differences of a unit in the last figure, in some cases, between the " final " values and the values obtained by h^2-extrapolation. Approximate corrections for interval length have also been applied to the values of $2\,(\partial u/\partial y)_0$ for which only results calculated by one step are available. From the general theory of the method of integration, it follows that the leading term in the error after one step $\delta x = \Delta$ is a quarter of that after two such steps. The error in the latter case can be estimated from the results of calculations with two small steps $\delta x = \Delta$ and one large step $\delta x = 2\Delta$, and hence a correction to the results calculated by one small step $\delta x = \Delta$ can be estimated. " Final " results involving corrections thus estimated are enclosed in brackets.

10. *The Separation Point.*—There are two ways of attempting to determine the position of the separation point from the results of integrations such as those considered in this report. One is to carry out the integrations up to points as near the separation point as possible, to determine the value of $(\partial u/\partial y)_0$, the velocity gradient at the boundary, at different sections, and to extrapolate, from these values, the value of x at which $(\partial u/\partial y)_0 = 0$. The other way is to carry the integration up to the separation point itself by altering slightly the trial and error process for finding the solution satisfying the required boundary conditions; instead of taking a given x-interval and adjusting $(\psi_2'')_0$ so that the solution satisfies the required condition at ∞, one can specify $(\psi_2'')_0 = 0$ and adjust the x-interval length. Both these methods have been used, and give closely consistent values for x_s, the value of x at the separation point.

The values of $2\,(\partial u/\partial y)_0 = 2\psi_0''$ at different sections are given in Table 5. The extrapolation of the separation point from these values of $2\,(\partial u/\partial y)_0$ would be most satisfactory and convincing if it were known how $(\partial u/\partial y)_0$ should vary near the separation point, so that the extrapolation would be simply a matter of determining constants in a known formula. But no analytical investigation of the nature of the singularity at separation, and of the flow upstream from separation, was available at first. Goldstein[3] had examined the flow *downstream* from an arbitrary *given* velocity distribution through the boundary layer, and had considered shortly the case when this velocity distribution was such that $(\partial u/\partial y)_0 = 0$ (Ref. 3, section 4.1), but this problem is rather different from the problem of the flow *upstream* from separation, in which the velocity distribution at separation is essentially not given, but has to be determined from the boundary-layer equation and the flow further upstream. Thus there was no theoretical formula with which to compare the results of the integration, and in the first instance the analysis of these results had to be more or less empirical. Different methods of making this analysis, either by plotting $\log [2 (\partial u/\partial y)_0]$ against $\log (x_s - x)$ for different values of x_s, or by plotting $[2 (\partial u/\partial y)_0]^{1/q}$ against x for different values of q, all showed that, near the separation point, the values of $2 (\partial u/\partial y)_0$ could be fitted closely by

$$2 (\partial u/\partial y)_0 \propto (x_s - x)^q , \qquad (29)$$

with x_s close to $0 \cdot 959$, and q definitely greater than $\frac{1}{2}$ and less than $\frac{3}{4}$. A value about $q = 0 \cdot 6$ seemed indicated, and indeed the whole set of values of $2 (\partial u/\partial y)_0$ from $x = 0 \cdot 8$ onwards are represented very closely by (29) with $q = 0 \cdot 6$.

At this stage, the results were discussed with Dr. Goldstein, who undertook further analytical examination of the flow upstream from separation. An account of this work has recently been published[14].

Goldstein found first that the boundary-layer equations have no solution giving a relation of the form (29) for the behaviour of $(\partial u/\partial y)_0$, with the index q greater than $\frac{1}{2}$ and less than $\frac{3}{4}$, and with u finite at $x = x_s$, $y = 0$. This showed that, as seemed possible from the first, (29) was simply an empirical formula fitting the available numerical data, but having no theoretical basis. Such an empirical fit can, of course, only suggest and not establish the limiting behaviour of $(\partial u/\partial y)_0$ as $x \rightarrow x_s$; if $(\partial u/\partial y)_0$ were really expressible, for example, in a power series in $(x_s - x)^{1/4}$ beginning with a term in $(x_s - x)^{1/2}$, then for the small, but not extremely small values of $(x_s - x)$ for which values of $(\partial u/\partial y)_0$ are available, the behaviour of the function defined by the power series might simulate quite closely the behaviour expressed by (29) with $y = 0 \cdot 6$.

From Goldstein's results for the flow *down*stream from a *given* velocity distribution at separation, it follows that either the singularity at separation is of such a kind that $(\partial u/\partial y)_0 = O\,[(x - x_s)^{1/4}]$, in which case the condition (7a) of the present report is violated at the separation point, or the singularity is of a less drastic kind in which $(\partial u/\partial y)_0 = O\,[(x - x_s)^{1/2}]$ and the conditions (7a), (7b) still hold at the separation point itself. Although, as already pointed out, these results are not directly applicable to the present problem, they suggest the kind of solution to be examined.

The values of $(\partial u/\partial y)_0$ calculated by integration seem to rule out a limiting behaviour described by an index $q = \frac{1}{4}$ in (29), hence only the index $\frac{1}{2}$ seemed possible, and this case was examined in detail by Goldstein for the flow upstream from separation. The results so far obtained, and as far as they concern the present discussion, are as follows.

Notation :—

$$U_s = \text{value of } U_a \text{ at separation},$$
$$U_s' = \text{value of } dU_s/dx_a \text{ at separation},$$
$$l_s = -\, U_s/U_s',$$
$$R_s = l_s U_s/\nu,$$
$$x_1 = [(x_s)_a - x_a]/l_s,$$
$$y_1 = R_s^{1/2}\, y_a/l_s,$$
$$u_1 = u_a/U_s,$$
$$p_1 = p_a/\rho_a U_s^2,$$

(a suffix a is used here to represent values of quantities in some dimensional system of measurements, since the symbols $x, y, u \ldots$ without suffixes are used in this report for non-dimensional quantities).

Let the pressure distribution near separation be such that

$$-\frac{\partial p_1}{\partial x_1} = 1 + P_1 x_1 + P_2 x_1^2 + P_3 x_1^3 \ldots, \quad \ldots \quad (30)$$

so that for $\quad U = b_0 - b_1 x, P_1 = 1, P_2 = P_3 = \ldots = 0$.

Also let $\quad \xi_1 = x_1^{1/4}, \quad \eta_1 = y_1/2^{1/2}\, x_1^{1/4}$,

and suppose that the velocity u_1 at separation is expansible in the form

$$(u_1)_{\xi=0} = a_2 y_1^2 + a_3 y_1^3 + a_4 y_1^4 + \ldots, \quad \ldots \quad (31)$$

and that the velocity gradient at the solid boundary, upstream from separation, is given by

$$\left(\frac{\partial u_1}{\partial y_1}\right)_{y_1=0} = 2^{3/2}\,(\alpha_1 \xi_1^2 + \alpha_2 \xi_1^3 + \alpha_3 \xi_1^4 + \ldots). \quad \ldots \quad (32)$$

Then $a_2 = \frac{1}{2}$ and $a_3 = 0$ (so that the relations (7a), (7b) still hold), α_1 is arbitrary, and the other coefficients are determined in terms of it by relations of which the first few are [Ref. 14, formulæ (29) and (31)–(34)]

$$\left.\begin{array}{lll} \alpha_2 = 1\cdot778\alpha_1^2 & & (a) \\ \alpha_3 = 3\cdot311\alpha_1^3 & & (b) \\ a_4 = -\frac{1}{6}\alpha_1^2 & & (c) \\ a_5 = -0\cdot135\alpha_1^3 & & (d) \\ a_6 = -0\cdot0595\alpha_1^4 - \dfrac{P_1}{360}\,. & & (e) \end{array}\right\} \quad \ldots \quad (33)$$

Since the boundary-layer equation, in the dimensionless reduced form (1) which has been taken as the basis of the present work, can be derived from the dimensional form by putting $\nu = 1$, $\rho = 1$, it follows that the relations between Goldstein's reduced variables and those of this paper, defined in section 2, can be obtained by putting $\nu = 1$, $\rho = 1$ in the definitions of his reduced variables and dropping the suffix a. Thus, for example, taking $x_s = 0\cdot959$ as the separation point to an adequate approximation for the present purpose, we have

$$-U_s' = \frac{1}{8}, \quad U_s = 1 - \frac{1}{8}(0\cdot959) = 0\cdot88, \quad l_s = 7\cdot04, \quad R_s = 6\cdot20. \quad \ldots \quad (34)$$

Now from (32)

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{R_s^{1/2} U_s}{l_s}\left(\frac{\partial u_1}{\partial y_1}\right)_{y_1=0} = \frac{2^{3/2} R_s^{1/2} U_s}{l_s} \times \left[\alpha_1\left(\frac{x_s - x}{l_s}\right)^{1/2} + \alpha_2\left(\frac{x_s - x}{l_s}\right)^{3/4} + \ldots\right],$$

or, on substitution for α_2, α_3 in terms of α_1 from (33) (a), (b),

$$\frac{2\,(\partial u/\partial y)_0}{(x_s - x)^{1/2}} = \frac{2^{5/2} R_s^{1/2} U_s}{l_s^{3/2}}\left[\alpha_1 + \frac{1\cdot778\alpha_1^2}{l_s^{1/4}}(x_s - x)^{1/4} + \frac{3\cdot311\alpha_1^3}{l_s^{1/2}}(x_s - x)^{1/2} + \ldots\right].$$

With the numerical values (34) appropriate to the present case, this becomes*

$$100 \cdot \frac{2\,(\partial u/\partial y)_0}{[100\,(x_s - x)]^{1/2}} = 6\cdot65\,[\alpha_1 + 0\cdot3453_1\alpha_1^2\{100\,(x_s - x)\}^{1/4}$$
$$+ 0\cdot1248\,\alpha_1^3\{100\,(x_s - x)\}^{1/2} + \ldots]. \quad \ldots \quad (35)$$

(The factors 100 are introduced for numerical convenience).

On the assumptions on which this formula is deduced, namely that there is a singularity at separation of such a nature that $(\partial u/\partial y)_0 = O\,[(x_s - x)^{1/2}]$ for $x \to x_s$, the velocity gradient at the boundary, upstream from the separation point but in the neighbourhood of it, must vary

* I am indebted to Dr. C. W. Jones for corrections to the values of the numerical coefficients in this equation. Consequent corrections have been made in Fig. 1, in the value (36) of x_1, and in Table 6 (Nov. 1948).

according to (35) for some value of α_1, and it is required to deduce the value of α_1 and x_s from the observed values of $(\partial u/\partial y)_0$ near the separation point. This is most conveniently done graphically by plotting $(\partial u/\partial y)_0/(x_s - x)^{1/2}$ against $(x_s - x)^{1/4}$ (or convenient multiples of these variables as indicated in (35)). This is done in Fig. 1. A set of curves, drawn according to (35) for different values of α_1, gives a set of possible variations of the right-hand side of (35) in the immediate neighbourhood of the separation point; these are shown by broken curves in the figure. Since x is not exactly known, it is necessary to use different trial values in evaluating the left-hand side of (35) from the values of $2(du/dy)_0$ obtained from the integration and given in Table 5; curves drawn through points so calculated for $x_s = 0{\cdot}9588$ and $0{\cdot}9590$ are shown by full lines in the figure.

The fit between the two kinds of curve is not perfect, but a fairly good fit is given by

$$\alpha_1 = 0{\cdot}47 \qquad \ldots \qquad (36)$$

approximately, of the set of curves given by (35), and the curve for a value of x slightly smaller than

$$x_s = 0{\cdot}9589\,. \qquad \ldots \qquad (37)$$

Such a curve is shown thus:— —·—·—·

The " bump " in the curve at about $x = 0{\cdot}95$ is curious, but seems real. The general agreement between the calculations for different values of x-interval length seems a good check against gross errors in the numerical work, and it does not seem at all probable that the results are subject to such errors. The smallness of the corrections for x-interval length makes it seem probable that, except perhaps at $x = 0{\cdot}958$, the values of $2\,(\partial u/\partial y)_0$ tabulated in Table 5, and used in plotting the results in Fig. 1, are not in error by more than $0{\cdot}0002$. At the bottom of the figure is a set of vertical lines showing the displacement in ordinate of plotted points at different values of x, for a difference of $0{\cdot}0005$ in the value of $2\,(\partial u/\partial y)_0$, and the errors in the plotted points should not be half the length of the corresponding lines; corrections for such errors (if they really existed) would not smooth out the " bump ". If the point at $x = 0{\cdot}956$ was omitted, a smoother fit can be made by taking $\alpha_1 = 0{\cdot}51$, $x_s = 0{\cdot}9587$ approximately; but there seems no other reason for rejecting the results at $x = 0{\cdot}956$.

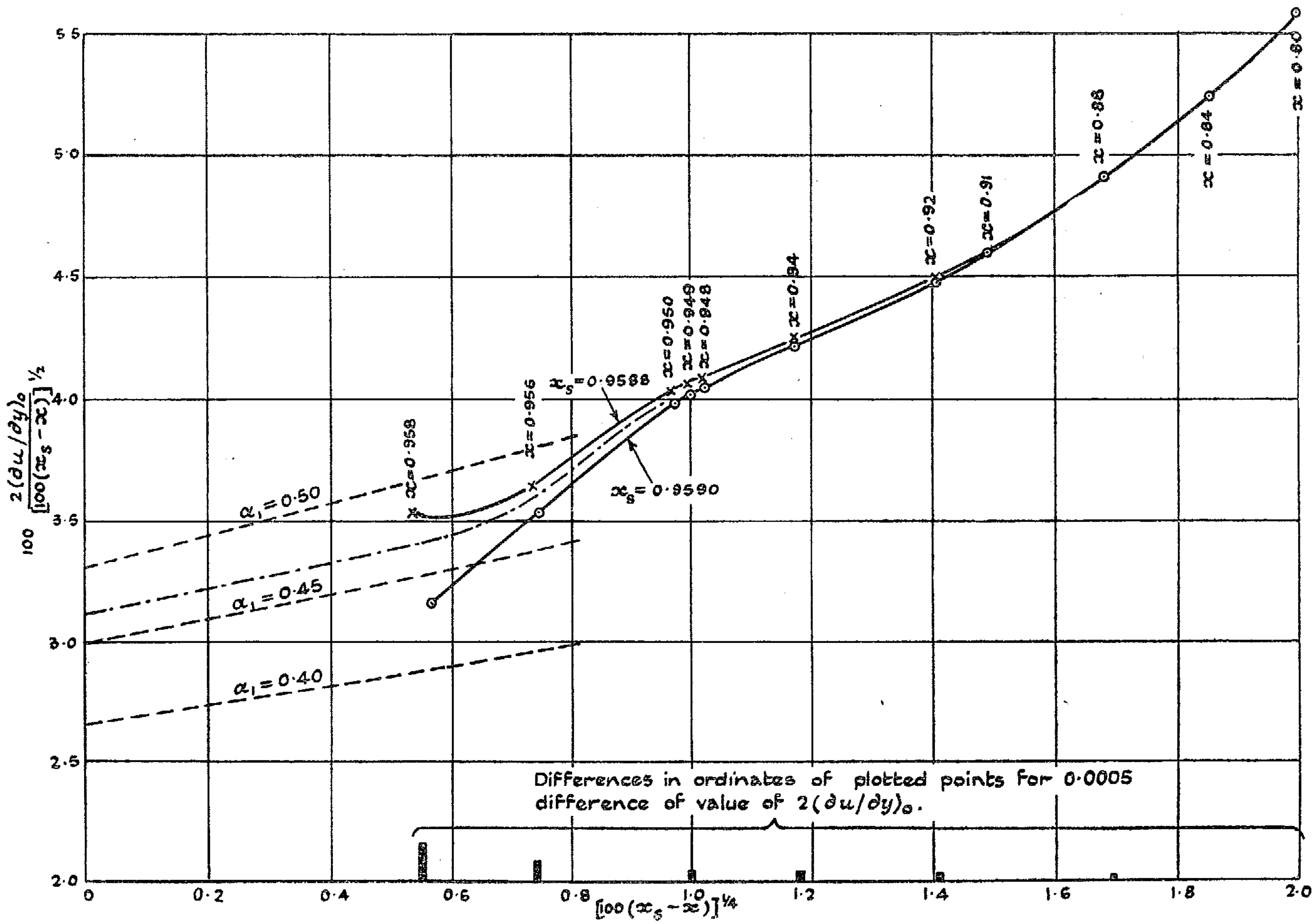
As already mentioned, an attempt was also made to carry the process of numerical integration up to the separation point itself, by taking $(\psi_2'')_0 = 0$ and adjusting the length of the x interval so that the boundary condition at infinity was satisfied; this was found to be quite practicable. The integration starting from $x_1 = 0{\cdot}94$, and going to the separation point (defined by $(\psi_2'')_0 = 0$) in one step, gave $x_s = 0{\cdot}9592$, whereas an integration also starting from $x = 0{\cdot}94$, taking one step to $x = 0{\cdot}95$ and another from there to separation, gave $x = 0{\cdot}9590$. The results are given in Table 6, and there are in surprisingly close agreement, considering the large correction for interval length at $x = 0{\cdot}958$, compared to that at $x = 0{\cdot}956$, already noted.

Values of $8\,(\partial\psi/\partial y)$ corrected for x-interval length are also given in Table 6; the correction is only approximate, as in the two-step integration the intervals were not of exactly the same length, and in any case it is not clear that Richardson's h^2-extrapolation process is valid in the present case, when the range of integration in x has been defined by ψ'', not in terms of x. But the correction is small and should be approximately correct. The agreement with the value (37) for the position of the separation point is excellent.

Using the relations (33) (c), (d), (e), and the value (36) of α_1, the velocity profile at separation becomes

$$8u = 0{\cdot}4403y^2 - 0{\cdot}0041y^4 - 0{\cdot}0005_5y^5 - 0{\cdot}0000_5y^6\,. \qquad \ldots \qquad (38)$$

Values of $8\partial\psi/\partial y = 8u$ at separation, calculated from this formula, are given in Table 6 for comparison with the results of the integration out to the separation point. The agreement is good out to about $y = 0{\cdot}8$, and this is about as far as any agreement in the fourth decimal place is to be expected, since in fitting the series (28) to the velocity distribution through the boundary layer at the separation point, it is usually found that terms of order y^7 and higher



Differences in ordinates of plotted points for 0·0005 difference of value of $2(\partial u/\partial y)_0$.
$100 \frac{2(\partial u/\partial y)_0}{[100(x_s - x)]^{1/2}}$
$[100(x_s - x)]^{1/4}$
$\alpha_1 = 0{\cdot}50$
$\alpha_1 = 0{\cdot}45$
$\alpha_1 = 0{\cdot}40$
$x_s = 0{\cdot}9588$
$x_s = 0{\cdot}9590$
$x = 0{\cdot}958$
$x = 0{\cdot}956$
$x = 0{\cdot}950$
$x = 0{\cdot}949$
$x = 0{\cdot}948$
$x = 0{\cdot}94$
$x = 0{\cdot}92$
$x = 0{\cdot}91$
$x = 0{\cdot}88$
$x = 0{\cdot}84$
$x = 0{\cdot}80$

FIG. 1.

give appreciable contributions to the fourth decimal in $8u$ at $y = 1$, and probably the same as the case here. In view of this, and of the doubt about the validity of the correction for x-interval length for the values of $8u$ at separation the agreement seems staisfactory.

In comparing these results, it must not be forgotten that, as pointed out in section 4, the process of integration imposes on the approximate solution ψ certain conditions, such as $\psi_0''' = u_0'' = \frac{dp}{dx}$, $\psi_0^{(iv)} = u_0''' = 0$, for all x. Thus even if there were a singularity of a kind for which these conditions were violated, the method of integration up to the separation point would fail to reveal its nature. But the values of $2\,(\partial u/\partial y)_0$ seem to indicate rather definitely that this quantity has not the behaviour $2\,(\partial u/\partial y)_0 = O\,[(x_s - x)^{1/4}]$ to be expected if there is such a singularity, and, if it has not, then the fact that the integration up to separation could not reveal such a singularity is no reason for suspecting the results of the integration in this case.

There are, however, two difficulties remaining.

First, a singularity of the type assumed would make the normal velocity v, at the separation point, become infinite like $(x_s - x)^{-1/2}$ (for $y \simeq 0$). Large normal velocities are to be expected at separation, and the appearance of formal infinities may simply be a sign of the breakdown of the assumptions of the boundary-layer theory (negligible normal accelerations and rates of shear).

Further, the expressions for the α 's such as (33) (a), (b), are found from the condition that the solution for the function f_{n+1} in the expansion

$$u_1 = 2\xi_1^{\,2}\,[f_0'(\eta_1) + \xi_1 f_1'(\eta_1) + \xi_1^{\,2} f_2'(\eta_1) + \ldots]$$

should not contain exponentially large terms in its asymptotic expansion for large η. For f_6, however, Goldstein found that this condition does not determine α_5, but gives a relation (Ref. 14, formula (35)) between the functions f_1 to f_5, and it was not clear whether this relation is satisfied. If it is not satisfied, the conclusion would seem to be that the singularity is not of the type assumed, and it is doubtful whether there is any other kind of singularity which gives a solution of the boundary-layer equations at separation. This point has been examined more recently by Dr. C .W. Jones[15], who comes to the conclusion that the condition is satisfied, so that there is a singularity of the kind supposed.

It is difficult from a purely numerical treatment of the solution of the equations to make absolutely certain of the existence of a singularity or separation, but all the evidence of the present work suggests that there is one. The main lines of evidence are as follows.

(*a*) The variaton of $2\,(\partial u/\partial y)_0$ with x near the separation point does not suggest a polynomial variation with $(x - x_s)$, as would be necessary to avoid a singularity.

(*b*) If there were no singularity, the correction for size of x-interval length would not be expected to increase very rapidly as the separation point is approached, as in fact it does (compare results at $x = 0{\cdot}956$ and $0{\cdot}958$ in Table 4).

(*c*) If there were no singularity at the separation point, the velocity distribution there would have no terms in y^3, y^4, y^5, and would be

$$8u = 0{\cdot}4400y^2 - 0{\cdot}0000_4 y^6,$$

which does not fit the velocity profile calculated by integration (compare (38) and Table 6).

(*d*) If there were no singularity at the separation point, no difficulty would be expected in taking the solution through this point, or in starting from it and working downstream. Actually both these processes have been tried fairly thoroughly, and in neither case has it been found possible to get any solution at all satisfying the boundary condition at $y = \infty$, for any starting value of $(\psi_2'')_0$. In this connection, it should be mentioned that Goldstein found (Ref. 14, p. 50 and 55) that the fact that a_4 in (31) is necessarily negative (*see* (33) (c)) means that there is no *real* solution of the boundary-layer equations downstream from separation.

These results all strongly suggest the presence of a singularity of a fairly severe kind at the separation point.

11. *Acknowledgments.*—I wish to acknowledge my thanks and indebtedness to Dr. S. Goldstein for his interest and for many valuable discussions during the course of this work, and for his permission to quote his results referred to in section 10. Also I wish to express my thanks to the Aeronautical Research Committee for a grant to enable me to obtain professional assistance in some of the extensive computing work involved, and to Dr. L. J. Comrie, Director of Scientific Computing Service Ltd., and his staff for their contributions to the progress of the work. Further, I wish to acknowledge the very substantial help I had from my fathei, the late Mr. W. Hartree, in the balance of the computing work, both in the exploratory work discussed in section 8 which established the possibility of using the method contemplated and subsequently, particularly in the rather tedious and trying work with small x-intervals used in approaching the separation point.

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TABLE 1

Trial Integration : $\xi = 0$ to $0\cdot4$ in One and Two Steps

Summary of Results

	Results using substitution (25a) for Q	Results using substitution (25b) for Q	Results calculated from Howarth's Tables
$\xi = 0\cdot2\left(\frac{\partial^2\varphi}{\partial\eta^2}\right)_0$	$1\cdot1238_5$	$1\cdot1211_5$	$1\cdot1208_5$
$\xi = 0\cdot4\left(\frac{\partial^2\varphi}{\partial\eta^2}\right)_0$			
one step	$0\cdot9167_5$	$0\cdot9063$	
two steps	$0\cdot9075$	$0\cdot9045_5$	$0\cdot9039_6$
h^2-extrapolated	$0\cdot9044$	$0\cdot9039_5$	
$\xi = 0\cdot4$ Maximum error in $\partial\varphi/\partial\eta$			
one step	$0\cdot0244$	$0\cdot0122$	
two steps	$0\cdot0069$	$0\cdot0031$	
h^2-extrapolated	$0\cdot0005$	$0\cdot0004$	

TABLE 2

Trial integration, $\xi = 0$ to $0 \cdot 4$ in One and Two Steps

Results at $\xi = 0 \cdot 4$, using substitution (25b) for Q, and comparison with results calculated from Howarth[4]s tables

	One step	Two steps	Final (h^2-extrapolated)	Howarth	(Final) —(Howarth) 4th decimal
$(\delta^2\varphi/\delta\eta^2)_0$	0·9063	0·9045$_5$	0·9039$_5$	0·9039$_5$	—0$_1$
η		Table of $2\partial\varphi/\partial\eta$			
0·0	0·0000	0·0000	0·0000	0·0000	0
0·1	0·1848$_5$	0·1846$_5$	0·1846	0·1846	0
0·2	0·3775$_5$	0·3768	0·3765$_5$	0·3765$_5$	0
0·3	0·5771$_5$	0·5759$_5$	0·5755$_5$	0·5755	+0$_5$
0·4	0·7830	0·7812	0·7806	0·7805$_5$	+0$_5$
0·5	0·9940	0·9914$_5$	0·9906	0·9905$_5$	+0$_5$
0·6	1·2086$_5$	1·2054	1·2043	1·2041	+2
0·7	1·4252	1·4211	1·4197	1·4194$_5$	+2$_5$
0·8	1·6416$_5$	1·6367$_5$	1·6351	1·6348	+3
0·9	1·8558$_5$	1·8500$_5$	1·8481	1·8478$_5$	+2$_5$
1·0	2·0656	2·0590$_5$	2·0568$_5$	2·0564$_5$	+4
1·1	2·2685$_5$	2·2612$_5$	2·2588$_5$	2·2584	+4$_5$
1·2	2·4624	2·4544$_5$	2·4518	2·4514	+4
1·3	2·6452	2·6366$_5$	2·6338	2·6335	+3
1·4	2·8150$_5$	2·8061$_5$	2·8032	2·8030	+2
1·5	2·9706	2·9615$_5$	2·9585$_5$	2·9584	+1$_5$
1·6	3·1108	3·1019	3·0989	3·0988	+1
1·7	3·2352$_5$	3·2266$_5$	3·2237$_5$	3·2237	+0$_5$
1·8	3·3440$_5$	3·3356$_5$	3·3328$_5$	3·3328$_5$	0
1·9	3·4371	3·4294	3·4268	3·4268	0
2·0	3·5156$_5$	3·5086$_5$	3·5063	3·5062$_5$	+0$_5$
2·1	3·5806	3·5744$_5$	3·5724		
2·2	3·6334	3·6280	3·6262	3·6262	0
2·3	3·6756	3·6710	3·6695		
2·4	3·7086$_5$	3·7048	3·7035	3·7034	+1
2·5	3·7339$_5$	3·7309	3·7299		
2·6	3·7531	3·7507$_5$	3·7499$_5$	3·7500	—0$_5$
2·7	3·7673	3·7655	3·7649		
2·8	3·7776	3·7761$_5$	3·7756$_5$	3·7755$_5$	+1
2·9	3·7849$_5$	3·7838	3·7834		
3·0	3·7900	3·7892	3·7889	3·7889	0
3·1	3·7935$_5$	3·7929$_5$	3·7927$_5$		
3·2	3·7959	3·7955	3·7954	3·7953	1
3·3	3·7974$_5$	3·7971$_5$	3·7970$_5$		
3·4	3·7984$_5$	3·7983	3·7982$_5$	3·7981$_5$	1
3·5	3·7990$_5$	3·7990	3·7990		
3·6	3·7994$_5$	3·7994	3·7994	3·7993	1
3·7	3·7996$_5$	3·7996$_5$	3·7996$_5$		
3·8	3·7998$_5$	3·7998	3·7998	3·7997$_5$	0$_5$
3·9	3·7999	3·7999	3·7999		
4·0	3·8000	3·7999$_5$	3·7999$_5$	3·7999	0$_5$
4·1	3·8000	3·8000	3·8000		
4·2	3·8000	3·8000	3·8000	3·8000	0

TABLE 3

Results at $\xi = 0 \cdot 8$ in One and Two Steps from $\xi = 0 \cdot 4$

	One step	Two steps	Difference (1 step)–(2 step) 4th decimal	Final (h^2-extrapolated and smoothed)
$(\partial^2\varphi/\partial\eta^2)_0$	$0 \cdot 4018_5$	$0 \cdot 3997$	21_5	$0 \cdot 3990$
η		Table of $2\partial\varphi/\partial\eta$		
0·0	0·0000	0·0000	0	0·0000
0·1	0·0877	0·0872	5	0·0870
0·2	0·1897	$0 \cdot 1885_5$	11_5	0·1881
0·3	0·3057	0·3037	20	0·3030
0·4	0·4355	0·4323	32	0·4312
0·5	0·5778	0·5734	44	0·5719
0·6	0·7317	0·7260	57	0·7241
0·7	0·8957	0·8888	69	0·8864
0·8	$1 \cdot 0682_5$	$1 \cdot 0600_5$	82	1·0573
0·9	1·2477	1·2382	95	1·2350
1·0	1·4318	1·4211	107	1·4175
1·1	1·6186	1·6065	121	1·6025
1·2	1·8056	1·7924	132	1·7880
1·3	1·9904	1·9764	140	1·9717
1·4	2·1708	2·1560	148	2·1512
1·5	2·3442	2·3292	150	2·3242
1·6	2·5089	2·4939	151	2·4888
1·7	2·6630	2·6482	148	2·6432
1·8	2·8050	2·7907	143	2·7859
1·9	2·9340	2·9203	137	2·9157
2·0	3·0491	3·0365	126	3·0323
2·1	3·1506	3·1391	115	3·1352
2·2	3·2383	3·2279	104	3·2246
2·3	3·3127	3·3039	88	3·3010
2·4	3·3753	3·3677	76	3·3652
2·5	3·4267	3·4203	64	3·4182
2·6	3·4684	3·4630	54	3·4613
2·7	3·5013	3·4971	42	3·4957
2·8	3·5271	3·5238	33	3·5227
2·9	3·5471	3·5444	27	3·5436
3·0	3·5621	3·5602	19	3·5595
3·1	3·5733	3·5718	15	3·5714
3·2	3·5815	3·5804	11	3·5801
3·3	3·5872	3·5866	6	3·5864
3·4	3·5916	3·5910	6	3·5908
3·5	3·5945	3·5940	5	3·5939
3·6	3·5963	3·5961	2	3·5960
3·7	3·5978	3·5975	3	3·5974
3·8	3·5987	3·5984	3	3·5983
3·9	3·5993	3·5991	2	3·5990
4·0	3·5996	3·5995	2	$3 \cdot 5994_5$
4·1	3·5999	$3 \cdot 5997_5$	1_5	3·5997
4·2	3·6000	3·5999	1	3·5999
4·3	3·6000	3·6000	0	3·6000
4·4	3·6000	3·6000	0	3·6000

TABLE 4

Results of Integration of Boundary-layer Equation for $U = 1 - \frac{1}{8}x$

	x=0·80	0·84 1 step	0·88 1 step	0·88 1 step	0·88 Final	0·91 1 step	0·92 1 step	0·94 1 step	0·94 2 step	0·94 Final
$2(\partial^2\psi/\partial y^2)_0$	0·2229	0·1808	0·1374	0·1383	0·1385	0·1024	0·0883	0·0586$_5$	0·0582$_5$	0·0582
y			Table of $8\partial\psi/\partial y$							
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·000	0·0000	0·0000	0·0000
0·1	0·0936	0·0768	0·0595	0·0597$_5$	0·0598$_5$	0·0453	0·039$_5$	0·0278$_5$	0·0277	0·0277
0·2	0·1962	0·1624	0·1278	0·1284	0·1286	0·0994	0·088	0·0645	0·0643	0·0642
0·3	0·3077	0·2570	0·2049	0·2058$_5$	0·2062	0·1624	0·145$_5$	0·1099	0·1096	0·1095
0·4	0·4280	0·3604	0·2907	0·2920$_5$	0·2925	0·2341	0·212	0·1640	0·1636$_5$	0·1635$_5$
0·5	0·5569	0·4726	0·3853	0·3869	0·3874$_5$	0·3145	0·287$_5$	0·2268	0·2264	0·2262$_5$
0·6	0·6942	0·5932	0·4883$_5$	0·4903	0·4909	0·4033	0·371	0·2982	0·2977	0·2975
0·7	0·8395	0·7221	0·5998$_5$	0·6020	0·6026$_5$	0·5006	0·463	0·3781	0·3774	0·3771$_5$
0·8	0·9926	0·8590	0·7196	0·7217$_5$	0·7225	0·6061	0·563$_5$	0·4663	0·4654	0·4651
0·9	1·1532	1·0036	0·8472	0·8494	0·8502	0·7196	0·671$_5$	0·5626	0·5615$_5$	0·5612
1·0	1·3209	1·1556	0·9823	0·9846	0·9855	0·8408	0·787$_5$	0·6669	0·6656$_5$	0·6652
1·1	1·4954	1·3145	1·1247	1·1271	1·1280	0·9695	0·911	0·7789	0·7774	0·7769
1·2	1·6762	1·4800	1·2740	1·2766	1·2774	1·1054	1·042	0·8984	0·8967	0·8961$_5$
1·3	1·8626	1·6517	1·4300	1·4325	1·4334	1·2482	1·180	1·0251	1·0233	1·0227
1·4	2·0540	1·8290	1·5921	1·5946	1·5955	1·3975	1·324$_5$	1·1587$_5$	1·1567$_5$	1·1561
1·5	2·2500	2·0114	1·7597	1·7624	1·7633	1·5529	1·475$_5$	1·2990	1·2968$_5$	1·2961$_5$
1·6	2·4498	2·1983	1·9326	1·9354	1·9363	1·7139	1·632	1·4454	1·4431$_5$	1·4424
1·7	2·6528	2·3890	2·1101	2·1131	2·1140	1·8802	1·744	1·5977	1·5953	1·5945
1·8	2·8580	2·5830	2·2907	2·2948	2·2958	2·0511	1·961	1·7553	1·7528	1·7520
1·9	3·0646	2·7797	2·4770	2·4800	2·4818	2·2263	2·132$_5$	1·9181	1·9154	1·9145$_5$
2·0	3·2720	2·9784	2·6654	2·6680	2·6690	2·4052	2·308	2·0854	2·0825$_5$	2·0816
2·1	3·4794	3·1781	2·8560	2·8584	2·8593	2·5872	2·486$_5$	2·2568	2·2537	2·2527
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2·7	4·6800	4·3585	4·0078	4·0097	4·0103	3·7085	3·595$_5$			
2·8	4·8654	4·5454	4·1949	4·1966	4·1972	3·8941	3·780$_5$			
2·9	5·0442	4·7277	4·3788	4·3803	4·3808		3·964			
3·0	5·2176	4·9047	4·5587	4·5600	4·5604		4·145$_5$			
3·1	5·3932	5·076	4·735	4·735	4·735		4·324			
3·2	5·5412	5·241	4·905	4·905	4·905		4·498$_5$			
3·3	5·6912	5·398$_5$	5·071	5·070$_5$	5·070		4·669			
3·4	5·8330	5·549$_5$	5·229	5·229	5·229		4·835			
3·5	5·9666	5·693	5·381	5·381$_5$	5·381$_5$		4·996			
3·6	6·0918	5·829	5·526	5·526$_5$	5·527		5·151$_5$			

TABLE 4—*continued*

	x=0·80	0·84		0·88		0·91	0·92
		1 step	1 step	2 step	Final	1 step	1 step
y		Table of $8\partial\varphi/\partial y$					
3·7	6·2082	5·956	5·664	5·664$_5$	5·665		5·301$_5$
3·8	6·3162	6·075$_5$	5·796	5·796	5·796		5·445
3·9	6·4158	6·187	5·920	5·920	5·920		5·581$_5$
4·0	6·5072	6·290$_5$	6·037	6·036$_5$	6·037		5·711
4·1	6·5906	6·386	6·145	6·144$_5$	6·145		5·834
4·2	6·6664	6·474	6·245	6·245$_5$	6·246		5·950
4·3	6·7348	6·555	6·337	6·339	6·339		6·058$_5$
4·4	6·7962	6·628	6·422	6·425	6·425		6·160
4·5	6·8510	6·694$_5$	6·501	6·503$_5$	6·504		6·254
4·6	6·8998	6·754$_5$	6·574	6·575$_5$	6·576		6·341
4·7	6·9430	6·808	6·639	6·641$_5$	6·642		6·421$_5$
4·8	6·9810	6·855$_5$	6·698	6·701	6·702		6·495$_5$
4·9	7·0142	6·897$_5$	6·752	6·754$_5$	6·755		6·563$_5$
5·0	7·0430	6·935	6·801	6·802$_5$	6·803		6·625$_5$
5·1	7·0680	6·968	6·844	6·845$_5$	6·846		6·681
5·2	7·0896	6·997	6·882	6·883$_5$	6·884		6·731
5·3	7·1082	7·022$_5$	6·914	6·917	6·918		6·776
5·4	7·1240	7·044$_5$	6·944	6·946$_5$	6·947		6·816$_5$
5·5	7·1374	7·063	6·969	6·972$_5$	6·973		6·852$_5$
5·6	7·1488	7·079	6·991	6·995	6·996		6·884$_5$
5·7	7·1584	7·093	7·010	7·014$_5$	7·016		6·912$_5$
5·8	7·1662	7·105	7·028	7·031	7·032		6·937
5·9	7·1726	7·115	7·043	7·045$_5$	7·046		6·958$_5$
6·0	7·1780	7·123	7·056	7·057$_5$	7·058		6·977
6·2	7·1860	7·135$_5$	7·076	7·077$_5$	7·078		7·006$_5$
6·4	7·1912	7·144	7·091	7·091$_5$	7·092		7·029
6·6	7·1946	7·149$_5$	7·101	7·101$_5$	7·102		7·045
6·8	7·1968	7·153$_5$	7·107	7·107$_5$	7·108		7·056$_5$
7·0	7·1982	7·156	7·112	7·112	7·112		7·064$_5$
7·2	7·1991	7·158	7·115	7·115	7·115		7·069$_5$
7·4	7·1996	7·159	7·117	7·117	7·117		7·073$_5$
7·6	7·1998	7·159$_5$	7·119	7·118$_5$	7·118$_5$		7·076$_5$
7·8	7·1999	7·160	7·120	7·119$_5$	7·119$_5$		7·078$_5$
8·0	7·2000	7·160	7·120	7·120	7·120		7·080

TABLE 4—*continued*

	x=0·948	0·949	0·956			0·958		
	1 step	1 step	1 step	2 step	Final	1 step	2 step	Final
$2(\partial^2\varphi/\partial y^2)_0$	0·0425	0·0402	0·0204	0·0196	0·0193	0·0125	0·0106	0·0100
y			Table of $8\partial\varphi/\partial y$					
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·1	0·0214	0·0205	0·0125	0·0122$_5$	0·0121$_5$	0·0093	0·0088	0·0086
0·2	0·0516	0·0498	0·0339	0·0333	0·0331	0·0275	0·0262$_5$	0·0259
0·3	0·0905	0·0878	0·0640	0·0631$_5$	0·0628$_5$	0·0545	0·0524$_5$	0·0517
0·4	0·1382	0·1345$_5$	0·1028	0·1017$_5$	0·1014	0·0902	0·0874	0·0865
0·5	0·1945	0·1899$_5$	0·1502	0·1490	0·1486	0·1345	0·1310$_5$	0·1299
0·6	0·2593	0·2539	0·2063	0·2048$_5$	0·2044	0·1874	0·1833	0·1819
0·7	0·3327$_5$	0·3264	0·2709	0·2692$_5$	0·2687	0·2489	0·2442	0·2426
0·8	0·4145	0·4072	0·3438	0·3420$_5$	0·3415	0·3188	0·3135	0·3117
0·9	0·5043	0·4962	0·4251	0·4231$_5$	0·4225	0·3970	0·3911	0·3891
1·0	0·6022	0·5932	0·5145	0·5123	0·5116	0·4833$_5$	0·4767$_5$	0·4745$_5$
1·1	0·7080	0·6982	0·6119	0·6094	0·6086	0·5777	0·5704	0·5680
1·2	0·8213	0·8106	0·7170	0·7143	0·7134	0·6799	0·6719$_5$	0·6693
1·3	0·9420	0·9304	0·8295	0·8267	0·8258	0·7896	0·7811	0·7783
1·4	1·0697	1·0573	0·9494	0·9464	0·9454	0·9067	0·8976	0·8946
1·5	1·2042	1·1910	1·0763	1·0731$_5$	1·0721	1·0309	1·0212	1·0180
1·6	1·3451	1·3312	1·2100	1·2065$_5$	1·2054	1·1619	1·1517	1·1482
1·7	1·4922	1·4775	1·3500	1·3464	1·3452	1·2994	1·2886	1·2850
1·8	1·6449	1·6296	1·4959	1·4922	1·4910	1·4431	1·4318	1·4280
1·9	1·8029	1·7869	1·6475	1·6437	1·6424			
2·0	1·9567	1·9490	1·8043	1·8004	1·7990			

TABLE 5

Displacement Thickness, Momentum Thickness and Velocity Gradient at Boundary, for $U = 1 - \frac{1}{8}x$

x	Displacement thickness $\int_0^\infty [1 - (u/U)]dy$	Momentum thickness $\int_0^\infty (u/U)[1 - (u/U)]dy$	$2(\delta u/\delta y)_0$ Small steps	Large step	Final
0·80	2·248	0·719			
0·84	2·290	0·746	0·1808		(0·1809)
0·88	2·546	0·772	0·1383	0·1374	0·1385
0·91			0·1023		(0·1022)
0·92	2·748	0·798	0·0883		(0·0882)
0·94			$0·0582_5$	$0·0586_5$	0·0582
0·948			0·0425		(0·0424)
0·949			0·0402		(0·0401)
0·950			0·0379		(0·0378)
0·956			0·0196	0·0204	0·0193
0·958			0·0106	0·0125	0·0100

TABLE 6

$U = 1 - \frac{1}{8}x$

Integration from $x = 0·94$ to Separation Point in One and Two Steps

	1 step	2 step	h^2-extrapolated	Calculated from formula (38)
x_s	0·9598	0·9591	0·9589	
y		Table of $8\delta\varphi/\delta y$		
0·0	0·0000	0·0000	0·0000	0·0000
0·1	0·0044	0·0044	0·0044	0·0044
0·2	0·0176	$0·0176_5$	$0·0176_5$	0·0176
0·3	0·0395	$0·0396_5$	0·0397	0·0396
0·4	0·0701	0·0704	0·0705	0·0703
0·5	0·1094	0·1098	0·1099	0·1098
0·6	0·1573	0·1579	0·1581	0·1579
0·7	0·2138	0·2145	0·2147	0·2147
0·8	0·2787	0·2796	0·2799	0·2799
0·9	0·3520	0·3530	0·3533	0·3536
1·0	0·4335	0·4347	0·4351	0·4356
1·1	0·5231	0·5244	0·5247	
1·2	0·6205	0·6220	0·6225	
1·3	0·7257	0·7272	0·7277	

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