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Problems of Longitudinal Stability below Minimum Drag Speed, and Theory of Stability under Constraint

By

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Summary.—Practical difficulties in cruising below minimum drag speed have long been known but not fully explained. The reasoning proposed by Painlevé in 1910 purported that flight below minimum drag speed should be fundamentally unstable, so that any speed error would lead to a divergence. This reasoning is shown to be invalid on the ground of the general theory of dynamic stability in uncontrolled flight, Painlevé's criterion being a grossly inadequate approximation to the condition of phugoid stability. However, the criterion may be fully vindicated for the case of flight controlled by the elevator in such a way as to maintain constant height. In this form, the criterion seems not only to explain qualitatively the troubles encountered in slow cruising, but also to lead to a good quantitative estimate of speed variation following an initial disturbance. The criterion also applies to the problem of ultimate height response to an elevator deflection.

The concept of stability of partially controlled flight is further developed, leading to a general theory of 'stability with constraint', *i.e.*, when a control (elevator, throttle, etc.) is used to suppress one component of the disturbance. The theory may be useful as giving approximate solutions of problems in which the pilot moves his control so as to keep one component of the disturbance always as small as possible. The principles of the theory are set out in section 4.1, and several examples given in sections 4.2, 4.3 and 4.4. Flight tests are needed to explore further the validity of this method of approximation.

1. *Introduction.*—As early as 1910, *i.e.*, a year before the publication of the pioneering Bryan's book on stability in aviation[2], the French mathematician Painlevé published a paper[1] on the theory of flight, in which a simple and interesting stability criterion was put forward for the first time. He asserted that level flight could only be stable in what he termed 'régime normal', *i.e.*, at speeds exceeding that corresponding to minimum drag. According to Painlevé, level flight at lower speeds would be unstable.

Painlevé's criterion was originally accepted in France and Germany, and it became widely known through a popular textbook 'Fluglehre' by von Mises[3] (first German edition in 1915). The criterion seemed plausible because, even in those early days of aviation, the danger of stalling was already fully recognised, and therefore a principle exposing the disadvantage of 'slow' flight had an excellent chance of success. However, the criterion was soon objected to by Fuchs and Hopf in their textbook on Aerodynamics[4], published in Germany in 1922. In their analysis, it was shown to be inconsistent with the full theory of dynamic stability developed on Bryan's lines. It appeared that a flight in the 'low-speed régime' (but above stalling speed) could be perfectly stable, and this conclusion was soon confirmed by flight tests. Since 1922, there were no serious attempts by continental aerodynamics to defend the principle.

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Painlevé's criterion seems to have never been analysed by British scientists, and the theory of aircraft stability has been steadily developed in this country on the firm basis of Bryan's work. However, reasoning similar to that of Painlevé was occasionally propounded by technicians. Recently, the 'minimum drag limit' has been revived and has become a matter of widespread interest in connection with the concept of 'comfortable cruising speed' suggested by flight tests[12, 13, 17]. In spite of serious difficulties in defining the 'comfort in cruising', there seems to exist a common belief that cruising can only be comfortable if the speed exceeds that corresponding to minimum drag by at least a certain fraction (to be determined experimentally).

In this paper, an attempt is made to analyse the matter from the point of view of the modern theory of longitudinal stability. The original ideas of Painlevé are first discussed and somewhat extended (sections 2.1, 2, 3), and the relationship between 'minimum drag' and 'minimum power' limits explained. It is then shown (section 2.4) that Painlevé's theory may be considered only as an imperfect attempt to approximate the condition for phugoid stability, the results being unsuccessful as far as uncontrolled flight is concerned. If, however, a flight can be partially controlled by the pilot so as to maintain constant height (or, more strictly, to keep a rectilinear flight path), then Painlevé's criterion becomes perfectly valid—section 2.5. The case of instability in such conditions is examined in more detail in section 2.6, and the results seem to agree reasonably with some flight tests evidence. An alternative interpretation of the minimum drag limit is then discussed (section 3), *viz.*, the criterion for ultimate height response of an aircraft after elevator deflection : this response is reversed at speeds below the minimum drag limit.

The reasoning which has vindicated Painlevé's criterion in the case of constant height, may be further developed so as to lead to a more general concept of restrained stability under partial control. The matter is investigated on general lines in section 4.1, and several simple examples given in section 4.2,3,4.

The Appendix deals with a certain simplified criterion of dynamic longitudinal stability as compared with that of Painlevé and with the full condition of phugoid stability.

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2. *Painlevé's Criterion. Stability at Strictly Constant Height.*—2.1. *Stability Criterion Based on Thrust Curves, or on Polar Diagram, for the Case of Constant Available Thrust.*—The simplest graphical method of calculating the ordinary level flight performance consists in tracing two curves representing the thrust required T_{re} and available T_{av} (or alternatively : power required P_{re} and available P_{av}) against the flight speed V (or against the corresponding kinetic pressure $q = \frac{1}{2}\rho V^2$). The two points of intersection of these curves give two different solutions V_1, V_2 (or q_1, q_2) corresponding to the fast or slow flight, respectively. Of four alternative sets of curves, we shall use the thrust curves plotted against kinetic pressure — T_{re} and T_{av} versus q (Fig. 1), as they are most convenient for explaining Painlevé's criterion. The reasoning is as follows :

Let us first assume that the thrust available (at a certain altitude and a certain throttle setting) is constant, *i.e.*, independent of speed, so that the curve of T_{av} is a straight horizontal line in Fig. 1. And let us consider the aircraft flying in equilibrium conditions corresponding to the point of intersection 1 (fast flight). Suppose that the speed is altered somewhat, *e.g.*, increased, owing to an accidental disturbance. As seen from Fig. 1, the thrust required (drag) then becomes greater than the thrust available so that there will be a decelerating resultant force, and the speed will tend to decrease back to its original value. Similarly, if the speed is accidentally decreased, there will be a resultant accelerating force, again tending to restore the original conditions. The equilibrium at 1 must therefore be considered as stable. Inversely, the flight in conditions corresponding to the point 2 (slow flight), while satisfying equilibrium conditions, must be considered as unstable, by similar reasoning. If the velocity is slightly increased above V_2, it must go on increasing until the value V_1 is reached : while an accidental decrease of the velocity below V_2 should lead to a further decrease, up to a stall.

Let us suppose now that the engine is throttled down, or the altitude increased, so that the propulsive thrust gradually diminishes. The curve of T_{re} remains unchanged, but the horizontal line representing T_{av} shifts downwards, and the two points of intersection (1 and 2) move along the curve and get gradually nearer to each other. Ultimately when T_{av} falls to a certain value $T_{av, min}$, these points coincide at m, and we have the conditions of flight at minimum thrust, or minimum drag. Such a flight should be considered as neutrally stable.

The conclusion is that the flight should be stable if it corresponds to a point of the T_{re}-curve lying to the right of m, and unstable in the opposite case. In other words, the condition of stability would be that the speed should exceed that corresponding to minimum drag. Analytically, this condition may be expressed by the inequality :

$$dT_{re}/dq > 0. \qquad (2.1.1)$$

It is more convenient to express this inequality in terms of the lift and drag coefficients C_L, C_D. To obtain this, let us consider the fundamental equilibrium equations for level flight :

$$C_D S q = D = T_{re}, \quad , \quad C_L S q = L = W, \qquad (2.1.2)$$

which can also be written :

$$T_{re} = W \,.\, C_D/C_L, \quad , \quad q = W/SC_L. \qquad (2.1.3)$$

The equations (2.1.3) had to be used, in fact, to determine the curve of T_{re} versus q from the polar curve of the aircraft :

$$C_D = F(C_L) \qquad (2.1.4)$$

(*see* Fig. 2). It is particularly convenient to replace the dimensional quantities T and q by the following non-dimensional ones :

$$\frac{T}{W} = \frac{C_D}{C_L}\frac{qS}{W} = \frac{1}{C_L}, \qquad (2.1.5)$$

and this has been done in Fig. 1 by simply modifying the scales. In such a way, the curve of T_{re} versus q becomes really a curve of C_D/C_L versus $(1/C_L)$. The condition of stability can now be written :

$$\frac{d(C_D/C_L)}{d(1/C_L)} > 0 \quad , \quad \text{or} \quad \frac{C_L\,dC_D - C_D\,dC_L}{-\,dC_L} > 0\,, \qquad (2.1.6)$$

or finally

$$\frac{C_D}{C_L} - \frac{dC_D}{dC_L} > 0\,. \qquad (2.1.7)$$

This inequality may be found even more simply by considering Fig. 2. The polar curve there corresponds to the T_{re}-curve in Fig. 1, while a straight line 012 (through origin 0, with the slope T_{av}/W), corresponds to the line of T_{av} of Fig. 1. The points 1 and 2 again represent two alternative solutions for a given value of the available thrust, giving fast and slow flight, respectively. Reducing the available thrust is represented by rotating the line 012 in Fig. 2 anti-clockwise, and the limiting case given by the tangent 0m from the origin to the polar curve. It is seen directly in the figure that, at the point m, we have $C_D/C_L = dC_D/dC_L$, and the inequality (2.1.7) is satisfied only along the lower arc 'am' of the polar curve, but not along the upper arc above m. The point m gives the minimum value of the ratio C_D/C_L, or the minimum value of the drag for a given weight. Painlevé's criterion is therefore often formulated in such a way that *the flight is stable only when the speed exceeds that corresponding to the minimum drag.*

The above theory (in an obsolete form, and different notation) was given by Painlevé as applicable to airscrew-propelled aircraft. Painlevé believed that the thrust of an airscrew driven by an orthodox aero-engine could be considered with sufficient accuracy as independent of speed. We now know that this is far from true and that, in such cases, the thrust decreases considerably as the speed increases. Remarkably enough, Painlevé's assumption applies fairly well to many modern jet engines.

It must be mentioned that the above reasoning would also apply, with very little change, to *inclined rectilinear flight*. The equations (2.1.5) would then be replaced by :

$$\frac{T}{W\cos\gamma} - \tan\gamma = \frac{C_D}{C_L} \quad , \quad \frac{qS}{W\cos\gamma} = \frac{1}{C_L} , \qquad \qquad (2.1.8)$$

γ being the angle between the flight path and the horizontal, assumed constant (positive for climb, negative for descent). It is seen that the diagram in Fig. 1 would remain unchanged, only with some modification in the meaning of the co-ordinates. The stability condition would remain (2.1.6) or (2.1.7). In the following sections, we shall always consider level flight ($\gamma = 0$) for simplicity, but the results will also apply to inclined flight.

2.2. *Modified Forms of Stability Criterion for the Case of Propulsive Airscrews* (*variable available thrust*).—A simple modification of Painlevé's criterion for the *case of fixed pitch airscrews* may be obtained by assuming that the available thrust is a *linear* decreasing function of the kinetic pressure q (Everling's assumption[7]), thus writing :

$$T_{\mathrm{av}} = T_0 - C_{\mathrm{AS}}Sq , \qquad \qquad (2.2.1)$$

where C_{AS} is supposed to be a constant 'airscrew drag coefficient', *cf.* Ref. 8, p. 31. This will be represented in Fig. 1 by the inclined straight line T'_{av}, and the points of intersection with the curve T_{re} will now be 1′ and 2′ (in Fig. 1, the points 2 and 2′ happen to coincide, but this is unimportant). Throttling down the engine will be represented again by a downward shift of the line T'_{av}, and the limiting conditions given by the line $T'_{\mathrm{av,\,min}}$ touching the T_{re} curve at m′, instead of m. Apart from this difference, the entire reasoning will be quite similar to that of the section 2.1, and the condition of stability (2.1.1) will be replaced by :

$$dT_{\mathrm{re}}/dq > dT_{\mathrm{av}}/dq , \qquad \qquad (2.2.2)$$

and hence we obtain, instead of (2.1.6) and (2.1.7) :

$$\frac{d(C_D/C_L)}{d(1/C_L)} > -\,C_{\mathrm{AS}} , \qquad \qquad (2.2.3)$$

or

$$\frac{C_D + C_{\mathrm{AS}}}{C_L} - \frac{dC_D}{dC_L} > 0. \qquad \qquad (2.2.4)$$

The inequality (2.2.4) may be again found in a particular simple way by considering the polar curve in Fig. 2. The equation (2.2.1) may be divided by W and written, in terms of C_D and C_L, thus :

$$\frac{C_D + C_{\mathrm{AS}}}{C_L} = \frac{T_0}{W} , \qquad \qquad (2.2.5)$$

and this is represented by the straight line 0′1′2′ through the new origin 0′, the segment 00′ being equal to C_{AS}. Throttling down the engine means rotating this line about 0′ anti-clockwise, and the critical point (m′) is obtained by simply drawing a tangent to the polar curve through 0′. We have obviously :

$$\frac{C_D + C_{\mathrm{AS}}}{C_L} = \frac{dC_D}{dC_L} \text{ at the point m}' ,$$

and therefore the inequality (2.2.4) defines the arc am′ on the curve, below m′.

Everling's assumption is certainly not exact, and the lines in Fig. 1 representing the available thrust against q are generally somewhat curved, while the corresponding lines in Fig. 2 are neither straight nor concurrent to a point. However, we can now define C_{AS} as

$$C_{AS} = -\frac{1}{S} \cdot \frac{\partial T_{av}}{\partial q}, \qquad (2.2.6)$$

this being not a constant but really a function of kinetic pressure, height and throttle position. The condition of stability will still be represented by (2.2.3) or (2.2.4), provided the appropriate value of C_{AS} is introduced. The correction involved is usually of little importance, and a rough estimate of the mean value of C_{AS} will generally be good enough.

An alternative method of analysis is often used, based on *power curves* (P_{re} and P_{av} versus q) instead of the thrust curves. It is usually presented in such a way as to lead to a stability criterion seemingly different from that obtained above. The results of the two methods must, however, be exactly identical, and it may be useful to show that it is so. Let us consider the simple case when the power available P_{av} does not depend on speed, so that the curve of P_{av} becomes a horizontal straight line, an assumption which holds, very nearly, in the case of constant-speed airscrews. This horizontal line again intersects the curve of P_{re} at two points corresponding to fast and slow flight, respectively. A reasoning, exactly similar to that of section 2.1, leads to the condition of stability in the form :

$$dP_{re}/dq > 0, \qquad (2.2.7)$$

analogous to (2.1.1). The power required is, in view of (2.1.3) :

$$P_{re} = V.T_{re} = WV\, C_D/C_L, \qquad (2.2.8)$$

and hence (2.2.7) becomes :

$$V\frac{d(C_D/C_L)}{dq} + \frac{C_D}{C_L}\frac{dV}{dq} > 0\,. \qquad (2.2.9)$$

We have, however :

$$q = \tfrac{1}{2}\rho V^2 \quad , \quad \text{hence } dq = \rho V\, dV\,, \qquad (2.2.10)$$

and

$$q = W/SC_L \quad , \quad \text{hence } dq = W/S.\, d(1/C_L), \qquad (2.2.11)$$

and substituting in (2.2.9), we obtain :

$$\frac{d(C_D/C_L)}{d(1/C_L)} + \frac{C_D}{C_L}\frac{W}{\rho S V^2} > 0 \quad , \quad \text{or } \frac{C_D\, dC_L - C_L\, dC_D}{dC_L} + \tfrac{1}{2}C_D > 0\,, \qquad (2.2.12)$$

or finally :

$$\underline{\frac{3}{2}\frac{C_D}{C_L} - \frac{dC_D}{dC_L} > 0\,.} \qquad (2.2.13)$$

The criterion obtained differs clearly from (2.1.7), and it is seen at once that the relevant critical point m″ on the polar curve (Fig. 2) lies higher than m. It has been shown before that m corresponds to the minimum value of the drag (or minimum of C_D/C_L), and it is easily shown that *m″ corresponds to the minimum of the power required.* We have indeed, from (2.2.7,10,11) :

$$P_{re} = W(2W/\rho S)^{1/2}\, C_D/C_L^{3/2}\,, \qquad (2.2.14)$$

and this becomes minimum when :

$$\frac{d(C_D/C_L^{3/2})}{dC_L} = 0 \quad , \quad \text{or } \frac{3}{2}\frac{C_D}{C_L} = \frac{dC_D}{dC_L} \qquad (2.2.15)$$

The stability required in the present case seems therefore to be that *the speed should exceed that corresponding to the minimum power*, or that C_L should be less than its respective value.

The difference between the two criteria (2.1.7) and (2.2.13) is easily explained. In the former case the assumption was that of constant thrust available, while in the latter the power has been assumed constant. It is important, however, that the requirement (2.2.13) is equivalent to the more general one (2.2.4), if the proper value of C_{AS} is introduced. In our present case, the thrust available is not constant but inversely proportional to V:

$$T_{av} = \frac{P_{av}}{V}, \qquad \qquad (2.2.16)$$

the corresponding airscrew drag coefficient C_{AS} is obtained as follows:

$$C_{AS} = -\frac{dT_{av}}{S\,dq} = -\frac{1}{\rho SV}\frac{dT_{av}}{dV} = \frac{P_{av}}{\rho SV^3} = \frac{T_{av}}{\rho SV^2} = \frac{D}{\rho SV^2},$$

and thus finally:

$$C_{AS} = \tfrac{1}{2}C_D. \qquad \qquad (2.2.17)$$

Substituting this into (2.2.4), we obtain (2.2.13), *q.e.d.*

The above reasoning can, of course, be generalized for the case of the power available P_{av} being an arbitrary function of speed, but it is obvious that the criterion (2.2.13) will apply generally, so that it suffices to determine C_{AS} in each particular case. Suppose, for instance, that P_{av} is proportional to an arbitrary power of speed, thus:

$$P_{av} = KV^p, \qquad \qquad (2.2.18)$$

where K and p are constants. We have then:

$$T_{av} = P_{av}/V = KV^{p-1}, \qquad \qquad (2.2.19)$$

hence:

$$C_{AS} = -\frac{1}{\rho SV}\frac{dT_{av}}{dV} = \frac{(1-p)KV^{p-2}}{\rho SV} = \frac{(1-p)T_{av}}{\rho SV^2} = \frac{(1-p)D}{\rho SV^2},$$

and finally:

$$C_{AS} = \frac{1-p}{2}C_D. \qquad \qquad (2.2.20)$$

Substituting this into (2.2.4), we get, as the requirement for stability:

$$\underline{\frac{3-p}{2}\frac{C_D}{C_L} - \frac{dC_D}{dC_L} > 0}. \qquad \qquad (2.2.21)$$

This inequality is a generalization of the two previous ones. In the case of constant thrust $p = 1$, and we obtain (2.1.7). If the power is assumed constant, then $p = 0$, and this gives (2.2.13). We may consider also the case $p = \frac{1}{2}$ which, according to Bryant and McMillan[9], applies fairly well to fixed-pitch airscrews. We have then:

$$C_{AS} = \tfrac{1}{4}C_D, \qquad \qquad (2.2.22)$$

and the stability criterion in the form:

$$\frac{5}{4}\frac{C_D}{C_L} - \frac{dC_D}{dC_L} > 0. \qquad \qquad (2.2.23)$$

This case is obviously an intermediate one between the two considered previously.

2.3. *Special Case of the Polar Curve Approximated by a Parabola.*—The polar curve is often approximated by a simple parabola:

$$C_D = C_{D0} + sC_L^2, \qquad \qquad (2.3.1)$$

where C_{D0} is a constant being the minimum value of C_D (at $C_L = 0$) and s is another constant coefficient. The parabola, with proper choice of the constants, may coincide with the true polar along its major and most important part, but the agreement always breaks down near the stall, *i.e.*, for C_L exceeding a certain considerable value (*cf.* Fig. 2). If this approximation is adopted, the analysis given in sections 2.1 and 2.2 takes a particularly simple form. If the curve of thrust available T_{av} versus q is assumed to be a straight line (equation 2.2.1 with constant C_{AS}, including the case of $T_{av} = T_0 =$ const when $C_{AS} = 0$), then even the performance calculation becomes very simple. The equilibrium condition $T_{av} = T_{re}$ is then :

$$T_0 - C_{AS}Sq = C_DSq, \qquad (2.3.2)$$

which, dividing by W and introducing $C_L = W/Sq$, becomes :

$$\frac{T_0}{W} = \frac{C_D + C_{AS}}{C_L}, \qquad (2.3.3)$$

or, making use of (2.3.1) and simplifying :

$$sC_L^2 - \frac{T_0}{W} C_L + (C_{D0} + C_{AS}) = 0. \qquad (2.3.4)$$

This is a quadratic equation for C_L, with the two solutions :

$$C_{L1,2} = \frac{T_0}{2Ws} \mp \sqrt{\left\{\left(\frac{T_0}{2Ws}\right)^2 - \frac{C_{D0} + C_{AS}}{s}\right\}}, \qquad (2.3.5)$$

giving the values of C_L for two points of intersection 1′ (or 1) and 2 in Fig. 2. When T_0 decreases, then the lower value rises and the higher one falls, until they coincide for :

$$T_{0,\min} = 2W\sqrt{\{(C_{D0} + C_{AS})s\}}, \qquad (2.3.6)$$

the common value C'_{Lm} (or C_{Lm}) at the point of contact m′ (or m) becoming :

$$C_{Lm}' = \sqrt{\left\{\frac{C_{D0} + C_{AS}}{s}\right\}}, \qquad (2.3.7)$$

with the corresponding value of C_D :

$$C_{Dm}' = 2C_{D0} + C_{AS}. \qquad (2.3.8)$$

It may be also noticed that

$$C_{L1}C_{L2} = C_{Lm}'^2 \quad , \quad \text{hence } V_1V_2 = V_m'^2. \qquad (2.3.9)$$

The Painlevé's stability criterion now reduces to $C_L < C'_{Lm}$, or $C_D < C'_{Dm}$, the latter condition taking the particularly simple form :

$$\underline{C_D < 2C_{D0} + C_{AS}} \text{ (or } C_D < 2C_{D0} \text{ for } C_{AS} = 0). \qquad (2.3.10)$$

It should be stressed that the simple performance calculation given above applies exclusively when $C_{AS} =$ const, in particular $C_{AS} = 0$ (jet engines), but not if C_{AS} depends on C_D, i.e., in usual cases of propulsive airscrews. In the latter cases, the graph of T_{av} is not a straight line but a curve, the equation (2.3.4) must be replaced by a more complicated one of higher degree, and the solution is more easily obtained graphically.

As to Painlevé's stability criterion, however, it is always represented by (2.3.10) if a parabolic polar curve is assumed. We have seen indeed that the criterion is always represented by the inequality (2.2.4), provided a proper value for C_{AS} is introduced. Now, the equation (2.3.1) gives :

$$dC_D/dC_L = 2sC_L, \qquad (2.3.11)$$

so that (2.2.4) becomes :

$$2sC_L^2 < C_D + C_{AS},$$

or, substituting sC_L^2 from (2.3.1) :

$$2(C_D - C_{D0}) < C_D + C_{AS}, \qquad (2.3.12)$$

which is identical with (2.3.10), *q.e.d.*

We may consider the case of the power available being proportional to an arbitrary power of speed (*cf.* 2.2.18). C_{AS} is then given by (2.2.20), and hence the stability criterion (2.3.10) becomes :

$$C_D < \frac{4C_{D0}}{1+p}, \qquad (2.3.13)$$

with the following particular cases :

$p = 1$ (jet engines) : $C_D < 2C_{D0} = C_{Dm}$ .. (2.3.14a)

$p = \frac{1}{2}$ (fixed-pitch airscrews) : $C_D < \frac{8}{3} C_{D0} = \frac{4}{3} C_{Dm}$ .. (2.3.14b)

$p = 0$ (constant-speed airscrews) : $C_D < 4C_{D0} = 2C_{Dm} = C_{Dm}''$. .. (2.3.14c)

It may be noticed that, for the point m″ of minimum power required, the drag coefficient is twice that corresponding to the point m of minimum drag. All limiting points defined by (2.3.14) lie well below the stalling region, so that the approximation of the polar by a parabola is justified.

Numerical example. In Fig. 2, a major part of the polar is approximated by the parabola :

$$C_D = 0 \cdot 009 + 0 \cdot 1 C_L^2 \quad , \quad \text{so that } C_{D0} = 0 \cdot 009 \quad , \quad s = 0 \cdot 1$$

For performance calculation, assume $S = 500$ sq ft, $W = 10{,}830$ lb, $\rho = 0 \cdot 0015625$ slugs/cu ft (corresponding to the altitude of about 14,000 ft), then for any C_L :

$$V = \sqrt{\left\{\frac{2W}{\rho S C_L}\right\}} = \sqrt{\left\{\frac{27724 \cdot 8}{C_L}\right\}}.$$

Assuming first *constant thrust available* ($C_{AS} = 0$) $T_{av} = T_0 = 0 \cdot 1275W$, we obtain from (2.3.5) :

$$C_{L1} = 0 \cdot 075 \quad , \quad C_{L2} = 1 \cdot 2,$$

and hence :

$$V_1 = 608 \text{ ft/sec} = 360 \text{ knots} \quad , \quad V_2 = 152 \text{ ft/sec} = 90 \text{ knots}.$$

The minimum thrust required (minimum drag) is, from (2.3.6) :

$$T_{0,\min} = 0 \cdot 06W,$$

the corresponding lift and drag coefficients, from (2.3.7,8,9) :

$$C_{Lm} = 0 \cdot 3 \quad , \quad C_{Dm} = 0 \cdot 018,$$

and the ' critical ' speed :

$$V_m = 304 \text{ ft/sec} = 180 \text{ knots}.$$

According to Painlevé's criterion, flying would be stable only for speeds exceeding 180 knots.

Assuming next *thrust available linearly decreasing with kinetic pressure* (*see* 2.2.1), with

$$T_0 = 0 \cdot 1325W \quad , \quad C_{AS} = 0 \cdot 006 \text{ (constant)},$$

we obtain similarly from (2.3.5) :

$$C'_{L1} = 0 \cdot 125 \quad , \quad C'_{L2} = 1 \cdot 2,$$

and hence

$$V'_1 = 470 \cdot 95 \text{ ft/sec} \simeq 279 \text{ knots}, \; V'_2 = 90 \text{ knots}.$$

The minimum value of T_0 required becomes, from (2.3.6) :

$$T_{0,\min} = 0 \cdot 07746W,$$

the corresponding lift and drag coefficients, from (2.3.7,8,9) :

$$C'_{Lm} = \sqrt{(0 \cdot 15)} = 0 \cdot 3873 \quad , \quad C'_{Dm} = 0 \cdot 024,$$

the ' critical ' speed :

$$V'_m = 267 \cdot 55 \text{ ft/sec} \simeq 158 \text{ knots},$$

and flying at speeds below this would be unstable.

If the *power available is assumed proportional to various powers of speed*, the limiting conditions re calculated simply from (2.3.14), and we get :

$$\text{for } p = 1\,, \quad C_D = 0\cdot018 = C_{D\,m}\,, \quad C_L = 0\cdot3\,, \quad V = 180 \text{ knots}\,,$$

$$\text{for } p = \tfrac{1}{2}\,, \quad C_D = 0\cdot024\,, \quad C_L = 0\cdot3873\,, \quad V = 158 \text{ knots}\,,$$

$$\text{for } p = 0\,, \quad C_D = 0\cdot036\,, \quad C_L = 0\cdot5196\,, \quad V = 137 \text{ knots}\,.$$

[he agreement of the values for $p = \frac{1}{2}$ with those found above for $C_{AS} = \text{const} = 0\cdot006$ is, of ourse, accidential and due to the value of C_{AS} being exactly $\frac{1}{4}$ of the corresponding C_D.)

2.4. *Fallacy of Painlevé's Criterion in Uncontrolled Flight; Comparison with the Criterion for 'hugoid Stability*.—The first criticism of the Painlevé's principle was published by Fuchs and [opf[4] in 1922. They pointed out that, in Painlevé's reasoning, the only flight disturbance dmitted was a change of speed along the path, and hence the aircraft was considered as a body ith only one degree of freedom. If this tacit assumption were true, the conclusions would be correct. [owever, an uncontrolled aircraft possesses many degrees of freedom and, even if the disturbance ; purely longitudinal, *i.e.*, confined to the vertical symmetry plane, there are three of them : 1 addition to speed variation, the aircraft may move in the direction normal to the original ight path, and also rotate about its lateral axis. The three component motions are interependent. Even if the initial disturbance affects only one degree of freedom, the two remaining nes become immediately involved. For example, if the aircraft has accelerated initially along s path, the lift increases producing a vertical velocity component ; this is followed by a change f incidence, and the resulting moment about the centre of gravity generates a rotation about the iteral axis (pitching). Each of the component disturbances gives rise to new forces and moments ffecting the entire disturbed motion in a favourable or adverse manner, and its ultimate course epends on the properties of all forces and moments, and not merely on the forces in the flight irection produced by the initial disturbance. The true problem to be solved is, therefore, much iore complicated than the simplified one envisaged by Painlevé. Furthermore, it would be isufficient to find out whether the forces and moments initially arising tend to modify the iotion favourably, *i.e.*, act against the initial disturbance. The tendency may change as the isturbance develops and, even if persisting, it may not bring the aircraft to the original state f equilibrium but induce oscillations about this equilibrium, not necessarily damped. It is seen hat the problem belongs essentially to the field of dynamics, and cannot be solved or even nderstood on purely static considerations. The correct approach consists in drawing up diferential equations of motion, and in finding and analyzing their solutions. The equations should e derived under the assumption that the accidential initial source of disturbance no longer cts, and no new similar sources appear, also that the pilot does not interfere through his ontrols. The equations have therefore an invariable form, independent of the initial disturbance. stability criterion, to be determined by analyzing the equations, must therefore be quite eneral and applicable in all cases, whatever the initial disturbance may have been.

Fuchs and Hopf discussed the differential equations of longitudinally disturbed motion, and iiled to find any connection between the conditions of dynamic stability based on these equations, nd Painlevé's criterion. They therefore condemned the latter as completely meaningless.

Although Fuchs and Hopf's criticism must still be considered as fundamentally correct, it may e worthwhile to re-consider the general equations of motion and to try to find a connection etween the true stability conditions and Painlevé's criterion. The equations, in the usual imensionless form (neglecting density changes with height), are[8, 18] :

$$\left.\begin{aligned} (D - x_u)\hat{u} \quad - x_w\hat{w} \quad + \tfrac{1}{2}C_L\theta &= 0 \\ - z_u\hat{u} + (D - z)\hat{w} \quad - \hat{q} \quad &= 0 \\ \varkappa\hat{u} + (\chi D + \omega)\hat{w} + (D + \nu)\hat{q} \quad &= 0 \\ - \hat{q} + D\theta &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.4.1)$$

where the meaning of symbols is as follows :—

$$\left.\begin{array}{lll} D & = & d/d\tau \text{ (differential operator) ; } \tau = t/\hat{t} \text{ (aerodynamic time)} \\ \hat{t} & = & W/g\rho SV = VC_L/2g \text{ (unit of aerodynamic time)} \\ \hat{u} & = & u/V \text{ (dimensionless velocity increment in } x\text{-direction)} \\ \hat{w} & = & w/V \text{ (dimensionless velocity increment in } z\text{-direction, or increment of the incidence } \alpha) \\ \theta & & \text{(angular displacement in pitch) ; } \hat{q} = q\hat{t} = D\theta \text{ (dimensionless rate of pitch)} \\ \mu & = & W/g\rho Sl = V\hat{t}/l \text{ (relative density)} \\ i_B & = & k_B^2/l^2 \text{ (coefficient of moment of inertia about lateral axis } y) \end{array}\right\} \quad \ldots \quad (2.4.2)$$

Force derivatives (neglecting effects of compressibility and elastic distortion) are :

$$x_u = -C_D - C_{AS}, \quad x_w = \tfrac{1}{2}(C_L - a.dC_D/dC_L), \quad z_u = -C_L, \quad z_w = -\tfrac{1}{2}(a + C_D), \quad \ldots \quad (2.4.3)$$

and 'compound' pitching-moment derivatives, similarly :

$$\omega = -\mu m_w/i_B, \quad \nu = -m_q/i_B, \quad \chi = -m_{\dot{w}}/i_B, \quad \varkappa = -\mu m_u/i_B \quad \ldots \quad \ldots \quad \ldots \quad (2.4.4)$$

The solution of the system (2.4.2) depends on finding four roots : $\lambda_1, \lambda_2 \, (= -R \pm iJ)$ and $\lambda_3, \lambda_4 \, (= -r \pm ij)$ of the determinantal stability quartic :

$$\Delta(\lambda) = \lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4.5)$$

where the coefficients are :

$$B_1 = N_1 + \nu + \chi, \quad C_1 = P_1 + N_1\nu + Q_1\chi + \omega, \quad D_1 = P_1\nu + R_1\chi + Q_1\omega - S_1\varkappa,$$
$$E_1 = R_1\omega - T_1\varkappa, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4.6)$$

with :

$$\left.\begin{array}{l} N_1 = -x_u - z_w = \tfrac{1}{2}(a + C_D) + C_D + C_{AS}, \quad P_1 = x_u z_w - x_w z_u = \tfrac{1}{2}(a + C_D)(C_D + C_{AS}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \tfrac{1}{2}C_L\left(C_L - a\dfrac{dC_D}{dC_L}\right) \\ Q_1 = -x_u = C_D + C_{AS}, \quad R_1 = -\tfrac{1}{2}C_L z_u = \tfrac{1}{2}C_L^2, \quad S_1 = \tfrac{1}{2}C_L - x_w = \tfrac{1}{2}a\dfrac{dC_D}{dC_L}, \\ \qquad\qquad\qquad\qquad\qquad\qquad T_1 = -\tfrac{1}{2}C_L z_w = \tfrac{1}{4}C_L(a + C_D) \end{array}\right\} \quad \ldots \quad (2.4.7)$$

The conditions of stability are that all four roots of the quartic are either real and negative or complex with negative real parts, and this requires that all coefficients be positive :

$$B_1 > 0, \quad C_1 > 0, \quad D_1 > 0, \quad E_1 > 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4.8)$$

and that the Routh discriminant be also positive :

$$\mathcal{R} = B_1(C_1D_1 - B_1E_1) - D_1^2 > 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4.9)$$

The conditions (2.4.8) have obviously no connection with Painlevé's criterion, it will suffice therefore to consider (2.4.9). D_1 and E_1 being normally small in comparison with B_1 and C_1, the inequality (2.4.9) may practically be replaced by the following simpler one :

$$C_1D_1 - B_1E_1 > 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4.10)$$

This is the approximate *condition for positive phugoid damping*, as may be seen from the known (Bairstow's) approximate factorisation of the quartic :

$$\Delta(\lambda) = (\lambda^2 + 2R\lambda + H)(\lambda^2 + 2r\lambda + h) = 0\,, \qquad \ldots \quad (2.4.11)$$

where $\quad 2R \simeq B_1, \quad H_1 \simeq C_1, \quad 2r \simeq D_1/C_1 - B_1E_1/C_1^2, \quad h \simeq E_1/C_1, \qquad \ldots \quad (2.4.12)$

and the first quadratic factor corresponds to the short-period oscillatory mode, the second factor the phugoid mode. r is the phugoid damping factor. By making use of (2.4.6,7) and neglecting unimportant terms, we may express r in terms of fundamental derivatives, as shown in Ref. 8. The conditions for positive phugoid damping may then be written in the approximate form* :

$$\frac{2r}{C_L} \simeq \frac{C_D + C_{AS}}{C_L} - \left(1 - \frac{2\omega}{2\omega + a\nu}\right)\frac{dC_D}{dC_L} + aC_L\frac{\nu(\nu + \chi) - \omega}{(2\omega + a\nu)^2} > 0\,. \qquad \ldots \quad (2.4.13)$$

Comparing this inequality with (2.2.4) which represents Painlevé's criterion, we notice at once that both terms of the latter do appear here, but there are two additional terms, of comparable magnitude and equal importance. It is seen that Painlevé's criterion simply omits the terms depending on the moment derivatives, ω, ν, χ, a not surprising fact as the moment equation was completely ignored in the derivation. The inequality (2.4.13) would reduce to (2.2.4) if a were infinite. In all practical cases the two criteria differ very considerably as shown in Figs. 7a to 7d, where $2r$ (from 2.4.13) has been plotted against ω for several values of ν, and horizontal straight lines representing the Painlevé's values :

$$2r_P = C_D + C_{AS} - C_L\,dC_D/dC_L \qquad \ldots \quad (2.4.14)$$

are shown for comparison. In all cases, we have assumed $a = 4$, $s = 0{\cdot}1$, $C_{D0} = 0{\cdot}01$. In Figs. 7a and 7b a low value $C_L = 0{\cdot}2$ was assumed, with $\chi = 0{\cdot}5\nu$ or 0 respectively, and in Figs. 7c and 7d, we have a large value $C_L = 1$. It is seen that the lines representing Painlevé's values are very different from the rigorous curves, and often, especially for high C_L, they show different signs. The final conclusion is that Painlevé's criterion is inapplicable for an uncontrolled aircraft.

2.5. *Validity of Painleve's Criterion when Height is Kept Constant by the Pilot Through Elevator Control.*—The main cause of failure of the Painlevé's criterion was found in the fact that an initial disturbance consisting in a velocity increment along its horizontal path produced a vertical acceleration and velocity components leading to short-period oscillations. A new line of investigation therefore presents itself : whether it is possible that the vertical velocity component can be effectively suppressed. This can obviously not happen without a pilot's (or auto-pilot's) appropriate action. Such an action, however, may be taken by the pilot with comparative ease, by simply trying to operate the elevator so as to maintain constant height. It is true that such an operation will never be completely (mathematically) successful, in particular immediately after a sharp initial disturbance involving change of height. Nevertheless, an experienced pilot may, if he wishes, quickly regain the original height with a good approximation, by observing his altimeter (or, less advisable, his rate of climb indicator), and countering any deviations from that height as best he can. Thereafter, he may keep the height practically constant by very slight elevator adjustments, until the next accidental disturbance. It should be understood that his initial action, after a sudden disturbance, is by no means completely determined (so as to be mathematically tractable). Two pilots, in exactly similar circumstances, will never perform it exactly in the same way, and each of them may improve his technique instinctively or consciously so as not only to counter the actual height deviation but also to anticipate and forestall overshooting in the opposite direction. The initial period of erratic height recovery may thus be considerably shortened. Once, however, this period has passed, and the skilled pilot perseveres in his determination, little will depend on his individual style. The motion will proceed *almost as if the height were maintained exactly constant.* This does not mean at all that the entire disturbance would have disappeared by then, because none of the variables $\hat{u}$, $\hat{w}$, θ and $\hat{q}$ need have

* The formula (2.4.13) is equivalent to the formula (20c) of Ref. 8, with the χ term omitted. The derivative dC_D/dC_L was, however, replaced there by $2sC_L$ (*see* 2.3.1.), as the polar was approximated by a parabola right from the start.

been brought down to zero yet. The further time history of the motion will be governed, with a fair accuracy, by the ordinary equations of motion (2.4.2), with two modifications. Firstly, we shall have an additional condition expressing the assumption that the height is maintained constant, or that the vertical velocity component is zero :

$$dH/dt = V\theta - w = V(\theta - \hat{w}) = 0, \quad \text{or } \theta = \hat{w} \qquad \text{(2.5.1)}$$

(*see, e.g.*, Ref. 11, Fig. 3 and equation (3.17)). Secondly, in our system of equations (2.4.1), the third (moments') equation must be omitted when considering the aircraft motion. The equation will apply in modified form, including on the right-hand side the term representing the effect of elevator movement, thus :

$$\varkappa\hat{u} + (\chi D + \omega)\hat{w} + (D + \nu)\hat{q} = -\delta\eta, \qquad \text{(2.5.2)}$$

where the compound derivative δ (elevator effect coefficient) is :

$$\delta = -\mu m_\eta / i_B = -\frac{\mu}{i_B}\frac{c}{2l}\frac{\partial C_m}{\partial \eta}. \qquad \text{(2.5.3)}$$

It must be clearly understood that the additional condition (2.5.1) restrains the previous freedom of the aircraft, depriving it of one degree, so that one of the equations of motion becomes unnecessary for investigating stability ; and the one to go is obviously the moments' equation, that directly affected by the elevator*.

The system of equations of motion now becomes :

$$\left.\begin{array}{l} (D - xu)\hat{u} \quad - x_w\hat{w} \quad + \tfrac{1}{2}C_L\theta = 0 \\ -z_u\hat{u} + (D - z_w)\hat{w} - \hat{q} \quad = 0 \\ \hat{w} \quad - \theta = 0 \\ -\hat{q} + D\theta = 0 \end{array}\right\} \qquad \text{(2.5.4)}$$

or, eliminating θ and $\hat{q}$ from the first two equations by means of the two last ones :

$$\left.\begin{array}{l} (D - x_u)\hat{u} + (\tfrac{1}{2}C_L - x_w)\hat{w} = 0 \\ -z_u\hat{u} \quad - z_w\hat{w} = 0 \end{array}\right\} \qquad \text{(2.5.5)}$$

Not surprisingly, the order of the system has been decreased, because the previous moments' equation, itself of the second order, has been replaced by the algebraic equality (2.5.1). What may seem surprising is that the order has fallen from fourth to merely first. This is because the differentials in the second equation have disappeared so that this differential equation has become algebraic. This equation now simply expresses the equilibrium of the normal force components or, true to the small quantities of first order, equilibrium of vertical force components (obviously necessary for keeping height constant).

The new system of equations (2.5.5) evidently does not involve a short-period oscillation any more, and thus represents only a modified sort of phugoid mode of motion which is no longer oscillatory but has degenerated into a simple subsidence or divergence. Eliminating $\hat{w}$ from (2.5.5), we obtain :

$$D\hat{u} - \left\{x_u + \frac{z_u}{z_w}(k - x_w)\right\}\hat{u} = 0 \qquad \text{(2.5.6)}$$

---

* The modified moments' equation, in the form (2.5.2), may be used, as shown later, to obtain the time history of η on the idealized assumption of the height deviation being strictly zero.

For a more general discussion of the principles of this calculation, the reader should refer to section 4.1.

and, still denoting by $(-2r)$ the only root of this equation (now replacing the two small complex roots), we obtain the solution in the form :

$$\hat{u} = \hat{u}_0 e^{-2r\tau}, \qquad (2.5.7)$$

where (*cf.* 2.4.3) :

$$2r = -x_u - \frac{z_u}{z_w}(k - x_w) = C_D + C_{AS} - \frac{C_L}{a + C_D}\frac{dC_D}{d\alpha}, \qquad (2.5.8)$$

so that the stability criterion, or condition for subsidence, becomes :

$$\underline{2r = C_D + C_{AS} - \frac{C_L.dC_D/dC_L}{1 + C_D/a} > 0}\ , \qquad (2.5.9)$$

and $\hat{u}_0$ is the constant of integration being the value of $\hat{u}$ at the instant after which the height remains constant. The variables $\hat{w}$ and θ are represented by exactly similar solutions :

$$\hat{w} = \theta = \hat{w}_0 e^{-2r\tau}, \qquad (2.5.10)$$

where $\hat{w}_0 = \theta_0$ is the value of $\hat{w}$ or θ at the same instant. The two constants $\hat{u}_0$ and $\hat{w}_0$ are not independent, as they must satisfy the condition $z_u\hat{u}_0 + z_w\hat{w}_0 = 0$, whence :

$$\hat{w}_0 = -\frac{2C_L}{a + C_D}\hat{u}_0 = \theta_0 . \qquad (2.5.11)$$

The formula (2.5.9), giving the damping factor and stability requirement, is almost exactly identical with the generalized Painlevé's criterion (2.2.4). The only difference lies in the correction term C_D/a in the denominator. The correction is due to z_w being exactly $-\frac{1}{2}(a+C_D)$, instead of $(-\frac{1}{2}a)$, and this subtlety has, of course, been overlooked in Painlevé's derivation. The correction is usually very small*, so that the *Painlevé's criterion has been vindicated for disturbances in which the pilot's elevator control ensures strictly constant height* **.

All conclusions deduced previously from the Painlevé's criterion also become true in these conditions. In particular, the flight becomes stable (with subsiding disturbances) or unstable (with diverging disturbances) according to whether it occurs at C_L below or above, respectively, the value corresponding to minimum drag or minimum power. *If C_L exceeds the critical value and the pilot tries to keep constant height, he is likely to gain (or loose) speed inordinately following any initial disturbance.* The phenomenon, of course, will never take place in ordinary conditions of high-speed flight, but it may become discomforting in cruising flight at too low speeds, and especially acute when flying with high C_L at great heights. It is interesting that such conditions have been reported from flight-testing quarters, *viz.*, by Cameron[12] (1942) and by the English Electric Co.[13] (1948) and, in broad lines, the above theory is encouragingly confirmed by these reports. It should be mentioned that, in the case of high-altitude cruising, the matter is seriously affected by the air density variation with height[11,13], so that our theory requires a considerable refinement on this score.

Another important case, where the instability at constant height may appear in strength, may well be that of an aircraft with parasite drag drastically reduced by full application of boundary-layer suction, as repeatedly suggested recently (*see, e.g.*, papers by Sir M. Jones and Head[14,15] (1951)). For such aircraft, C_{D0} may be many times smaller than the present values,

* Except near to or at the stall ; in the latter case $a=0$ and $dC_D/dC_L=\infty$, and in these particular circumstances the condition of stability at constant height becomes, from (2.5.8) :

$$2r = C_D + C_{AS} - \frac{C_L}{C_D}\frac{dC_D}{d\alpha} > 0\ , \qquad (2.5.12)$$

and is practically never satisfied.

** In this paper, only horizontal flight is considered, but it can be easily shown that the criterion still applies, in an exactly similar way, to a climbing or descending flight in which the pilot tries to maintain steady rectilinear path. The essential point consists in suppressing deviations from the original flight direction.

and hence (*see* form 2.3.7) the critical value $C'_{L\,m}$ will also be considerably smaller, so that a large part of the polar curve may become 'unstable' in Painlevé's sense. We shall come back to this point in section 3.

Two further remarks may be made as to the equations of this section.

(*a*) The formula (2.5.11) shows that, in a disturbance at constant height, the incidence increment $\hat{w}$ (hence θ) always has an opposite sign to $\hat{u}$, and its numerical value is normally much less than that of $\hat{u}$—except near or at the stall, where $\hat{w}$ may become much larger than $\hat{u}$. This is easily understood as displaying the manner in which the lift increase due to excess speed is compensated by the lift decrease through slightly reduced incidence.

(*b*) The time history of the elevator deflection η required to maintain constant height may be obtained from (2.5.2), substituting $\hat{q} = D\theta = D\hat{w}$, in the following operational form :

$$\delta\eta = -\{D^2 + (\nu + \chi)D + \omega\}\hat{w} - \varkappa\hat{u} \qquad \ldots \qquad (2.5.13)$$

and, using the solutions (2.5.7) and (2.5.10), in the final algebraic form :

$$\eta = -\frac{1}{\delta}\left\{\left(\omega - \frac{a + C_D}{2C_L}\varkappa\right) - 2r(\nu + \chi) + 4r^2\right\}\hat{w}_0\,e^{-2r\tau}. \qquad \ldots \qquad (2.5.14)$$

As, however, r is always small, and $\varkappa$ rather small and relatively unimportant, we may write this more simply, with sufficient approximation :

$$\eta \doteq -\frac{\omega}{\delta}\hat{w}_0\,e^{-2r\tau} = -\frac{\omega}{\delta}\hat{w}\,. \qquad \ldots \qquad (2.5.15)$$

It may be also interesting to see what will happen if the pilot operates two controls simultaneously, *e.g.*, if, in addition to keeping height constant by means of the elevator, he also tries to keep speed constant ($\hat{u}$ = small constant) by applying his throttle. We have then $D\hat{u} = 0$, and (2.5.5.) becomes :

$$\left.\begin{aligned}(C_D + C_{AS})\hat{u} + \tfrac{1}{2}a\,dC_D/dC_L.\hat{w} &= 0\\ C_L\hat{u} + \tfrac{1}{2}(a + C_D)\,\hat{w} &= 0\end{aligned}\right\}. \qquad \ldots \qquad (2.5.16)$$

These linear algebraic equations can be simultaneously satisfied only if :

either, at arbitrary incidence, $\hat{u} = \hat{w} = 0$ which means that the pilot has checked the disturbance completely ;

or, if $2r$ (from 2.5.10) is zero, *i.e.*, that the aircraft is flying at the critical incidence in Painlevé's sense—being neutrally stable in this sense ; in such a case $\hat{u}$ may have an arbitrary small constant value, and $\hat{w}$ will also be small constant, in the appropriate proportion to $\hat{u}$ (*see* 2.5.11).

2.6. *Case of Instability at Strictly Constant Height.*—The case of instability at strictly constant height is particularly interesting. In this case $2r < 0$, and the solution (2.5.7) implies that the speed error would increase indefinitely, *i.e.*, that the speed would either rise to infinity or fall to $(-\infty)$, according to whether u_0 is positive or negative. Both conclusions are obviously absurd, and their fallacy is due to our initial assumptions which no longer hold for large $\hat{u}$. The equations of motion cannot now be linearised as before, and some or all derivatives such as a, C_L, C_D, C_{AS}, dC_D/dC_L may not be constant. The true motion will be approximated by the solution (2.5.7) for only a short while and, subsequently, it will proceed in a different way, according to non-linearised equations of motion. It is not difficult to guess roughly what may happen. If the initial speed error is positive ($\hat{u}_0 > 0$), the speed will increase and C_L decrease, and the values of C_L, C_D will follow the polar curve (Fig. 2) from a point slightly below 2 towards the second point of equilibrium 1′ (or 1) where the polar is intersected again by the line of available thrust. We may therefore expect that the motion will ultimately converge to a new steady flight with greatly increased, but finite, speed. If, on the contrary, the initial speed error is negative ($\hat{u}_0 < 0$), the speed will further decrease and C_L rise, the values of C_L C_D, following the polar from a point slightly above 2 (in Fig. 2) towards the stalling point. It will then become impossible for the pilot to maintain constant height, and unless he checks the stall in time, he cannot expect anything but the usual troubles.

The second alternative, divergence towards the stall, would present some difficulties to analytical treatment. The first alternative, however, accelerated flight converging to the rapid steady régime, may be dealt with analytically in a comparatively simple way, at least with some reasonable simplifying assumptions. We have to start from first principles and handle non-linear equations, but we may assume that the height is kept constant permanently by a suitable elevator manoeuvre so that only two equations are to be considered, *viz.*, the static condition of equilibrium for vertical forces, and the dynamic equation for horizontal acceleration. If, further, the point 2 does not lie too high on the polar curve (not too near to the stall), then we may still assume that the relevant part of the polar curve can be approximated by a parabola (2.3.1), and that the lift slope a is constant throughout. We may also treat C_{D2} as small in comparison with a. The equations should lead to finding a functional relationship between the speed V and time t, so as to enable us to estimate the time needed to approach more or less closely the new steady speed. This time should probably be comparatively long, the acceleration being necessarily moderate ; but it may be quite interesting to know what order of magnitude we have to reckon with.

Let us consider the aircraft in steady flight (Fig. 3a) with speed V_2 at incidence α_2, corresponding to the point 2 in Fig. 2. It is subject to the forces W, L_2, T_2 and D_2, the thrust T_2 being, in general, inclined at a moderate angle φ_2 to the horizontal. The equilibrium equations are (replacing $\sin\varphi_2$ by φ_2, and $\cos\varphi_2$ by 1) :

$$L_2 + T_2\varphi_2 = W, \qquad (2.6.1)$$

$$T_2 = D_2. \qquad (2.6.2)$$

Supposing the speed has accidentally increased slightly, say to $V_2(1 + \varepsilon_2)$, and the height is maintained constant, the aircraft incidence and speed will both vary and, at an arbitrary time t, the conditions will be represented by Fig. 3b. The incidence will now be $(\alpha_2 + \Delta\alpha)$, the inclination of thrust $(\varphi_2 + \Delta\alpha)$, and the forces W, L, T and D. The vertical force components will still be in equilibrium :

$$L + T(\varphi_2 + \Delta\alpha) = W\ , \qquad (2.6.3)$$

where $\Delta\alpha$ has been considered as sufficiently small* for replacing $\sin(\varphi_2 + \alpha)$ by $(\varphi_2 + \alpha)$. The resultant of horizontal components will produce an acceleration dV/dt, so that

$$T - D = \frac{W}{g}\frac{dV}{dt}\ , \qquad (2.6.4)$$

where $\cos(\varphi_2 + \alpha)$ has again been replaced by 1.

We now obtain from (2.6.1,2) :

$$L_2 + D_2\varphi_2 = W \quad , \quad \text{or} \quad L_2 = W\Big/\left(1 + \frac{C_{D2}}{C_{L2}}\varphi_2\right). \qquad (2.6.5)$$

Let us now consider, for simplicity, only the case when the thrust T does not depend on speed (thus $C_{AS} = 0$), and hence :

$$T = T_2 = D_2 = L_2C_{D2}/C_{L2}, \qquad (2.6.6)$$

also :

$$L = L_2V^2C_L/C_{L2}V_2^2, \qquad (2.6.7)$$

$$D = D_2V^2C_D/C_{D2}\,V_2^2 = L_2V^2C_D/C_{L2}\,V_2^2\ . \qquad (2.6.8)$$

* If, as assumed, the speed increases, *i.e.*, $\varepsilon_2 > 0$, then $\Delta\alpha$ will really be negative throughout the disturbed motion, It has been shown as positive in Fig. 3b, merely for facilitating the derivation,

As a is considered as constant, we may also write :

$$\Delta\alpha = (C_L - C_{L2})/a . \quad (2.6.9)$$

We now introduce (2.6.5-9) into (2.6.3,4) and obtain, respectively, after some re-arrangement and simplification :

$$C_L\left(\frac{V^2}{V_2^2} + \frac{C_{D2}}{a}\right) = C_{L2}\left(1 + \frac{C_{D2}}{a}\right) ; \quad (2.6.10)$$

$$C_{D2} - C_D\frac{V^2}{V_2^2} = \frac{C_{L2} + C_{D2}\varphi_2}{g}\frac{dV}{dt} . \quad (2.6.11)$$

These two equations are still quite rigorous, but now a welcome simplification suggests itself, *viz.*, neglecting C_{D2}/a in (2.6.10) as small in comparison with 1 or to V^2/V_2^2 and for consistency if not for any clear analytical advantage, neglecting $C_{D2}\varphi_2$ in (2.6.11) as small in comparison with C_{L2}. We then get, as a still very good approximation :

$$C_L V^2 = C_{L2}V_2^2 , \quad (2.6.12)$$

$$C_{D2}V_2^2 - C_D V^2 = C_{L2}\frac{V_2^2}{g}\frac{dV}{dt} . \quad (2.6.13)$$

It may be noticed that the equation (2.6.12) simply means $L = L_2$, which would, of course, apply in strictly horizontal flight if neglecting small vertical components of the thrust. The equation (2.6.13) similarly means that the drag difference produces the acceleration, a small effect of thrust inclination being again neglected. An important observation is that the equations (2.6.10,11) or (2.6.12,13) can be linearised by assuming the variation of V, C_L and C_D as limited to a small range, *i.e.*, by putting :

$$V = V_2(1 + \hat{u}) , \quad C_L = C_{L2} + a\hat{w} , \quad C_D = C_{D2} + \frac{dC_D}{d\alpha}\hat{w} , \quad (2.6.14)$$

and considering $\hat{u}$, $\hat{w}$ as small, and a and $dC_D/d\alpha$ as constants. The equations (2.6.12,13) then reduce to the form :

$$\left.\begin{aligned} \left(\frac{d}{d\tau} + C_{D2}\right)\hat{u} + \tfrac{1}{2}\frac{dC_D}{d\alpha}\hat{w} &= 0 \\ C_{L2}\,\hat{u} \qquad + \tfrac{1}{2}a\hat{w} &= 0 \end{aligned}\right\} , \quad (2.6.15)$$

thus becoming identical with (2.5.5), except for the small correction term C_D in the expression for z_w (*see* 2.4.3) which, of course, is due to the small variations of the vertical thrust components. If a similar procedure is applied to the equations (2.6.10,11), then this correction term re-appears, along with another correction term $C_{D2}\varphi_2$ which is due to the initial thrust inclination (this refinement has never been introduced in the general linearised equations (2.4.1), and it has really very little significance). All these corrections are normally quite small and can certainly be neglected here. We therefore use the simpler system (2.6.12,13), and eliminate C_L and C_D from it by taking into account the relationship :

$$C_D = C_{D0} + sC_L^2 \text{ (in particular, } C_{D2} = C_{D0} + sC_{L2}^2\text{)}, \quad (2.6.16)$$

i.e., the parabolic approximation of the polar curve. We obtain :

$$C_{L2}\frac{V_2^2}{g}\frac{dV}{dt} = C_{D2}V_2^2 - C_{D0}V^2 - \frac{sC_{L2}^2V_2^4}{V^2} , \quad (2.6.17)$$

or, eliminating C_{D2} :

$$C_{L2}\frac{V_2^2}{g}\frac{dV}{dt} = \frac{(V^2 - V_2^2)(sC_{L2}^2V_2^2 - C_{D0}V^2)}{V^2} . \quad (2.6.18)$$

We have, however, from (2.3.7, 9), with $C_{AS} = 0$:

$$sC_{L2} = C_{D0}/C_{L1}\,, \qquad (2.6.19)$$

where C_{L1} corresponds to the alternative equilibrium speed V_1 (point 1 in Fig. 2) and, in view of (2.6.12), is given by :

$$C_{L1} = C_{L2}V_2^2/V_1^2. \qquad (2.6.20)$$

Substituting (2.6.19, 20), the equation (2.6.18) finally becomes :

$$\frac{C_{L2}}{C_{D0}}\frac{dV}{g\,dt} = \frac{(V^2 - V_2^2)(V_1^2 - V^2)}{V_2^2V^2}. \qquad (2.6.21)$$

It is seen that the acceleration becomes 0 at $V = V_2$ and $V = V_1$, i.e., at the two possible equilibrium speeds. Therefore, it must reach a maximum value somewhere in between. It is easily found by differentiating (2.6.21) that this happens at the speed :

$$V_m = \sqrt{(V_1V_2)}\,, \quad \text{or} \quad C_{Lm} = \sqrt{(C_{L1}C_{L2})}\,, \qquad (2.6.22)$$

i.e., as could be anticipated, at the minimum drag point (m in Fig. 2), and the maximum acceleration is :

$$\left(\frac{dV}{dt}\right)_{\max} = g\,\frac{C_{D0}}{C_{L2}}\left(\frac{V_1}{V_2} - 1\right)^2 \qquad (2.6.23)$$

Taking, for instance, the first numerical example of section 2.3 (the constant thrust case), we have $C_{D0} = 0{\cdot}009$, $C_{L2} = 1{\cdot}2$, $C_{L1} = 0{\cdot}075$, and hence :

$$(dV/dt)_{\max} = 0{\cdot}0675g = 2{\cdot}17 \text{ ft/sec}^2\,.$$

The accelerations thus have reasonable small values. An illustrative graph of acceleration versus speed, according to (2.6.21) is given in Fig. 4, where it was assumed $V_1/V_2 = 4$, thus $C_{L2}/C_{L1} = 16$, as in the above example. The vertical scale on the left refers to the non-dimensional expression in (2.6.21), that on the right gives the values of the acceleration in ft/sec² for the particular example considered. There are also two self-explanatory horizontal scales. For any alternative value of the ratio V_1/V_2 a different curve should be drawn, but all curves will be very similar although with varying quantitative characteristics.

It remains to integrate the differential equation (2.6.21). Separating the variables, we obtain :

$$\frac{C_{D0}\,g\,dt}{C_{L2}\,V_2^2} = \frac{V^2\,dV}{(V^2 - V_2^2)(V_1^2 - V^2)}\,, \qquad (2.6.24)$$

and it is obvious that the limits of integration for V should be $V_2(1 + \varepsilon_2)$ and $V_1(1 - \varepsilon_1)$, where a small fraction ε_2 denotes the initial error (excess) of speed over the original equilibrium value, and ε_1 similarly the ultimate error (deficiency) of speed below the new equilibrium value. Integrating between these two limits we find easily :

$$\frac{C_{D0}\,gt_u}{C_{L2}\,V_2} = \frac{V_2^2}{2(V_1^2 - V_2^2)}\left|\ln\frac{V - V_2}{V + V_2} + \frac{V_1}{V_2}\ln\frac{V_1 + V}{V_1 - V}\right|_{V_2(1+\varepsilon_2)}^{V_1(1-\varepsilon_1)}$$

$$= \frac{V_2^2}{2(V_1^2 - V_2^2)}\left[\ln\left(\frac{V_1 - V_2 - V_1\varepsilon_1}{V_1 + V_2 - V_1\varepsilon_1}\cdot\frac{2 + \varepsilon_2}{\varepsilon_2}\right)\right.$$

$$\left.+\frac{V_1}{V_2}\ln\left(\frac{V_1 - V_2 - V_2\varepsilon_2}{V_1 + V_2 + V_2\varepsilon_2}\cdot\frac{2 - \varepsilon_1}{\varepsilon_1}\right)\right]. \qquad (2.6.25)$$

It will suffice to assume that the initial and ultimate per cent errors are equal:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon \text{ , say ,} \qquad (2.6.26)$$

and a further simplification is achieved by expanding in ε, and neglecting second and higher powers:

$$t_u = \frac{C_{L2}V_2}{2C_{D0}g(V_1/V_2 - 1)}\left[\ln\left(\frac{2}{\varepsilon}\cdot\frac{V_1 - V_2}{V_1 + V_2}\right) - \frac{V_1 + V_2}{V_1 - V_2}\cdot\frac{\varepsilon}{2}\right]. \qquad (2.6.27)$$

For reasonable values of V_1/V_2 and ε, the second term in bracket may normally be neglected in comparison with the large logarithmic term. Using, in addition, the relationships (*see* 2.4.3):

$$C_{L2}V_2 = 2g\hat{t}_2 \text{ ,} \quad V_1/V_2 = \hat{t}_2/\hat{t}_1 \text{ ,} \qquad (2.6.28)$$

where $\hat{t}_1$ and $\hat{t}_2$ are units of aerodynamic time for two equilibrium conditions, respectively, we may bring (2.6.27) to the form:

$$t_u = \frac{1}{C_{D0}}\cdot\frac{\hat{t}_1\hat{t}_2}{\hat{t}_2 - \hat{t}_1}\ln\left(\frac{2}{\varepsilon}\cdot\frac{\hat{t}_2 - \hat{t}_1}{\hat{t}_2 + \hat{t}_1}\right). \qquad (2.6.29)$$

It is seen that t_u has been expressed in terms of these two units, minimum drag coefficient, and speed error ratio.

In our previous example $C_{D0} = 0{\cdot}009$, $C_{L2} = 1{\cdot}2$, $V_2 = 152$ ft/sec, $V_1/V_2 = 4$, and hence the formula (2.6.27) gives:

$$\hat{t}_u = 104{\cdot}9\left(\ln\frac{1{\cdot}2}{\varepsilon} - \frac{5}{6}\varepsilon\right),$$

and hence

for ε (per cent)	=	5	4	3	2	1	0·5	0·1
we have t_u (sec)	=	329	353	384	428	501	574	744

The units of aerodynamic time are, in this case, $\hat{t}_1 = 0{\cdot}708$ sec, $\hat{t}_2 = 2{\cdot}832$ sec and, for $\varepsilon = 0{\cdot}05$, we obtain from (2.6.29) $t_u = 333$ sec, with only 4 sec error. The exact formula (2.6.25) gives again $t_u = 329$ sec. For smaller ε, the errors by neglecting the correction term will be even smaller.

The values of t_u are seen to be between 5 and 13 minutes in this example. It is interesting to compare this with the results of flight tests at comparatively low speed and great height, made by the English Electric Co. Ltd., as described in Ref. 15 (last paragraph of the Addendum).

It should be mentioned that, in the second alternative, divergence towards the stall, the time needed to reach stalling conditions may be much shorter than that calculated above. A calculation in this case would require some awkward analytical approximation for the polar curve in the stalling region, and would be rather complicated. However, the general formula for the acceleration, easily obtained from (2.6.12,13).

$$\frac{dV}{dt} = g\left(\frac{C_{D2}}{C_{L2}} - \frac{C_D}{C_L}\right) \qquad (2.6.30)$$

shows that, in this case, the acceleration would be increasingly negative, never becoming 0 again, and more or less of the same order of magnitude as in the previous case. However, the difference between the initial speed V_2 and the stalling speed V_s will be generally small (in our numerical example $V_2 = 152$ ft/sec, $V_s \simeq 140$ ft/sec), and therefore the time required would also be much smaller, nearer to the order of 1 minute, or even less. It is seen that, trying to keep constant height at high incidence may lead to a stall quite rapidly, if the initial error of speed is negative.

3. *Modified Painlevé's Criterion for Ultimate Equilibrium after Elevator Deflection.*—There exists another problem, of considerable practical importance, for which a criterion very similar to Painlevé's has been known for a long time. The problem was much discussed in the early period of aviation, for instance by von Mises[3] (1915), Fuchs and Hopf[4] (1922), and others. It has been re-examined by von Mises in 1945 (Ref. 6, *see* especially Section XIV.4). A usual qualitative picture, based on graphs such as in our Fig. 1, can be described briefly as follows.

Suppose that an aircraft is flying level at one of the two equilibrium speeds compatible with a given throttle position (points 1 or 2), and then the elevator is deflected through a small angle $\eta_\times$, say positive (elevator down), so as to induce a small negative (nose-down) pitching moment. A complicated disturbance will follow, including both short-period and phugoid oscillations but, if the aircraft is stable and the elevator is kept deflected, a new steady flight will be eventually reached, with a somewhat lower incidence than before. The new steady flight path will not be horizontal, however, but slightly inclined. The new (reduced) incidence must be such that the ensuing positive moment just counterbalances that produced by the elevator deflection, and the new speed must be somewhat greater than before, so that the lift may still balance the weight. Fig. 1 then shows that, if the initial conditions correspond to the point 1, the thrust required will exceed the available one, and the deficiency must be made up by a small positive component of the weight in the direction of the new path, the latter necessarily pointing slightly downwards. The aircraft will therefore lose height steadily which is an expected response to a push-down manouvre. If, however, the initial conditions correspond to the point 2 then, after a similar elevator movement, the thrust required will fall below the available one, the excess will be able to compensate an adverse weight component, and hence the new steady path will point slightly upwards. The ultimate response will consist in the aircraft gaining height steadily (against expectation). It must be emphasized that the initial response was certainly conformable to the pilot's intention—this was often overlooked by earlier writers. The curious phenomenon begins by a momentary descent, followed by a hesitant oscillation, and finally resolves itself into a steady climb (' ballooning ' in old pilots' slang).

An exactly similar reasoning applies, *mutatis mutandis*, to the inverse case of small negative $\eta_\times$ (elevator up), followed by an increased incidence and reduced speed. If the initial steady flight corresponds to the point 1, then both the immediate and ultimate response involve climbing. If, however, the point 2 represents the original flight conditions, then a momentary climb, through a hesitant oscillation, finally leads to ' sinking '.

In either case ($\eta_\times \gtrless 0$), the point of minimum drag m in Figs. 1 and 2 (or, according to the properties of the propulsive system, that of minimum power m') clearly separates two regions displaying different behaviour. If the critical point itself depicts the original steady flight, then both up-and-down elevator displacements, supposed really small, eventually lead to another steady level flight, with slightly increased (decreased) incidence and decreased (increased) speed. An elevator deflection of appreciable magnitude will, however, be followed by an ultimate slow loss of height.

Although the above descriptive reasoning does not account for all details of the phenomenon (such as, *e.g.*, the speed effect on pitching moment, represented by the derivative m_u), yet no fundamental error can be detected here. Flight experience, too, amply confirms the conclusions. The matter seems never to have been examined analytically in any detail, but this can be done quite easily in the following way. The motion following a small elevator deflection $\eta_\times$ will still be governed by the linearised equations of motion (2.4.2), the only alteration being that the third equation must now be written :

$$\varkappa\hat{u} + (\chi D + \omega)\hat{w} + (D + \nu)\hat{q} = -\,\delta\eta_\times, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

where δ, as defined by (2.6.3), is a measure of elevator effectiveness. A complete solution of the modified system would provide a picture of the entire response, which it is not proposed to attempt here. The ultimate response, however, can be determined quite simply by assuming that a new equilibrium status has been reached (theoretically at $t \to \infty$), so that $\hat{u}$, $\hat{w}$ and θ have become constant, and $\hat{q} = 0$. All terms containing the differential operator D now disappear, and (2.4.2) reduces to a simple system of three simultaneous algebraic equations of the first degree :

$$\left.\begin{aligned} -x_u\hat{u} - x_w\hat{w} + \tfrac{1}{2}C_L\theta &= 0 \\ -z_u\hat{u} - z_w\hat{w} \qquad\quad &= 0 \\ \varkappa\hat{u} + \omega\hat{w} \qquad\quad &= -\delta\eta_\times \end{aligned}\right\} \qquad (3.2)$$

The solution of this system is :

$$\frac{\hat{u}}{\delta\eta_\times} = -\frac{C_L z_w}{2E_1}, \quad \frac{\hat{w}}{\delta\eta_\times} = \frac{C_L z_u}{2E_1}, \quad \frac{\theta}{\delta\eta_\times} = -\frac{P_1}{E_1}, \quad (\text{at } \tau \to \infty) \qquad (3.3)$$

where the auxiliary symbols E_1, P_1 are as in (2.4.7,8). It should be remembered now that z_u and z_w are normally negative, and P_1 positive. It is seen at once that, if the static margin is positive ($E_1 > 0$) then, for a positive $\eta_\times$, the ultimate value of $\hat{u}$ is positive, and those of $\hat{w}$ and θ negative. Thus, the speed is finally increased, the incidence decreased, and the nose brought down after a push-down manoeuvre, and *vice versa* after a pull-up, irrespective of the initial flight conditions (below the stall). This is a simple analytical confirmation of the previous general reasoning, with one additional piece of information about the attitude which, although intuitively plausible, would not be easily demonstrated without algebra.

The only really interesting further conclusion refers to the ultimate value of the vertical velocity component. This, according to (2.6.1) and (3.3), becomes :

$$\frac{dH}{dt} = V(\theta - \hat{w}) = -\frac{\delta\eta_\times V(kz_u + P_1)}{E_1}$$

$$= -\frac{\delta\eta_\times V}{2E_1}\left\{(a + C_D)(C_D + C_{AS}) - aC_L\frac{dC_D}{dC_L}\right\}. \qquad (3.4)$$

The expression in brackets is seen to be exactly proportional to the value of $2r$ in (2.6.9) and changes sign with it. Hence, Painlevé's dubious criterion for stability now re-appears as a generally applicable firm *criterion for the elevator-height response.* This may be formulated briefly as follows :

The ultimate height response of an aircraft with positive static margin, after the elevator has been deflected and kept so, occurs in the naturally desired direction, i.e., in agreement with the initial response, if the original speed exceeds that corresponding to minimum drag (or minimum power) conditions. At lower speeds, the opposite is true. This obviously does not depend on the rate and mode of the elevator's initial movement, provided it is ultimately kept deflected and fixed.

A more detailed numerical analysis shows that the expression (3.4) assumes, on the average, greater negative values at high speeds than positive ones at low speeds. The curious phenomenon of reversed ultimate response at low speeds will, therefore, have a rather mild form generally, but even lack of the normal response, let alone reversal, is unwelcome and disconcerting. The matter may, however, become even more serious in the case of aircraft with exceptionally low drag (*cf.* end of section 2.5), for which a very large part of the polar curve belongs to the region of erratic response. If such aircraft become an engineering reality then the reversed height response will dominate over the greater part of the speed range. It is true that this feature is always present in the final stages of landing of orthodox aircraft, but then the pilot's movements occur in quick succession, and there is little chance for the adverse effect to be fully operative. In cruising flight, there has always been a strong tendency to maintain the speed above the critical (hence the notion of 'minimum comfortable cruising speed'—*see* Refs. 12, 13 and 17). For aircraft with distributed suction, this minimum comfortable cruising speed may become very large indeed, and the matter deserves attention.

4. *General Theory of Stability Under Constraint, and Alternative Examples.*—4.1. *Principles of the Theory.*—The reasoning of section 2.5 has led to a new concept of stability under partial control, where a control is deliberately operated in such a way as to keep one chosen element of the disturbance equal to 0 (or, more generally, constant). In other words, the control is applied so as to remove one or more degrees of freedom. In sections 2.5 and 2.6, only the case of the vertical velocity component being kept = 0 (or height being kept constant) through a suitable elevator movement has been considered, and the results seem to agree well with the available flight test evidence. Now, it is obvious that this is not the only case worth studying, and that a considerable number of analogous problems may be considered, some of them presenting a definite practical interest. The general idea remains exactly the same in all cases, *i.e.*, it is assumed that one of the controls is operated so as to keep one of the usual variables or a linear combination of them, equal to 0 (or constant), while other elements of the disturbance are still free to vary according to the (suitably modified) system of dynamic equations. The modification of the system will consist in one equation (that directly affected by the control in operation) being removed, while the remaining equations simplify in a manner consistent with the assumption. The simplification will usually be very considerable, the order dropping by at least one degree but often by more than that, and the solutions will often be surprisingly easy and simply interpreted. The omitted equation can be used afterwards to find '*a posteriori*' what the control movement must have been to produce the result as anticipated '*a priori*'. The practical value of such an investigation obviously depends on whether the particular case is likely to occur often in flight, at least approximately. We shall use the term '*stability under constraint*' (or shorter 'constrained stability') in such problems, to distinguish it from the stability in the generally accepted sense (*i.e.*, in uncontrolled flight).

The notion of constrained stability may seen unfamiliar and different from the general idea of stability of mechanical systems, as universally adopted in theoretical dynamics and engineering practice. It seems never to have been applied to aircraft nor, as far as is known to the present author, to any other vehicles or mechanisms*. The new concept may be subject to criticism, and the following arguments in its favour should help to dispel the doubts :

(*a*) In each particular case, a system of dynamic equations equivalent to that defined above may be obtained by assuming that the relevant control is actuated by a simple *autopilot* producing a control deflection proportional to the change of the chosen disturbance element, assuming that the coefficient of proportionality (autopilot's strength) tends to infinity. This assumption is virtually equivalent to putting the respective disturbance element equal to zero right from the start, and hence the solutions must be identical in both cases. A true autopilot, of course, never has an infinite strength but, if the strength is sufficiently large, the stability characteristics differ little from the limiting ones. That is why, when dealing with autopilots of variable strength and plotting frequencies and damping factors against strength, one usually considers also the limiting case which is illustrated by asymptotes in stability diagrams[8, 16]. It is quite possible for a human pilot to act similarly to an autopilot of considerable strength.

(*b*) In fact, a human pilot *may do better* than this. Instead of operating the control according to a definite rigid function or the response of a detecting instrument (as an autopilot does), he operates, to quote Hopkin and Dunn[16], 'according to some personal and variable function of the response of his senses (directly or through instruments, or both)'. In such a way, he may keep the relevant disturbance element permanently suppressed in a more efficient way than the strongest autopilot, not only countering every small deviation, but also anticipating and forestalling the next deviation in the opposite direction. That this is really possible, can be seen by observing how a cyclist or motorist is able to follow a prescribed path with a great accuracy by almost imperceptible movements of his handlebar or steering wheel, irrespective of accidental

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*However, a somewhat similar concept was used by Mitchell, Thorpe and Frayne[10], 1945, who introduced, as a standard of comparison for various cases of lateral response, the response to aileron deflection with sideslip suppressed by an appropriate rudder action. This may be called a 'restrained response.' A theoretical stability calculation, given recently by Lean[17] in connection with his analysis of deck landing, leads to the same results as those obtained in section 2.5 of this paper, and some ideas of 'restrained stability' may be discovered there.

concussions. Although an aircraft, with its many degrees of freedom, may be not quite as docile, it should still be managed easily enough for the theory of restrained stability to apply with a reasonable accuracy. This is certainly so, *e.g.*, in the case when the pilot tries to maintain a fixed horizon position on his windscreen, and similarly in many other cases. The accuracy will undoubtedly be better if the control used is one of the main control surfaces which act directly and almost instantaneously, and poorer when using throttles, trimming tabs, etc., which act indirectly and with more lag. The matter may and should be investigated experimentally.

(*c*) In each case, a short transitory period following a more or less violent initial disturbance is not subject to our theory, as mentioned already in section 2.5. This period is usually short, however, especially for an able and experienced pilot, and very soon the conditions envisaged by the theory will be reached and maintained.

Before proceeding to examine some more particular cases of interest, we shall give the following discussion of constrained stability in general terms, on lines suggested by S. B. Gates :

Let $x_1, x_2, x_3, \ldots$ be small increments, from their equilibrium values, of the dynamical co-ordinates of a system whose stability is to be examined. And let the linearised equations of uncontrolled motion be :

$$\left.\begin{array}{l} f_1(x_1) + f_2(x_2) + f_3(x_3) \;\ldots\ldots\ldots = 0 \\ g_1(x_1) + g_2(x_2) + g_3(x_3) \;\ldots\ldots\ldots = 0 \\ h_1(x_1) + h_2(x_2) + h_3(x_3) \;\ldots\ldots\ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_1(x_1) + F_2(x_2) + F_3(x_3) \;\ldots\ldots\ldots = 0 \end{array}\right\} \qquad (4.1.1)$$

where the symbols $f, g, h, \ldots F$ represent linear homogeneous functions of the respective variables and their time derivatives, the number of equations being, of course, equal to that of the variables. Suppose the last equation is the only one that is affected when the system becomes subject to a certain external control whose deflection will be denoted by η. If this control is operated, the last equation is modified by the inclusion of the control term on the right-hand side :

$$F_1(x_1) + F_2(x_2) + F_3(x_3) + \ldots\ldots = \delta\eta, \qquad (4.1.2)$$

where δ is a constant coefficient. In general, there may be an arbitrary initial disturbance represented by the values $x_{10}, x_{20}, x_{30}, \ldots$ at $t = 0$.

Suppose now that human operator is instructed to actuate his control in such a way as to bring one of the variables, say x_1, down to zero, and thereafter maintain it permanently suppressed. His instruction is merely to observe an instrument measuring x_1 and to try to bring its reading to zero and keep it there, as best he can. Accordingly, his η will really be related to the observed x_1, however small. Symbolically, we may write :

$$\eta = H(x_1), \qquad (4.1.3)$$

where H is an undetermined function dependent on the psychological features of the operator. The full system of equations governing the motion is then :

$$\left.\begin{array}{l} f_1(x_1) + f_2(x_2) + f_3(x_3) \;\ldots\ldots\ldots = 0 \\ g_1(x_1) + g_2(x_2) + g_3(x_3) \;\ldots\ldots\ldots = 0 \\ h_1(x_1) + h_2(x_2) + h_3(x_3) \;\ldots\ldots\ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_1(x_1) + F_2(x_2) + F_3(x_3) \;\ldots\ldots = \delta\eta \\ \eta = H(x_1) \end{array}\right\} \qquad (4.1.4)$$

This is merely a formal presentation of what happens and, in view of the indeterminate nature of the function H, it is not suggested that any analytical procedure may be applied to solve the system (4.1.4). However, we may assume that the operator is so successful in keeping the x_1 reading very small (at least, after a lapse of time short compared with the duration of the entire disturbed motion) that the terms $f_1(x_1)$, $g_1(x_1)$, $h_1(x_1)$, . . . $F_1(x_1)$ have become small in comparison with the remaining terms of the equations. The motion will then be governed, with sufficient approximation, by the simpler system of equations :

$$\left.\begin{array}{r} x_1 = 0 \\ f_2(x_2) + f_3(x_3) \;\ldots\ldots = 0 \\ g_2(x_2) + g_3(x_3) \;\ldots\ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots \quad (4.1.5)$$

with the control equation :

$$F_2(x_2) + F_3(x_3) \ldots\ldots = \delta\eta. \quad \ldots \quad (4.1.6)$$

The concept of restrained stability thus consists in using the system of equations (4.1.5) instead of (4.1.1). Equation (4.1.6) plays no part in the solution of the stability problem. To make its meaning quite clear we should consider what would happen in practice. The η supplied by the operator would still depend on x_1 (however small) according to (4.1.3), whilst (4.1.6) corresponds to the idealised (humanly impossible) case of x_1 being strictly 0 throughout.

It may be noted that, if the motion represented by (4.1.5) happens to be stable, then η ultimately tends to 0. If, however, the motion is unstable, η will diverge (or oscillate with increasing amplitude), just as other variables x_2, x_3, ...

In the particular case of section 2.5, the quantity suppressed was not one of the dynamical co-ordinates but a linear combination of two of them, *viz.* $(\theta - \hat{w})$, but clearly a mere change of variables brings the analysis into line with the above general scheme.

With the above arguments and restrictions in mind, a few interesting particular cases will be examined in the following sections 4.2, 4.3, 4.4 and 4.5, the range falling far short of exhausting all the many possibilities.

4.2. *Stability at Constant Attitude Under Elevator Control.*—This is the case when the aircraft is not allowed to rotate in pitch, and thus defined by the assumption :

$$\theta = 0 \quad , \quad \text{hence } \hat{q} = 0, \quad \ldots \quad (4.2.1)$$

which means that the longitudinal attitude remains fixed relative to the ground, so that *the horizon maintains a constant position on the windscreen.* The system of equations (2.4.1) becomes :

$$\left.\begin{array}{r} (D - x_u)\hat{u} \qquad - x_w\hat{w} = 0 \\ - z_u\hat{u} + (D - z_w)\hat{w} = 0 \end{array}\right\}, \quad \ldots \quad (4.2.2)$$

i.e., is reduced to the second order. The determinantal equation becomes a quadratic :

$$\lambda^2 + N_1\lambda + P_1 = 0, \text{ hence } \lambda_{1,2} = -\frac{N_1}{2} \mp \sqrt{\left\{\frac{N_1^2}{4} - P_1\right\}}, \quad \ldots \quad (4.2.3)$$

(*cf.* formulae 2.4.7), and identical with the analogous equation for an aircraft with autopilot of θ-type, of infinite strength, as given in Ref. 8 (section 5.1). Except for conditions at or beyond the stall, or sometimes in the transonic range, the coefficients N_1, P_1 are always positive, so that the motion is fundamentally stable. In addition, we have normally :

$$P_1 << \tfrac{1}{4}N_1^2, \quad \ldots \quad (4.2.4)$$

so that the two stability roots are real and well approximated by :

$$\lambda_1 \simeq - N_1 \quad , \quad \lambda_2 \simeq - P_1/N_1 \, . \qquad \ldots \qquad (4.2.5)$$

The exact complete solutions of the system (4.2.2), assuming the initial values of $\hat{u}$, $\hat{w}$ to be $\hat{u}_0$, $\hat{w}_0$, are easily found :

$$\left.\begin{aligned} \hat{u} &= \hat{u}_0 e^{\lambda_2 \tau} + \frac{x_w \hat{w}_0 + (x_u - \lambda_2)\hat{u}_0}{\lambda_2 - \lambda_1} (e^{\lambda_2 \tau} - e^{\lambda_1 \tau}) \\ \hat{w} &= \hat{w}_0 e^{\lambda_1 \tau} - \frac{x_u - \lambda_2}{x_w} \frac{x_w \hat{w}_0 + (x_u - \lambda_1)\hat{u}_0}{\lambda_2 - \lambda_1} (e^{\lambda_2 \tau} - e^{\lambda_1 \tau}) \end{aligned}\right\} \qquad \ldots \qquad (4.2.6)$$

These formulae may be simplified, using the approximation (4.2.5) and the expressions (2.4.3), and neglecting x_u when accompanying a, and x_u^2 when accompanying C_L^2. We then obtain :

$$x_u - \lambda_1 \simeq \lambda_2 - \lambda_1 \simeq \tfrac{1}{2}a \quad , \quad x_w = \frac{1}{2}\left(C_L - \frac{dC_D}{d\alpha}\right) \simeq \frac{a}{2C_L}(x_u - \lambda_2), \qquad \ldots \qquad (4.2.7)$$

and hence

$$\left.\begin{aligned} \hat{u} &= \hat{u}_0 e^{\lambda_2 \tau} - \frac{C_L - dC_D/d\alpha}{a}\left(\hat{w}_0 + \frac{2C_L}{a}\hat{u}_0\right)(e^{\lambda_2 \tau} - e^{\lambda_1 \tau}) \\ \hat{w} &= \hat{w}_0 e^{\lambda_1 \tau} - \frac{2C_L}{a}\left(\frac{C_L - dC_D/d\alpha}{a}\hat{w}_0 + \hat{u}_0\right)(e^{\lambda_2 \tau} - e^{\lambda_1 \tau}) \end{aligned}\right\} \qquad \ldots \qquad (4.2.8)$$

These approximate solutions still satisfy the initial conditions exactly. The root λ_1 being large and negative, and $(C_L - dC_D/d\alpha)$ being very much smaller than a, the disturbance will be well described after a short time by the simple equations :

$$\hat{u} \simeq \hat{u}_0 e^{\lambda_2 \tau} \quad , \quad \hat{w} \simeq - \frac{2C_L}{a}\hat{u}_0 e^{\lambda_2 \tau} \qquad \ldots \qquad (4.2.9)$$

The expressions (4.2.9) satisfy, approximately, the second of the equations (4.2.2), if we neglect the small acceleration term $D\hat{w}$. We have thus :

$$2C_L \hat{u} + a\hat{w} \simeq 0, \qquad \ldots \qquad (4.2.10)$$

which means that, during the later stage, there is practically no incremental lift.

The motion clearly consists of two subsidences, one of which is heavily damped while the other one persists much longer. The mere fact that there are no oscillations (there is only one change of sign of $\hat{u}$ and $\hat{w}$, or none at all) shows sufficiently that this sort of disturbance is much 'calmer' than the usual free phugoid oscillation. In addition, even the smaller damping factor $(-\lambda_2)$ is considerably greater than the usual phugoid damping factor r, as given by (2.4.13). We have, from (4.2.5) and (2.4.7), with a sufficient accuracy :

$$- \lambda_2 \simeq C_D + C_{AS} - C_L \frac{dC_D}{dC_L} + \frac{C_L^2}{a} \qquad \ldots \qquad (4.2.11)$$

Examining Figs. 7a to 7d we see that this value lies between $2r_{as}$ and $2r_{max}$. Hence the damping factor for an aircraft kept 'at fixed horizon' is always positive and normally at least twice that for an aircraft in uncontrolled flight. This seems to agree well, at least qualitatively, with flight experience.

The formula (4.2.11) may be represented in a modified form (assuming for simplicity $C_{AS} = 0$). The resultant aerodynamic force in disturbed flight can be resolved into *components along the z and x axes reversed,* instead of L and D. These may be called 'normal' and 'tangential', respectively, and the corresponding dimensionless coefficients denoted by C_n and C_t (Fig. 5).

We have then :

$$\left.\begin{aligned} C_n &= C_L \cos \hat{w} + C_D \sin \hat{w} \simeq C_L + C_D \hat{w} \\ C_t &= C_D \cos \hat{w} - C_L \sin \hat{w} \simeq C_D - C_L \hat{w} \end{aligned}\right\} , \quad \ldots \quad (4.2.12)$$

hence

$$\frac{dC_n}{d\alpha} = a + C_D \simeq a \quad , \quad \frac{dC_t}{d\alpha} = \frac{dC_D}{d\alpha} - C_L \ , \quad \ldots \quad (4.2.13)$$

and accordingly, for $\hat{w} \to 0$:

$$\frac{C_t}{a}\frac{dC_n}{d\alpha} - \frac{C_n}{a}\frac{dC_t}{d\alpha} = -\lambda_2 \ . \quad \ldots \quad (4.2.14)$$

The condition for stability ($\lambda_2 < 0$)then becomes :

$$\frac{C_t}{C_n} - \frac{dC_t}{dC_n} > 0, \quad \ldots \quad (4.2.15)$$

which presents an interesting analogy to (2.1.7). We find an analogous graphical illustration, introducing a C_n vs. C_t '*polar curve*', instead of the usual C_L vs. C_D polar. It must be realised however, that only one C_L vs. C_D curve exists for a given aircraft, while there may be as many C_n vs. C_t curves as alternative initial equilibrium conditions. The original polar curve being given, we may draw, through arbitrary points of it, the corresponding C_n vs. C_t curves. This is shown in Fig. 6 where the original polar curve is the same as in Fig. 2. It is seen that the slope dC_t/dC_n is usually negative, and certainly smaller than C_t/C_n, throughout the normal incidence range. The inequality (4.2.15) is therefore always satisfied.

4.3. *Stability at Constant Speed Under Elevator Control.*—Suppose now that the pilot operates his elevator so as to maintain constant speed, so that :

$$\hat{u} = 0 \ . \quad \ldots \quad (4.3.1)$$

The system of equations (2.4.2) becomes :

$$\left.\begin{aligned} -x_w \hat{w} + \tfrac{1}{2} C_L \theta &= 0 \\ (D - z_w)\hat{w} - D\theta &= 0 \end{aligned}\right\} , \quad \ldots \quad (4.3.2)$$

i.e., is reduced to the first order. The only stability root is (*see* 2.4.4) :

$$\lambda = \frac{z_w C_L}{C_L - 2x_w} = -\tfrac{1}{2} C_L \frac{1 + C_D/a}{dC_D/dC_L} \quad \ldots \quad (4.3.3)$$

or, introducing the symbol s from (2.5.7) :

$$\underline{\lambda = -\frac{1 + C_D/a}{4s}} \text{ or, with sufficient accuracy, } \lambda = -1/4s. \quad \ldots \quad (4.3.4)$$

It is seen that the stability root is always negative, and has generally surprisingly high values. If, for instance, $s = 0{\cdot}1$, as in Fig. 2, then $\lambda = -2{\cdot}5$. The parameter s, being a measure of the induced drag, decreases considerably when the aspect ratio increases. We have therefore to deal with a strong subsidence, especially rapid for aircraft with large aspect ratio.

The exact complete solutions of the system (4.3.2), assuming the initial error in incidence to be $\hat{w}_0$, are easily found :

$$\hat{w} = \hat{w}_0 e^{\lambda\tau} \quad , \quad \theta = (1 - 2as)\hat{w}_0 e^{\lambda\tau} \ . \quad \ldots \quad (4.3.5)$$

It may be interesting to find the vertical velocity component (*see* 2.6.1) :

$$dH/dt = V(\theta - \hat{w}) = -2asV\hat{w}_0 e^{\lambda t/\hat{t}} , \quad \ldots \quad (4.3.6)$$

whence the gain of height, after a time t, is :

$$\Delta H = \frac{2asVt\hat{w}_0}{\lambda}(1 - e^{\lambda\tau}) \quad .. \quad (4.3.7)$$

The exponential term decreases rapidly, and the ultimate gain or loss of height, practically reached after quite a short time, becomes simply (using 4.3.4 and 2.4.3) :

$$(\Delta H)_{t\to\infty} = -8as^2Vt\hat{w}_0 = -\frac{4as^2V^2C_L}{g}\hat{w}_0 \quad .. \quad (4.3.8)$$

This expression assumes very small values in normal cases. Let us assume, for instance :

$$a = 4\,,\ s = 0{\cdot}1\,,\ V = 400\ \text{ft/sec},\ C_L = 1\,,\ g = 32\ \text{ft/sec}^2, \quad .. \quad (4.3.9)$$

then :

$$(\Delta H)_{t\to\infty} = -800\hat{w}_0,$$

where $\hat{w}_0$ is measured in radians. Hence, we obtain :

$$(\Delta H)_{t\to\infty} = -14\ \text{ft per one degree initial error in incidence.}$$

An interesting conclusion is that, if the pilot keeps the speed constant by means of his elevator, the height remains practically constant with a very good accuracy.

4.4. *Some Examples of Restrained Stability Under Throttle Control.*—The throttle control is generally less efficient than that by the elevator, because of unavoidable lag and rather sluggish response. It is also undesirable to juggle continuously with the engine controls (*cf.* Ref. 13). With these reservations, it may still be interesting to investigate a few examples.

(*a*) Assuming first that the *speed is maintained constant*, we must omit the first of the equations (2.4.2), and the terms with $\hat{u}$ in the remaining ones. We then obtain, replacing z_w by $(-\frac{1}{2}a)$:

$$\left.\begin{aligned}(D + \tfrac{1}{2}a)\hat{w} \qquad\qquad - \hat{q} &= 0\\ (\chi D + \omega)\hat{w} + (D + \nu)\hat{q} &= 0\end{aligned}\right\}, \quad .. \quad (4.4.1)$$

and this is, of course, the familiar system of equations of the pure short-period oscillation, with the stability quadratic :

$$\underline{\lambda^2 + (\nu + \chi + \tfrac{1}{2}a)\lambda + (\omega + \tfrac{1}{2}a\nu) = 0} \quad .. \quad (4.4.2)$$

(*cf.*, for instance, Ref. 18). Maintaining the speed constant obviously eliminates the phugoid oscillation. The short-period oscillation is normally well damped, except some peculiar cases in the transonic range.

(*b*) Let us assume next that the *attitude is maintained constant* (' keeping horizon '), *i.e.*, that :

$$\theta = 0, \quad \hat{q} = 0\,, \quad .. \quad (4.4.3)$$

and we must again omit the first of the equations (2.4.2). The remaining equations become :

$$\left.\begin{aligned}-z_u\hat{u} + (D - z_w)\hat{w} &= 0\\ \varkappa\hat{u} + (\chi D + \omega)\hat{w} &= 0\end{aligned}\right\}, \quad .. \quad (4.4.4)$$

the system being of the first order. The only stability root is :

$$\lambda = \frac{z_w\varkappa - z_u\omega}{\varkappa + z_u\chi} \quad .. \quad (4.4.5)$$

or, replacing z_w by $(-\frac{1}{2}a)$, and z_u by $(-C_L)$:

$$\underline{\lambda = \frac{\omega - a\varkappa/2C_L}{\varkappa/C_L - \chi}}\,. \quad .. \quad (4.4.6)$$

The numerator is proportional to the static margin, and will be normally positive. It is therefore necessary for stability that the denominator be negative :

$$\chi > \varkappa/C_L \,. \qquad (4.4.7)$$

We shall have to deal, therefore, with a subsidence if $\varkappa$ is negative, zero, or very small positive, but with a divergence if $\varkappa$ is large and positive. The derivatives $\varkappa$ and χ being usually the most uncertain ones, it is seen that the stability in this case is doubtful.

(*c*) If, finally, the *height is maintained constant*, i.e.:

$$\theta = \hat{w} \quad , \quad \hat{q} = D\hat{w} \qquad (4.4.8)$$

(*see* section 2.6), we obtain :

$$\left.\begin{aligned} -z_u\hat{u} \qquad\qquad\qquad\qquad - z_w\hat{w} &= 0 \\ \varkappa\hat{u} + \{D^2 + (\nu + \chi)D + \omega\}\hat{w} &= 0 \end{aligned}\right\} , \qquad (4.4.9)$$

with the stability quadratic :

$$\underline{\lambda^2 + (\nu + \chi)\lambda + (\omega - a\varkappa/2C_L) = 0 \,.} \qquad (4.4.10)$$

We have obtained a quadratic rather similar to (4.4.2), with the damping factor considerably reduced, and with the constant term (thus frequency) depending on the static margin, instead of the manoeuvre margin. This represents a modified form of short-period oscillations.

LIST OF SYMBOLS

a	Lift curve slope
B_1, C_1, D_1, E_1	Coefficients of stability quartic, *see* (2.4.5)
C_{AS}	Airscrew drag coefficient, *see* (2.2.1) and (2.2.5)
C_D	Drag coefficient
C_{D0}	Value of C_D at zero lift
C_L	Lift coefficient
C_m	Pitching-moment coefficient
C_n	' Normal force ' coefficient, *see* (4.2.12) and Fig. 5
C_t	' Tangential force ' coefficient, *see* (4.2.12) and Fig. 5
c	Wing mean chord
D	Drag, lb
D	Differential operator, *see* (2.4.2)
g	Gravity constant, ft/sec^2
H	Height, ft
H, h	Free terms of quadratic factors of stability quartic, *see* (2.4.11)
i_B	Inertia coefficient, *see* (2.4.2)
K	Constant, *see* (2.2.18)
k_B	Radius of gyration of aircraft about lateral axis, ft

L	Lift, lb
l	Representative length (usually tail arm), ft
m_q	Rotary damping derivative in pitch, dimensionless
m_u	Pitching-moment derivative due to u, dimensionless
m_w	Pitching-moment derivative due to w, dimensionless
$m_{\dot{w}}$	Pitching-moment derivative due to rate of change of w, dimensionless
m_η	Pitching-moment derivative due to elevator displacement, dimensionless
$\left.\begin{matrix} N_1, P_1, Q_1, \\ R_1, S_1, T_1 \end{matrix}\right\}$	Shorthand constants, *see* (2.4.7)
P	Power, lb ft per sec
P_{av}	Power available
P_{re}	Power required
p	Exponent, *see* (2.2.18)
q	Kinetic pressure, lb/sq ft, in sections 2.1, 2.2, 2.3
q	Rate of pitch, in radians per sec, in form (2.4.2.) only
$\hat{q}$	Rate of pitch, dimensionless, *see* (2.4.2)
R	Damping factor of short-period oscillation, dimensionless
$\mathscr{R}$	Routh discriminant, *see* (2.4.9)
r	Damping factor of phugoid oscillation, dimensionless
S	Gross wing area, sq ft
s	*See* (2.3.1)
T	Thrust, lb
T_{av}	Thrust available
T_{re}	Thrust required
T_0	Thrust at zero speed, *see* (2.2.1)
t	Time, sec
t_u	Time needed for an aircraft flying at constant height, disturbed from unsteady equilibrium, to reach steady equilibrium, *see* (2.6.25)
$\hat{t}$	Unit of aerodynamic time, sec, *see* (2.4.2)
u	Longitudinal increment of velocity in disturbed flight, ft/sec
$\hat{u}$	Longitudinal increment of velocity in disturbed flight, dimensionless, *see* (2.4.2)
V	Velocity of aircraft in undisturbed flight, ft/sec
W	Weight of aircraft, lb
w	Normal increment of velocity in disturbed flight, ft/sec
$\hat{w}$	Normal increment of velocity in disturbed flight, dimensionless, *see* (2.4.2)

x_u, x_w	Longitudinal force derivatives due to u or w, dimensionless, *see* (2.4.4)
z_u, z_w	Normal force derivatives due to u or w, dimensionless, *see* (2.4.4)
α	Wing incidence, radians
α_0	Effective incidence in undisturbed flight, *see* (A.7)
δ	Compound pitching-moment derivative due to elevator displacement, *see* (2.5.3)
δ	Rotary damping derivative in von Kármán and Biot's notation, *see* (A.7)
ε, ε_1, ε_2	Small fractions denoting initial or final errors of speed, *see* (2.6.25)
η	Angular displacement of elevator, radians
θ	Angular displacement of aircraft in pitch from equilibrium position, radians
$\varkappa$	Compound pitching-moment derivative due to u, dimensionless, *see* (2.4.5)
μ	Relative density of aircraft, *see* (2.4.3)
ν	Compound rotary damping derivative, dimensionless, *see* (2.4.5)
ρ	Air density, slugs/cu ft
σ	Parameter of static stability in von Kármán and Biot's notation, *see* (A.7)
τ	Aerodynamic time, dimensionless, *see* (2.4.3)
χ	Compound pitching-moment derivative due to rate of change of w, dimensionless, *see* (2.4.5)
ω	Compound pitching-moment derivative due to w, dimensionless, *see* (2.4.5)

REFERENCES

No.	*Author*	*Title, etc.*
1	P. Painlevé	Étude sur le régime normal d'un aéroplane. *La Technique Aeronautique*, Vol. I, pp. 3 to 11. Paris, 1910.
2	G. H. Bryan	*Stability in aviation.* MacMillan and Co., Ltd. London, 1911.
3	R. von Mises	*Fluglehre.* Springer-Verlag, Berlin. 1915.
4	R. Fuchs and L. Hopf	*Aerodynamik.* Springer-Verlag, Berlin. 1922. Second edition, 1934, Vol. I, Ch.V.
5	Th. von Kármán and M. A. Biot	*Mathematical methods in engineering.* McGraw-Hill Book Co., Inc. New York and London, 1940, Ch. VI, Sect. 8, pp. 249 to 255.
6	R. von Mises	*Theory of flight.* McGraw-Hill Book Co., Inc. New York and London, 1945.
7	E. Everling	Der Aufstieg von Flugzeuge. *Z.F.M.*, Vol. 7, p. 124. 1910.
8	S. Neumark	The disturbed longitudinal motion of an uncontrolled aircraft and of an aircraft with automatic control. R. & M. 2078. January, 1943.

REFERENCES—*continued*

No.	Author	Title, etc.
9	L. W. Bryant and G. A. McMillan ..	Further analysis of the experiments on longitudinal stability of a twin-engined monoplane with airscrews running. R. & M. 2310. 1940.
10	K. Mitchell, A. W. Thorpe and E. M. Frayne	The theoretical response of a high speed aeroplane to the application of ailerons and rudder. R. & M. 2294. May, 1945.
11	S. Neumark	Dynamic longitudinal stability in level flight, including the effects of compressibility and variation of atmospheric characteristics with height. R.A.E. Report Aero. 2265. A.R.C. 11726. May, 1948.
12	D. Cameron	The minimum power and speed for continuous cruising. A. & A.E.E. Report Res/163. A.R.C. 5743. January, 1942.
13	English Electric Co., Ltd.	Minimum comfortable cruising speed. Aerodynamic Tech. Note Ae.26. January, 1948, with an Addendum, August, 1951.
14	M. R. Head	The boundary layer with distributed suction. R. & M. 2783. April, 1951.
15	M. Jones and M. R. Head	The reduction of drag by distributed suction. Proceedings, 3rd Anglo-American Aeronautical Conference, Brighton. September, 1951.
16	H. R. Hopkin, R. W. Dunn and Staff of Automatic Control Section, Instruments Division of I.A.P.	Theory and development of automatic pilots, 1937–1947. R.A.E. Report IAP. 1459. A.R.C. 13,825. August, 1947.
17	D. Lean	The carrier deck-landing properties of five jet-propelled aircraft (Appendix IV). R.A.E. Report Aero. 2465. A.R.C. 15,494. June, 1952.
18	S. Neumark	Analysis of short-period longitudinal oscillations of an aircraft: interpretation of flight tests. R. & M. 2940. September, 1952.

APPENDIX

Remarks on von Kármán and Biot's Criterion of Longitudinal Stability

In connection with section 2.4 it may be worthwhile to analyse a curious simple criterion of longitudinal dynamic stability, proposed by von Kármán and Biot in their book on *Mathematical Methods in Engineering*[5] (1940), Section VI.8, pp. 249 to 255. Their investigation follows the usual lines, although the variables used are partly different from those in our equations (2.4.1). There is, however, one unusual simplifying assumption (page 250) that '*the drag can be neglected or that it is balanced at every instant by a propeller thrust of equal magnitude.*' This assumption does not lower the order of the determinantal equation which still remains a quartic; however, its coefficients become much simpler. A further simplification is introduced by the moment derivatives $m_{\dot{w}}$ and m_u being tacitly neglected. Instead of following the analysis of Ref. 5 (with its entirely different notation), it will be simpler to see what effect the simplifying assumptions have on our results of section 2.4. We have to put $C_D = C_{AS} = 0$, $dC_D/dC_L = 0$, and also to neglect χ and $\varkappa$. The coefficients (2.4.6) of the stability quartic then become:

$$B_1 = \tfrac{1}{2}a + \nu\ ,\quad C_1 = \omega + \tfrac{1}{2}a\nu + \tfrac{1}{2}C_L{}^2\ ,\quad D_1 = \tfrac{1}{2}C_L{}^2\nu\ ,\quad E_1 = \tfrac{1}{2}C_L{}^2\omega, \quad \ldots \quad (A.1)$$

and the approximate expression for phugoid damping (2.4.13) takes the strikingly simple form:

$$\frac{2r}{C_L} \simeq aC_L \frac{\nu^2 - \omega}{(2\omega + a\nu)^2} \quad \ldots \quad (A.2)$$

The condition for dynamic stability (positive phugoid damping) thus becomes :

$$\omega < \nu^2 , \quad \ldots \quad (A.3)$$

and this is almost identical with von Kármán and Biot's criterion given by their inequality (8.19). The latter contains a small correction term due to the fact that they used the exact Routh discriminant (2.4.9). Following their procedure, we obtain from (A.1) :

$$\mathscr{R} = B_1(C_1D_1 - B_1E_1) - D_1^2 = \tfrac{1}{4}aC_L^2\{(\nu + \tfrac{1}{2}a)(\nu^2 - \omega) + \tfrac{1}{2}C_L^2\nu\} , \quad \ldots \quad (A.4)$$

and the condition for stability becomes :

$$\omega < \nu\left(\nu + \frac{C_L^2}{2\nu + a}\right) . \quad \ldots \quad (A.5)$$

This is, apart from notation, completely identical with the inequality (8.19) of Ref. 5*.

If the procedure of Ref. 5 were followed but the derivative m_w' (hence χ) retained, the formula (A.4) and inequalities (A.3) and (A.5) would be modified :

$$\mathscr{R} = \tfrac{1}{4}aC_L^2[(\nu + \chi + \tfrac{1}{2}a)\{\nu(\nu + \chi) - \omega\} + \tfrac{1}{2}C_L^2(\nu + \chi)] , \quad \ldots \quad (A.8)$$

$$\omega < (\nu + \chi)\nu , \quad \ldots \quad (A.9)$$

$$\omega < (\nu + \chi)\left(\nu + \frac{C_L^2}{2\nu + 2\chi + a}\right) . \quad \ldots \quad (A.10)$$

To discuss properly these results, we must notice first that the correction terms depending on C_L^2 in (A.5) and (A.10) are practically negligible. Let us consider, *e.g.*, the cases of Figs. 7a to 7d, where $a = 4$. Suppose that $\nu = 3$ and $\chi = 0$ or $1 \cdot 5$. Then the inequalities (A.3,5) give :

$$\omega < 9 \qquad \text{or} \qquad \omega < 9 + 0 \cdot 3C_L^2 ,$$

and (A.9,10) similarly :

$$\omega < 13 \cdot 5 \qquad \text{or} \qquad \omega < 13 \cdot 5 + 0 \cdot 346C_L^2 .$$

It is seen that the effect of χ (neglected in Ref. 5) may be quite considerable but that of the correction terms depending on C_L^2 is very small. This is illustrated in Fig. 8 where the inequalities (A.3,5,9,10) are represented by curves (von Kármán and Biot's stability boundaries). Only the cases $C_L = 1$ and $C_L = 0$ are illustrated, and it is seen that the corresponding curves are practically the same.

Irrespective of this negligible correction, the stability criterion under discussion leads to results greatly different from those given by the fuller theory. In the stability condition (2.4.13) there are three terms, and von Kármán and Biot's simplification leads to neglecting the two first terms entirely. It is easily seen, however, that the neglected terms are generally of the same order of magnitude as the only one retained, and they may often be greater. Neglecting the first two terms, the damping factor would be given by :

$$2r = aC_L^2\frac{\nu(\nu + \chi) - \omega}{(2\omega + a\nu)^2} , \quad \ldots \quad (A.11)$$

and the corresponding curves are shown as broken lines in Figs. 7a to 7d. It is seen that these lines deviate so considerably from the correct full lines that they cannot be regarded as valid approximations.

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*This inequality was given in the form :

$$\sigma < \delta^2 + \frac{2\delta\alpha_0}{1 + \delta\alpha_0} , \quad \ldots \quad (A.6)$$

and it is easily found that the meaning of the symbols σ, δ and α_0 is :

$$\sigma = 4\omega/C_L^2 \quad , \quad \delta = 2\nu/C_L \quad , \quad \alpha_0 = C_L/a, \quad \ldots \quad (A.7)$$

so that (A.5) and (A.6) are equivalent.

The condition of stability given by (A.3) or (A.9) implies that the aircraft becomes dynamically unstable if ω exceeds some moderate positive value, *i.e.*, if the 'static stability' is too large. This is really a warning against large static margins, and the limit imposed would be quite low in most cases, cutting across the usual range of these margins in quite an unexpected way. Now, the diagrams corresponding to the fuller theory show that the phugoid damping decreases with rising ω only in a very narrow range, and, with further increasing ω, improves gradually, so that :

$$(2r)_{\omega \rightarrow \infty} \rightarrow C_D + C_{AS}. \quad \text{(A.12)}$$

The drag coefficient is thus mainly responsible for the phugoid damping of aircraft with great static stability.

There is no doubt that von Kármán and Biot treated their solution only as an academic example of mathematical technique, and were far from propounding any general requirement of very small static margins. In fact, they added a somewhat vague remark to the effect that, if the neglected terms were re-instated, a somewhat different stability boundary would be obtained. However, the remark may easily be overlooked, and that is what apparently happened in the well known textbook on *Theory of Flight* by von Mises[6] (1945). His analysis is essentially the same as that of von Kármán and Biot, and the following statement concludes the analysis : 'An aeroplane is stable in level flight with respect to longitudinal disturbances if it is statically stable and if the static stability does not surpass a limit determined by the damping factor and the other force derivatives'. The parabolic stability boundary re-appears in von Mises' presentation, and any doubts are dismissed by the remark that 'the more precise formula does not change the procedure considerably'. The truth is that, if the neglected drag terms are re-instated, the stability boundary is altered completely. Using our formula (2.4.13), we would obtain the equation of the stability boundary in the form :

$$4(C_D + C_{AS})\,\omega^2 + 4a(C_D + C_{AS} - \tfrac{1}{2}C_L\, dC_D/dC_L)\omega\nu$$

$$+ a^2\left(C_D + C_{AS} - C_L dC_D/dC_L + C_L^2\,\frac{1+\zeta}{a}\right)\nu - aC_L^2\,\omega = 0, \quad \text{(A.13)}$$

where ζ is the assumed value of the ratio χ/ν. The boundary is practically almost always an ellipse, and a few examples, corresponding to Figs. 7a to 7d, are shown in Fig. 8. The phugoid damping is positive outside the boundary. It is seen that, even if the rotary damping derivative $\nu = 0$, the damping will be positive if

$$\omega > \frac{aC_L^2}{4(C_D + C_{AS})}, \quad \text{(A.14)}$$

and this is usually quite a moderate value. The position is completely different from that indicated by (A.3) or (A.9).

One additional remark may be made. The formula (2.4.13) may be written in the form :

$$\frac{2r}{C_L} = \left(\frac{C_D + C_{AS}}{C_L} - \frac{dC_D}{dC_L}\right) + \frac{2\omega}{2\omega + a\nu}\frac{dC_D}{dC_L} + a\,C_L\,\frac{\nu(\nu+\chi) - \omega}{(2\omega + a\nu)^2} > 0, \quad \text{(A.15)}$$

and it is seen then that the first term (in brackets) alone represents the original Painlevé's theory, while the third one corresponds to that of von Kármán and Biot. The former theory takes into account only the drag derivatives, the latter neglects them and emphasises the moment derivatives. The fuller formula (A.15) combines the two terms, but there is in addition the 'interference term' which involves both drag and moment derivatives.

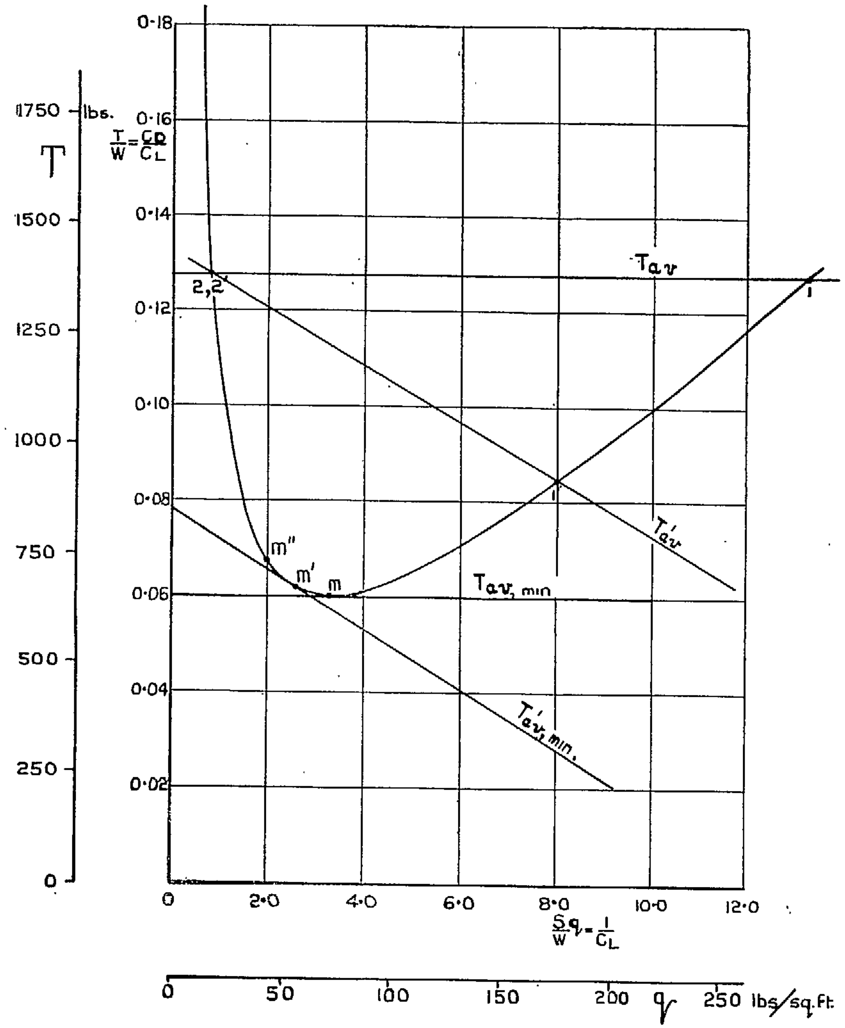

FIG. 1. Curves of thrust required and available for determining performance illustrating Painlevé's stability criterion. $C_{D0} = 0 \cdot 009$, $s = 0 \cdot 1$.

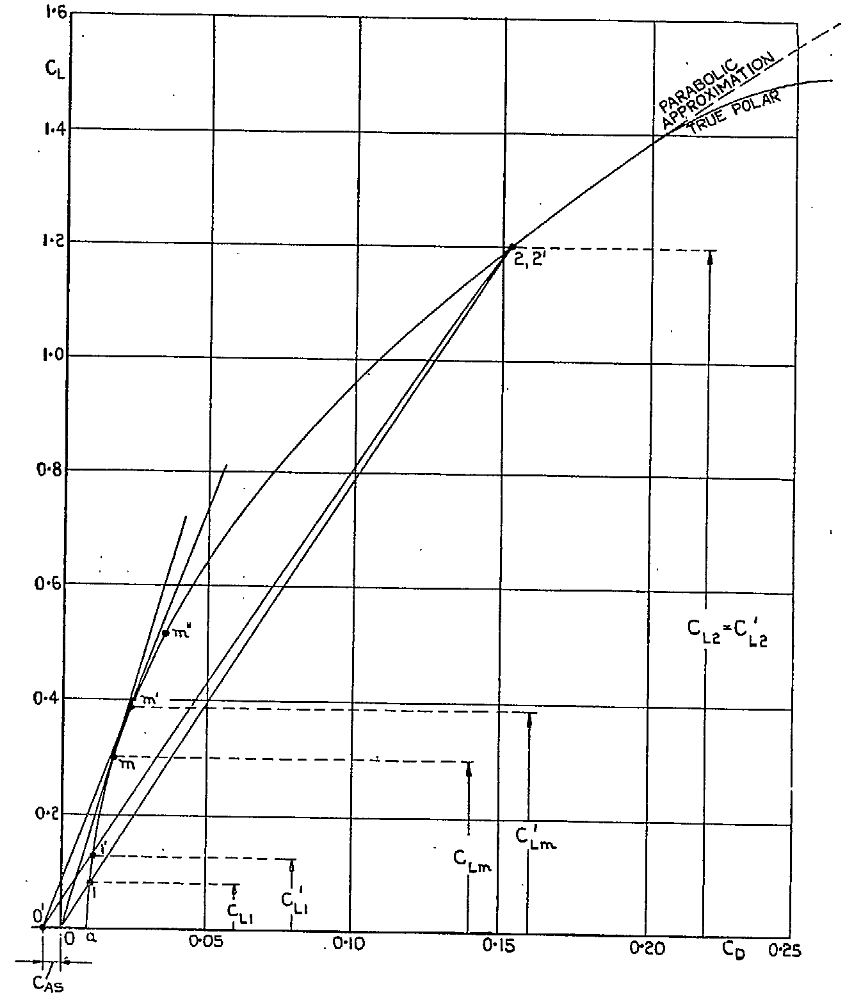

FIG. 2. Polar curve for determining performance, illustrating Painlevé's stability criterion. $C_{D0} = 0 \cdot 009$, $s = 0 \cdot 1$.

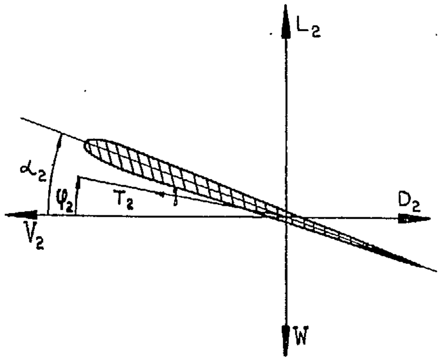

FIG. 3a. Aircraft in steady flight.

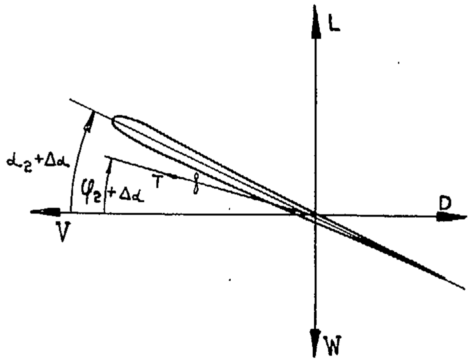

FIG. 3b. Aircraft at modified speed and incidence.

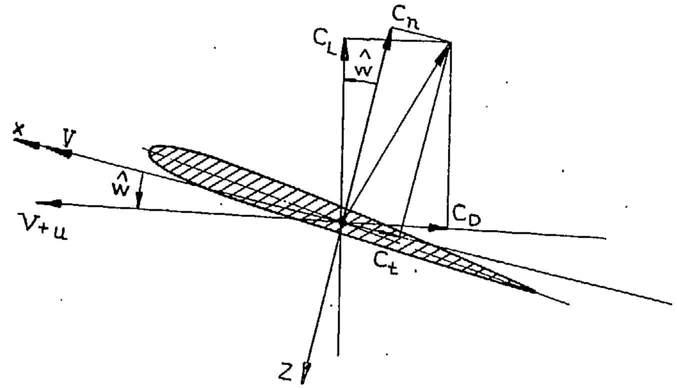

FIG. 5. ' Normal ' and ' tangential ' force coefficients C_n, C.

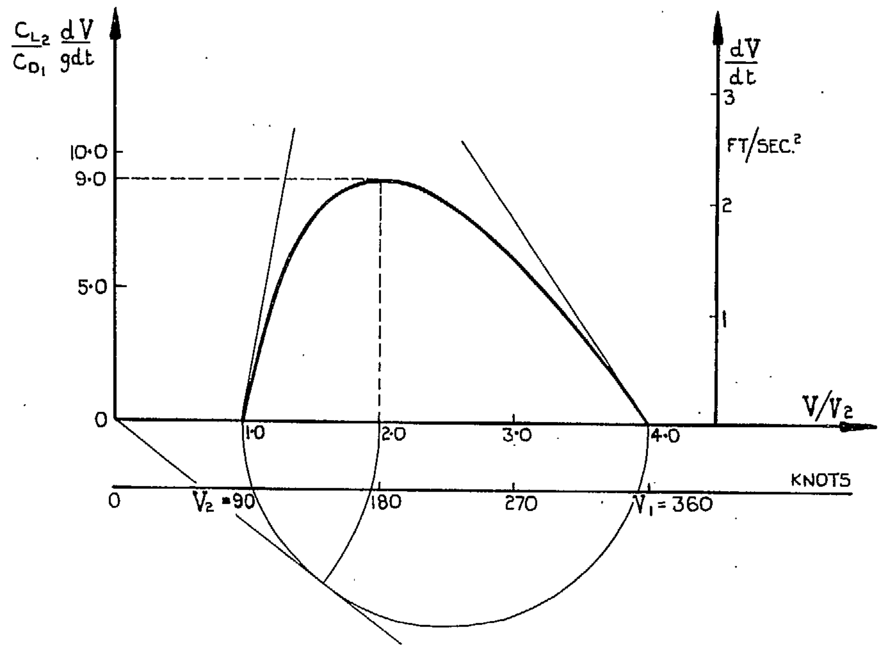

FIG. 4. Acceleration versus speed, at constant height.

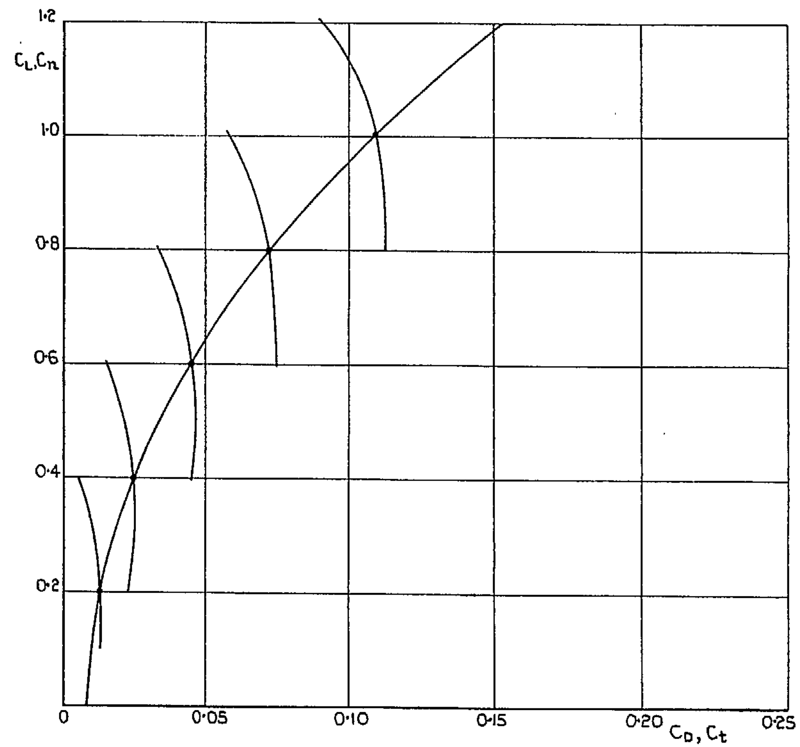

FIG. 6. Usual C_D vs. C_L polar curve and C_t vs. C_n polar curves.

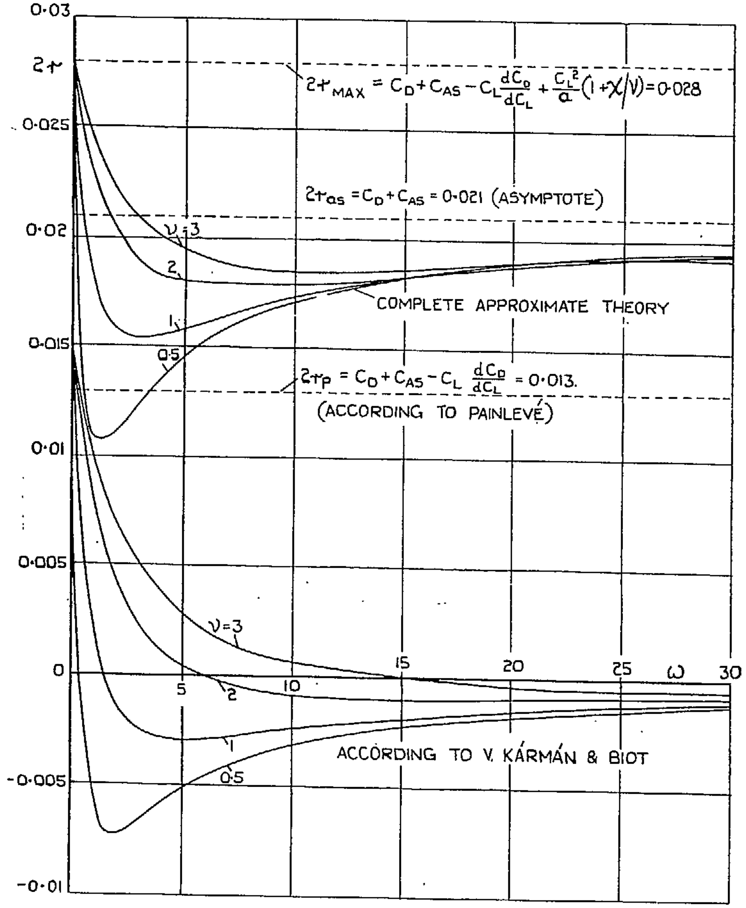

FIG. 7a. Phugoid damping factor for varying ω and ν, at low incidence. $C_L = 0{\cdot}2$, $C_D = 0{\cdot}014$, $C_{AS} = 0{\cdot}007$, $a = 4$, $\chi/\nu = 0{\cdot}5$.

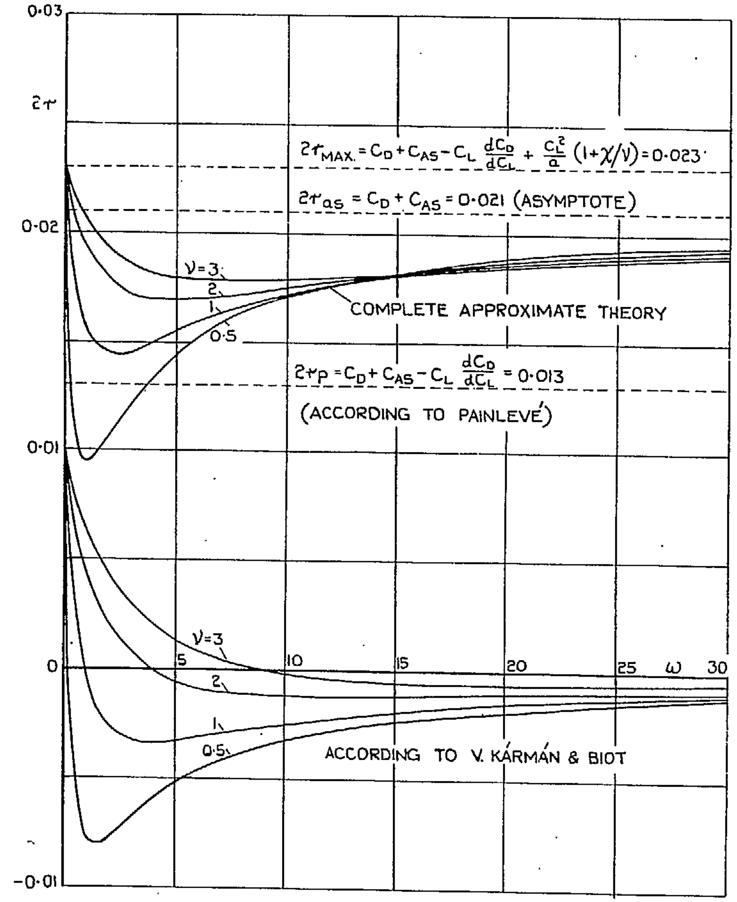

FIG. 7b. Phugoid damping factor for varying ω and ν, at low incidence. $C_L = 0{\cdot}2$, $C_D = 0{\cdot}014$, $C_{AS} = 0{\cdot}007$, $a = 4$, $\chi = 0$.

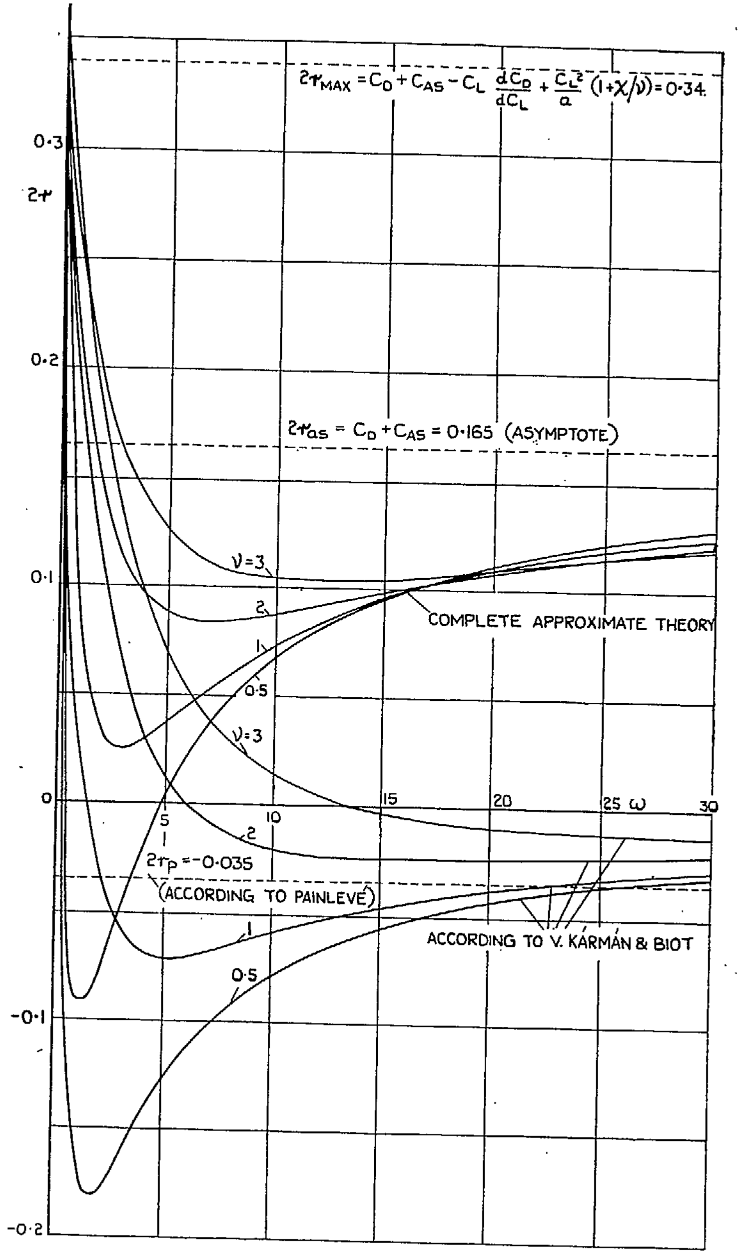

FIG. 7c. Phugoid damping factor for varying ω and ν, at high incidence. $C_L = 1$, $C_D = 0{\cdot}11$, $C_{AS} = 0{\cdot}055$, $a = 4$, $\chi/\nu = 0{\cdot}5$.

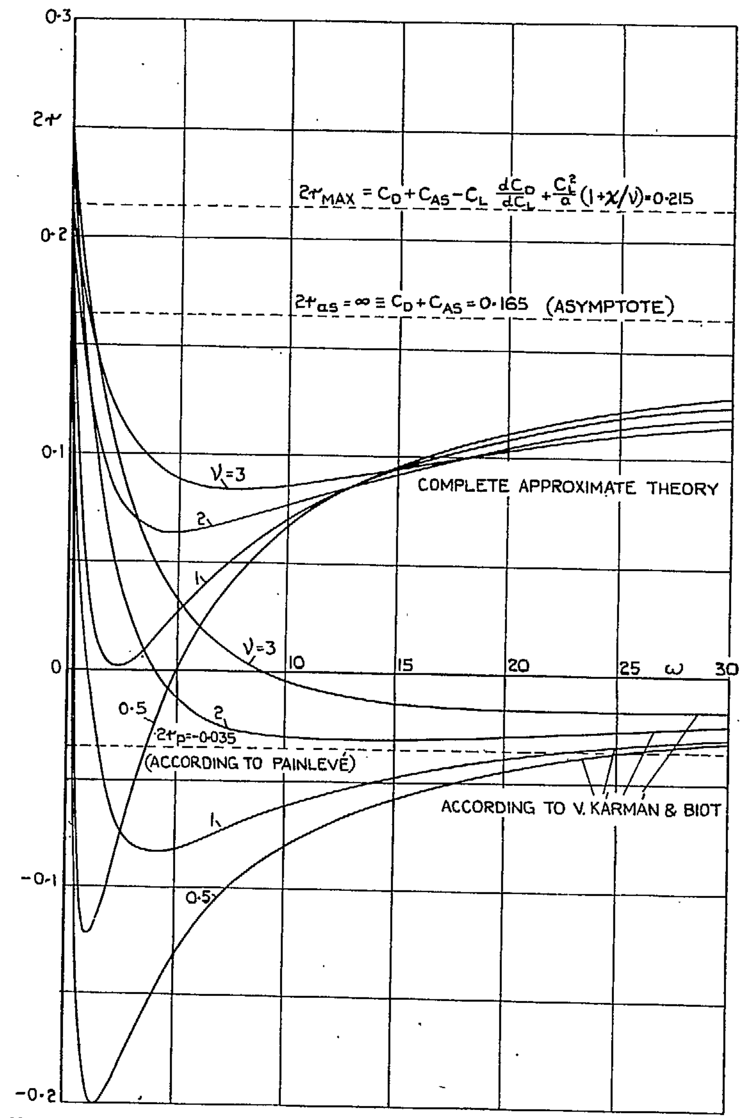

FIG. 7d. Phugoid damping factor for varying ω and ν, at high incidence. $C_L = 1$, $C_D = 0{\cdot}11$, $C_{AS} = 0{\cdot}055$, $a = 4$, $\chi = 0$.