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Summary.—This report develops statistical methods of choosing allowable design stresses for annealed and heat-treated glass. The results are easy to apply but additional fundamental knowledge of some properties of glass is needed before they can be used to the best advantage. The report draws attention to these gaps in existing knowledge and makes recommendations for further research.

The report discusses the influence of the known causes of strength variations between nominally identical specimens in relation to two types of glass typical of those used by the aircraft industry, and shows that improved control of heat-treatment processes offers the best hope of a big increase in the useful strength of glass. Chemical protection of the glass surfaces, or changes of composition which increase the intrinsic strength and chemical stability of the glass, would increase the useful strength of both annealed and heat-treated glasses. The potential benefits for heat-treated glass are small compared with those obtainable by improved control of the heat-treatment processes but are nevertheless important.

1. *Introduction.*—The current requirements[1] for the strength of glass panels for military aircraft were drawn up by a Sub-Committee of the Joint Ministry of Supply/Society of British Aircraft Constructors Airworthiness Committee in 1949. The requirements in force at that time gave no guidance on allowable stresses or acceptable test procedure. A fatal accident, attributed to the failure of a pilot's windstreen, made it necessary to issue new interim requirements without waiting for the completion of research that was progressing under M.o.S contract, and the Sub-Committee decided to recommend a test procedure as the basis of design approval. Knowledge of the strength of glass components was scanty and the Sub-Committee was aware that the proposed test procedure was probably conservative ; the Sub-Committee therefore recommended that the requirements should be reconsidered when more data became available.

Tests of nominally identical glass components reveal a wide variation of strength. The stress to which glass is subjected in service, therefore, must be small compared with the average ultimate strength of the material or the weaker specimens will fail. The test procedure specified in the current strength requirements[1] was designed to ensure this ; six components of each type are tested to destruction and the permissible factored design load for heat-treated glass is one-third of the average ultimate load so found. Service experience to date shows that components which satisfy these requirements are likely to be safe, but the requirements have not been in force long enough for firm conclusions to be drawn. Also it is difficult, from Service experience alone, to judge whether the glass components are overstrength and therefore overweight. The aim of the present report, therefore, is to consider how the structural efficiency of glass components can be increased. There are two possibilities:

(*a*) That more use could be made of the strength already available.

(*b*) That the useful strength of the material could be increased by closer control of manufacturing processes.

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The report develops methods of choosing allowable design stresses for glass which take into account the known causes of strength variations. The results are easy to apply, but additional fundamental knowledge of the strength of glass is needed before they can be used to the best advantage. The report draws attention to these gaps in existing knowledge and makes recommendations for further research.

2. *Available Glasses*.—There has been little development of special glasses for aircraft, and much of the glass now used is commercial plate glass specially selected for its optical quality. This is a soda-lime-silica glass composed approximately of 72 per cent silica, 15 per cent soda, 10 per cent lime and 2 per cent magnesia. When the best possible light transmission is essential special white plate glass is used. Several types of special white plate are in use; all have slightly greater alkali contents than commercial plate glass but contain less of the iron impurity which gives the latter its slight green colour. Plate glass is mechanically polished with abrasives after being rolled and its surfaces contain very fine grinding and polishing scratches. These scratches are not present in sheet glass whch is usually made by free drawing from a tank of molten glass; the surfaces of sheet glass, however, are neither plane nor parallel and its optical quality, therefore, is not good enough for aircraft transparencies.

Most of the properties described in the following sections are common to glasses of all types, but the discussion is mainly concerned with the soda-lime-silica glasses.

3. *Thermal Properties of Glass*[2].—3.1. Glass has no definite freezing point; as it cools from the molten state its viscosity increases steadily until it becomes, for all practical purposes, a brittle solid. The increasing viscosity impedes atomic movements in the glass and, in a certain temperature range which varies with the composition of the glass and with the rate of cooling, further movement becomes practically impossible; on further cooling the glass retains the atomic structure appropriate to this temperature range. Thus glass is not in 'structural equilibrium' at low temperatures and identical specimens of glass cooled at different rates from the same high temperature will have different structures. This is important because both the strength and the chemical activity of the glass may change with change of atomic structure.

3.2. If glass is rapidly cooled from an initial temperature below the annealing temperature the stresses set up by the thermal gradient vanish when the temperature of the glass again becomes uniform. If the initial temperature is above the annealing temperature, however, there are permanent residual stresses in the glass when its temperature again becomes uniform. For a uniform sheet cooled simultaneously and equally on both surfaces these stresses are compressive at the surfaces and tensile near the central plane of the sheet.

4. *Strength Properties of Glass*.—4.1. The coefficient of variation of strength for annealed glass can be as large as $0 \cdot 25$, and one hundred nominally identical specimens may include some having four times the strength of others of the same group. For such a material normal probability theory predicts that one specimen in about 30,000 should have zero strength or less. This is clearly impossible and the distribution of strength, therefore, must be skew. The analytical difficulties associated with non-normal distributions, however, are very great, and it is necessary to assume that the basic strength distributions are approximately normal in order to simplify the analysis. The implications of this assumption are discussed in section 8.3.

There are theoretical reasons for expecting glass to be very much stronger than experiments show it to be, and Griffith[3] attributed this discrepancy between theory and experiment to the presence of minute cracks which act as stress-raisers. It is supposed that these cracks spread under stress and that failure occurs when the glass is sufficiently weakened. The probability that a defect of a given size will exist in the region of maximum stress is less for small specimens than for large ones, and small specimens are stronger, on the average, than large ones of the same type and material. For the same reason the strength found for a given specimen will depend upon the distribution of the applied stress.

4.2. If glass is subjected to a stress which does not cause immediate failure delayed fracture may eventually occur. For a given type of specimen the time lapse between application of the load and failure of the glass increases with decrease of applied load. The endurance of drawn sheet glass under sustained loads in normal atmospheric conditions was investigated by Holland and Turner[4] whose results are given in Table 1 and illustrated in Fig. 1.

Gurney and Pearson[5] found that the endurance of soda-lime-silica glass under a constant stre s increased when the tests were made in a vacuum. If the specimens were first baked in a vacuum and then tested either in a vacuum or in air free from water and carbon dioxide their endurance again increased and most of the specimens either broke while being loaded or did not break at all. Gurney and Pearson concluded that delayed fracture of glass is mainly due to chemical attack by water and carbon dioxide on the highly-stressed material at the ends of the Griffith cracks; they also concluded that delayed fracture may occur when these compounds are present as constituents of the atmosphere or of capillary liquid in the Griffith cracks or are absorbed in the surface of the glass. The polyvinyl butyral interlayer used for laminating aircraft windows, therefore, is unlikely to prevent delayed fracture because it is not applied under conditions which ensure complete and permanent removal of capillary liquids or absorbed gases.

In an earlier paper Gurney[6] suggested that changes of atomic structure at the ends of the Griffith cracks might cause a delayed fracture effect but the experiments showed that these changes were not important for glass of the type tested. Theoretically such changes can occur because stress increases the local mobility of the atoms and thus facilitates further progress towards structural equilibrium (*see* section 3.1). Gurney and Pearson[7] also made comparative tests of glass under static loading and under cyclic loading at two different frequencies. The endurance under cyclic loading was not significantly different from the static endurance in either case; the number of cycles of loading, therefore, has little influence on the fatigue of glass and the total duration of loading is the important parameter.

Murgatroyd and Sykes[8] compared the strength under rapid loading of a large number of glass rods. Half the rods had previously been subjected to a sustained load sufficient to break about 20 per cent of their number; the remainder had no previous loading experience. To ensure a true comparison the results for the weakest 20 per cent of the specimens tested without previous loading were rejected from the calculations. Two groups of specimens were tested in each condition. Within the limits of experimental error the average strengths of all four groups were the same. It can be concluded, therefore, that the strength of glass is little affected by a sustained load smaller than that which will cause failure, and Gurney[9] showed that this result is compatible with the theory that delayed fracture is caused by spreading of the Griffith cracks.

Clearly Murgatroyd and Sykes comparison rests on the reasonable assumption that the earliest failures in sustained loading tests are of the weakest specimens. The probability of agreement being found if this were not so cannot be estimated but is likely to be small. These experiments, therefore, provide grounds for confidence in the truth of this assumption.

A natural consequence of the delayed fracture effect is the variation of strength with rate of loading which is found for glass. This was investigated by Black[10] whose results are given in Table 2 and illustrated in Fig. 2.

4.3. Fractures in glass usually start from a free surface where the material is exposed to atmospheric attack, and there is evidence that they are caused only by tensile stress. If the surfaces are put into compression by chilling them from a temperature above the annealing temperature the apparent strength of the glass at the surfaces is increased by an amount equal to the residual compressive stress. Glass so treated is known by several different trade names; for the sake of generality, therefore, it is referred to as 'heat-treated' glass in the present report. The possible degree of pre-stressing is limited, especially for laminated panels, by the need for avoiding excessive distortion of the glass during chilling.

The stress distribution in a specimen of heat-treated glass subjected to a bending moment is shown diagrammatically in Fig. 3. The maximum tensile stress occurs in a plane below the surface of the glass; the possibility therefore arises that fracture may originate in this plane and not at the surface. However, there is evidence[12] from numerous tests of glass in the condition used for British aircraft that fracture does originate at the surface.

Small random variations of furnace temperature, time of exposure, rate of chilling, etc., combine to cause variations of residual stress from one specimen to another. The variation of apparent strength found from tests of heat-treated glass, therefore, is the sum of the variation of the intrinsic strength of the glass and the variation of the residual compressive stress. The intrinsic strength of heat-treated glass is unknown but may be different from that of the annealed plate glass from which it is made because:

(*a*) the heat treatment tends to eliminate the grinding and polishing scratches from the surfaces

(*b*) the rapid chilling causes the glass near the surfaces to retain an atomic structure which is different from that of the more slowly cooled annealed plate glass.

The chemical activity of rapidly cooled glass may also be greater than that of annealed glass (section 3.1) and the endurance of the glass under sustained loads, therefore, may be less

Holland and Turner[13] found that flaws in the edges of small beams of annealed sheet glass were only partly eliminated when the beams were heated for six hours at 570 deg C. This is about 80 deg C less than the heat-treatment temperature for glass, but the total time of exposure during heat-treatment is only a few minutes and the glass only momentarily attains the highest temperature. It is unlikely, therefore, that heat-treatment causes major changes in the surface flaws; in this case variations in the extent to which the flaws are modified due to random fluctuations of furnace temperature should be negligibly small. Both conclusions could readily be checked by tests of specimens which have been heat-treated and then carefully re-annealed. Similarly it is unlikely that changes of atomic structure due to rapid chilling will cause major changes of strength, and again variations due to small temperature fluctuations should be negligibly small. For a given type of glass and heat-treatment, therefore, it is unlikely that strong correlation will exist between the intrinsic strength and the residual compressive stress. Research is needed to confirm or modify this conclusion.

4.4. Different workers have reported widely differing effects of temperature on the strength and endurance of glass. Jones and Turner[14] found that the strength of small beams of annealed sheet glass was practically constant in the temperature range 20 deg to 480 deg C; above this range the strength diminished as the temperature approached the softening point for the glass. Smekal[15] (whose results were summarised by Holland[16]) and Vonnegut and Glathart[17], however, found that the strength of round rods of annealed glass diminished with increase of temperature up to about 200 deg C and then increased until softening occurred. Smekal, Vonnegut and Glathart, and Holland[16] all found an increase of strength with decrease of temperature below normal laboratory temperature.

Jones and Turner also found that the average endurance of their specimens under a constant sustained stress increased with increase of temperature. It follows that the variation of strength with rate of loading should decrease with increase of temperature. Comparison of Smekal's results for two rates of loading supports this conclusion and also suggests that the variation of strength with rate of loading decreases with decrease of temperature below normal laboratory temperature. These results show general agreement with those found by Vonnegut and Glathart except in respect of the temperature at which the variation is greatest; according to Vonnegut and Glathart this is 200 deg C. This discrepancy can probably be explained by differences in the size and type of specimen tested, the surface finish of the specimens and the rate of loading. Jones and Turner and Holland were very careful to ensure that the tensile surfaces of their specimens were free from artificial flaws whereas Vonnegut and Glathart deliberately roughened the surfaces of their specimens in an attempt to eliminate natural surface flaws. Smekal tested both roughened and undamaged specimens.

If, as seems well established, the variation of strength with rate of loading varies with temperature, it follows that the variation of strength with temperature must change with rate of loading, and such a trend is apparent in Vonnegut and Glathart's results. Moreover, for any nominal rate of loading, the actual rate of increase of stress at the critical point depends upon the surface condition at that point. It is probable, therefore, that the reported differences in the effects of temperature on the strength of glass can be explained by the experimental differences already mentioned. Now Jones and Turner, and Holland, tested specimens closely representative of the glass used for aircraft. They also tested many more specimens under each set of conditions than either Smekal or Vonnegut and Glathart and, for this reason, their results are statistically more significant. For the annealed glass normally used for aircraft, therefore, it is reasonable to conclude that:

(*a*) no significant loss of strength occurs when the temperature is raised at least up to 450 deg C

(*b*) the strength is greater at low temperatures than at normal laboratory temperature

(*c*) the endurance increases with change of temperature away from normal laboratory temperature.

There is a tendency at all temperatures for viscous flow to relieve the residual stresses in heat-treated glass and the amount of stress relief occurring at any temperature will depend upon the time of exposure to that temperature. This aspect of the strength of glass has not been studied in detail and there is no general agreement as to the temperature at which viscous flow becomes important. It is probable, however, that this temperature is about 300 deg C for commercial plate glass. Research is needed to confirm this estimate.

4.5. Table 3 compares the strength of heat-treated glass panels tested immediately after manufacture with that of similar panels tested after one year's storage in air at normal temperature. The results show that the stored specimens were significantly weaker than the freshly manufactured specimens. These data were obtained from an investigation [18] of the effect of edge and surface damage on the strength of heat-treated glass. The stored specimens were not damaged before being tested but the edges of the panels tested immediately after manufacture were deliberately chipped before the tests were made. Examination of the fragments after each test, however, showed that the failures did not start from the damaged areas, and it is unlikely that the initial damage affected the results. Moreover, the initial damage could only reduce the strength of the specimens tested immediately after manufacture and thus reduce the difference between these specimens and those tested after storage. It is possible, of course, that the stored specimens suffered accidental surface damage during manufacture and test and further work is needed to check this result.

If the loss of strength by the stored specimens was not due to accidental damage, it seems reasonable to assume that it was due to chemical attack by moisture and carbon dioxide. Heat-treated glass, however, is further removed from structural equilibrium at normal temperature than the glass tested by Gurney and Pearson[5] and therefore structural changes in the glass at the ends of the Griffith cracks may more readily occur under the influence of the residual stresses. The distinction between these processes is important. Atmospheric attack reduces the intrinsic strength of the glass, but the structural changes cause a volume shrinkage which diminished the residual compressive stress at the ends of the cracks and thus leads to increased tensile stress in these regions when the glass is subsequently loaded. Any further work, therefore, should aim to determine the part played by each of these processes in bringing about the observed loss of strength.

4.6. Heat-treated glasses break into smaller fragments than annealed glass because fracture is accompanied by a large release of strain energy. Fracture of the surfaces also exposes glass subject to tensile stress to attack by moisture and carbon dioxide, and this causes further fragmentation. The final size of the fragments depends upon the magnitude of the residual stress; and the number of fragments included within one square inch is frequently used as a quality control measurement for heat-treated glass[19].

5. *A Method of Interpreting Sustained Loading Data.*—5.1. Holland and Turner's experiments[4] (Table 1) showed that there is no simple relationship between the applied load and the endurance of annealed sheet glass under that load. The range of endurance for any loading is very wide, and the results suggest that the distribution of endurance is skew and that the skewness is different for different loadings. Moreover the number of results for any loading is small from the statistical point of view and a reliable estimate of the probability of any particular endurance cannot readily be made; this is particularly true of the lowest loadings which are the ones of greatest interest. The mean strength and the coefficient of variation of strength for the control specimens, however, are known and there is a good reason for believing that the earliest failures in sustained loading tests are of the weakest specimens (*see* section 4.2). From the number of specimens which broke in Holland and Turner's experiments within a chosen time after application of a chosen load, therefore, a probable upper limit can be found for the 'initial strength'* of the strongest broken specimen (*i.e.*, the last specimen to break in the chosen time). This procedure can be repeated for each interval of time for which Holland and Turner recorded their results. If these probable maximum initial strengths are then divided by the applied loads a series of coefficients is obtained and these can be plotted on a time basis. A curve drawn through the points for any one applied load will define, for any chosen endurance, the minimum ratio which the initial strength of a specimen must bear to the applied load in order that the specimen shall have at least the chosen endurance. A method of estimating the probable maximum initial strengths of selected specimens is given in Appendix I to this report, and the results of an analysis of Holland and Turner's data by this method are given in Table 4†.

The results from Table 4 are plotted in Fig. 4. The circle surrounding each point shows the reliability of the estimate for that point; the probability that the point should lie above or below the circle is less than 1 in 20. Each point is plotted at the end of the time interval in which the corresponding failure occurred. This requires justification because the time at which the last failure occurred in any interval is unknown. Appendix I shows, however, that the probable upper limit of the strength of the strongest broken specimen coincides with the probable lower limit of the strength of the weakest surviving specimen (*i.e.*, the first specimen to fail in the following interval). Clearly, therefore, the point must be plotted at the common boundary of the two intervals.

Fig. 4 shows that weak specimens fail earlier than strong specimens *when all are loaded to the same fraction of their initial strengths*. It might be suspected, therefore, that even shorter endurances would be found for specimens which initially are very weak. If this were so the curves for applied loadings lower than 40 per cent of the mean initial strength of the whole sample would lie below the 40 per cent curve. The fact that Holland and Turner found no failures when the applied load was 30 per cent of the mean initial strength is not necessarily evidence to the contrary. The rapid decrease of failure rate with decrease of applied load shows that specimens weak enough to fail under loads lower than 40 per cent of the mean initial strength must be rare; a sample of 100 specimens, therefore, would probably not contain one. An estimate of the position of the endurance curve for specimens loaded to 30 per cent of their mean initial strength can be made, however, by assuming conservatively that the weakest specimen of the sample was about to break when the experiment ended. A point found on this assumption is included in Fig. 4, and lies very close to the 40 per cent curve. This, and the small slope of all the curves at high endurances, leads to the reasonable conclusion that the lower boundary curve shown in Fig. 4 can be used with confidence to predict endurances up to 1,000 hours, and that the errors introduced by extrapolating it to higher endurances are unlikely to be large.

Baker and Preston[20] found that, on average, the stresses which would just break heavily and lightly scratched specimens at room temperature diminished in approximately the same proportion for a given increase of time under load. Later work by Vonnegut and Glathart[17] supported

* The initial strength of a specimen is defined for this purpose as the strength which it would exhibit if broken by being loaded at a uniform rapid rate.

† These calculations have not been extended beyond 1,000 hours endurance because only a limited number of the specimens which survived this period were tested for longer periods.

this conclusion. The curves given in Fig. 4, therefore, should apply equally well to sheet and plate glass of the same chemical composition. The maximum stress to which annealed sheet or plate glass can be subjected without risk of premature failure, therefore, can be expressed as

$$f_m = kf_i \quad \ldots \quad (1)$$

where k is a coefficient depending upon the endurance required and f_i is the initial strength of the specimen. Now f_i is identified with a particular rate of loading but clearly f_m is independent of the manner in which f_i is determined. A change of loading rate for the control specimens, therefore, must lead to achange of k such that

$$\frac{k_1}{k_2} = \frac{f_{i2}}{f_{i1}} \quad \ldots \quad (1A)$$

where the suffices $_1$ and $_2$ refer to the different rates of loading.

5.2. The apparent strength f_a of the material at the surfaces of heat-treated glass is the sum of the intrinsic strength of the glass in these regions and the residual compressive stress f_c. Therefore, provided that the method of loading is such that failure starts in the surface of the glass, the maximum stress which a specimen of heat-treated glass will safely sustain can be written as

$$f_m = f_c + Kf_i \quad \ldots \quad (2)$$

where f_i is the initial intrinsic strength for that specimen and K is a coefficient which, for reasons given in section 4.3, may be different from k. This expression is true even if f_c varies with time provided that the statistical distribution of f_c is found from aged specimens.

6. *The Choice of Allowable Design Stresses.*—6.1. The maximum allowable stresses defined by equations (1) and (2) are variable and allowable design stresses (*i.e.*, stresses which must not be exceeded when the design loads are applied) must be found statistically. Usually it will be necessary to ensure that not more than a small proportion of all specimens of a given type will either:

(*a*) fail under the design load within the expected life of the component,

or

(*b*) have static strength safety factors less than a specified minimum.

6.2. For annealed glass the choice of an allowable design stress which will satisfy the requirements for endurance is straightforward if the mean $\bar{F}_i$ and standard deviation σ_i of f_i are known. The mean and standard deviation of f_m are given by

$$\bar{F}_m = k\bar{F}_i ,$$

and

$$\sigma_m = k\sigma_i ,$$

respectively, and the distribution of f_m is normal if the distribution of f_i is normal. In this case the allowable design stress is

$$f_d = \bar{F}_m(1 - bv_m) \quad \ldots \quad (3)$$

where $v_m = \sigma_m/\bar{F}_m = \sigma_i/\bar{F}_i$, and b is a constant depending upon the chosen acceptable risk of failure.

The distribution of the maximum allowable stress for heat-treated glass is derived in Appendix II. This appendix assumes that the distribution of apparent strength f_a and the distribution of f_i are known; but the theory applies with only minor alterations when f_a and f_c, or f_c and f_i are known. The mean and standard deviation of f_m are given by

$$\bar{F}_m = \bar{F}_a - (1 - K)\bar{F}_i \quad \ldots \quad (4)$$

and

$$\sigma_m = \sqrt{[\sigma_a^2 - (1 - K^2)\sigma_i^2 + 2(1 - K)\varrho\sigma_i\{\varrho\sigma_i - \sqrt{(\sigma_a^2 - (1 - \varrho^2)\sigma_i^2)}\}]} \quad (5)$$

where $\bar{F}_a$ is the mean and σ_a the standard deviation of the apparent strength of the glass, and ϱ is the coefficient of correlation of f_c with f_i. For the special case of $\varrho = 0$ equation (5) reduces to

$$\sigma_m = \sqrt{[\sigma_a^2 - (1 - K^2)\sigma_i^2]} \,. \qquad \text{(5A)}$$

The allowable design stress in either case is given by equation (3) as before but

$$v_m = \frac{\sigma_m}{\bar{F}_m} \neq \frac{\sigma_i}{\bar{F}_i} \,.$$

6.3. It has been shown (section 4.2) that the strength of glass is little affected by a sustained load smaller than that necessary to cause failure. Therefore, if the applied stress is only very slightly smaller than the stress chosen to satisfy the endurance requirement, all specimens which satisfy that requirement will have inherent safety factors equal to or greater than

$$n = \frac{f_c + f_i}{f_c + Kf_i} \qquad \text{(6)}$$

throughout their design lives. It follows that such specimens will have safety factors of N or greater if the chosen stress is multiplied by n/N. A method of calculating the distribution of n for heat-treated glass is given in Appendix III; the method for annealed glass is self-evident.

7. *The Effect of Thickness Variations*.—7.1. The actual thickness of plate glass of a given nominal thickness varies considerably and therefore the stress induced by identical loads in otherwise identical specimens will vary from one specimen to another. To a first approximation the stress will be proportional to $(\bar{T}/t)^2$, where $\bar{T}$ is the mean thickness, and t the true thickness, of the glass; it is convenient, however, to work in terms of the mean thickness of the glass and a fictitious distribution of maximum allowable stress defined by

$$f_m' = f_m \left(\frac{t}{\bar{T}}\right)^2 .$$

Clearly the same conclusions will be drawn regarding the strength of any specimen whichever approach is made.

It is shown in Appendix IV that, if the distribution of t is normal, the mean and standard deviation of the fictitious maximum allowable stress are given by

$$\bar{F}_m' = \bar{F}_m(1 + v_t^2) \qquad \text{(7)}$$

and

$$\sigma_m' = (1 + v_t^2)\sqrt{\{\sigma_m^2 + (v_{t^2})^2 (\sigma_m^2 + \bar{F}_m^2)\}} \,, \qquad \text{(8)}$$

where v_t and v_{t^2} are the coefficients of variation of t and t^2, respectively. Appendix IV also shows that the distribution of f_m' is skew if the distributions of f_m and t are normal. The skewness is positive, however (*i.e.*, the long tail extends to the right of the mean), and conservative estimates of allowable design stress are obtained if the distribution of f_m' is treated as normal. Allowable design stresses, therefore, can be found from the expression

$$f_d' = \bar{F}_m'(1 - bv_m') \qquad \text{(9)}$$

where

$$v_m' = \frac{\sigma_m'}{\bar{F}_m'} \,.$$

7.2. For any specimen the ratio $n = (f_c + f_i)/(f_c + Kf_i)$ is independent of the thickness of the glass. The statistical distribution of n, therefore, is independent of thickness variations and the methods of section 6.3 remain valid.

7.3. Random variations of thickness may also affect the apparent strength of heat-treated glass because the temperature attained by the glass prior to chilling will depend upon the thickness. Analysis of the data from Ref. 18, however, shows no correlation of apparent strength with thickness. It is reasonable, therefore, to neglect this possible effect of thickness variations at least for thicknesses up to $\frac{5}{8}$ in., which was the maximum tested in these experiments, and probably for all thicknesses because the proportional tolerance on thickness is less for thicker glasses.

8. *Discussion*.—8.1. The importance of each of the variables considered in the preceding sections can best be discussed in relation to typical examples. Curves showing allowable design stresses for annealed and heat-treated glasses have therefore been drawn with each of the variables as parameter. These curves are included as Figs. 5 to 12; they are intended to be illustrative only and should not be used for design purposes.

A mean apparent modulus of rupture of 25,000 lb/in.^{2} has been assumed in preparation of the curves for heat-treated glass; this is typical of glass in the condition known as 'toughened for laminating'. It has been further assumed that there is no correlation between f_c and f_i. The stresses shown have been chosen to satisfy an assumed requirement for indefinite endurance and do not include a safety factor other than the inherent safety factor defined by equation (6).

As far as possible the parameters have been given values which span their estimated practical ranges of variation and the curves, therefore, can be compared directly one with another. Thus the values for v_a span a range estimated from tests of a number of separate batches of heat-treated glass, and the values for k cover the endurance range 10 minutes to 10,000 hours, approximately. The values for v_t were found by assuming that t is normally distributed, that the manufacturing tolerance is $\pm\frac{1}{32}$ in. for all thicknesses used for aircraft (*i.e.*, not less than $\frac{3}{16}$ in.) and that not more than $0\cdot1$ per cent of the glass falls outside these limits. In this case v_t is approximately $0\cdot05$ for glass of $\frac{3}{16}$ in. nominal thickness and approximately $0\cdot01$ for glass of 1 in. nominal thickness. The probable ranges of variation for $\bar{F}_i$ and v_i are unknown and the ranges chosen are those found from tests of annealed sheet and plate glass which has not had furnace treatment. Similarly the range for K has been assumed to be the same as that for k. The sheet glass data were taken from Ref. 4. The specimens to which they relate were small and their strength, therefore, was high; consequently the range ascribed to $\bar{F}_i$ is probably somewhat wide. On the other hand the assumption that K takes the same values as k may be optimistic. In both cases, however, the estimates are the best that can be made from the data now available.

Equations (3) to (8) show that the influence of any one of the variables on the allowable design stress depends upon the values taken by one or more of the remainder. In each illustration, therefore, average values have been given to all the variables held constant except K which has been taken as $0\cdot4$ throughout. This exception has been made because the greatest importance is usually attached to design for long life and $0\cdot4$ is the value of k corresponding to indefinite endurance of annealed glass.

The current requirements[1] for the strength of glass panels for military aircraft allow a maximum design stress of about 4,500 lb/in.^{2} for glass in the 'toughened for laminating' condition; the corresponding stress for annealed glass is about 800 lb/in.^{2}. These stresses are indicated in Figs. 6 to 12 for comparison with the allowable stresses derived in the present report. It appears that the stress now permitted may be too high in some cases; components designed to the current requirements have behaved satisfactorily in service (section 1.1), however, and it is more likely that the ranges ascribed to some of the variables in the present report are too wide.

8.2. *The Importance of the Individual Variables*.—8.2.1. Fig. 5 shows that the allowable design stress for annealed plate glass is most affected by variation of k; v_t is important for thin glass but is unimportant for glass thicker than $\frac{7}{16}$ in. ($v_t < 0\cdot025$). The values for k have been chosen to illustrate the effects of sustained loading or fatigue. Clearly, however, the curves also show the advantages of using glasses having better sustained loading properties than the soda-lime-silica types (*i.e.*, glasses which are chemically more stable), or of protecting the surfaces of the glass from atmospheric attack. Curves showing the influence of $\bar{F}_i$ and v_i are not included but the effects of variation of either can easily be found from equation (3).

8.2.2. The curves of allowable design stress in Figs. 6, 7 and 8 are drawn for constant values of v_a and $\bar{F}_a$ and illustrate the importance of the assumptions that must be made regarding v_i and $\bar{F}_i$. As can be seen in Fig. 8 changes of v_i and $\bar{F}_i$ within the range considered could reduce the allowable design stress by 40 per cent; clearly, therefore, there is a need for determination of the intrinsic strength of glass in the heat-treated state. Assumptions have also been made regarding K; Fig. 9 shows that the effects of errors in these assumptions are significant but not of major importance.

The effects of thickness variations on the allowable design stress are shown in Fig. 10. As before these effects are important for thin glass but decrease in importance for glass thicker than $\frac{7}{16}$ in.

Fig. 11 is drawn for constant values of v_c and $\bar{F}_c$ and shows what benefits would result from the use of stronger and more consistant glasses if such materials were available, or from chemical protection of existing types of glass. In this connection it is important to observe that any process which increases K by restricting atmospheric attack on the glass surfaces will probably also increase $\bar{F}_i$ and reduce v_i. The potential benefits of such a process, therefore, are substantial.

Figs. 6, 7, 9 and 10 also show that the influence of v_a on the allowable design stress greatly exceeds that of any of the other variables. The greatest increase in the structural efficiency of heat-treated glass therefore, could be achieved by control of v_a. Now

$$\sigma_a^2 = \sigma_c^2 + 2\varrho\sigma_c\sigma_i + \sigma_i^2 \text{ (Appendix II).}$$

Therefore
$$v_a = \frac{1}{\bar{F}_a}\sqrt{\{(\bar{F}_c v_c)^2 + 2\varrho\bar{F}_c\bar{F}_i v_c v_i + (\bar{F}_i v_i)^2\}}\,.$$

For a given glass and heat-treatment $\bar{F}_a$ and $\bar{F}_i$ are fixed; similarly v_i is fixed except to the extent that random variations in the heat-treatment may modify it. Moreover, $\bar{F}_c$ is fixed because $\bar{F}_a$ and $\bar{F}_i$ are fixed and v_a, therefore, can be reduced only if v_c and v_i are reduced, *i.e.* if control of the heat-treatment is improved. Clearly, therefore, a study of methods of improving control of the heat-treatment is urgently required.

The importance of the loss of strength found from tests of stored specimens is shown by Fig. 12. Allowable design stresses are shown for two different assumptions:

(*a*) That the strength loss was due to atmospheric attack on the glass

(*b*) That the strength loss was due to either local or general relaxation of the residual compressive stress.

In both cases it is assumed that K remains unchanged; this may be an optimistic assumption (*see* section 4.3). The curves show that the allowable design stress is seriously reduced in either case. Further work is therefore required to confirm that the loss of strength is real and to determine the cause, and it would be unwise to lower the standards of strength specified for military aircraft before completion of this work. Clearly, if atmospheric attack is responsible, the potential value of chemical protection of the glass surfaces is greatly enhanced.

Fig. 13 shows a typical distribution of the inherent safety factor n. Each of the two branches represents one-half of the population, the upper branch corresponding to those specimens for which f_i predominates and the lower to those specimens for which f_c predominates. The lower branch should always be used to determine design stresses.

8.3. *The Effects of Skewness on the Accuracy of Strength Estimates.*—The theory developed in the Appendices assumes that the variables obey the normal law of errors. To check the validity of this assumption Geary and Pearson's[22] tests of normality have been applied to the strength distributions found for three different types of glass. The results are summarised in Table 5; with one exception they show that the apparent deviations from normality are within the probable limits of sampling error. It is likely, therefore, that any real non-normality of these distributions is small within the ranges covered by the experiments and that errors of strength estimation due to this cause will also be small.

The skewness test gave an inconclusive result for Holland and Turner's[4] sheet-glass data. The errors due to skewness, therefore, should be larger for this distribution than for either of the other two. Now sheet glass is never used for aircraft transparencies and it is unlikely that heat-treatment causes the intrinsic strength of plate glass to approach that of Holland and Turner's specimens. Comparison of the strength ranges within which certain numbers of specimens broke in Holland and Turner's experiments with the corresponding ranges found from normal probability theory, therefore, should give a conservative picture of the errors that may arise in a typical case. This comparison is made in Table 6. The maximum error is less than 7 per cent and, for the lower tail of the distribution which is the range of greatest interest, the true strength is greater than the estimated strength.

The corresponding errors for heat-treated glass can be found by considering separately:

(*a*) the effects of skewness of the distributions of intrinsic strength and residual stress on estimates of the maximum allowable stress for particular specimens whose apparent strengths under rapidly applied loading are known

(*b*) the effect of skewness of the distribution of apparent strength on estimates of this quantity.

The first problem is considered in Appendix V and errors estimated on the conservative assumption that the distribution of intrinsic strength for heat-treated glass is the same as for Holland and Turner's sheet-glass specimens are given in Table 7. The maximum error due to this cause is again about 7 per cent but the theory over-estimates the strengths of the weaker specimens.

Errors due to skewness of the distribution of apparent strength should be conservative. This is so because, for reasons given in section 4.1, the distribution is likely to be positively skew, and because the effect of thickness variations is to increase this skewness (*see* section 7.1). Usually, therefore, the total errors of strength estimation for heat-treated glass should be much less than 7 per cent.

8.4. *Limitations of the Theory.*—8.4.1. The maximum errors due to skewness would easily be absorbed by the usual design safety factor. Provided that predictions from the theory are limited to events of moderate improbability, therefore, no further limitation is necessary on this account. A restriction to probabilities not less than 1 in 1,000 seems reasonable in the light of the foregoing discussion.

Precise measurements of the intrinsic strength properties of heat-treated glass will be difficult to make. The influence of K on the allowable design stress is small, however, and good estimates of v_i and $\bar{F}_i$ can be obtained from tests of glass which have been heat-treated and then carefully re-annealed. In this case only an unlikely adverse combination of errors in estimating these quantities can lead to large errors in estimating allowable design stresses. It does not seem likely, therefore, that lack of precise knowledge of the intrinsic strength properties of heat-treated glass need seriously restrict application of the methods of the present report.

8.4.2. The statistical difficulties, and the difficulties due to lack of fundamental knowledge of the properties of glass, arise from the need for prediction of the behaviour of the material under long-sustained loads. It may be argued, therefore, that a direct experimental attack on this problem using specimens of heat-treated glass would be simpler than the approach adopted in the present report. Such an approach, however, would introduce its own difficulties. Unless very large numbers of specimens were tested some generalisation of the results would be necessary and this would be hampered by the same lack of fundamental knowledge. Furthermore, simple beams of the type used by Holland and Turner[4] would not be suitable because they cannot be given the same heat-treatment as large panels and because edge effects would tend to confuse the results. The experimental effort entailed in large numbers of sustained load tests on large panels would be much greater than that needed to advance fundamental knowledge enough for the methods of this report to be used with confidence.

9. *Conclusions*.—The main conclusions reached in this report are summarised as follows:

(*a*) A substantial increase in the useful strength of heat-treated glass is possible and improved control of the heat-treatment processes offers the best hope of achieving this increase

(*b*) Chemical protection of the glass which restricts atmospheric attack on the surfaces, or changes of composition which increase the strength and chemical stability of the glass, would increase the useful strength of both annealed and heat-treated glasses. The potential benefits for heat-treated glass are small compared with those obtainable by improved control of the heat-treatment processes but are nevertheless important.

(*c*) There is a need for further research to determine the effects of storage on the useful strength of heat-treated glass, and it would be unwise to lower the standards of strength now specified[1] for military aircraft windows before this work is completed

(*d*) There is a need for research to determine the intrinsic strength of heat-treated glass, and the correlation between the intrinsic strength and the residual stress. Information on the sustained loading properties of rapidly chilled glass is also required

(*e*) There is a need for determination of the temperature above which stress relaxation causes a significant loss of strength for heat-treated glass.

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LIST OF SYMBOLS

f	Stress
$\bar{F}$	Mean stress
t	Thickness
$\bar{T}$	Mean thickness
σ	Standard deviation
v	Coefficient of variation
ϱ	Correlation coefficient
b	A constant depending upon a chosen acceptable risk of failure
k	A coefficient relating the maximum allowable stress and the initial strength for annealed glass
K	A coefficient relating the maximum allowable stress and the initial intrinsic strength for heat-treated glass
β_1, β_2	Shape parameters for a frequency distribution
n	The inherent safety factor for a particular specimen as defined by equation (6)
N	A specified minimum safety factor

Suffices

$_i$	Initial strength of annealed glass as defined in section 5.1, or initial intrinsic strength of heat-treated glass as defined in section 5.2
$_c$	Residual compressive stress for heat-treated glass
$_a$	Apparent strength of heat-treated glass
$_m$	Maximum allowable stress as defined in sections 5.1 and 5.2
$_d$	Allowable design stress as defined in section 6.1

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APPENDIX I

The Probable Limits of the Strength of the Strongest of the n Weakest Specimens in a Random Sample N

Consider a normal distribution having mean value $\bar{X}$ and standard deviation σ. Let $\bar{X}_N$ and σ_N be the mean value and standard deviation for a sample of N individuals taken at random from this distribution. It can be shown[21] that $\bar{X}_N$ is normally distributed about $\bar{X}$ with standard deviation $\sigma/\sqrt{N}$; similarly, in large samples (with which we are here concerned) the distribution of σ_N tends to normality with σ as mean value and standard deviation $\sigma/\sqrt{(2N)}$. The probable ranges of variation for $\bar{X}_N$ and σ_N, therefore, are:

$$\bar{X} - \frac{t\sigma}{\sqrt{N}} < \bar{X}_N < \bar{X} + \frac{t\sigma}{\sqrt{N}}$$

and

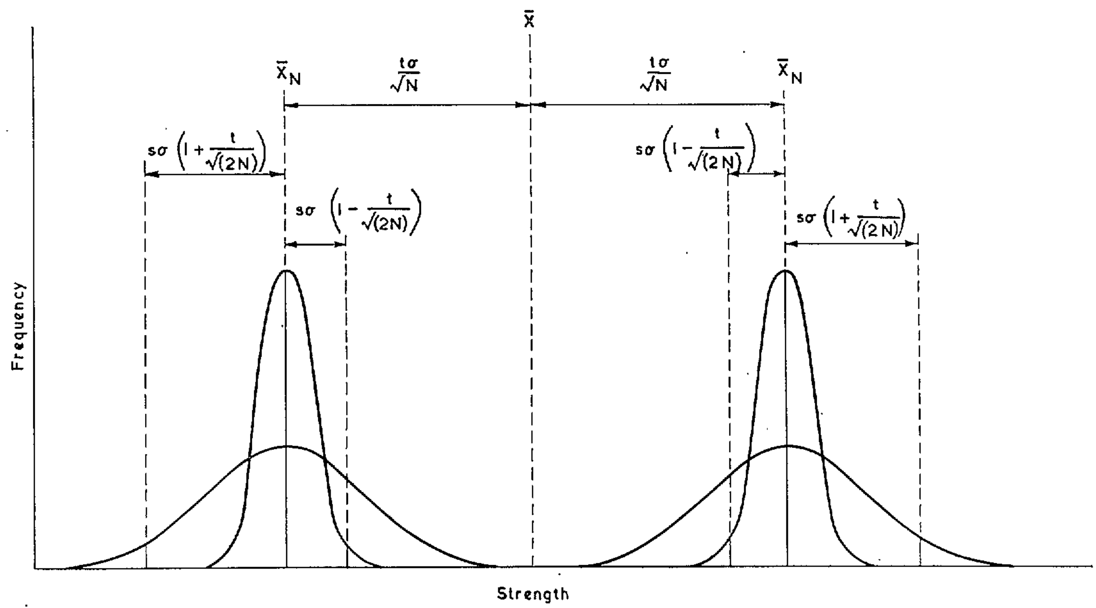
$$\sigma - \frac{t\sigma}{\sqrt{(2N)}} < \sigma_N < \sigma + \frac{t\sigma}{\sqrt{(2N)}}$$

where t is so chosen that the risk of $\bar{X}_N$ or σ_N falling outside these limits is acceptably small.

In any sample n individuals will lie below $\bar{X}_N - s\sigma_N$, where s is determined by the ratio n/N and can be found from tables of normal probability. The probable upper and lower limits of the strength of the strongest of the n weakest specimens, therefore, are:

$$\bar{X} - s\sigma \pm \frac{t\sigma}{\sqrt{N}} \left\{1 + \frac{s}{\sqrt{2}}\right\}$$

as will be clear from the sketch below.



The sketch also shows that the probable limits of the strength of the weakest of the n strongest specimens are:

$$\bar{X} + s\sigma \pm \frac{t\sigma}{\sqrt{N}} \left\{1 + \frac{s}{\sqrt{2}}\right\}.$$

It is also apparent that the probable limits of the strength of the weakest of the $(N - n)$ strongest specimens are the same as those for the strongest of the n weakest specimens.

APPENDIX II

The Distribution of $f_c + Kf_i$

Let x and y be two normally distributed variables with zero means, standard deviations σ_x and σ_y respectively, and correlation coefficient ϱ. Let $z = x + y$.

The joint distribution of x, y is[23]

$$\frac{dx\,dy}{2\pi\sigma_x\sigma_y\sqrt{(1-\varrho^2)}}\exp\left[-\frac{1}{2(1-\varrho^2)}\left(\frac{x^2}{\sigma_x^2}-\frac{2\varrho xy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y^2}\right)\right].$$

Now $x = z - y$ and $dx = dz$ when y is held constant.

Therefore the joint distribution of z and y is

$$\frac{dz\,dy}{2\pi\sigma_x\sigma_y\sqrt{(1-\varrho^2)}}\exp\left[-\frac{1}{2(1-\varrho^2)}\left\{\frac{(z-y)^2}{\sigma_x^2}-\frac{2\varrho(z-y)y}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y^2}\right\}\right].$$

The expression in $\{\ \}$ brackets =

$$\left(\frac{y^2}{\sigma_x^2}+\frac{2\varrho y^2}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y^2}\right)-\left(\frac{2z}{\sigma_x^2}+\frac{2\varrho z}{\sigma_x\sigma_y}\right)y+\frac{z^2}{\sigma_x^2}$$

$$=\frac{a^2}{\sigma_x^2\sigma_y^2}\left[y^2-2zy\sigma_y\frac{(\sigma_y+\varrho\sigma_x)}{a^2}+z^2\sigma_y^2\frac{(\sigma_y+\varrho\sigma_x)^2}{a^4}+\frac{z^2\sigma_y^2}{a^2}\left\{1-\frac{(\sigma_y+\varrho\sigma_x)^2}{a^2}\right\}\right],$$

where

$$a^2 = \sigma_x^2 + 2\varrho\sigma_x\sigma_y + \sigma_y^2 .$$

Now

$$\left\{1-\frac{(\sigma_y+\varrho\sigma_x)^2}{a^2}\right\}=\frac{(1-\varrho^2)\sigma_x^2}{a^2}.$$

The joint distribution of z and y, therefore, is

$$\frac{dz\,dy}{2\pi\sigma_x\sigma_y\sqrt{(1-\varrho^2)}}\exp\left[-\left\{\frac{a^2(y-b)^2}{2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right\}\right]\exp\left[-\left\{\frac{z^2}{2a^2}\right\}\right],$$

where

$$b=\frac{z\sigma_y(\sigma_y+\varrho\sigma_x)}{a^2}.$$

Integrating over $y(-\infty, +\infty)$, the distribution of z is

$$\frac{dz}{a\sqrt{(2\pi)}}\exp\left[-\left\{\frac{z^2}{2a^2}\right\}\right].$$

Thus z is distributed normally with $\sigma_z^2 = a^2 = \sigma_x^2 + 2\varrho\sigma_x\sigma_y + \sigma_y^2$. Thus if x and y are normally distributed with correlation ϱ, their sum, $z = x + y$, is also normally distributed with $\sigma_z^2 = \sigma_x^2 + 2\varrho\sigma_x\sigma_y + \sigma_y^2$.

If now y and z are given as normal variables it follows that x will be normally distributed, and similarly that $x + Ky$, where K is a constant, will be normally distributed.

If

$$\omega = x + Ky,$$

$$\sigma_\omega^2 = \sigma_x^2 + 2\varrho K\sigma_x\sigma_y + K^2\sigma_y^2 .$$

Now

$$\sigma_x^2 = \sigma_z^2 - 2\varrho\sigma_x\sigma_y - \sigma_y^2 .$$

therefore

$$\sigma_\omega^2 = \sigma_z^2 - 2\varrho\sigma_x\sigma_y(1 - K) - \sigma_y^2(1 - K^2) .$$

Also

$$\sigma_x^2 + 2\varrho\sigma_x\sigma_y + \varrho^2\sigma_y^2 = \sigma_z^2 - \sigma_y^2 + \varrho^2\sigma_y^2 .$$

Therefore

$$(\sigma_x + \varrho\sigma_y) = \sqrt{\{\sigma_z^2 - (1 - \varrho^2)\sigma_y^2\}} .$$

Therefore

$$\sigma_x = \sqrt{\{\sigma_z^2 - (1 - \varrho^2)\sigma_y^2\}} - \varrho\sigma_y .$$

Therefore

$$\sigma_\omega^2 = \sigma_z^2 - \sigma_y^2(1 - K^2) + 2\varrho\sigma_y(1 - K)[\varrho\sigma_y - \sqrt{\{\sigma_z^2 - (1 - \varrho^2)\sigma_y^2\}}] .$$

Giving means of distributions values other than zero

$$\bar{\omega} = \bar{x} + K\bar{y}$$

$$= \bar{z} - (1 - K)\bar{y} .$$

Therefore, in the notation of the present report,

$$f_a = f_c + f_i ;$$

the mean value of $f_c + Kf_i = \bar{F}_m = \bar{F}_a - (1 - K)\bar{F}_i$; and the standard deviation of $f_c + Kf_i = \sigma_m$

$$= \sqrt{(\sigma_a^2 - \sigma_i^2(1 - K^2) + 2\varrho\sigma_i(1 - K)[\varrho\sigma_i - \sqrt{\{\sigma_a^2 - (1 - \varrho^2)\sigma_i^2\}}])} .$$

In the special case of $\varrho = 0$

$$\bar{F}_m = \bar{F}_a - (1 - K)\bar{F}_i$$

$$\sigma_m = \sqrt{\{\sigma_a^2 - (1 - K^2)\sigma_i^2\}} .$$

APPENDIX III

The Distribution of $\dfrac{f_c + f_i}{f_c + Kf_i}$

Let x and y be two normally distributed variables with zero means, standard deviations σ_x and σ_y, respectively, and correlation coefficient ϱ.

The joint distribution of x and y is [23]

$$\frac{dx\,dy}{2\pi\sigma_x\sigma_y\sqrt{(1 - \varrho^2)}} \exp\left[-\frac{1}{2(1 - \varrho^2)}\left\{\frac{x^2}{\sigma_x^2} - \frac{2\varrho xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right\}\right] .$$

Consider the distribution of $z = (x + x_0)/(y + y_0)$,

where x_0 and y_0 are the means of the distributions of which the quotient is required.

$$x + x_0 = z(y + y_0) \,.$$

Therefore $$dx = dz(y + y_0) \text{ when } y \text{ is held constant}$$

and

$$x = zy + A$$

where

$$A = zy_0 - x_0 \,.$$

The joint distribution of y, z, therefore, is

$$\frac{dy\,dz(y + y_0)}{2\pi\sigma_x\sigma_y\sqrt{(1 - \varrho^2)}} \exp\left[-\frac{1}{2(1 - \varrho^2)}\left\{\frac{(zy + A)^2}{\sigma_x^2} - \frac{2\varrho y(zy + A)}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right\}\right].$$

Now

$$\frac{(zy + A)^2}{\sigma_x^2} - \frac{2\varrho y(zy + A)}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}$$

$$= \frac{1}{\sigma_x^2\sigma_y^2}\left\{B^2y^2 + 2A\sigma_y y(z\sigma_y - \varrho\sigma_x) + A^2\sigma_y^2\right\}$$

$$= \frac{1}{\sigma_x^2\sigma_y^2}\left\{By + \frac{A}{B}\sigma_y(z\sigma_y - \varrho\sigma_x)\right\}^2 + \frac{A^2}{\sigma_x^2} - \frac{A^2}{B^2\sigma_x^2}(z\sigma_y - \varrho\sigma_x)^2 ,$$

where

$$B = \sqrt{(z^2\sigma_y^2 - 2\varrho z\sigma_x\sigma_y + \sigma_x^2)} \,.$$

Now

$$B^2 - (z\sigma_y - \varrho\sigma_x)^2 = (1 - \varrho^2)\sigma_x^2 \,.$$

Therefore $$\frac{(zy + A)^2}{\sigma_x^2} - \frac{2\varrho y(zy + A)}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2} = \frac{1}{\sigma_x^2\sigma_y^2}\left\{By + \frac{A}{B}\sigma_y(z\sigma_y - \varrho\sigma_x)\right\}^2 + \frac{A^2}{B^2}(1 - \varrho^2) \,.$$

Therefore joint distribution of y and z is

$$\frac{dy\,dz(y + y_0)}{2\pi\sigma_x\sigma_y\sqrt{(1 - \varrho^2)}} \exp\left[-\frac{\{By + \frac{A}{B}\sigma_y(z\sigma_y - \varrho\sigma_x)\}^2}{2(1 - \varrho^2)\sigma_x^2\sigma_y^2}\right] \exp\left[-\frac{A^2}{2B^2}\right].$$

Put $t = By + \frac{A}{B}\sigma_y(z\sigma_y - \varrho\sigma_x)$ and consider the joint distribution of t and z.

Both A and B are functions of z and independent of y and t.

Therefore $$dt = B\,dy$$

and

$$y + y_0 = \frac{t}{B} + \frac{C}{B^2} ,$$

where

$$C = B^2y_0 - A\sigma_y(z\sigma_y - \varrho\sigma_x) \,.$$

Then joint distribution of t and z is

$$\frac{dz\,dt}{2\pi B\sigma_x\sigma_y\sqrt{(1 - \varrho^2)}}\left(\frac{t}{B} + \frac{C}{B^2}\right) \exp\left[-\frac{A^2}{2B^2}\right] \exp\left[-\frac{t^2}{2(1 - \varrho^2)\sigma_x^2\sigma_y^2}\right].$$

In deriving this expression it has been assumed that $(x + x_0)$ and $(y + y_0)$ are both positive. This means that $z > 0$ and $t/B + C/B^2 > 0$, i.e., $t > -C/B$. In the original distribution of x, y both are distributed over a range from $-\infty$ to $+\infty$. Thus t also covers the range $-\infty$ to $+\infty$.

When $t < -C/B$, $y + y_0 < 0$ and it is necessary to choose a negative sign for the expression for the probability distribution in order to retain a positive probability.

The distribution of z is found by integrating the joint probability for z and t over the appropriate full range of the variable t. This gives the distribution of z as the sum of two integrals:

$$-\int_{-\infty}^{-C/B} dt\left(\frac{t}{B}+\frac{C}{B^2}\right)\exp\left(-\frac{t^2}{2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)\frac{dz}{2\pi B\sigma_x\sigma_y\sqrt{(1-\varrho^2)}}\exp\left(-\frac{A^2}{2B^2}\right)$$
$$+\int_{-C/B}^{+\infty} dt\left(\frac{t}{B}+\frac{C}{B^2}\right)\exp\left(-\frac{t^2}{2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)\frac{dz}{2\pi B\sigma_x\sigma_y\sqrt{(1-\varrho^2)}}\exp\left(-\frac{A^2}{2B^2}\right).$$

Now

$$-\int_{-\infty}^{-C/B} dt\,\frac{t}{B}\exp\left(-\frac{t^2}{2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)$$
$$=\frac{(1-\varrho^2)\sigma_x^2\sigma_y^2}{B}\left[\exp\left(-\frac{t^2}{2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)\right]_{-\infty}^{-C/B}$$
$$=\frac{(1-\varrho^2)\sigma_x^2\sigma_y^2}{B}\exp\left(-\frac{C^2}{2B^2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right).$$

Similarly

$$\int_{-C/B}^{\infty} dt\,\frac{t}{B}\exp\left(-\frac{t^2}{2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)=\frac{(1-\varrho^2)\sigma_x^2\sigma_y^2}{B}\exp\left(-\frac{C^2}{2B^2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right).$$

Also

$$-\int_{-\infty}^{-C/B}\frac{C\,dt}{B^2}\exp\left(-\frac{t^2}{2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)$$
$$=-C\,\frac{\sqrt{(2\pi)}\sqrt{(1-\varrho^2)}}{B^2}\,\sigma_x\sigma_y\,\mathrm{erf}\left(-\frac{C}{B\sigma_x\sigma_y\sqrt{(1-\varrho^2)}}\right),$$

where

$$\mathrm{erf}\,p=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{p}\exp\left(-\frac{p^2}{2}\right)dp\,.$$

and similarly

$$\int_{-C/B}^{\infty}\frac{C\,dt}{B^2}\exp\left(-\frac{t^2}{2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)$$
$$=\frac{C\sqrt{(2\pi)}\sqrt{(1-\varrho^2)}}{B^2}\,\sigma_x\sigma_y\left\{1-\mathrm{erf}\left(-\frac{C}{B\sigma_x\sigma_y\sqrt{(1-\varrho^2)}}\right)\right\}.$$

Therefore the distribution of z is

$$\frac{dz}{2\pi B\sigma_x\sigma_y\sqrt{(1-\varrho^2)}}\left[\frac{C\sqrt{(2\pi)}\sqrt{(1-\varrho^2)}}{B^2}\,\sigma_x\sigma_y\left\{1-2\,\mathrm{erf}\left(-\frac{C}{B\sigma_x\sigma_y\sqrt{(1-\varrho^2)}}\right)\right\}\right.$$
$$\left.+\frac{2(1-\varrho^2)\sigma_x^2\sigma_y^2}{B}\exp\left(-\frac{C^2}{2B^2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)\right]\exp\left(\frac{A^2}{2B^2}\right)$$
$$=\frac{C\,dz}{B^3\sqrt{(2\pi)}}\left\{1-2\,\mathrm{erf}\left(-\frac{C}{B\sigma_x\sigma_y\sqrt{(1-\varrho^2)}}\right)\right\}\exp\left(-\frac{A^2}{2B^2}\right)$$
$$+\frac{\sigma_x\sigma_y\,dz\,\sqrt{(1-\varrho^2)}}{\pi B^2}\exp\left[-\left(\frac{A^2}{2B^2}+\frac{C^2}{2B^2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)\right].$$

Now

$$C = B^2 y_0 - A\sigma_y(z\sigma_y - \varrho\sigma_x)$$

and

$$B^2 = z^2\sigma_y^2 - 2\varrho z\sigma_x\sigma_y + \sigma_x^2 .$$

Therefore

$$A^2(1-\varrho^2)\sigma_x^2\sigma_y^2 + C^2$$

$$= B^2\{B^2y_0^2 - 2Ay_0\sigma_y(z\sigma_y - \varrho\sigma_x) + A^2\sigma_y^2\}$$

$$= B^2\{y_0^2\sigma_x^2 - 2\varrho x_0 y_0\sigma_x\sigma_y + x_0^2\sigma_y^2\} .$$

Write

$$\frac{y_0^2\sigma_x^2 - 2\varrho x_0 y_0\sigma_x\sigma_y + x_0^2\sigma_x^2}{(1-\varrho^2)\sigma_x^2\sigma_y^2} = a^2 ,$$

where a is a constant independent of z.

Then

$$\frac{A^2}{B^2} + \frac{C^2}{B^2(1-\varrho^2)\sigma_x^2\sigma_y^2} = a^2 ,$$

and

$$\frac{\sigma_x\sigma_y\,dz\,\sqrt{(1-\varrho^2)}}{\pi B^2}\exp\left[-\left(\frac{A^2}{2B^2} + \frac{C^2}{2B^2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)\right]$$

$$= \frac{\sigma_x\sigma_y\,dz\,\sqrt{(1-\varrho^2)}}{\pi B^2}\exp\left(-\frac{a^2}{2}\right).$$

Write $A/B = q$, where q, A, B are all functions of z, then

$$dq = \frac{B\,dA - A\,dB}{B^2} .$$

Now

$$A = zy_0 - x_0 .$$

Therefore

$$dA = y_0\,dz .$$

Also

$$B^2 = z^2\sigma_y^2 - 2\varrho z\sigma_x\sigma_y + \sigma_x^2 .$$

Therefore

$$B\,dB = \sigma_y\,dz\,(z\sigma_y - \varrho\sigma_x)$$

$$\frac{dq}{dz} = \frac{B^2y_0 - A\sigma_y(z\sigma_y - \varrho\sigma_x)}{B^3} = \frac{C}{B^3}$$

$$dz = \frac{B^3\,dq}{C} .$$

Therefore

$$\frac{\sigma_x\sigma_y\,dz\,\sqrt{(1-\varrho^2)}}{\pi B^2}\exp\left[-\left(\frac{A^2}{2B^2} + \frac{C^2}{2B^2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)\right]$$

$$= \frac{B\sigma_x\sigma_y\,dq\,\sqrt{(1-\varrho^2)}}{C\pi}\exp\left(-\frac{a^2}{2}\right).$$

But

$$\frac{C^2}{B^2(1-\varrho^2)\sigma_x^2\sigma_y^2} = a^2 - q^2 .$$

Therefore $\frac{\sigma_x \sigma_y \, dz \sqrt{(1-\varrho^2)}}{\pi B^2} \exp\left[-\left(\frac{A^2}{2B^2} + \frac{C^2}{2B^2(1-\varrho^2)\sigma_x^2\sigma_y^2}\right)\right]$

$$= \frac{dq}{\pi\sqrt{(a^2-q^2)}} \exp\left(-\frac{a^2}{2}\right).$$

Also

$$\frac{C\,dz}{B^3} = dq\,.$$

Therefore the distribution of q becomes

$$\frac{dq}{\sqrt{(2\pi)}}\left[1 - 2\,\text{erf}\,\{-\sqrt{(a^2-q^2)}\}\right]\exp\left(-\frac{q^2}{2}\right) + \frac{dq}{\pi\sqrt{(a^2-q^2)}}\exp\left(-\frac{a^2}{2}\right).$$

where

$$q = \frac{A}{B} = \frac{zy_0 - x_0}{\sqrt{(z^2\sigma_y^2 - 2\varrho z\sigma_x\sigma_y + \sigma_x^2)}},$$

and

$$a = \sqrt{\left\{\left(\frac{1}{1-\varrho^2}\right)\left(\frac{x_0^2}{\sigma_x^2} - \frac{2\varrho x_0 y_0}{\sigma_x\sigma_y} + \frac{y_0^2}{\sigma_y^2}\right)\right\}}.$$

Substitution of average values for x_0, y_0, σ_x, and σ_y gives $a = 7 \cdot 1$, which is large in comparison with the required values of q, and $[1 - 2\,\text{erf}\,\{-\sqrt{(a^2-q^2)}\}] = 1 \cdot 0$ very nearly. The distribution of q, therefore, may be approximated by the first term

$$\frac{dq}{\sqrt{(2\pi)}}\exp\left(-\frac{q^2}{2}\right),$$

which is a normal distribution of zero mean and standard deviation $1 \cdot 0$.

Now

$$q^2 = \frac{(zy_0 - x_0)^2}{z^2\sigma_y^2 - 2\varrho z\sigma_x\sigma_y + \sigma_x^2}$$

and when $z = -\infty$,

$$q^2 = \left(\frac{y_0}{\sigma_y}\right)^2.$$

Let $q = q_1$ when $z = z_1$, then the probability that z lies between $-\infty$ and z_1 is the same as the probability that q lies between y_0/σ_y and q_1. Thus by choosing q_1 so that the probability that q lies between y_0/σ_y and q_1 has any desired value, the corresponding value of z can be found.

In the notation of the present report

$$z = n = \frac{f_a}{f_m},$$

and

$$q^2 = \frac{(n\bar{F}_m - \bar{F}_a)^2}{n^2\sigma_m^2 - 2\varrho n\sigma_m\sigma_a + \sigma_a^2}.$$

APPENDIX IV

The Distribution of $f_m\left(\frac{t}{\bar{T}}\right)^2$

The notation used in this appendix is as follows:

f_m	Maximum allowable stress
$\bar{F}_m$	Mean value of f_m
t	Actual thickness
$\bar{T}$	Mean value of t
$f_m' =$	$f_m\left(\frac{t}{\bar{T}}\right)^2$
N, n	Numbers of individuals in the distributions of f_m and f_m', respectively
v_t, v_{t^2}	Coefficients of variation of t and t^2, respectively
v_m, v_m'	Coefficients of variation of f_m and f_m', respectively
σ_m, σ_m'	Standard deviations of f_m and f_m', respectively
μ_1', μ_2', etc.	Moments about zero of the distribution of f_m
μ_2, μ_3, etc.	Moments about mean of the distribution of f_m
M_1'	First moment about zero of the distribution of f_m'
M_2, M_3, etc.	Moments about mean of the distribution of f_m'
λ_2, λ_3, etc.	Moments about mean of the distribution of f_{mp}'
β_1, β_2	Shape parameters of the distribution of f_{mp}'
B_1, B_2	Shape parameters of the distribution of f_m'

Suffices $_p$, $_q$ denote the general terms of the distributions of f_m and f_m', respectively.

It is assumed that f_m and t are normally distributed about their mean values $\bar{F}_m$ and $\bar{T}$, respectively.

Multiplication by $(t/\bar{T})^2$ transforms each point f_{mp} in the normal distribution of f_m into a small distribution f_{mp}'. Because the distribution of t is normal, the distribution of t^2 and, therefore, of f_{mp}', will be skew. Since f_m is constant for each distribution f_{mp}', $v_{mp}' = v_{t^2}$.

Now

$$v_t^2 = E(t - \bar{T})^2/\bar{T}^2 .$$

Therefore

$$E(t^2) = A\bar{T}^2$$

where

$$A = 1 + v_t^2 .$$

Therefore mean value of

$$f_{mp}' \equiv E(f_{mp} \,.\, t^2/\bar{T}^2) = Af_{mp} . \qquad \text{(1)}$$

Therefore

$$M_1' = \frac{1}{nN}\sum_{p=1}^{N}\sum_{q=1}^{n} f_{mpq}' = \frac{A}{N}\sum_{p=1}^{N} f_{mp} \text{ from (1)},$$

$$= \underline{A\bar{F}_m}\,.$$

$$M_2 = \frac{1}{nN}\sum_{p=1}^{N}\sum_{q=1}^{n}(f_{mpq}' - A\bar{F}_m)^2$$

$$= \frac{1}{nN}\sum_{p=1}^{N}\sum_{q=1}^{n}\{(f_{mpq}' - Af_{mp}) + A(f_{mp} - \bar{F}_m)\}^2$$

$$= \frac{1}{nN}\left[\sum_{p=1}^{N}\sum_{q=1}^{n}(f_{mpq}' - Af_{mp})^2 + 2A\sum_{p=1}^{N}\left\{(f_{mp} - \bar{F}_m)\sum_{q=1}^{n}(f_{mpq}' - Af_{mp})\right\} + A^2\sum_{p=1}^{N}\sum_{q=1}^{n}(f_{mp} - \bar{F}_m)^2\right].$$

Now

$$v_{t^2}{}^2 = v_{mp}'^2 = \sum_{q=1}^{n}\frac{(f_{mpq}' - Af_{mp})^2}{nA^2 f_{mp}{}^2}\,.$$

Therefore

$$\sum_{q=1}^{n}(f_{mpq}' - Af_{mp})^2 = nA^2\, v_{t^2}{}^2\, f_{mp}{}^2\,. \qquad \ldots \quad (2)$$

Also

$$\sum_{q=1}^{n}(f_{mpq}' - Af_{mp}) = 0\,. \qquad \ldots \quad (3)$$

Therefore

$$M_2 = A^2(v_{t^2}{}^2\mu_2' + \mu_2)\,.$$

Now

$$\mu_2 \equiv \frac{1}{N}\sum_{p=1}^{N}(f_{mp} - \bar{F}_m)^2 = \mu_2' - \bar{F}_m{}^2\,. \qquad \ldots \quad (4)$$

Therefore

$$M_2 = \mathrm{A}^2(\mu_2 + v_{t^2}{}^2)(\mu_2 + \bar{F}_m{}^2)\,.$$

Therefore

$$\underline{\sigma_m' \equiv \sqrt{M_2} = (1 + v_t{}^2)\sqrt{\{\sigma_m{}^2 + v_{t^2}{}^2(\sigma_m{}^2 + \bar{F}_m{}^2)\}}\,.}$$

$$M_3 = \frac{1}{nN}\sum_{p=1}^{N}\sum_{q=1}^{n}(f_{mpq}' - A\bar{F}_m)^3$$

$$= \frac{1}{nN}\sum_{p=1}^{N}\sum_{q=1}^{n}\{(f_{mpq}' - Af_{mp}) + A(f_{mp} - \bar{F}_m)\}^3$$

Therefore

$$M_3 = \frac{1}{nN}\sum_{p=1}^{N}\sum_{q=1}^{n}\{(f_{mpq}' - Af_{mp})^3 + 3A(f_{mpq}' - Af_{mp})^2(f_{mp} - \bar{F}_m) + 3A^2(f_{mpq}' - Af_{mp})(f_{mp} - \bar{F}_m)^2 + A^3(f_{mp} - \bar{F}_m)^3\}\,.$$

Because f_m is normally distributed

$$\mu_3 \equiv \sum_{p=1}^{N}(f_{mp} - \bar{F}_m')^3 = 0\,. \qquad \ldots \quad (5)$$

Therefore substituting from (2), (3) and (5)

$$M_3 = \frac{1}{N}\left\{3A^3 v_{t^2}^{\,2} \sum_{p=1}^{N} f_{mp}^{\,2}(f_{mp} - \bar{F}_m) + \sum_{p=1}^{N} \lambda_3\right\}.$$

Now

$$\beta_1 = \frac{\lambda_3^{\,2}}{\lambda_2^{\,3}}\text{; and } \lambda_2 = v_{t^2}^{\,2}A^2 f_{mp}^{\,2} \text{ from (2).}$$

Therefore

$$\lambda_3 = \sqrt{(\beta_1)}\, v_{t^2}^{\,3} A^3 f_{mp}^{\,3}. \quad \ldots \quad (6)$$

Also

$$\mu_3' = \mu_3 + 3\mu_1'\mu_2 + \mu_1'^3. \quad \ldots \quad (7)$$

Therefore substituting from (4), (5), (6) and (7)

$$\underline{M_3 = \{1 + v_t^{\,2}\}^3\{6v_{t^2}^{\,2}\bar{F}_m\mu_2 + \sqrt{(\beta_1)}\, v_{t^2}^{\,3}(3\bar{F}_m\mu_2 + \bar{F}_m^{\,3})\}.}$$

$$M_4 = \frac{1}{nN}\sum_{p=1}^{N}\sum_{q=1}^{n}(f_{mpq}' - A\bar{F}_m)^4 = \frac{1}{nN}\sum_{p=1}^{N}\sum_{q=1}^{n}\{(f_{mpq}' - Af_{mp}) + A(f_{mp} - \bar{F}_m)\}^4$$

$$= \frac{1}{nN}\sum_{p=1}^{N}\sum_{q=1}^{n}\{(f_{mpq}' - Af_{mp})^4 + 4A(f_{mpq}' - Af_{mp})^3(f_{mp} - \bar{F}_m)$$
$$+ 6A^2(f_{mpq}' - Af_{mp})^2(f_{mp} - \bar{F}_m)^2$$
$$+ 4A^3(f_{mpq}' - Af_{mp})(f_{mp} - \bar{F}_m)^3$$
$$+ A^4(f_{mp} - \bar{F}_m)^4\}.$$

Therefore substituting from (2), (5) and (6)

$$M_4 = \frac{1}{N}\sum_{p=1}^{N}\left\{\lambda_4 + 4A^4\sqrt{(\beta_1)}\, v_{t^2}^{\,3} f_{mp}^{\,3}(f_{mp} - \bar{F}_m) + 6A^4 v_{t^2}^{\,2} f_{mp}^{\,2}(f_{mp} - \bar{F}_m)^2\right\} + A^4\mu_4.$$

Now

$$\beta_2 = \frac{\lambda_4}{\lambda_2^{\,2}}.$$

Therefore

$$\lambda_4 = \beta_2\lambda_2^{\,2} = \beta_2\, v_{t^2}^{\,4} A^4 f_{mp}^{\,4}.$$

Also

$$\mu_4 = 3\mu_2^{\,2} \text{ for a normal distribution.}$$

Therefore

$$M_4 = A^4\{\beta_2 v_{t^2}^{\,4}\mu_4' + 4\sqrt{(\beta_1)}\, v_{t^2}^{\,3}(\mu_4' - \bar{F}_m\mu_3')$$
$$+ 6v_{t^2}^{\,2}(\mu_4' - 2\bar{F}_m\mu_3' + \bar{F}_m^{\,2}\mu_2') + 3\mu_2^{\,2}\}.$$

Now

$$\mu_4' = \mu_4 + 4\mu_1'\mu_3 + 6\mu_1'^2\mu_2 + \mu_1'^4$$
$$= 3\mu_2^{\,2} + 6\bar{F}_m^{\,2}\mu_2 + \bar{F}_m^{\,4}, \quad \ldots \quad (8)$$

since $\mu_3 = 0$ and $\mu_4 = 3\mu_2^{\,2}$.

Therefore substituting from (4), (7) and (8)

$$\underline{M_4 = \{1 + v_t^{\,2}\}^4\{\beta_2 v_{t^2}^{\,4}(3\mu_2^{\,2} + 6\bar{F}_m^{\,2}\mu_2 + \bar{F}_m^{\,4})}$$
$$\underline{+ 12\sqrt{(\beta_1)}\, v_{t^2}^{\,3}\mu_2(\mu_2 + \bar{F}_m^{\,2}) + 6v_{t^2}^{\,2}\mu_2(3\mu_2 + \bar{F}_m^{\,2}) + 3\mu_2^{\,2}\}.}$$

The shape parameters for f_m' are, therefore:

$$B_1 = \frac{M_3^2}{M_2^3} = \frac{\{6v_{t^2}^2\bar{F}_m\mu_2 + \sqrt{(\beta_1)}\, v_{t^2}^3(3\bar{F}_m\mu_2 + \bar{F}_m^3)\}^2}{\{\mu_2 + v_{t^2}^2(\mu_2 + \bar{F}_m^2)\}^3},$$

and,

$$B_2 = \frac{M_4}{M_2^2}$$

$$= \frac{\{\beta_2 v_{t^2}^4(3\mu_2^2 + 6\bar{F}_m^2\mu_2 + \bar{F}_m^4) + 12\sqrt{(\beta_1)}v_{t^2}^3\mu_2(\mu_2 + \bar{F}_m^2) + 6v_{t^2}^2\mu_2(3\mu_2 + \bar{F}_m^2) + 3\mu_2^2\}}{\{\mu_2 + v_{t^2}^2(\mu_2 + \bar{F}_m^2)\}^2}.$$

If $v_{t^2} = 0$ then $B_1 = 0$ and $B_2 = 3$; these are the values for a normal curve as would be expected. The general values of B_1 and B_2 depend mainly upon v_{t^2}; therefore, the smaller v_{t^2} the more closely the distribution approximates to the normal curve.

If t is normally distributed the distribution of t^2 and, therefore, the distribution of $f_m(t/\bar{T})^2$, have the same shape parameters as χ^2 with one degree of freedom, i.e., $\beta_1 = 8$ and $\beta_2 = 15$. Now v_{t^2} is unlikely to exceed $0 \cdot 10$; therefore, substituting this value in the expressions for B_1 and B_2 and taking typical values of $\bar{F}_m$ and $\mu_2 = 17{,}500$ lb/in.² and 25,660,000 lb/in.², respectively, we find $B_1 = 0 \cdot 086$ and $B_2 = 3 \cdot 65$. The distribution of $f_m(t/\bar{T})^2$, therefore, is of Pearson's Type IV; this is a positively skew distribution and, therefore, estimates of the probability of occurrence of weak specimens based on normal distribution theory will be conservative. For example, when the true probability that a specimen is weaker than a certain value is 0·05 the 'normal' probability is 0·059; similarly when the true probability is 0·005, the 'normal' probability is 0·006.

Because t can be measured directly it is convenient to express v_{t^2} in terms of v_t. This can be done as follows:

$$v_{t^2}^2 = \frac{E\{t^2 - \bar{T}^2(1 + v_t^2)\}^2}{\bar{T}^4(1 + v_t^2)^2}.$$

Now

$$E(t^4) = \frac{1}{\sigma_t\sqrt{(2\pi)}}\int_{-\infty}^{\infty} t^4 \exp\left[-\frac{(t - \bar{T})^2}{2\sigma_t^2}\right] dt.$$

$$= \bar{T}^4 + 6\sigma_t^2\bar{T}^2 + 3\sigma_t^4.$$

But

$$\sigma_t^2 = v_t^2\bar{T}^2.$$

Therefore

$$E(t^4) = \bar{T}^4(1 + 6v_t^2 + 3v_t^4).$$

Therefore

$$v_{t^2}^2 = \frac{2v_t^2(2 + v_t^2)}{(1 + v_t^2)^2}.$$

APPENDIX V

The Effect of Skewness of the Distributions of Intrinsic Strength and Residual Stress on the Accuracy of Strength Estimates for Heat-treated Glass

Let F be the strength which a particular specimen of heat-treated glass would exhibit if broken by rapidly applied loading.

Let f_{ie} and f_{ce}, respectively, be the estimates of its initial intrinsic strength and residual surface compressive stress derived from normal probability theory, and let f_i and f_c be the corresponding true values.

Let f_{me} and f_m, respectively, be the estimated and true values of the maximum allowable stress for this specimen.

Then $F = f_c + f_i = f_{ce} + f_{ie}$; f_{ce}, therefore, is known if f_{ie} is known and it is necessary to consider only the skewness of f_i.

If $f_{ie} = f_i(1 - q)$,
then
$$f_m = f_c + Kf_i = F - (1 - K)f_i,$$
and
$$f_{me} = F - (1 - K)f_{ie} = F - (1 - K)(1 - q)f_i.$$
Therefore
$$f_{me} - f_m = (1 - K)qf_i.$$

The maximum allowable stress for this particular specimen, therefore, is overestimated by an amount $(1 - K)qf_i$. Now the allowable design stress is defined to be the maximum allowable stress for a specimen of chosen improbability (*see* section 6). Therefore, if F is the strength of this specimen under rapidly applied loading, the error of estimation for the allowable design stress is the same as that for the maximum allowable stress.

TABLE 1

The Effect of Sustained Loading on the Breaking Strength of Annealed Sheet Glass

(After Holland and Turner—Ref. 4)

3-point bending tests on specimens approximately 3·94 in. × 0·33 in. × 0·11 in.

Applied load as percentage of mean breaking load*	100	90	80	70	60	50	40	30
Modulus of rupture (lb/in.²)	12,670	11,400	10,120	8,860	7,600	6,330	5,060	3,800
Number unbroken after 1,000 hours ..	0	0	0	0	7	44	68	100
Number fractured before application of full load	57	23	3	1	0	0	0	0
Time to produce fracture (seconds)	Number of specimens fractured							
0 to 10	20	26	31	7	6	—	—	—
11 to 100	14	23	25	16	10	—	—	—
101 to 1,000	9	27	29	33	19	10	—	—
1,001 to 10,000	—	1	12	35	24	14	10	—
10,001 to 100,000	—	—	—	6	23	22	14	—
100,001 to 1,000,000	—	—	—	2	9	6	8	—
1,000,001 to 10,000,000	—	—	—	—	2	4	—	—
Mean time to cause fracture**	34 sec	1 min 37 sec	6 min 4 sec	1 hr 30 min	16 hr 14 min	39 hr 44 min	33 hr 50 min	

* Mean modulus of rupture for specimens loaded at 454 lb/in.²/sec = 12,670 lb/in.²
Coefficient of variation = 0·116.

** Fractured specimens only included.

TABLE 2

The Effect of Rate of Loading on the Breaking Strength of Annealed Plate Glass
(After Black—Ref. 10)

3-point bending tests on specimens 10 in. × 2 in. × $\frac{7}{64}$ in.

Rate of loading* (lb/in.^{2}/sec)	Modulus of rupture (lb/in.^{2})
1,540	10,765
510	9,042
171	7,701
58·6	7,044
19·2	6,913
6·8	6,494

Each result is the average for ten specimens.

* Estimated from published results.

TABLE 3

The Effect of Storage on the Strength of Heat-treated Glass

Bursting tests on panels 12 in. × 12 in. × $\frac{3}{8}$ in.

Condition of specimens	Number in batch	Modulus of rupture (lb/in.^{2})	Coefficient of variation
Edges damaged ; tested immediately after manufacture ..	23	25,290	0·18
Edges undamaged ; tested after one year's storage	24	20,130	0·22

Calculated value of $t = 3 \cdot 93$ for 45 degrees of freedom.

Value of t for 5 per cent level of significance (from tables of normal probability) $= 2 \cdot 02$.

Therefore the mean strengths of the two samples are significantly different.

TABLE 4

The Stress which will just cause Delayed Failure of Annealed Glass Expressed as a Fraction of the Probable Initial Strength

Time (sec)	Applied stress as percentage of mean breaking strength under rapid loading	Probable initial strength of strongest broken specimen expressed as percentage of mean breaking strength under rapid loading	Applied stress expressed as percentage of probable initial strength of strongest broken specimen
10	100	108·4 ± 1·0	92·4 ± 1·0
	90	100·0 ± 2·3	90·0 ± 2·0
	80	95·0 ± 3·0	84·1 ± 2·5
	70	83·5 ± 4·6	83·9 ± 4·5
	60	81·9 ± 4·9	73·4 ± 4·2
100	100	115·5 ± 4·5	86·5 ± 3·0
	90	106·5 ± 1·4	84·5 ± 1·1
	80	102·8 ± 1·9	77·9 ± 1·5
	70	91·8 ± 3·4	76·4 ± 2·9
	60	88·5 ± 3·9	67·9 ± 2·9
1,000	90	126·9 ± 6·1	71·0 ± 3·2
	80	113·6 ± 4·2	70·4 ± 2·4
	70	102·9 ± 1·2	68·1 ± 0·9
	60	95·5 ± 2·9	62·8 ± 1·8
	50	85·0 ± 4·5	58·9 ± 2·9
10,000	70	115·9 ± 4·6	60·4 ± 2·3
	60	102·4 ± 2·0	58·6 ± 1·0
	50	91·8 ± 3·4	54·4 ± 1·9
	40	85·0 ± 4·5	47·1 ± 2·3
100,000	70	123·9 ± 5·7	56·5 ± 2·4
	60	110·7 ± 3·8	54·2 ± 1·8
	50	98·8 ± 2·5	50·6 ± 1·2
	40	91·8 ± 3·4	43·6 ± 1·6
1,000,000	60	115·7 ± 4·5	51·9 ± 2·0
	50	100·5 ± 2·3	50·0 ± 1·3
	40	94·5 ± 3·1	42·4 ± 1·4
1,000 hours	60	117·2 ± 4·7	51·1 ± 1·7
	50	101·7 ± 2·6	49·2 ± 1·3
	40	94·6 ± 1·5	42·3 ± 0·7
	30	<73·0 ± 6·1	>41·0 ± 3·1

TABLE 5

Summary of Distribution Data for Annealed and Heat-Treated Glass

Distribution	Source	Number of specimens	Experimental range	$\sqrt{\beta_1}$	β_2	a	Significance level of $\sqrt{\beta_1}$ (per cent)	Significance level of β_2 or a (per cent)	Non-normality
Strength of annealed sheet glass under rapid loading	Ref. 4	400	$-2 \cdot 96\sigma$ to $+3 \cdot 78\sigma$	0·192	3·109	—	$\simeq 5$	> 5	Doubtful significance
Strength of annealed plate glass under rapid loading	Ref. 18	49	$-2 \cdot 14\sigma$ to $+1 \cdot 56\sigma$	0·222	—	0·841	> 5	$\simeq 10$	Not significant
Apparent strength of heat-treated plate glass	Ref. 18	322	$-2 \cdot 62\sigma$ to $+2 \cdot 22\sigma$	0·014	3·140	—	> 5	> 5	Not significant

TABLE 6

Comparison of the Theoretical and Experimental Ranges of Strength for Given Probabilities of Failure of Annealed Sheet Glass

Range of strength* (lb/in.²)	Experimental* frequency of failure below upper limit of strength range	Experimental probability of failure below upper limit of strength range	Theoretical upper limit of strength range for same probability of failure (lb/in.²)	Percentage by which normal probability theory under-estimates upper limit of strength range
8,550 to 9,119·9 ..	1	0·0025	8,549	6·26
8,550 to 9,689·9 ..	4	0·0100	9,254	4·50
8,550 to 10,259·9 ..	17	0·0425	10,140	1·20
8,550 to 10,829·9 ..	40	0·1000	10,790	0·41
8,550 to 11,399·9 ..	81	0·2025	11,450	−0·41
8,550 to 11,969·9 ..	133	0·3325	12,040	−0·54
8,550 to 12,539·9 ..	195	0·4875	12,630	−0·68
8,550 to 13,109·9 ..	265	0·6625	13,290	−1·36
8,550 to 13,679·9 ..	321	0·8025	13,920	−1·76
8,550 to 14,249·9 ..	358	0·8950	14,510	−1·85
8,550 to 14,819·9 ..	379	0·9475	15,050	−1·58
8,550 to 15,389·9 ..	391	0·9775	15,610	−1·45
8,550 to 15,959·9 ..	396	0·9900	14,090	−0·79
8,550 to 16,529·9 ..	398	0·9950	16,450	0·46
8,550 to 17,099·9 ..	399	0·9975	16,800	1·74
8,550 to 17,669·9 ..	400	1·0000	—	—

* The authors are indebted to Dr. A. J. Holland for these hitherto unpublished details of the strengths found for the control specimens of Ref. 4.

TABLE 7

The Effect of Skewness of the Distributions of Intrinsic Strength and Residual Stress on Estimates of the Maximum Allowable Stress for Specimens of Heat-treated Glass

Intrinsic strength f_i in terms of $\bar{F}_i$ and σ_i	q—positive when strength is under-estimated	Percentage error in estimating allowable design stresses—positive when strength is under-estimated		
		f_m = 4,800 lb/in.²	f_m = 6,000 lb/in.²	f_m = 7,000 lb/in.²
$\bar{F}_i - 2{\cdot}420\sigma_i$	+ 0·0626	− 7·15	− 5·71	− 4·90
$\bar{F}_i - 2{\cdot}029\sigma_i$	+ 0·0450	− 5·45	− 4·36	− 3·74
$\bar{F}_i - 1{\cdot}640\sigma_i$	+ 0·0120	− 1·54	− 1·23	− 1·05
$\bar{F}_i - 1{\cdot}253\sigma_i$	+ 0·0041	− 0·56	− 0·44	− 0·38
$\bar{F}_i - 0{\cdot}864\sigma_i$	− 0·0041	+ 0·58	+ 0·47	+ 0·40
$\bar{F}_i - 0{\cdot}476\sigma_i$	− 0·0054	+ 0·81	+ 0·65	+ 0·55
$\bar{F}_i - 0{\cdot}089\sigma_i$	− 0·0068	—	+ 0·85	+ 0·73
$\bar{F}_i + 0{\cdot}299\sigma_i$	− 0·0136	—	+ 1·79	+ 1·53
$\bar{F}_i + 0{\cdot}687\sigma_i$	− 0·0176	—	+ 2·40	+ 2·06
$\bar{F}_i + 1{\cdot}075\sigma_i$	− 0·0185	—	+ 2·64	+ 2·26
$\bar{F}_i + 1{\cdot}461\sigma_i$	− 0·0158	—	+ 2·34	+ 2·01
$\bar{F}_i + 1{\cdot}850\sigma_i$	− 0·0145	—	—	+ 1·91
$\bar{F}_i + 2{\cdot}240\sigma_i$	− 0·0079	—	—	+ 1·08
$\bar{F}_i + 2{\cdot}625\sigma_i$	+ 0·0046	—	—	− 0·65
$\bar{F}_i + 3{\cdot}015\sigma_i$	+ 0·0174	—	—	− 2·55

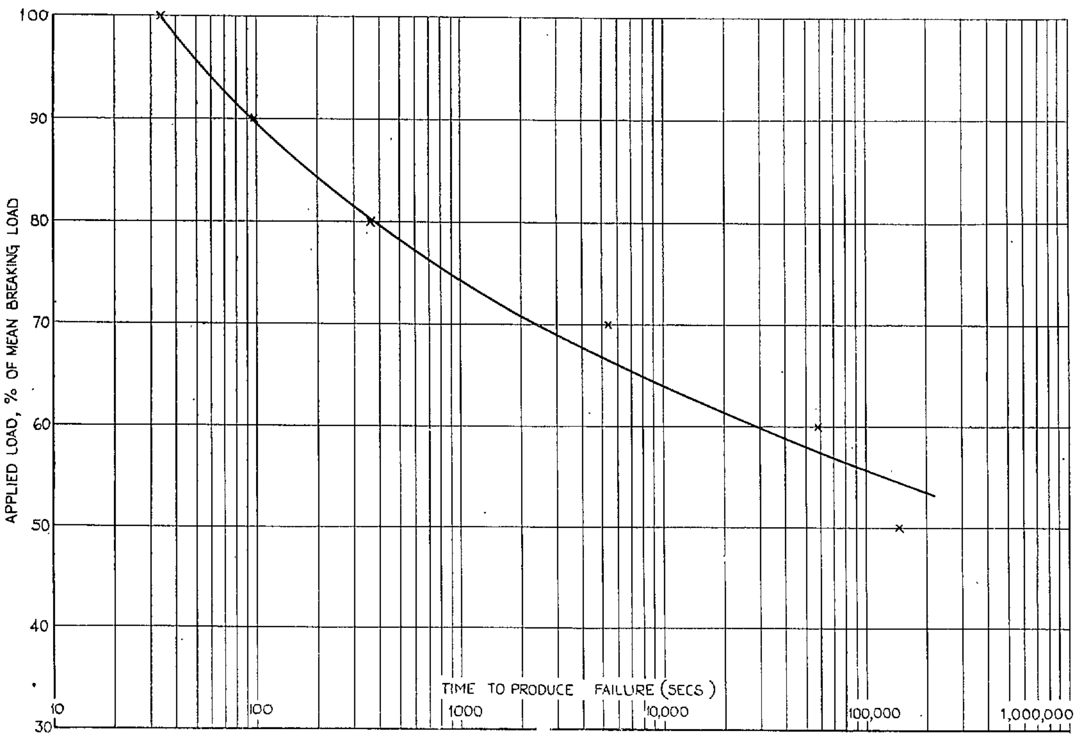


FIG. 1. The effect of sustained loading on the breaking strength of annealed sheet glass. After Holland and Turner (Ref. 4).

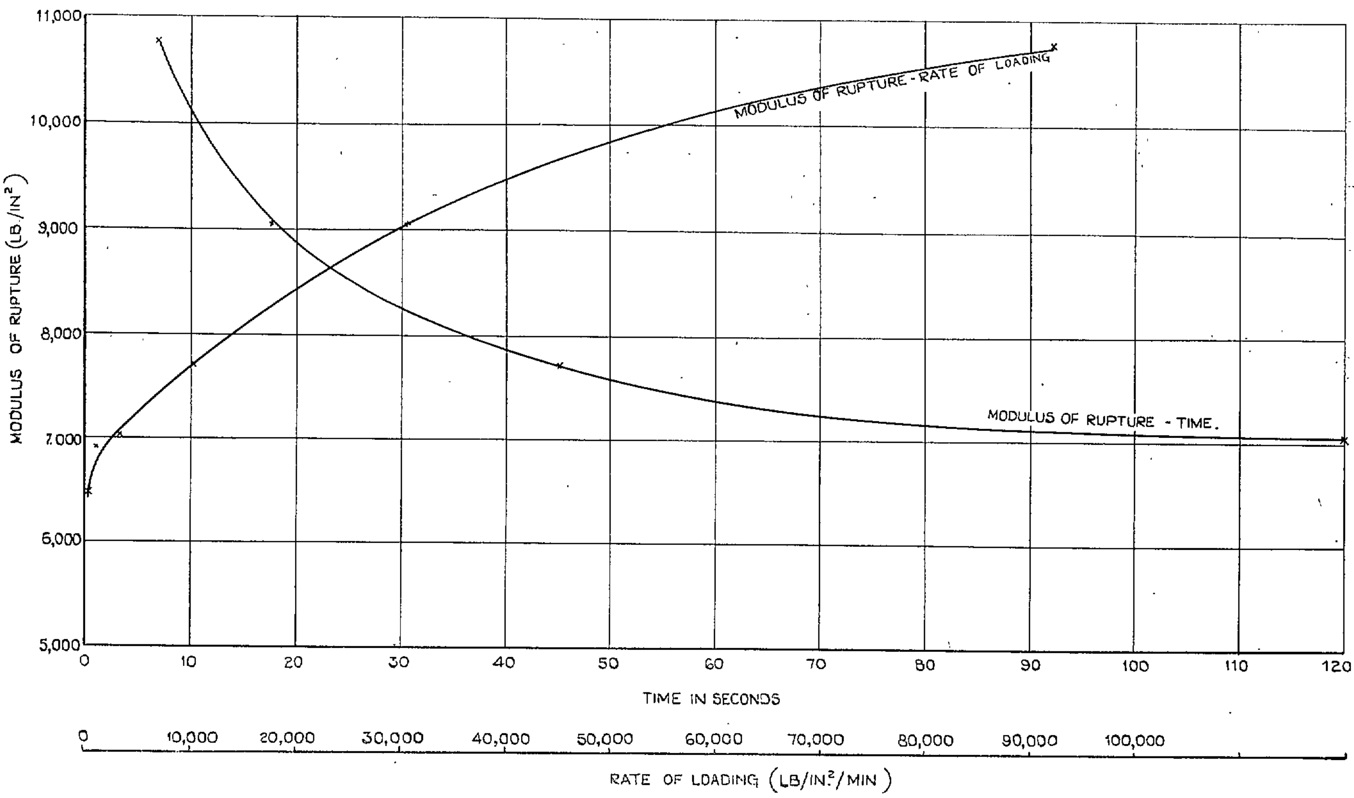


FIG. 2. The effect of rate of loading on the breaking strength of annealed plate glass. After Black (Ref. 10).

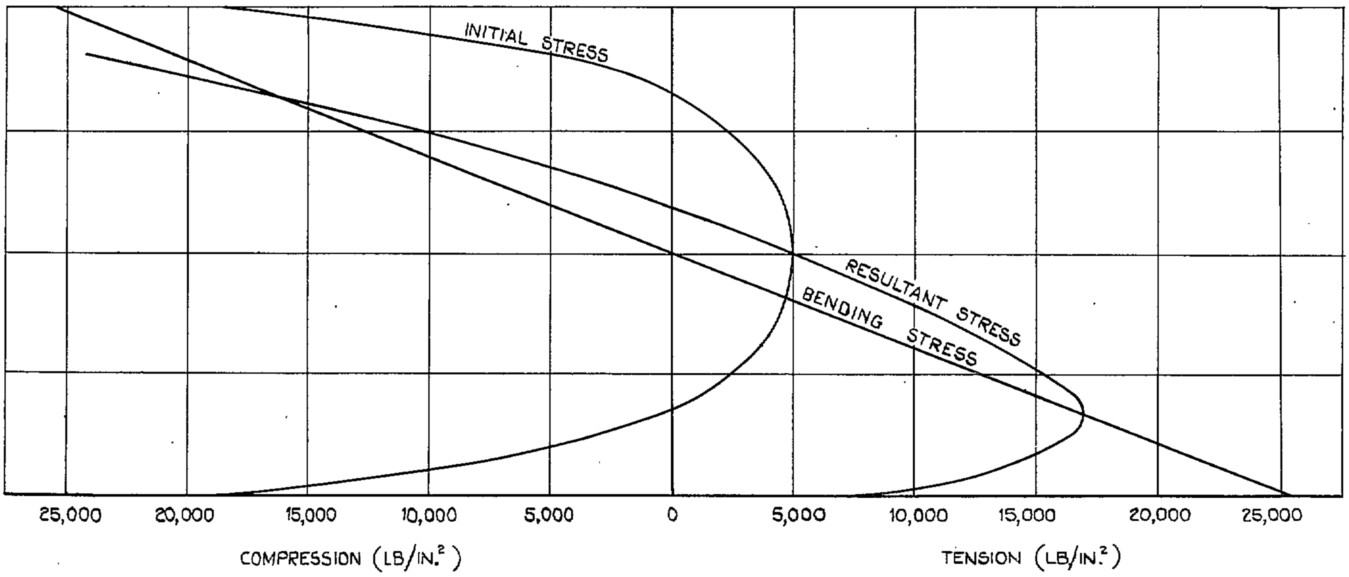


FIG. 3. The distribution of stress in a specimen of heat-treated glass subjected to a bending moment. After Littleton (Ref. 11).

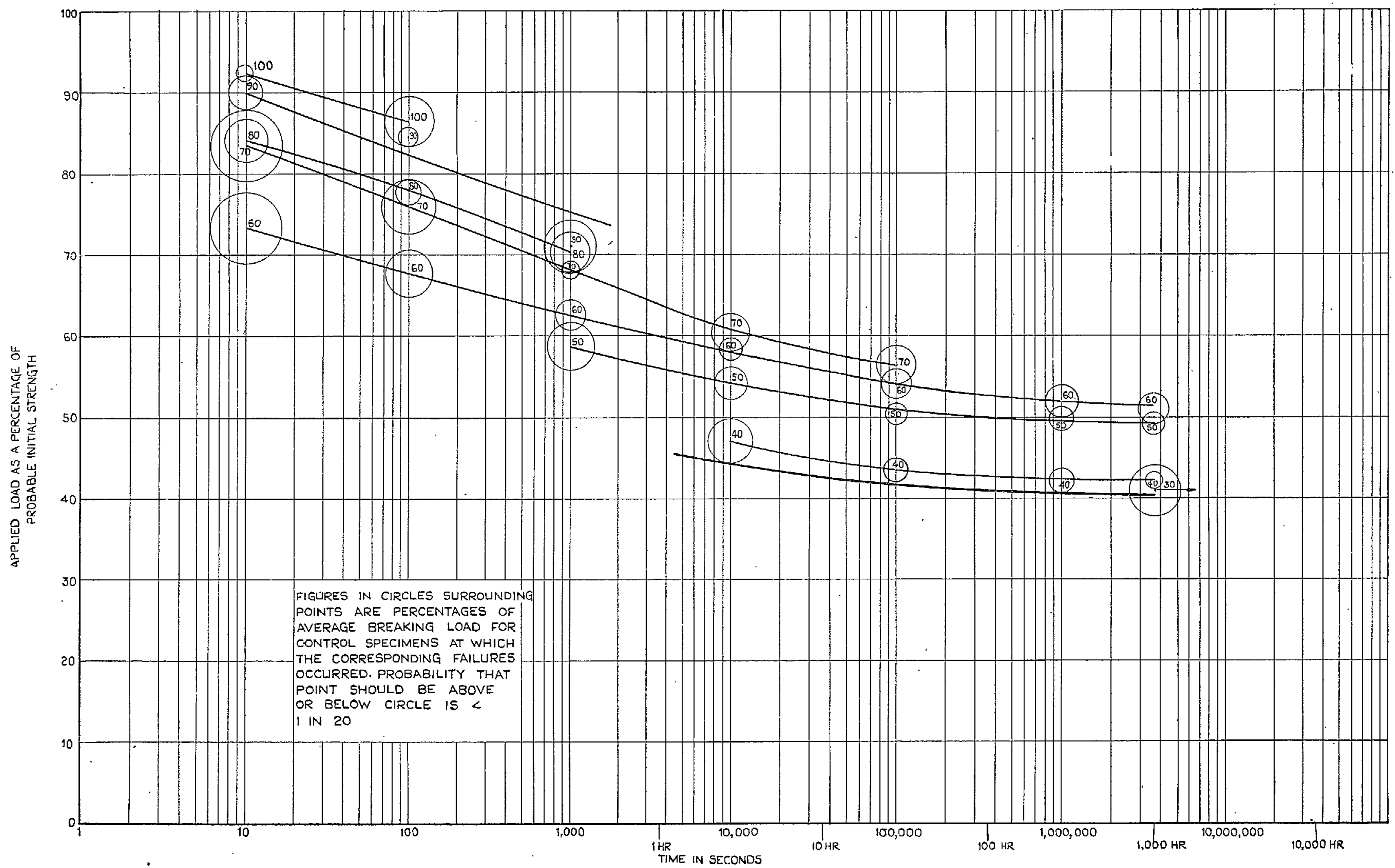


FIG. 4. The stress which will just cause delayed failure of annealed glass expressed as a fraction of the probable initial strength.

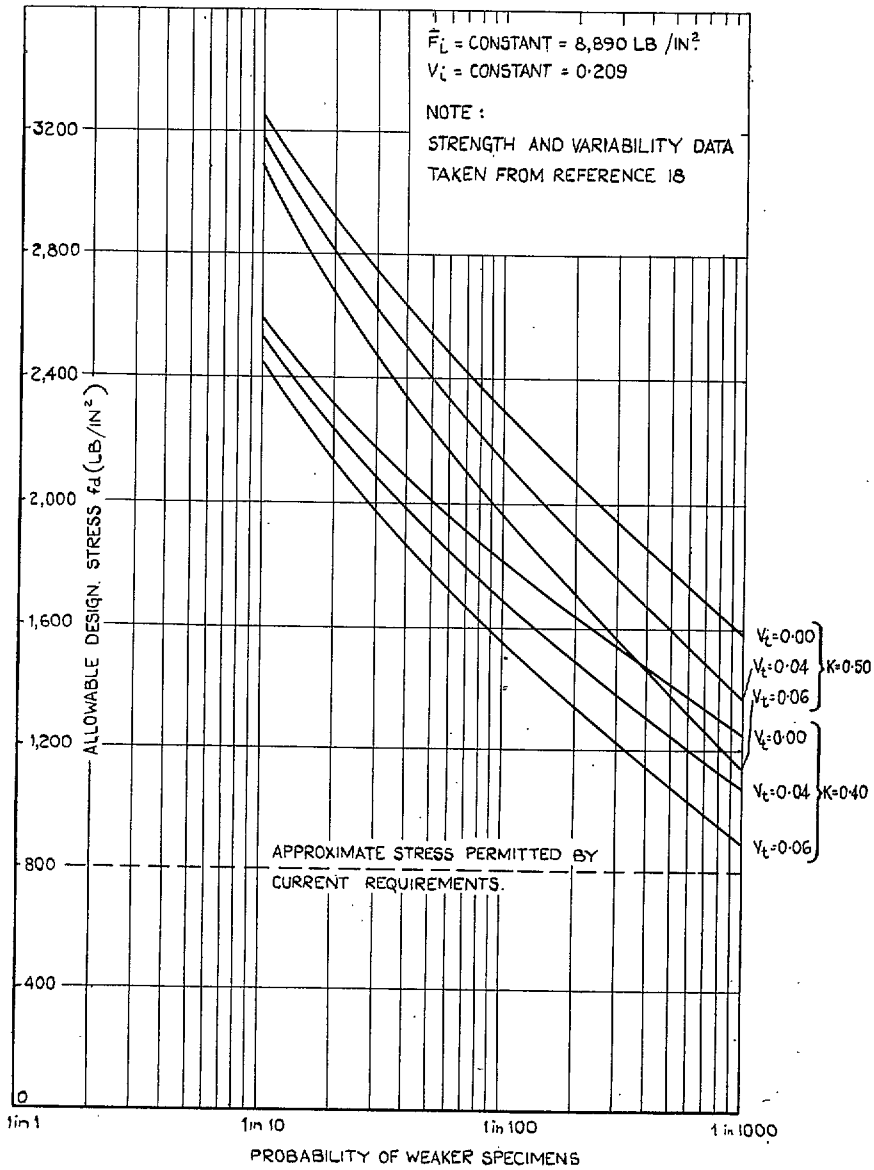


FIG. 5. Allowable design stresses for annealed glass.

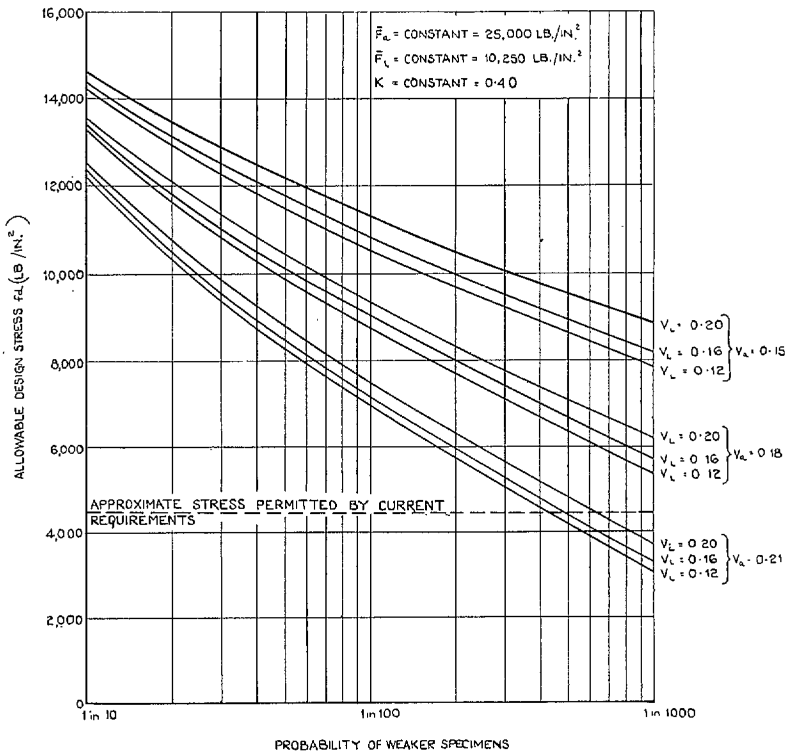


FIG. 6. The influence of v_a and v_i on the allowable design stress for heat-treated glass.

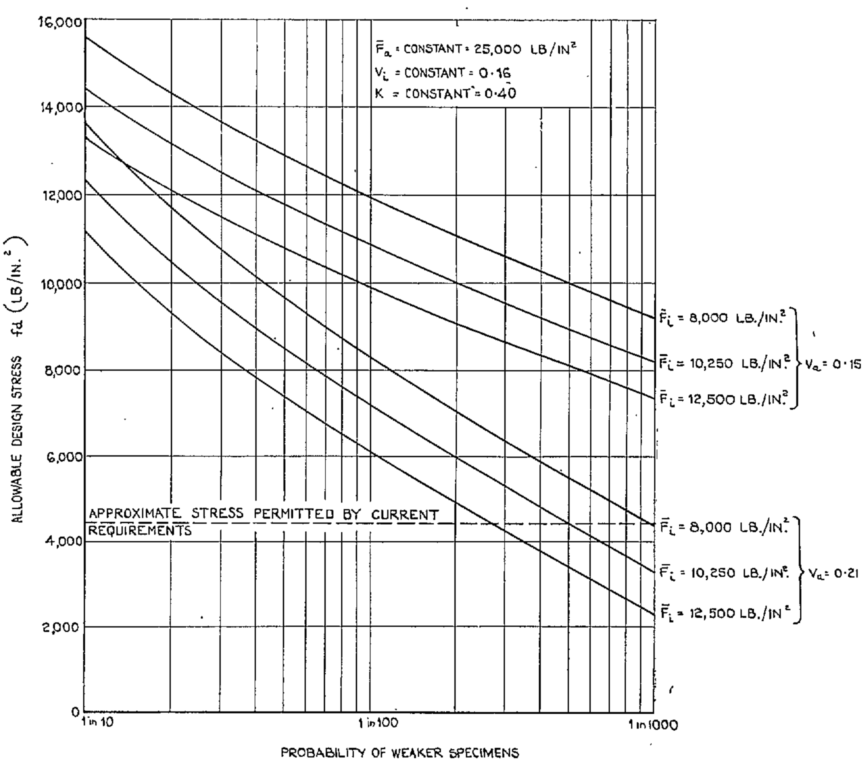


FIG. 7. The influence of v_a and $\bar{F}_i$ on the allowable design stress for heat-treated glass.

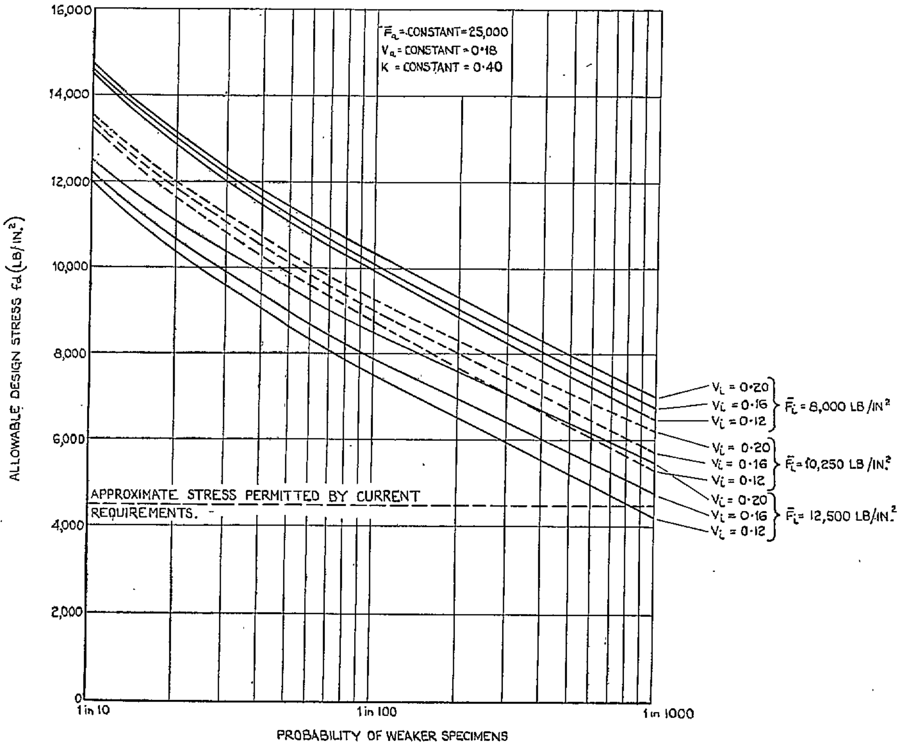


FIG. 8. The influence of v_i and $\bar{F}_i$ on the allowable design stress for heat-treated glass. ($\bar{F}_a$ and v_a constant.)

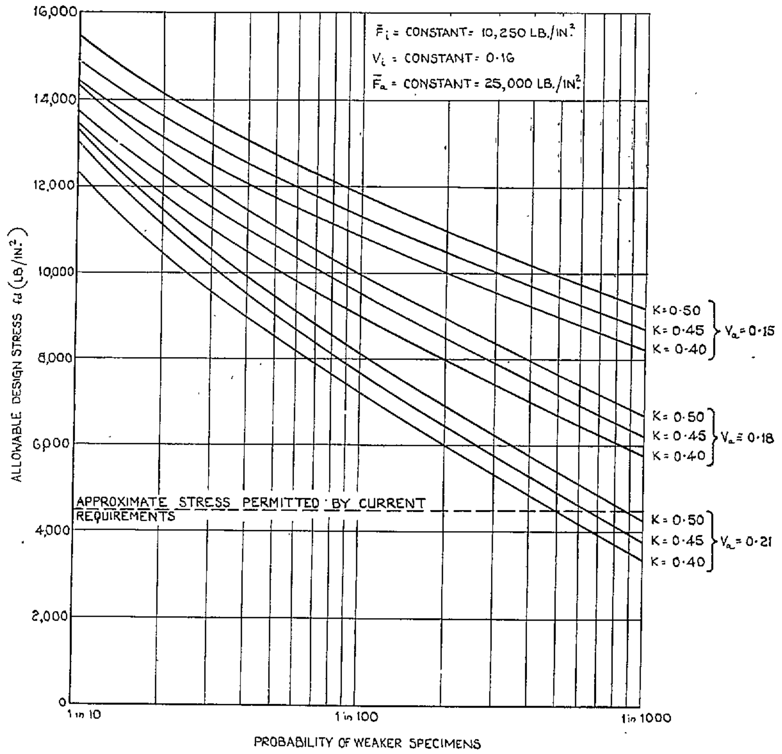


FIG. 9. The influence of K and v_a on the allowable design stress for heat-treated glass.

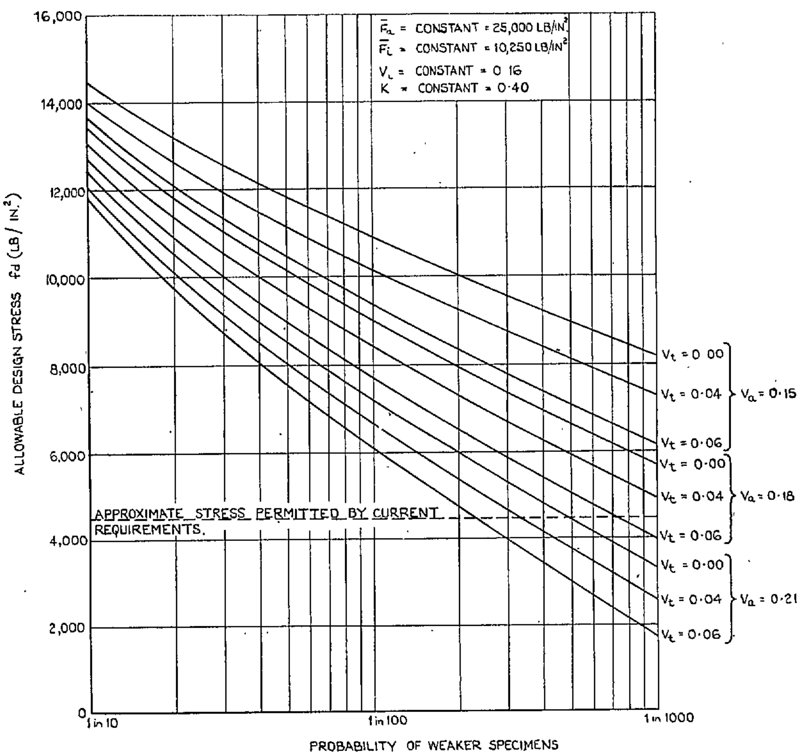


FIG. 10. The influence of v_t and v_a on the allowable design stress for heat-treated glass.

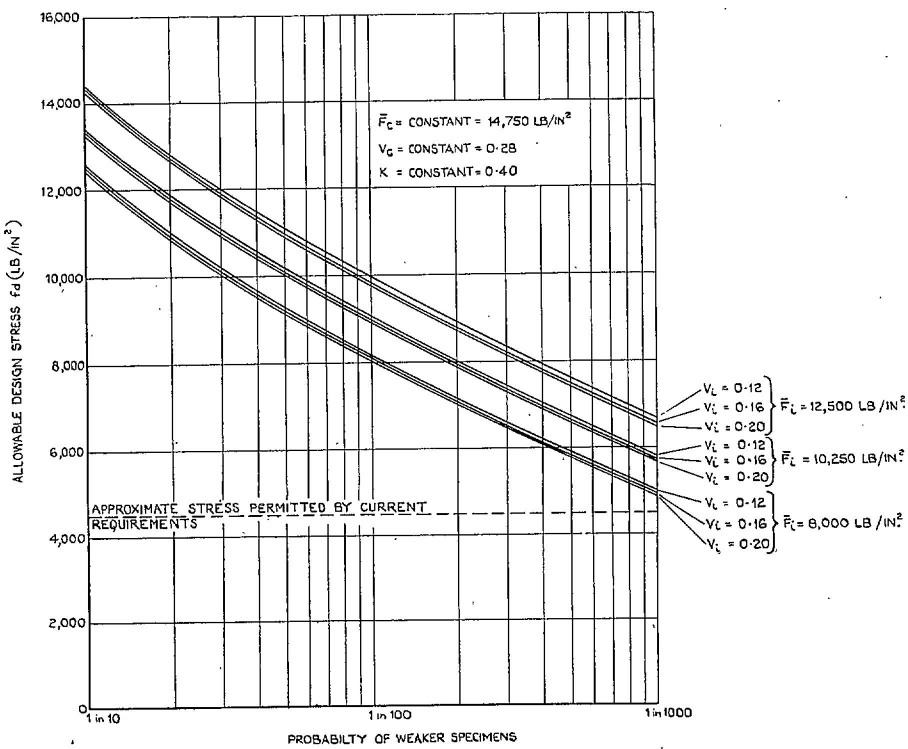


FIG. 11. The influence of v_t and $\bar{F}_t$ on the allowable design stress for heat-treated glass. ($\bar{F}_c$ and v_c constant.)

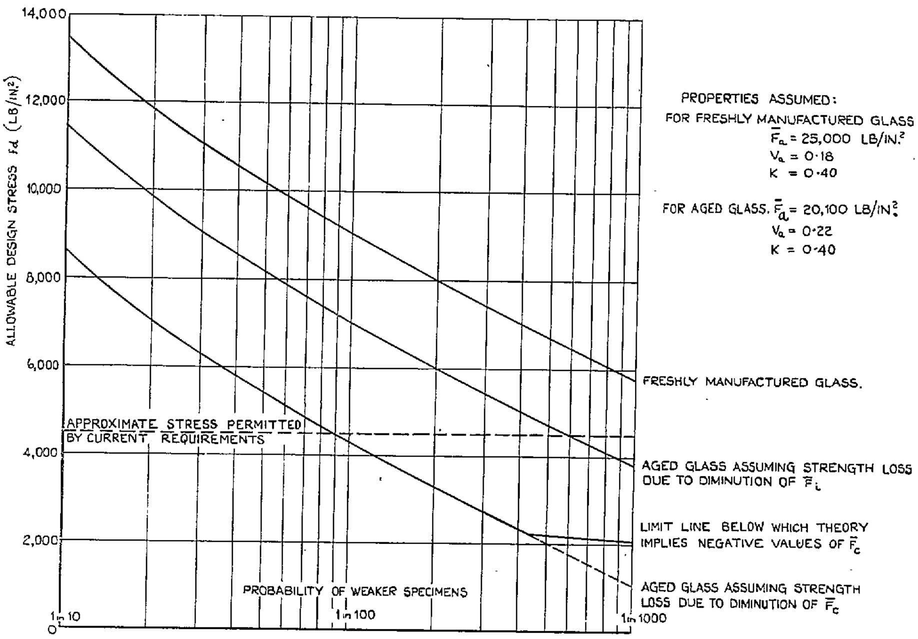


FIG. 12. The effect of age on the allowable design stress for heat-treated glass.

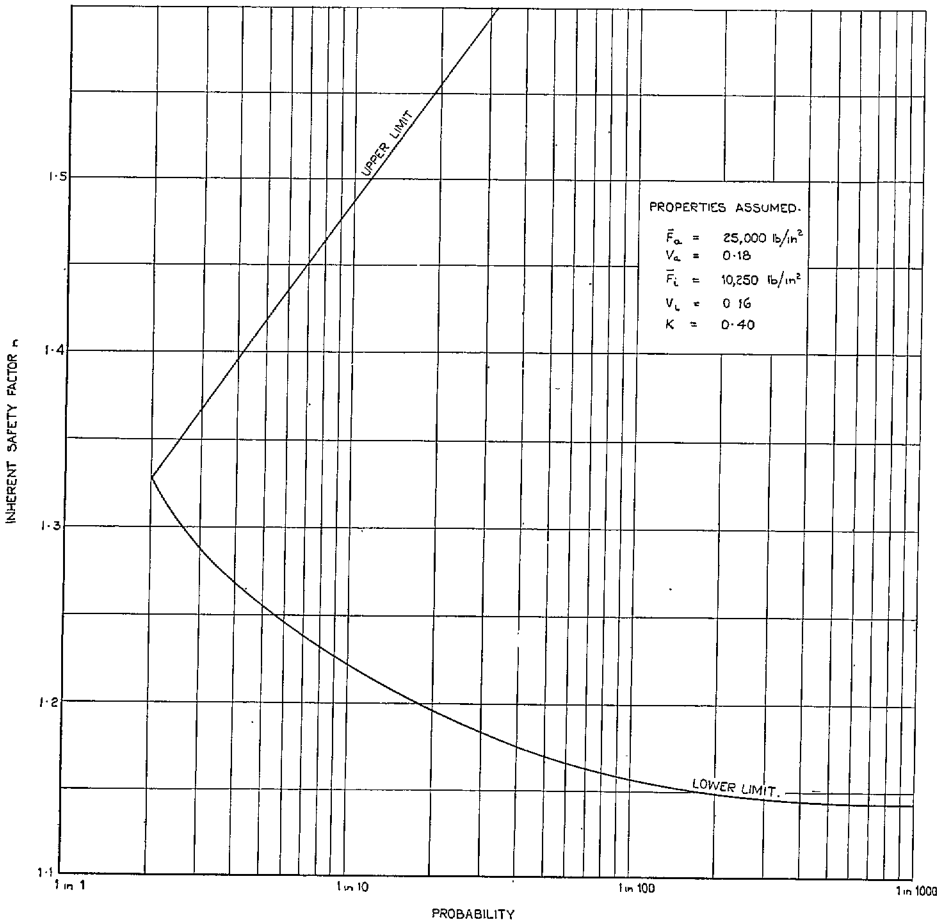


FIG. 13. The inherent safety factor n for heat-treated glass. $n = \dfrac{f_c + f_i}{f_c + Kf_i}$.

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