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Summary.—Expressions are derived for the sideslip derivatives on the assumptions of the linearised theory of flow for a delta wing with small dihedral flying at supersonic speeds. A discussion is included in the Appendix on the relation between two methods that have been evolved for the treatment of aerodynamic force problems of the delta wing lying within its apex Mach cone.

When the leading edges are within the Mach cone from the apex, the pressure distribution and the rolling moment are independent of Mach number but dependent on aspect ratio.

When the leading edges are outside the apex Mach cone, the non-dimensional rolling derivative is, in contrast to the other case, dependent on Mach number and independent of aspect ratio : the other derivatives and the pressure, however, are dependent on both variables.

1. *Introduction.*—The present paper, in which the aerodynamic derivatives with respect to sideslip are calculated, is one of a series dealing with the force coefficients acting on a delta wing at supersonic speeds. The investigation will be confined to the case of small deviations from the neutral position of a wing at zero incidence, so that in particular it may be assumed that if the wing is initially wholly within the Mach cone emanating from its apex it will remain so in the disturbed condition, and vice versa.

The problem divides into the two cases in which the wing protrudes through its apex Mach cone and in which it is entirely enclosed within it. In the former the task simplifies to integrating a uniform distribution of supersonic sources, since the motion ahead of the trailing edge above the wing is independent of that below the wing. In the latter case recourse is made to a method based on that introduced by Stewart[1] in his solution of the basic lift problem, except that the expression relating the pressure distribution to the boundary conditions is derived in a different manner.

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Robinson[2] solved the lift problem by other means, and a comparison of the two techniques employed is made in the Appendix to this paper.

2. *Notation.*

$\bar{V}$	Free stream velocity
$\bar{v}$	Sideslip velocity
ρ	Air density
M	Mach number
β	$\sqrt{(M^2 - 1)}$
λ	$\beta \tan \gamma$
L	Rolling moment
N	Yawing moment (referred to vertex)
Y	Side force
δ	Dihedral angle
γ	Semi-vertex angle
c	Maximum chord
S	$= c^2 \tan \gamma$; the wing area
s	$= c \tan \gamma$; the semi span
l_v	$= L/\rho \bar{v} \bar{V} S s$; the non-dimensional rolling derivative
n_v	$= N/\rho \bar{v} \bar{V} S s$; the non-dimensional yawing derivative
y_v	$= Y/\rho \bar{v} \bar{V} S$; the non-dimensional sideslip derivative.

3. *Results* —A thin flat delta wing of small dihedral is travelling at supersonic speed $\bar{V}$ with sideslip $\bar{v}$ with vertex into wind (*see* Fig. 3a).

The forces due to sideslip at zero incidence are :—

	Inside Mach Cone ($\lambda < 1$)	Outside Mach Cone ($\lambda > 1$)
L	$+\frac{2}{3} \rho \bar{v} \bar{V} \delta c^3 \tan^3 \gamma.$	$+\frac{2}{3\beta} \rho \bar{v} \bar{V} \delta c^3 \gamma^2.$
N	$-\frac{8}{3\pi} \rho \bar{v} \bar{V} \delta^2 c^3 \tan^2 \gamma.$	$-\frac{8}{3\pi} \rho \bar{v} \bar{V} \delta c^3 \tan^2 \gamma \frac{\sec^{-1}\lambda}{\sqrt{(\lambda^2 - 1)}}$
Y	$-\frac{4}{\pi} \rho \bar{v} \bar{V} \delta^2 c^2 \tan^2 \gamma$	$-\frac{4}{\pi} \rho \bar{v} \bar{V} \delta^2 c^2 \tan^2 \lambda \frac{\sec^{-1}\lambda}{\sqrt{(\lambda^2 - 1)}}$

The non-dimensional aerodynamic derivatives with respect to sideslip are :—

	Inside Mach Cone ($\lambda < 1$)	Outside Mach Cone ($\lambda > 1$)
l_v	$\frac{2}{3} \delta \tan \gamma.$	$+\frac{2}{3}\frac{\delta}{\beta}$
n_v	$-\frac{8}{3\pi} \delta^2$	$-\frac{8}{3\pi} \delta^2 \frac{\sec^{-1}\lambda}{\sqrt{(\lambda^2)} - 1}$
y_v	$-\frac{4}{\pi} \delta^2 \tan \gamma.$	$-\frac{4}{\pi} \delta^2 \tan \gamma \frac{\sec^{-1}\lambda}{\sqrt{(\lambda^2 - 1)}}$

It will be noted that the above quantities are continuous on transition from one case to the other.

In Fig. 1 the quantities $\beta l_v/\delta$, n_v/δ^2 and $\beta y_v/\delta^2$ for zero incidence are plotted against the parameter λ.

In Fig. 2 the quantities l_v/δ n_v/δ^2 and y_v/δ^2 for zero incidence are plotted against Mach number for different aspect ratios. It will be seen that the values of l_v/δ obtained for the higher aspect ratios, when the leading edges are within the Mach cone, are comparable with those obtained in incompressible flow.

The pressure distributions are :—

(a) leading edges within the Mach cone,

$$\frac{2}{\pi}\rho\bar{v}\bar{V}\delta\,\frac{y\tan\gamma}{\sqrt{(x^2\tan^2\gamma - y^2)}}$$

(b) leading edges outside the Mach cone

(i) at a point outside the Mach cone,

$$\rho\bar{v}\bar{V}\delta\,\frac{\tan\gamma}{\sqrt{(\lambda^2-1)}}\,,$$

(ii) at a point inside the Mach cone,

$$\frac{2}{\pi}\rho\bar{v}\bar{V}\delta\,\frac{\tan\gamma}{\sqrt{(\lambda^2-1)}}\tan^{-1}\left\{y\cot\gamma\sqrt{\left(\frac{\lambda^2-1}{x^2-\beta^2y^2}\right)}\right\}.$$

4. *Delta Wing Enclosed within the Apex Mach Cone.*—4.1. *Relating the Pressure Distribution to the Boundary Conditions.*—In the linearised supersonic theory excess pressure is proportional to the induced velocity in the free stream direction. Since the angle of dihedral is small, the boundary conditions can be expressed by equating the velocity normal to the yawing plane to the component of the sideslip velocity along the normal to the aerofoil itself.

Using the cartesian axes indicated in Fig. 3(a), we will establish for the class of problems to which our present one belongs that the induced velocity components u, v and w in the x, y and z-directions can be expressed as the real parts of functions U, V and W of a complex variable τ, and that there exist relations of the form

$$\frac{dU}{d\tau} = f_1(\tau)\frac{dW}{d\tau} \quad\text{and}\quad \frac{dV}{d\tau} = f_2(\tau)\frac{dW}{d\tau}$$

The problem therefore reduces to determining a suitable transformation from the x, y, z-space to the τ-plane, and a suitable function $dW/d\tau$, so that $w = R(W)$ takes up the known values at the boundaries. This is essentially the method of Stewart[1] but our derivation of the relations between U, V and W will be somewhat different.

The flow at any point ahead of the trailing edge is uninfluenced by the trailing edge, so that if we replace the aerofoil by one of the same shape but of different size the flow at such a point will be unaltered. Hence the flow at any point along a ray through the vertex is the same. The induced velocity is therefore of degree zero in x, y, z; this type of flow is called conical, a term introduced by Busemann.

In the linearised supersonic theory the equation of continuity is the Prandtl-Glauert equation

$$-\beta^2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(1)}$$

For irrotational flow curl $(u, v, w) = 0$, and there exists a velocity potential Φ.

It will therefore be seen that u, v, w and Φ satisfy the equation :—

$$-\beta^2\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \qquad \text{(2)}$$

Under the transformation $(x', y', z') = (x, i\beta y, i\beta z)$ every solution of Laplace's equation in x', y', z', is also a solution of equation (2) in x, y, z and vice versa.

It was established by Donkin in 1857 that the most general solution of Laplace's equation of zero degree in three dimensions is of the form :—

$$F_1\left(\frac{y' + iz'}{x' + r}\right) + F_2\left(\frac{y' - iz'}{x' + r}\right), \qquad \ldots \quad (3)$$

where $r^2 = x^2 + y^2 + z^2$.

Hence any analytic function of ω is a solution of equation (2) of degree zero, where

$$\omega = \eta + i\zeta = \beta \frac{y + iz}{x + r},$$

and where $r^2 = x^2 - \beta^2 y^2 - \beta^2 z^2$.

Therefore we take u, v, w to be the real parts of $U(\omega)$, $V(\omega)$, $W(\omega)$, satisfying both equation (2) and Laplace's equation in η, ζ. It will be noted that the velocity potential is not of degree zero and cannot therefore be put in this form.

It will be seen that for conical flow the induced velocity potential is of the form $\Phi = r\psi(\eta, \zeta)$, so that

$$\left.\begin{aligned}
u &= \eta \frac{\partial \psi}{\partial \eta} + \zeta \frac{\partial \psi}{\partial \zeta} - \frac{1 + \eta^2 + \zeta^2}{1 - \eta^2 - \zeta^2}\psi, \\
v &= -\tfrac{1}{2}\beta(1 + \eta^2 - \zeta^2)\frac{\partial \psi}{\partial \eta} - \beta\eta\zeta\frac{\partial \psi}{\partial \zeta} + \frac{2\beta\eta}{1 - \eta^2 - \zeta^2}\psi, \\
w &= -\beta\eta\zeta\frac{\partial \psi}{\partial \eta} - \tfrac{1}{2}\beta(1 - \eta^2 + \zeta^2)\frac{\partial \psi}{\partial \zeta} + \frac{2\beta\zeta}{1 - \eta^2 - \zeta^2}\psi.
\end{aligned}\right\} \qquad \ldots \quad (4)$$

The equation of continuity (1) becomes

$$\left\{1 - \eta^2 - \zeta^2\right\}^2 \left\{\frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial^2 \psi}{\partial \zeta^2}\right\} - 8\psi = 0 \qquad \ldots \quad (5)$$

Now since u is the real part of $U = U(\omega)$, the Cauchy-Riemann equations give

$$\frac{dU}{d\omega} = \frac{\partial u}{\partial \eta} - i\frac{\partial u}{\partial \zeta},$$

and similarly for V and W. Therefore

$$\begin{aligned}
\frac{dU}{d\omega} &= \eta\frac{\partial^2 \psi}{\partial \eta^2} - i\omega\frac{\partial^2 \psi}{\partial \eta\,\partial \zeta} - i\zeta\frac{\partial^2 \psi}{\partial \zeta^2} \\
&\quad - 2\frac{\eta^2 + \zeta^2}{1 - \eta^2 - \zeta^2}\left\{\frac{\partial \psi}{\partial \eta} - i\frac{\partial \psi}{\partial \zeta}\right\} - 4\frac{\eta - i\zeta}{(1 - \eta^2 - \zeta^2)^2}\psi
\end{aligned} \qquad \ldots \quad (6)$$

$$\begin{aligned}
\frac{dV}{d\omega} &= -\tfrac{1}{2}\beta(1 + \eta^2 - \zeta^2)\frac{\partial^2 \psi}{\partial \eta^2} + \tfrac{1}{2}i\beta(1 + \omega^2)\frac{\partial^2 \psi}{\partial \eta\,\partial \zeta} + i\beta\eta\zeta\frac{\partial^2 \psi}{\partial \zeta^2} \\
&\quad + \beta\left\{\frac{2\eta}{1 - \eta^2 - \zeta^2} - \omega\right\}\left\{\frac{\partial \psi}{\partial \eta} - i\frac{\partial \psi}{\partial \zeta}\right\} + 2\beta\frac{1 + (\eta - i\zeta)^2}{(1 - \eta^2 - \zeta^2)^2}
\end{aligned} \qquad (7)$$

and

$$\begin{aligned}
\frac{dW}{d\omega} &= -\beta\eta\zeta\frac{\partial^2 \psi}{\partial \eta^2} - \tfrac{1}{2}\beta(1 - \omega^2)\frac{\partial^2 \psi}{\partial \eta\,\partial \zeta} + \tfrac{1}{2}i(1 - \eta^2 - \zeta^2)\frac{\partial^2 \psi}{\partial \zeta^2} \\
&\quad + \beta\left\{\frac{2\zeta}{1 - \eta^2 - \zeta^2} + i\omega\right\}\left\{\frac{\partial \psi}{\partial \eta} - i\frac{\partial \psi}{\partial \zeta}\right\} - 2i\beta\frac{1 - (\eta - i\zeta)^2}{(1 - \eta^2 - \zeta^2)^2}\psi
\end{aligned} \qquad (8)$$

Hence
$$\left.\begin{aligned}&\beta(1-\omega^2)\frac{dU}{d\omega}-2i\omega\frac{dW}{d\omega}\\&\quad=\beta\eta(1-\eta^2-\zeta^2)\left\{\frac{\partial^2\psi}{\partial\eta^2}+\frac{\partial^2\psi}{\partial\zeta^2}\right\}-\frac{8\beta\eta}{1-\eta^2-\zeta^2}\psi,\\ \text{and}\quad&(1-\omega^2)\frac{dV}{d\omega}+i(1+\omega^2)\frac{dW}{d\omega}\\&\quad=-\tfrac{1}{2}\beta\left(1-(\eta^2+\zeta^2)^2\right)\left\{\frac{\partial^2\psi}{\partial\eta^2}+\frac{\partial^2\psi}{\partial\zeta^2}\right\}+4\beta\frac{1+\eta^2+\zeta^2}{1-\eta^2-\zeta^2}\psi,\end{aligned}\right\}\quad\ldots\quad(9)$$

so that by equation (5)

$$\left.\begin{aligned}&\frac{dU}{d\omega}=\frac{1}{\beta}\cdot\frac{2i\omega}{1-\omega^2}\cdot\frac{dW}{d\omega},\\ \text{and}\quad&\frac{dV}{d\omega}=-i\cdot\frac{1+\omega^2}{1-\omega^2}\cdot\frac{dW}{d\omega}.\end{aligned}\right\}\quad\ldots\quad(10)$$

On the Mach cone $r^2 = x^2 - \beta^2(y^2+z^2) = 0$, so that $|\omega|^2 = \beta^2\frac{(y^2+z^2)}{(r+x)^2} = 1$. At the aerofoil $z = 0$, so $\zeta = 0$, and at a leading edge $y = \pm x\tan\gamma$, so $\eta = \frac{\pm\beta\tan\gamma}{1+\sqrt{(1-\beta^2\tan^2\gamma)}} = \frac{\pm k'}{1+k}$, where $k^2 = 1 - k'^2 = 1 - \beta^2\tan^2\gamma$.

The Mach cone and its interior are, therefore, represented in the ω-plane by the unit circle and its interior, while the aerofoil becomes the real axis between $\pm k'/(1+k)$. (Fig. 3(b) refers).

Consider the transformation $\mathrm{cn}\,(\tau, k) = 2i\omega/(1-\omega^2)$ where $\mathrm{cn}\,(\tau, k)$ is the Jacobian elliptic function of modulus k in Glaisher's notation.

The interior of the unit circle in the ω-plane is traced on the τ-plane in the rectangle, vertices $\pm 2iK'(k)$, $K(k) \pm 2iK'(k)$. In Fig. 3(c) the imaginary axis AA′ between $\tau = \pm 2iK'$ represents the Mach cone, while the aerofoil becomes the parallel line BB′ between $\tau = K \pm 2iK'$, such that CQ is the lower surface, $z = -0$, $y < 0$, QB the upper surface $z = +0$, $y < 0$, CQ′ the lower surface, $z = -0$, $y > 0$ and Q′B′ the upper surface $z = +0$, $y > 0$. The leading edges become the points Q, Q′. The point C corresponds to the wing axis on the lower surface and the points B, B′ both to the axis on the upper surface. The line OC represents the portion of the zx-plane, $y = 0$, $z < 0$, between the Mach cone and the aerofoil, while AB, A′ B′ both correspond to the similar section above the aerofoil : the line PQ corresponds to that part of the xy-plane, $y < 0$, $z = 0$ between the Mach cone and the leading edge, and the line P′ Q′ to the similar part, $y > 0$, $z = 0$.

$$\text{In the } \tau\text{-plane}\quad\frac{dU}{d\tau}=\frac{1}{\beta}\,\mathrm{cn}\,\tau\,\frac{dW}{d\tau}\quad\ldots\quad(11)$$

$$\text{and}\quad\frac{dV}{d\tau}=-i\,\mathrm{sn}\,\tau\,\frac{dW}{d\tau}.$$

4.2. *Calculation of Derivatives with respect to Sideslip.*—As already indicated we assume that the kinematic boundary conditions are fulfilled at the normal projection of the aerofoil on the xy-plane rather than at the aerofoil itself. The boundary condition for a sideslip velocity $\bar{v}$ and dihedral δ reduces to $w = \bar{v}\delta$ for $y > 0$ and $w = -\bar{v}\delta$ for $y < 0$.

From the asymmetry of the configuration it follows that $w = 0$ at the zx-plane. In addition $w = 0$ at the Mach cone.

From physical considerations $dU/d\tau$, $dV/d\tau$ and $dW/d\tau$ must be finite at the Mach cone. Furthermore the aerodynamic forces must be finite, so that any infinity of u at the aerofoil must be such that the integral of u with respect to area is finite.

We have to choose $dW/d\tau$ so that $dU/d\tau$, $dV/d\tau$, u, w fulfil these conditions and so that u, v, w are single valued.

In order that $dW/d\tau$ may be finite on the Mach cone and w zero on the Mach cone and the zx-plane, $dW/d\tau$ must be regular and real on AA′ and be imaginary on OC, AB and A′B′ with no singularities other than poles ; the residues of such poles must be zero or real except at C, B and B′ where there are discontinuities in w. Since $dU/d\tau (= 1/\beta \operatorname{cn} \tau \, dW/\tau)$ and $dV/d\tau (= - i \operatorname{sn} \tau \, dW/d\tau)$ are to be also finite on the Mach cone, $dW/d\tau$ must have at least a simple zero at the points P and P′ ($\tau = \pm iK'$). Since w is to be constant over the two halves of the aerofoil, $dW/d\tau$ must be real on BB′ and have no singularities which contribute to w except, as before, at C, B and B′. In integrating $dW/d\tau$ along OCB, w must jump in value by an amount $+ \bar{v}\delta$ at C and $- \bar{v}\delta$ in integrating along OCB′. Clearly, therefore $dW/d\tau$ must have a simple pole at C of residue of imaginary part $2\bar{v}\delta/\pi$. Similarly $dW/d\tau$ must have simple poles of residue of imaginary part $- 2\bar{v}\delta/\pi$ at B and B′, so that w may return to zero on AB and A′B′. In order that u, v, w may be single valued $dU/d\tau$, $dV/d\tau$ $dW/d\tau$ must be regular within the rectangle.

Functions satisfying these conditions and equation (11) are :—

$$\left.\begin{aligned} \frac{dW}{d\tau} &= \frac{2i\bar{v}\delta k'^3}{\pi} \operatorname{sc} \tau \operatorname{nd}^2 \tau \\ \frac{dV}{d\tau} &= \frac{2\bar{v}\delta k'^3}{\pi} \operatorname{sd}^2 \tau \operatorname{nc} \tau \\ \frac{dU}{d\tau} &= \frac{2i\bar{v}\delta k'^3}{\pi\beta} \operatorname{sn} \tau \operatorname{nd}^2 \tau \end{aligned}\right\} \quad \ldots \quad (12)$$

Now $dU/d\tau$ is regular except for a double pole at $\tau = K \pm iK'$, so

$$u = \frac{2\bar{v}\delta k'^3}{\pi\beta} R \int_0^\tau i \operatorname{sn} \tau \operatorname{nd}^2 \tau \, d\tau$$

$$= \frac{2}{\pi} \bar{v}\delta \tan \gamma R \, (- i \operatorname{cd} \tau)$$

On the aerofoil $\operatorname{cn} \tau = \dfrac{2i\omega}{1-\omega^2} = \dfrac{i\beta y}{\sqrt{(x^2 - \beta^2 y^2)}}$

and so $\quad \operatorname{cd} \tau = - \dfrac{iy}{\sqrt{(x^2 \tan^2 \gamma - y^2)}}$ on the xy-plane for $z = +0$
and is of opposite sign for $z = -0$.

Therefore for $\quad z = +0$ we have

$$u = \frac{2}{\pi} \bar{v}\delta \tan \gamma . \frac{y}{\sqrt{(x^2 \tan^2 \gamma - y^2)}} \quad \ldots \quad (13)$$

In the linearised theory the pressure p = const. $- \rho u \bar{V}$, so that the rolling moment due to sideslip is :—

$$L = \iint 2\rho \bar{V} u y \, dy \, dx,$$

where the integration is over the whole wing

$$= \frac{4}{\pi} \rho \bar{v} \bar{V} \delta \tan \gamma \iint \frac{y^2 \, dy \, dx}{\sqrt{(x^2 \tan^2 \gamma - y^2)}},$$

$$= \frac{8}{\pi} \rho \bar{v} \bar{V} \delta \tan^3 \gamma \int_0^{ct} \int_0^1 q^2 \sqrt{(1 - t^2)} \frac{dt}{t^3} \, dq,$$

where $\quad x = q/t \quad y = q \tan \gamma \dfrac{\sqrt{(1 - t^2)}}{t}$

$$L = + \tfrac{2}{3} \rho \bar{v} \bar{V} \delta c^3 \tan^3 \gamma.$$

Hence the derivative

$$l_v = \frac{L}{\rho \bar{v} \bar{V} S s} = \tfrac{2}{3} \delta \tan \gamma.$$

The sideforce due to the pressure distribution over the aerofoil resulting from a sideslip is :—

$$Y = - \iint 2\rho \bar{V} \delta |u| dy \, dx$$

$$= - \frac{4}{\pi} \rho \bar{v} \bar{V} \delta^2 \tan \gamma \iint \frac{|y| dy \, dx}{\sqrt{(x^2 \tan^2 \gamma - y^2)}},$$

$$= - \frac{8}{\pi} \rho \bar{v} \bar{V} \delta^2 \tan^2 \gamma \int_0^{ct} \int_0^1 \frac{q}{t^2} \, dt \, dq,$$

$$= - \frac{4}{\pi} \rho \bar{v} \bar{V} \delta^2 c^2 \tan^2 \gamma.$$

and $\quad y_v = \dfrac{Y}{\rho \bar{v} V S} = - \dfrac{4}{\pi} \delta^2 \tan^2 \gamma.$

The corresponding yawing moment is similarly :—

$$N = - \iint 2\rho \bar{V} |u| \delta . \, x \, dy \, dx$$

$$= - \frac{8}{3\pi} \rho \bar{v} \bar{V} \delta^2 c^3 \tan^2 \gamma$$

and $\quad n_v = - \dfrac{8}{3\pi} \delta^2.$

5. *Delta Wing with Leading Edges Outside Mach Cone.*—The boundary condition at the aerofoil is $w = \bar{v}\delta$ on one half and $- \bar{v}\delta$ on the other. When considering the upper surface, $y > 0$, where $w = v\delta$ we may take $w = - \bar{v}\delta$ on the corresponding lower surface, since the flow above the aerofoil is independent of the flow below it in the case under consideration. In this artificial condition there is a jump of $- 2\bar{v}\delta$ in the value of $\partial \Phi / \partial n$ at the surface, so that the surface can be replaced by a uniform supersonic source distribution of density $- \bar{v}\delta/\pi$; the other half of the aerofoil, $y < 0$, where $w = - \bar{v}\delta$, can be likewise replaced by a source distribution of density $\bar{v}\delta/\pi$.

Hence $\Phi(x, y, o) = - \dfrac{\bar{v}\delta}{\pi} \displaystyle\iint \frac{\sigma \, dx_0 \, dy_0}{\sqrt{[(x - x_0)^2 - \beta^2 (y - y_0)^2]}}$

where $\sigma = + 1$, when $y > 0$, and $\sigma = - 1$, when $y < 0$.

so $\quad \Phi = - \dfrac{\bar{v}\delta}{\pi} \displaystyle\iint \sigma \, d\rho \, d\psi$, where $x_0 = x - \beta\rho \cosh \psi$, and $y_0 = y - \rho \sinh \psi$.

In Fig. 3(d) P is the point (x,y), OL_1 and OL_2 are the leading edges, and PL_1 and PL_2 are the boundaries where $(x - x_0)^2 - \beta^2(y - y_0)^2 = 0$.

The values of ρ, ψ vary as follows :—

when (x_0, y_0) is on (i) PL_1, $\psi = -\infty$

(ii) PL_2, $\psi = +\infty$

(iii) OP, $\psi = \tanh^{-1}\dfrac{\beta y}{x} = \varepsilon$

(iv) OX, $\rho = \rho_0 = y \operatorname{cosech}\psi$

(v) OL_1, $\rho = \rho_1 = \dfrac{x\tan\gamma - y}{\lambda\cosh\psi - \sinh\psi}$

(vi) OL_2, $\rho = \rho_2 = \dfrac{x\tan\gamma + y}{\lambda\cosh\psi + \sinh\psi}$

When P is inside the Mach cone from the apex, we have

$$\Phi = -\frac{\bar{v}\delta}{\pi}\left\{\int_{-\infty}^{\varepsilon}\rho_1 d\psi + \int_{\varepsilon}^{\infty}\rho_0 d\psi - \int_{\varepsilon}^{\infty}(\rho_2 - \rho_0)\,d\psi\right\}$$

so that $$u = \frac{\bar{v}\delta}{\pi}\left\{\int_{-\infty}^{\varepsilon}\frac{\partial\rho_1}{\partial x}d\psi - \int_{\varepsilon}^{\infty}\frac{\partial\rho_2}{\partial x}d\psi\right\},$$

since $\dfrac{\partial\rho_0}{\partial x} = 0$ and $\rho_0 = \rho_1 = \rho_2$, when $\psi = \varepsilon$.

$$u = \frac{\bar{v}\delta}{\pi}\int_{-\infty}^{\varepsilon}\frac{\tan\gamma\,d\psi}{\lambda\cosh\psi - \sinh\psi} - \frac{\bar{v}\delta}{\pi}\int_{\varepsilon}^{\infty}\frac{\tan\gamma\,d\psi}{\lambda\cosh\psi + \sinh\psi}$$

$$= \frac{2\bar{v}\delta}{\pi}\int_{-1}^{\tau}\frac{\tan\gamma\,dt}{\lambda(1+t^2) - 2t} - \frac{2\bar{v}\delta}{\pi}\int_{\tau}^{1}\frac{\tan\gamma\,dt}{\lambda(1+t^2) + 2t}$$

where $t = \tanh\frac{1}{2}\psi$, $\tau = \tanh\frac{1}{2}\varepsilon$

$$u = \frac{2\bar{v}\delta\tan\gamma}{\pi\sqrt{(\lambda^2 - 1)}}\left\{\tan^{-1}\frac{\lambda\tau - 1}{\sqrt{(\lambda^2 - 1)}} + \tan^{-1}\frac{\lambda\tau + 1}{\sqrt{(\lambda^2 - 1)}}\right\}$$

$$= \frac{2\bar{v}\delta\tan\gamma}{\pi\sqrt{(\lambda^2 - 1)}}\tan^{-1}\left\{y\cot\gamma\sqrt{\left(\frac{\lambda^2 - 1}{x^2 - \beta^2 y^2}\right)}\right\}.$$

When P is outside the apex Mach cone

$$\Phi = -\frac{v\delta}{\pi}\int_{-\infty}^{\infty}\rho_1 d\psi, \quad y > 0$$

so that $u = \dfrac{\bar{v}\delta\tan\gamma}{\sqrt{(\lambda^2 - 1)}}$, by putting $\varepsilon = \infty$ in the above. When $y < 0$, u changes sign.

Hence the rolling moment due to sideslip is

$$L = \iint 2\rho\bar{V}uy\,dy\,dx$$

$$= \frac{4\rho\bar{v}\bar{V}\delta\tan\gamma}{\sqrt{(\lambda^2 - 1)}}\left\{\int_0^{c\sec\theta}\int_{\cot^{-1}\beta}^{\gamma} r^2\sin\theta\,d\theta\,dr\right.$$

$$\left.\frac{2}{\pi}\int_0^{\frac{c}{\beta}\operatorname{sech}\psi}\int_0^{\infty}\beta\tan^{-1}\left[\frac{\sqrt{(\lambda^2 - 1)}}{\lambda}\sinh\psi\right]q^2\sinh\psi\,d\psi\,dq\right\},$$

where $x = r\cos\theta$, $y = r\sin\theta$ in the first integral and $x = q\beta \cosh\psi$, $y = q\sinh\psi$ in the second integral

$$L = \frac{4\rho\bar{v}\bar{V}\delta c^3 \tan\gamma}{3\sqrt{(\lambda^2-1)}}\left\{\int_{\cot^{-1}\beta}^{\gamma} \tan\theta \sec^2\theta \, d\theta + \frac{2}{\pi\beta^2}\int_0^\infty \tan^{-1}\left[\frac{\sqrt{(\lambda^2-1)}}{\lambda}\sinh\psi\right]\tanh\psi \operatorname{sech}^2\psi \, d\psi\right\}$$

$$= \frac{2\rho\bar{v}\bar{V}\delta c^3 \tan\gamma}{3\sqrt{(\lambda^2-1)}}\left\{\tan^2\gamma - \frac{1}{\beta^2} + \frac{2}{\pi\beta^2}\left[\frac{\pi}{2} - \int_0^\infty \frac{\lambda\sqrt{(\lambda^2-1)}\cosh\psi \tanh^2\psi \, d\psi}{\sqrt{(\lambda^2-1)}\sinh^2\psi + \lambda^2}\right]\right\}$$

$$= \frac{2\rho\bar{v}\bar{V}\delta c^3 \tan\gamma}{3\sqrt{(\lambda^2-1)}}\left\{\tan^2\gamma - \frac{2}{\pi\beta^2}\int_0^\infty \frac{\lambda\sqrt{(\lambda^2+1)}t^2 dt}{(1+t^2)[(\lambda^2-1)t^2+\lambda^2]}\right\}$$

where $t = \sinh\psi$

$$= \frac{2\rho\bar{v}\bar{V}\delta c^3 \tan\gamma}{3\sqrt{(\lambda^2-1)}}\left\{\tan^2\gamma + \frac{2\lambda\sqrt{(\lambda^2-1)}}{\pi\beta^2}\left[\tan^{-1}t - \frac{\lambda}{\sqrt{(\lambda^2-1)}}\tan^{-1}\frac{t\sqrt{(\lambda^2-1)}}{\lambda}\right]_0^\infty\right\}$$

$$= \frac{2\rho\bar{v}\bar{V}\delta c^3 \tan^2\tau}{3\beta}$$

Hence $l_v = \dfrac{L}{\rho\bar{v}\bar{V}Ss} = +\dfrac{2\delta}{3\beta}$

The side force due to sideslip is

$$Y = -\iint 2\rho\bar{V}|u|\delta \, dy \, dx$$

$$= -\frac{4\rho\bar{v}\bar{V}\delta^2 \tan\gamma}{\sqrt{(\lambda^2-1)}}\left\{\int_0^{c\sec\theta}\int_{\cot^{-1}}^{\gamma} r \, dr \, d\theta + \frac{2}{\pi}\int_0^c\int_0^\infty \beta \tan^{-1}\left[\frac{\sqrt{(\lambda^2-1)}}{\lambda}\sinh\psi\right] q \, d\psi \, dq\right\}$$

$$= -\frac{2\rho\bar{v}\bar{V}\delta^2 c^2 \tan\gamma}{\sqrt{(\lambda^2-1)}}\left\{\tan\gamma - \frac{1}{\beta} + \frac{2}{\pi\beta}\int_0^\infty \tan^{-1}\left[\frac{\sqrt{(\lambda^2-1)}}{\lambda}\sinh\psi\right]\operatorname{sech}^2\psi \, d\psi\right\}$$

$$= -\frac{2\rho\bar{v}\bar{V}\delta^2 c^2 \tan\gamma}{\sqrt{(\lambda^2-1)}}\left\{\tan\gamma - \frac{1}{\beta} + \frac{2}{\pi\beta}\left[\frac{\pi}{2} - \int_0^\infty \frac{\lambda\sqrt{(\lambda^2-1)}\cosh\psi \tanh\psi \, d\psi}{(\lambda^2-1)\sinh^2\psi + \lambda^2}\right]\right\}$$

$$= -\frac{2\rho\bar{v}\bar{V}\delta^2 c^2 \tan\gamma}{\sqrt{(\lambda^2-1)}}\left\{\tan\gamma - \frac{2}{\pi\beta}\int_1^\infty \frac{\lambda\sqrt{(\lambda^2-1)}dt}{(\lambda^2-1)t^2+1}\right\}, \quad t = \cosh\psi$$

$$= -\frac{4}{\pi}\rho\bar{v}\bar{V}\delta^2 c^2 \tan^2\gamma \frac{\sec^{-1}\lambda}{\sqrt{(\lambda^2-1)}}$$

$$y_v = \frac{Y}{\rho\bar{v}\bar{V}S} = -\frac{4}{\pi}\delta^2 \tan\gamma \frac{\sec^{-1}\lambda}{\sqrt{(\lambda^2-1)}}$$

The yawing moment due to sideslip is similarly

$$N = -\iint 2\rho\bar{V}|u| x\delta \, dy \, dx$$

$$= -\frac{8}{3\pi}\rho\bar{v}\bar{V}\delta^2 c^3 \tan^2\gamma \frac{\sec^{-1}\lambda}{\sqrt{(\lambda^2-1)}}$$

$$n_v = -\frac{N}{\rho\bar{v}\bar{V}Ss} = -\frac{8\delta^2}{3\pi} \cdot \frac{\sec^{-1}\lambda}{\sqrt{(\lambda^2-1)}}$$

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No.	Author	Title, etc.
1	H. J. Stewart	The Lift of a Delta Wing at Supersonic Speeds. *Quarterly of Applied Mathematics*, October, 1936.
2	A. Robinson	Aerofoil Theory of a Flat Delta Wing at Supersonic Speeds. R.A.E. Report No. Aero. 2151 (A.R.C. 10222). 1946. (To be published).

APPENDIX

The Relation between Two Methods of Treating Aerodynamic Force Problems of a Delta Wing at Supersonic Speeds

1. *Introduction.*—Solutions to the problem of the lift at supersonic speeds of a flat delta wing lying within its apex Mach cone were obtained independently by Stewart[1] and by Robinson[2] by methods which at first sight appear very different. A transformation will be derived that links the two under conditions of conical flow.

2. *Hyperboloido-conal Co-ordinates.*—The co-ordinates developed in Ref. 2 were as follows :—

$$\left.\begin{aligned} x &= \frac{r\mu\nu}{k} \\ y &= \frac{r\sqrt{\{(\mu^2 - k^2)(\nu^2 - k^2)\}}}{\beta k k'} \\ z &= \frac{r\sqrt{\{(\mu^2 - 1)(1 - \nu^2)\}}}{\beta k'} \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \ldots \quad (1)$$

where $k'^2 = 1 - k^2 = \beta^2 \tan^2\gamma$

$0 \leqslant r < \infty$

$1 \leqslant \mu < \infty$

$k \leqslant \nu < 1$

The family of surfaces constituting the system are :—

$$\left.\begin{aligned} x^2 - \beta^2(y^2 + z^2) &= r^2 \\ \frac{x^2}{\mu^2} - \frac{\beta^2 y^2}{\mu^2 - k^2} - \frac{\beta^2 z^2}{\mu^2 - 1} &= 0 \\ \frac{x^2}{\nu^2} - \frac{\beta^2 y^2}{\nu^2 - k^2} + \frac{\beta^2 z^2}{1 - \nu^2} &= 0 \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \ldots \quad (2)$$

It will be observed that these co-ordinates are analogous to sphero-conal co-ordinates; in fact they correspond under the transformation $(x', y', z') = (x, i\beta y, i\beta z)$.

As $\mu \to 1$, the cones of the second family of surfaces approximate to the delta wing from both sides, and as $\mu \to \infty$ they tend to the Mach cone.

The equation $\quad -\beta^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$ .. .. (3)

now becomes :—

$$\left.\begin{aligned}&\sqrt{\{(\mu^2-k^2)(\mu^2-1)\}}\frac{\partial}{\partial\mu}\left\{\sqrt{\{(\mu^2-k^2)(\mu^2-1)\}}\frac{\partial\phi}{\partial\mu}\right\}\\&+\sqrt{\{(\nu^2-k^2)(1-\nu^2)\}}\frac{\partial}{\partial\nu}\left\{\sqrt{\{(\nu^2-k^2)(1-\nu^2)\}}\frac{\partial\phi}{\partial\nu}\right\}\\&-(\mu^2-\nu^2)\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)=0.\end{aligned}\right\} \quad \text{.. ..} \quad (4)$$

Writing $\quad \bar{\rho}=\int_\mu^\infty\frac{dt}{\sqrt{\{(t^2-k^2)(t^2-1)\}}},\ \bar{\sigma}=\int_\nu^k\frac{dt}{\sqrt{\{(t^2-k^2)(1-t^2)\}}}$

i.e.

$$\left.\begin{aligned}\mu&=ns(\bar{\rho},k)\\\nu&=knd(\bar{\sigma},k')\end{aligned}\right\} \quad \text{.. .. ..} \quad (5)$$

we have $\quad \frac{\partial^2\phi}{\partial\bar{\rho}^2}+\frac{\partial^2\phi}{\partial\bar{\sigma}^2}-(\mu^2-\nu^2)\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)=0$ .. .. (6)

Hence for conical flow $\frac{\partial^2\phi}{\partial\bar{\Phi}^2}+\frac{\partial^2\phi}{\partial\bar{\sigma}^2}=0$, where ϕ is a velocity.

As $\bar{\rho}$ varies from 0 to $K(k)$, μ varies from ∞ to 1. As $\bar{\sigma}$ varies from $-2K'(k)$ to $-K'(k)$, ν varies from k to 1 and back to k as $\bar{\sigma}$ continues through to zero, repeating as $\bar{\sigma}$ increases to $2k'$.

Equations (1) and (5) give

$$\left.\begin{aligned}x&=r\,\mathrm{ns}\,(\bar{\rho},k)\,\mathrm{nd}\,(\bar{\sigma},k')\\y&=\frac{r}{\beta}\,\mathrm{ds}\,(\bar{\rho},k)\,\mathrm{sd}\,(\bar{\sigma},k')\\z&=\frac{r}{\beta}\,\mathrm{cs}\,(\bar{\rho},k)\,\mathrm{cd}\,(\bar{\sigma},k')\end{aligned}\right\} \quad \text{.. .. ..} \quad (7)$$

To each value of $\bar{\rho}$, $\bar{\sigma}$ in the specified intervals of variation there corresponds just one triplet x, y, z for constant r on the right-hand sheet of the hyperboloid $x^2-\beta^2y^2-\beta^2z^2=r^2$. Previously we traced the (x, y, z)-plane on the ω-plane $\left(\omega=\eta+i\,\zeta=\beta\frac{y+iz}{x+r}\right)$, so that evidently there is a one to one correspondence between the points inside $|\omega|=1$ in the ω-plane and the points in the $\bar{\tau}$-plane ($\bar{\tau}=\bar{\rho}+i\bar{\sigma}$) within the specified intervals of variation of $\bar{\rho}$ and $\bar{\sigma}$.

Equation (6) shows that a function ϕ which satisfies equation (3) and is of degree zero in x, y, z satisfies Laplace's equation in $\bar{\rho}$, $\bar{\sigma}$, but any function which satisfies Laplace's equation in the ω-plane is of zero degree in x, y, z and satisfies equation (3). Hence every potential function in the ω-plane is a potential function in the $\bar{\tau}$-plane, provided the ω-plane is traced on the latter by means of the transformations given by $\omega=\beta\,(y+iz)/(x+r)$ and equations (1) and (5). Therefore the transformation is conformal.

By a transformation based on Stewart's method we previously transformed a set of points in the ω-plane into the rectangle, vertices $\tau=\pm\,2ik'$, $K\pm 2iK'$, but that set of points corresponds to the points in the (x, y, z)-plane which become, by the transformation of the previous paragraph, the " same " rectangle in the τ-plane with the vertices corresponding. It therefore follows from the general theory of conformal representations that the two transformations are identical.

We have shown that Stewart's τ-plane is connected to the system of hyperboloido-conal co-ordinates by the simple relations of equations (5). Furthermore we have given at equations

(7) a direct co-ordinate transformation between (x, y, z) and (ρ, σ), by which Stewart's relation between U, V and W as functions of τ could be established in the same manner as the relation between them as functions of the intermediate variable ω was established.

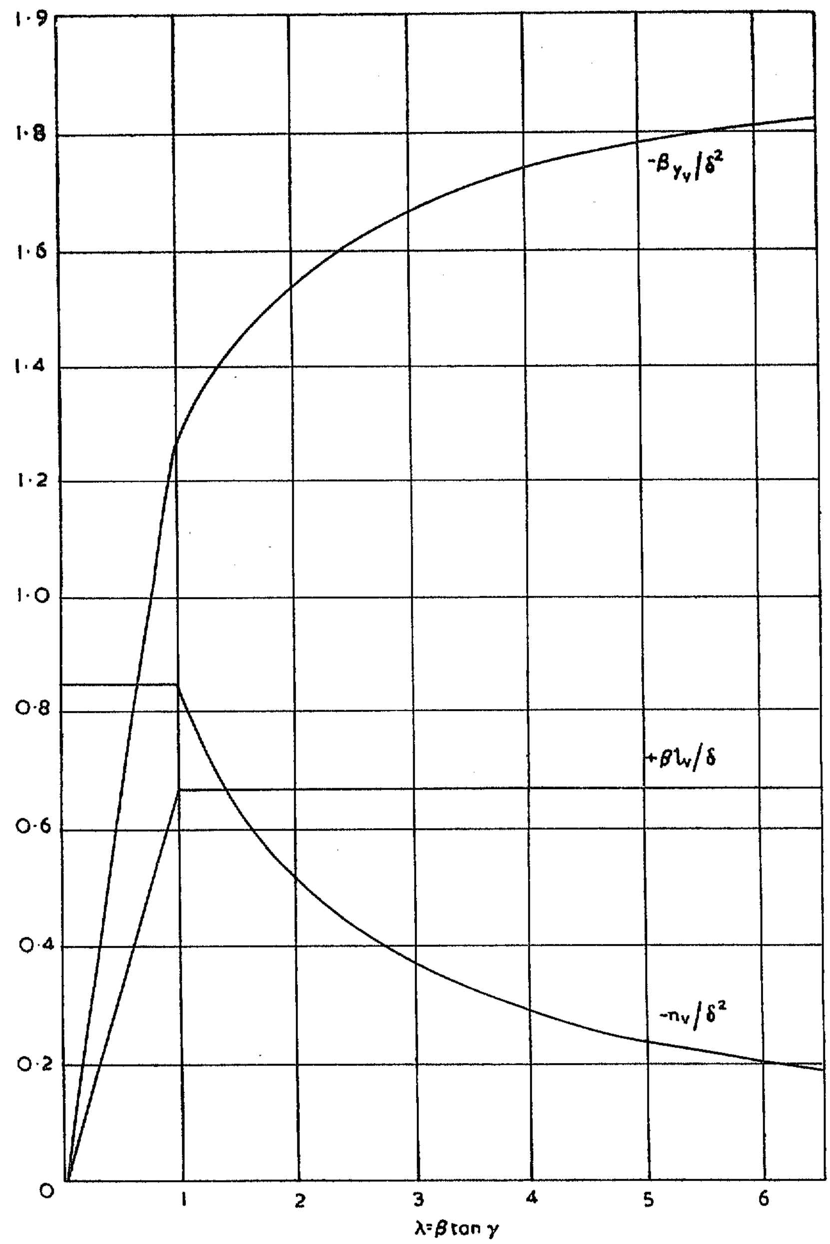


FIG. 1. Variation of derivative l_v, n_v, y_v, at zero incidence with the parameter λ.

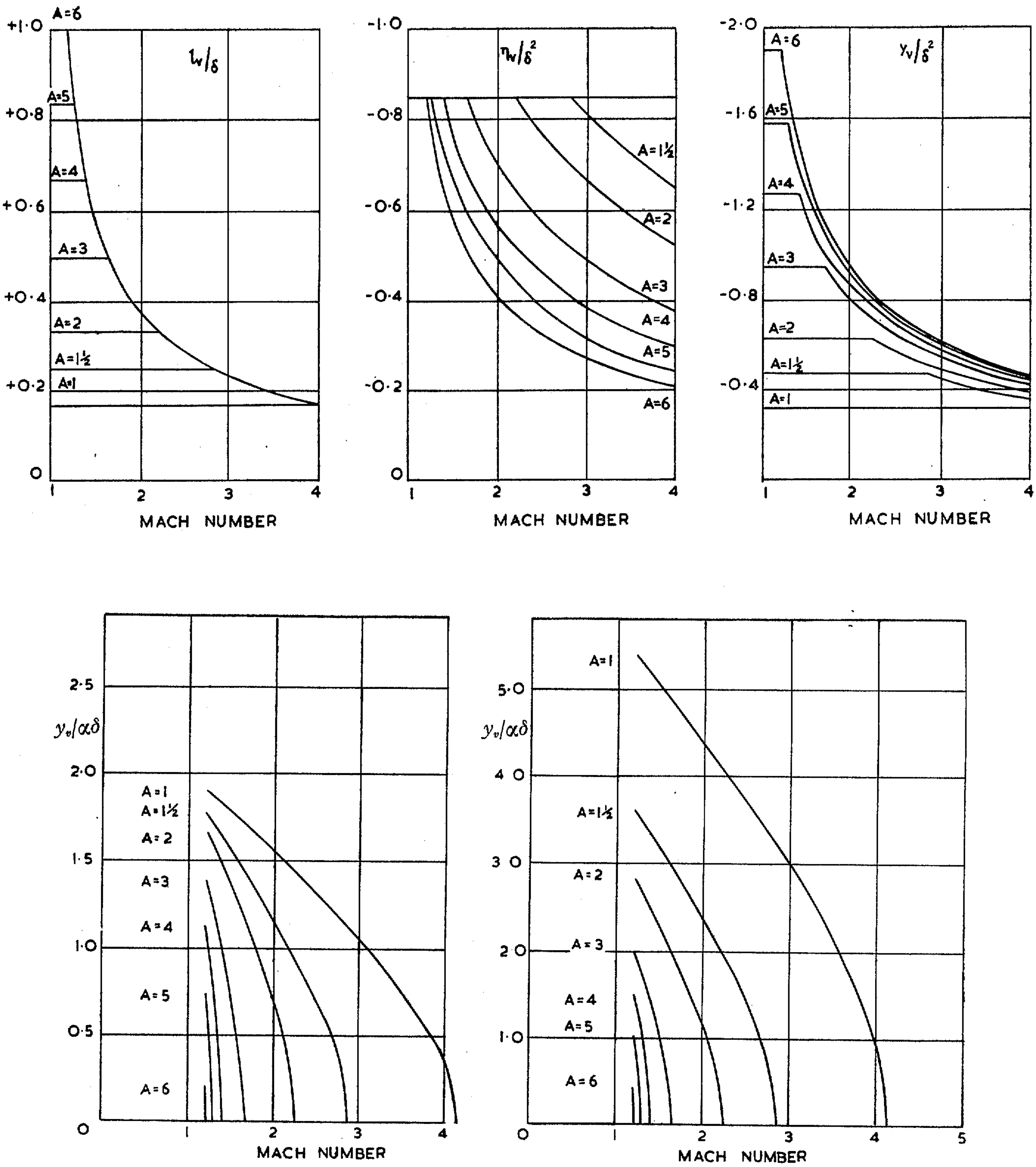
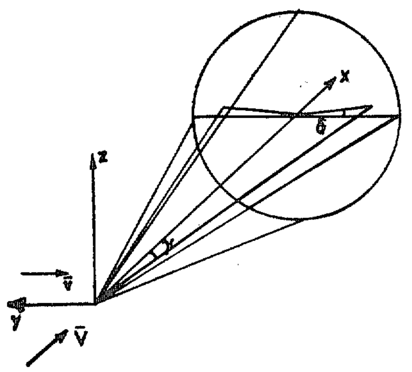
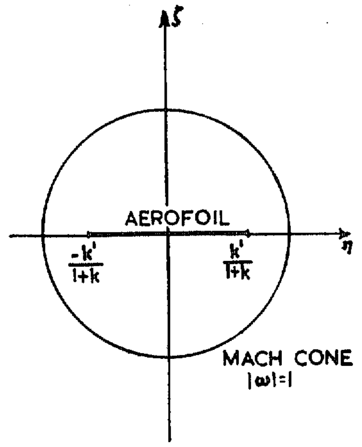


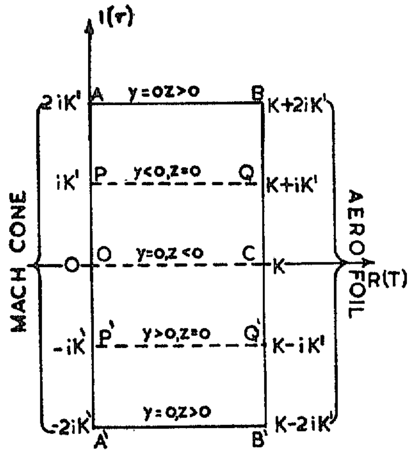
FIG. 2. Variation of derivatives l_v, n_v, y_v, at zero incidence with Mach number and aspect ratio.



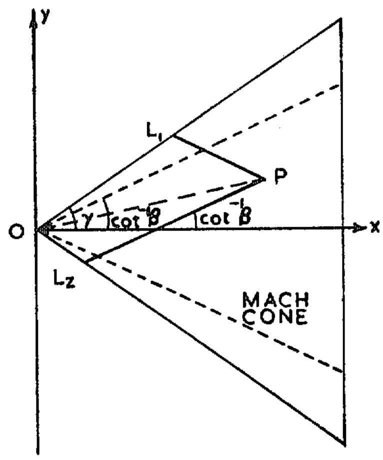
(a) THE AEROFOIL IN THE (x,y,z) FIELD



(b) THE ω-PLANE.



(c) THE τ-PLANE



(d) THE AEROFOIL FOR λ>1

FIG. 3.

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