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# An Analysis of the Longitudinal Stability and Control of a Single-Rotor Helicopter

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# An Analysis of the Longitudinal Stability and Control of a Single-Rotor Helicopter

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*Summary.*—The general theory of longitudinal stability and control for a single-rotor helicopter is presented in a form similar to that for fixed-wing aircraft. It is shown to be possible to establish for the helicopter in forward flight, in the same way as for fixed-wing aircraft, stick-fixed static and manoeuvre margins, on which the stability and handling qualities depend to a marked extent. If the static margin $K_n > 0$ the helicopter is mathematically statically stable, and the pilot requires a forward stick displacement to hold increased speed and conversely. If the manoeuvre margin $H_m > 0$, the helicopter is unlikely to be subject to rapid divergence in a disturbance, and the pilot requires a backward stick displacement for positive normal acceleration in a pull-out. Theoretical relations are derived for $K_n$ and $H_m$, in a general form covering the case of a tailplane linked to the rotor control. Relations are given also for determining $K_n$ and $H_m$ from measured control changes to trim.

An analysis is given of the growth of acceleration in a pull-out and assessment of estimated acceleration curves in terms of the National Advisory Committee for Aeronautics ' divergence requirement ' suggests that the latter may be satisfied if $H_m$ has a small positive value. Further evidence on this point will be obtained in tests now being made on a number of helicopters to study the correlation of stability and control characteristics and pilots' impressions of the handling qualities.

Extension of the theory to stick-free longitudinal stability depends on knowledge of the rotor forces on the control plane and the analysis of these forces is being considered.

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1. *Introduction.*—Increased attention has been given recently to the definition of desirable handling qualities for helicopters and to the development of stability theory in a form suitable for application in design. Methods of assessing the handling qualities of helicopters in flight tests have also been under consideration.

Work on these subjects at N.A.C.A. in America has led to the formulation of criteria for satisfactory stick-fixed longitudinal manoeuvre stability, defined in terms of the shape of the acceleration-time curve in a pull-out (Ref. 1) ; acceptable handling characteristics are claimed to depend on satisfying a ' divergence requirement ' that the normal acceleration-time curve must become concave downwards within 2 seconds of the control displacement to initiate the pull-out, and an ' anticipation requirement ' that the slope of the curve must be positive until the maximum acceleration is achieved. Recently the ' divergence requirement ' has been theoretically analysed and put in terms of relations between the aerodynamic characteristics, for the guidance of designers (Ref. 2). This first attempt to assess helicopter manoeuvre qualities in quantitative terms marks a definite advance both for design and test purposes. The criteria employed, however, are complex in comparison with that, for example, for the absence of divergent instability on a fixed-wing aircraft, namely that the manoeuvre margin should be positive (Ref. 3).

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It appears desirable to assess the handling qualities of an aircraft by the simplest adequate tests, starting with steady trimmed flight, following with steady accelerations in a pull-out or turn, and proceeding only to study the variation of acceleration with time in a pull-out, or finally full dynamic stability, if these are found to be required. This line of thought has led to an investigation being made to determine how far the static and manoeuvre margin theory can be applied to a single rotor helicopter. The theory has now been brought to a practical form for stick-fixed forward flight conditions and is given in this report.

An investigation is being made in flight tests of different helicopters to establish the correlation between the values of the static and manoeuvre margins and pilots' impressions of the handling qualities. Sufficient results from these tests are not yet available for definite conclusions to be drawn, but an indication of the correlation is obtained by using the N.A.C.A. test (which is based on pilots' impressions) to assess estimated curves of the growth of acceleration with time, for a range of values of the manoeuvre margin.

A list of symbols is given at the end of the report.

2. *General Theory for Longitudinal Motion.*—The motion is considered for a single-rotor helicopter with hinged blades. A diagram of the rotor layout in the pitching plane is given in Fig. 1. The blades are mounted on flapping hinges offset by a small distance, $e$, from the centre of the hub of rotation. The longitudinal cyclic pitch application is determined by the longitudinal tilt, $B_1$, of the control plane relative to the plane normal to the axis of rotation, forward tilt being produced by forward movement of the pilot's azimuth stick ; the no-feathering axis of the blades is normal to the control plane. The blades flap up relative to the control plane at forward speed, and the tip-path plane is tilted back due to flapping by an angle $a_1$, the value of $a_1$ depending on the rotor operating conditions. The rotor thrust is assumed to be approximately perpendicular to the tip-path plane and to act through the centre of the rotor hub. The rotor thrust in a given flight state depends on the collective pitch, $\theta$, of the rotor blades and the rotor speed, $\Omega$, which are determined by the pilot's collective pitch and throttle controls.

The analysis of the longitudinal motion of the helicopter is made on generally the same basis as for a fixed-wing aircraft in R. & M. 2075 (Ref. 4). A system of axes is taken with the origin at the centre of gravity, which is at a distance, $h$, below the hub and $k$ forward of the hub axis. The $x$-axis is forward along the wind direction in the equilibrium condition and fixed in the helicopter during the disturbed motion ; the $z$-axis is downwards in the plane of symmetry of the helicopter and perpendicular to the $x$-axis, while the $y$-axis is to starboard.

In the equilibrium condition the aircraft is moving with velocity $V_e(> 0)$ along the $x$-axis at an angle $\chi_e$ to the horizontal, $\chi_e$ being positive in climbing flight. The disturbance velocities (for the longitudinal plane only) are $u$, $w$, along the $x$- and $z$-axes respectively, and an angular velocity $q$ about the $y$-axis. It is assumed that the longitudinal motion can be considered separately from that in the lateral and directional planes ; in practice control action may be necessary by the pilot to prevent disturbances developing in the other planes.

The incidence of the helicopter is specified by the angle of incidence, $\alpha$, of the plane normal to the rotational axis to the flight path. This plane is selected as datum in preference to the control plane which is commonly used, because it appears to simplify consideration of control plane displacements ; it also appears more suitable for analysis of stick-free conditions, in which a fixed aircraft datum is preferable to the floating control plane. The angle of incidence in the equilibrium condition is $\alpha_e$ and in a disturbance $(\alpha_e + w/V_e)$. The angle of incidence of the control plane also occurs in the analysis in connection with the determination of the aerodynamic characteristics of the rotor and this is $\alpha_{pe}(= \alpha_e - B_{1e})$ in the equilibrium condition and $(\alpha_{pe} - B_{1d} - w/V_e)$ in the disturbed motion, where $B_{1d}$ is the angular displacement of the control plane from the value $B_{1e}$ in the equilibrium condition.

The aerodynamic forces may be expressed generally in the standard form. Account is taken of velocity but not of acceleration derivatives, and of the effect of changes in $B_1$ and $\theta$; the rotor speed however is assumed to be constant (the adequacy of this assumption is discussed in section 3.2) and no terms in $\Omega$ are included. Thus

$$X = X_e + X_u u + X_w w + X_q q + X_{B1} B_{1d} + X_\theta \theta$$
$$Z = Z_e + Z_u u + Z_w w + Z_q q + Z_{B1} B_{1d} + Z_\theta \theta$$
$$M = M_e + M_u u + M_w w + M_q q + M_{B1} B_{1d} + M_\theta \theta$$

where $X_e$, $Z_e$, $M_e$ are the values of $X$, $Z$, $M$ in the equilibrium conditions and $X_u$ represents $\partial X/\partial u$ and so on. The terms in $B_{1d}$ and $\theta$ represent the forces or moments due to changes from the equilibrium values, of the control plane angle and collective blade pitch. For example, $M_{B1} B_{1d}$ represents the moment applied by displacement of the control plane, which (unlike elevator displacement on a fixed-wing aircraft) also has a significant effect on $X$ and $Z$. The effects of a control plane displacement can be more clearly understood by considering firstly the effect of the change in $B_1$ at constant control plane incidence, and secondly the effect due to the change in control plane incidence. Thus

$$M_{B_1} = (M_{B_1})_{\alpha_p} + (M_{\alpha_p})_{B_1} \frac{\partial \alpha_p}{\partial B_1}.$$

Since $\delta\alpha_p = \delta(\alpha - B_1)$, $(M_{\alpha_p})_{B_1} = M_\alpha$ and $\partial\alpha_p/\partial B_1 = -1$.

Hence $M_{B_1} = (M_{B_1})_{\alpha_p} - M_\alpha$ .. .. .. .. .. .. .. (1)

In the same way, neglecting the effect of fuselage incidence changes (at constant control plane incidence) on $X$ and $Z$,

$$X_{B1} = -X_\alpha; \quad Z_{B_1} = -Z_\alpha.$$

The relations above are directly applicable only to the stick-fixed condition, but a similar development is possible for the azimuth stick-free case (the collective pitch control is normally irreversible and fixed) when the rotor and control system are mass balanced, using the condition that there is zero hinge moment on the control plane: There is as yet little information on rotor control force characteristics, but on the simplest general basis the control plane hinge-moment coefficient would be expressed as a linear function of the relevant parameters, including $B_1$ and $f$, where $f$ is a trimmer or bias spring setting. From the condition that this coefficient should be zero, $B_1$ can be expressed in terms of $f$, and $M_{B_1}$ in terms of $M_f$ and $M_\alpha$.

For generality all the derivatives included in the forms given for $X$, $Z$ and $M$ are retained at this stage in considering the motion, but several are not normally significant and could be neglected in most cases. The equations of motion are obtained by equating the sum of the inertia, gravity and aerodynamic forces or moments to zero, and by subtracting the corresponding equations for the equilibrium conditions. The final equations are of similar form to those for fixed wing aircraft with the substitution of the $B_1$ and $\theta$ terms for the elevator contributions*; they can be expressed in a similar non-dimensional form, taking $S$ to be the rotor area and the representative distance (on fixed-wing aircraft the distance from the c.g. to the tailplane) to be the rotor radius, $R$. Assuming $u$, $w$, and $q$ are proportional to $e^{\lambda\tau}$, where $\tau$, the aerodynamic time, $= t\rho SV_e/m$, $m$ being the helicopter mass, it follows from the equations that the motion depends on the roots of a stability quartic in the standard form,

$$\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \quad \text{.. .. .. .. ..} \quad (2)$$

where, with controls fixed ($B_1$, $\theta = 0$), the coefficients $B$, $C$, $D$, $E$ are corresponding functions of the inertia and aerodynamic characteristics to those defined in Ref. 4. The fundamental mathematical conditions for a stable dynamic motion are that the coefficients $B$, $C$, $D$ and $E$ should be positive and that the Routhian discriminant

$$\mathscr{R} = BCD - D^2 - B^2E \quad \text{.. .. .. .. .. .. ..} \quad (3)$$

should be positive.

* It should be noted that there would be an additional equation if rotor speed variation were included in the analysis.

The coefficient $E$ is of importance for static stability, the condition for the latter being simply that $E$ should be positive, while the coefficient $C$ is of special importance in manoeuvrability and in the response to disturbances. It is with the evaluation of these two coefficients for helicopters in stick-fixed conditions, and the consideration of their significance for stability and control, that the remainder of this report is mainly concerned. The basic approach to the subject and the method of analysis are generally similar to those adopted by Gates and Lyon for fixed-wing aircraft in R. & M. 2027 (Ref. 3) and in parts the phraseology is freely borrowed from the latter work.

3. *Static Stability*.—3.1. *Conditions for Static Stability*.—Static stability depends on the sign of $E$, which following Ref. 4, assuming $\chi$ small and in addition that the lift force is approximately equal to the rotor thrust (in effect, assuming $\alpha_p$ to be small), is given by

$$E = \frac{\mu_1}{i_E}\frac{C_T}{2}(m_w z_u - m_u z_w)$$

where

$$z_u = Z_u/\rho S V_e\,; \quad m_w = M_w/\rho S R V_e\,; \quad \text{and so on}$$

$$\mu_1 = m/\rho S R\,; \quad i_E = I/mR^2\,; \quad C_T = T/\tfrac{1}{2}\rho V_e^2 S.$$

$$I = \text{pitching moment of inertia.}$$

Alternatively,

$$E = \frac{\mu_1}{i_E}\frac{C_T}{2(\rho S V_e)^2 R}(M_w Z_u - M_u Z_w).$$

For stick-fixed conditions in steady near-level flight, since

$$Z = -T + W = 0 \text{ approximately,}$$

$$Z_u + Z_\alpha \frac{d\alpha}{dV} = 0.$$

Thus since $\alpha = \alpha_e + w/V_e$,

$$M_w Z_u - M_u Z_w = \frac{Z_\alpha}{V_e}\frac{dM}{dV}.$$

Now $dC_T/dV = -2C_T/V$ in steady flight, and for trimmed conditions, therefore, $(C_M = 0)$,

$$\frac{dM}{dV} = -C_T \rho S V_e R \left(\frac{dC_M}{dC_T}\right)_{B_1,\theta}$$

where

$$C_M = \frac{M}{\frac{1}{2}\rho V_e^2 S R}.$$

Hence, with

$$Z_\alpha = -\tfrac{1}{2}\rho V_e^2 S \frac{\partial C_T}{\partial \alpha},$$

$$E = -\frac{\mu_1}{i_E}\frac{C_T^2}{4}\frac{\partial C_T}{\partial \alpha}\left(\frac{dC_M}{dC_T}\right)$$

where both the derivatives $\partial C_T/\partial\alpha$ and $dC_M/dC_T$ are measured with $B_1$ and $\theta$ constant.

This is of similar form to the expression for $E$ for fixed-wing aircraft. In general $\partial C_T/\partial\alpha$ is positive and so $dC_M/dC_T$ must be negative for positive static stability. From the pilot's standpoint the degree of static stability determines the change of stick position to trim in steady flight (with constant $\theta$) at a speed differing from the trimmed speed. This will be shown (section 3.3) to be proportional to the out-of-balance pitching moment which would occur if the stick position could be kept constant at the changed speed ; that is, it depends on,

$$\frac{dC_M}{dV} = -\frac{2C_T}{V}\frac{dC_M}{dC_T}.$$

Thus the value of $(-\,dC_M/dC_T)$ may be taken as a measure of the degree of static stability at a given speed or $C_T$, and the static margin for a helicopter can be defined in the same way as for a fixed-wing aircraft as,

$$K_n = -\left(\frac{dC_M}{dC_T}\right)_{B_1,\,\theta} = -\frac{\partial C_M}{\partial \alpha}\frac{d\alpha}{dC_T} - \frac{\partial C_m}{\partial V}\frac{dV}{dC_T}$$

subject to the relationship,

$$\frac{dV}{dC_T} = -\frac{V}{2C_T}\,.$$

3.2. *Static Margin Analysis*.—The forces and moments on the helicopter determining the pitching moment are shown in Fig. 1. They include the rotor thrust $T$, normal to the tip-path plane ; the thrust vector is tilted forwards by an angle $B_1$ because of longitudinal pitch control application, and backwards by an angle $a_1$ because of blade flapping. There is also a transverse force $H$ in the plane of the rotor and a rotor pitching moment, $\frac{1}{2}bJe(B_1 - a_1)$, where $b$ is the number of blades and $J$ is the centrifugal force on a blade. The fuselage drag is taken to act through the centre of gravity and to be independent of fuselage incidence, but in the general case a fuselage and tailplane pitching moment is assumed in the form

$$M_F = \tfrac{1}{2}\rho V^2 SR\{C_B - a_T\,\bar{V}_T\,(\alpha + \eta_T - \varepsilon)\}$$

$C_B$ = fuselage body pitching-moment coefficient

$a_T$ = slope of the tailplane lift-coefficient curve

$\bar{V}_T$ = tail volume coefficient $S_T L/SR$, $L$ being the distance of the tailplane from the c.g., and $S_T$ the tailplane area

$\eta_T$ = tailplane setting to the fuselage

$\varepsilon$ = allowance for rotor downwash effect

If the tailplane is linked to the cyclic pitch control, so that

$$\eta_T = \eta_0 + \eta_B B_1\,,$$

then

$C_{F_0} = C_B - a_T\bar{V}_T\eta_0$, where $C_{F_0}$ is the fuselage and tailplane pitching-moment coefficient for $B_1 = 0$. Thus putting $\alpha = \alpha_p + B_1$,

$$M_F = \tfrac{1}{2}\rho V^2 SR[C_{F_0} - a_T\bar{V}_T\{\alpha_p - \varepsilon + (1 + \eta_B)B_1\}].$$

Putting $C_H = H/\frac{1}{2}\rho V^2 S$ and $C_J = \frac{1}{2}bJ/\frac{1}{2}\rho V^2 S$, the pitching-moment equation is therefore

$$C_M = 0 = -\,C_T\left[(B_1 - a_1)\frac{h}{R} + \frac{k}{R}\right] + C_H\frac{h}{R} - C_J\frac{e}{R}(B_1 - a_1) + C_{F_0}$$

$$-\,a_T\bar{V}_T\{\alpha_p - \varepsilon + (1 + \eta_B)\,B_1\}$$

$$= -\,B_1 C_T\frac{h_\eta}{R} + a_1 C_T\frac{h_e}{R} - C_T\frac{k}{R} + C_H\frac{h}{R} + C_{F_0} - a_T\bar{V}_T\,(\alpha_p - \varepsilon) \quad .. \quad (4)$$

where

$$\frac{h_e}{R} = \frac{h}{R} + \frac{C_J}{C_T}\frac{e}{R}$$

$$\frac{h_\eta}{R} = \frac{h_e}{R} + (1 + \eta_B).\frac{a_T\bar{V}_T}{C_T}\,.$$

For stick-fixed conditions, $B_1$ and $\theta$ are constant. Thus differentiating

$$K_n = -[C_M']_{B_1,\theta} = B_1\left(\frac{h}{R} + C_J'\frac{e}{R}\right) - a_1\left(\frac{h}{R} + C_J'\frac{e}{R}\right)$$

$$- a_1'.C_T\frac{h_e}{R} + \frac{k}{R} - C_H'\frac{h}{R} + a_T\bar{V}_T(\alpha_p' - \varepsilon') \quad \ldots \quad (5)$$

The values of the aerodynamic derivatives to be used in determining the static margin depend on the rotor operating conditions. For stick-fixed stability, the cyclic pitch and the collective pitch are constant, and strictly, the throttle setting should also be constant. With constant throttle setting, however, the rotor speed varies to some extent with the flight state, and although N.A.C.A. estimates (Ref. 2) have shown the effect in, for example, a pull-out to be small (partly due to the effect of rotor inertia), it is more satisfactory to assume in the analysis that the rotor speed is kept constant by throttle adjustments made either manually by the pilot or by an automatic throttle governor. The assumption of constant rotor speed has the advantage that although the small changes in engine torque due to throttle action will not have a marked effect on the aerodynamic derivatives, a more straightforward comparison is possible between steady flight states in which the speed, or normal acceleration, differ because of control displacement.

The values of the aerodynamic derivatives are considered in the Appendix for a simple rotor system with blades of constant chord and angle along their length ; blade and control circuit distortion and unsteady aerodynamic effects are neglected. In particular, it is shown that $C_J' = C_J/C_T$, and eliminating $B_1$ by means of (4),

$$K_n = -a_1\frac{h_e}{h_\eta}\frac{(1+\eta_\beta)a_T\bar{V}_T}{C_T} - a_1'\,C_T\frac{h_e}{R} + \frac{k}{h_\eta}\frac{(1+\eta_\beta)a_T\bar{V}_T}{C_T}$$

$$-\frac{h}{R}\left(C_H' - \frac{C_H}{C_T}\frac{h_e}{h_\eta}\right) + \frac{C_{F0}}{C_T}\frac{h_e}{h_\eta} + a_T\bar{V}_T\left[\alpha_p' - \varepsilon' - \frac{h_e}{h_\eta}\frac{\alpha_p - \varepsilon}{C_T}\right].$$

If there is no tailplane and the fuselage pitching moments are small, this reduces to,

$$K_n = -a_1'C_T\frac{h_e}{R} - \frac{h}{R}\left[C_H' - \frac{C_H}{C_T}\right].$$

It is shown in the Appendix that, for a rotor with blades of constant chord and angle,

$$a_1' = -\frac{a_1}{2C_T}\frac{1-\mu^2}{1+\mu^2}$$

$$C_H' = \left[\frac{C_H}{2C_T}(1-\tfrac{3}{2}\mu^2) + \frac{3\mu\sigma C_D}{4C_T}\right]\frac{1}{(1+\tfrac{3}{2}\mu^2)}$$

where $C_D$ is the mean blade profile-drag coefficient,

$\sigma = bc/\pi R$, the rotor solidity, $c$ being the blade chord.

Thus

$$K_n = \frac{a_1}{2}\frac{h_e}{R}\frac{1-\mu^2}{1+\mu^2} + \frac{h}{R}\left[\frac{C_H}{C_T}(1+3\mu^2) - \frac{3\mu\sigma C_D}{4C_T(1+\tfrac{3}{2}\mu^2)}\right]$$

Formulae for $a_1$ and $C_H$ are given in the Appendix.

3.3. *Static Margin in Terms of Control Changes to Trim.*—In steady trimmed flight, the rate of change of pitching moment, with controls fixed, due to variation of speed and incidence, must be balanced by the rate of change of pitching moment due to control application, which may include changes in both control plane angle and collective pitch. The steady flying qualities of the helicopter however are normally assessed with constant collective pitch, and the effect of control plane displacement only is considered for determination of the static margin. Thus,

$$(C_M')_{B_1,\theta} = \frac{\partial C_M}{\partial B_1}\left(\frac{dB_1}{dC_T}\right)_{C_{M'} = 0,\ \theta = \text{constant}}$$

and so from (4),

$$K_n = -\, C_T \frac{h_\eta}{R}\left(\frac{dB_1}{dC_T}\right)_{C_{M'},\theta} \quad \ldots \quad (6)$$

The static margin can therefore be determined from flight measurements of cyclic pitch to trim. The value of $h_\eta/R$ can be found from the difference, $\delta B_1$, in control position to trim in level flight at the same weight and speed, with two differing longitudinal centre of gravity positions, at $k$ and $(k + \delta k)$. If the variation of fuselage drag with incidence is neglected, it follows from the pitching moment equation ($\alpha_p$ being unchanged) that

$$\frac{h_\eta}{R} = -\,\frac{\delta k/R}{\delta B_1}.$$

Thus

$$K_n = C_T\left(\frac{\delta k/R}{\delta B_1}\right)\left(\frac{dB_1}{dC_T}\right)_{C_{M'},\theta}.$$

Equation (6) may also be written in the form

$$\frac{dB_1}{dV} = \frac{2}{V}\,\frac{K_n}{\dfrac{h}{R} + \dfrac{C_J}{C_W}\dfrac{e}{R} + (1 + \eta_B)\,\dfrac{a_T\bar{V}_T}{C_W}}$$

where $C_W = W/\frac{1}{2}\rho V^2 S$.

This relation gives the rate of change of longitudinal cyclic pitch application with speed in terms of the static margin. The corresponding azimuth stick control movement depends on the gear ratio between the control stick and the rotor control plane. Forward movement of the stick corresponds to increasing $B_1$, and it will be seen that a positive static margin results in a forward stick displacement for a higher speed.

4. *Approximate Theory of Response to Control in a Pull-out.*—4.1. *Condition for Stability in a Pull-out Manoeuvre.*—Attention is now turned to a pull-out manoeuvre to determine the relationship between the pilot's control action and the normal acceleration developed in the pull-out. It is assumed that the manoeuvre is made at constant speed in near level flight and that the change in the normal component of weight during the manoeuvre is insignificant. The effect of both cyclic and collective pitch applications is considered in the first instance. With the system of axes defined in section 2, the equations of motion at constant speed for which $u$, $\dot{u}$ are zero and $Z_q$ is neglected, are

$$-\, m(\dot{w} - V_e q) + Z_w w + Z_B B_1 + Z_\theta \theta = 0$$

$$-\, I\dot{q} + M_w w + M_q q + M_B B_1 + M_\theta \theta = 0\,.$$

Assuming $w$, $q$ proportional to $e^{\lambda\tau}$ shows the motion with controls fixed to be governed by the roots of the equation,

$$\lambda^2 + B\lambda + C = 0$$

where $B$ and $C$ are effectively the same as the corresponding coefficients in the stability quartic, and are given by

$$B = -\, z_w + \frac{m_q}{i_E}$$

$$C = \left(\frac{z_w m_q}{i_E} - \frac{m_w}{i_E}\,\mu_1\right).$$

Assuming the motion approaches a steady condition, this is a steady circle in which

$$Z_a = -mV_e \frac{dq}{d\alpha} . \qquad (7)$$

Thus,

$$C = -\frac{1}{I} \frac{dM(\alpha, q)}{d\alpha} \left(\frac{m}{\rho S V_e}\right)^2$$

$$= -\frac{\mu_1}{2i_E} \frac{\partial C_T}{\partial \alpha} \left[\frac{dC_M(\alpha, q)}{dC_T}\right]_{B_1, 0} .$$

This is of a similar form to that for a fixed-wing aircraft. It can also be shown from the above analysis that a pitching divergence is not to be expected in a quick manoeuvre if $C$ is positive, and it is reasonable therefore to define the manoeuvre margin for a helicopter in the standard form,

$$H_m = -\left[\frac{dC_M(\alpha, q)}{dC_T}\right]_{B_1, 0}$$

with the equilibrium condition for the steady circle at constant speed, from (7),

$$\frac{dq}{dC_T} = \frac{V_e}{R} \frac{1}{2\mu_1} .$$

4.2. *Manoeuvre Margin Analysis.*—The manoeuvre margin can be obtained from the pitching-moment equation (4) in the same way as the static margin. Thus

$$H_m = -(C_M')_{B_1, 0} = -\left[\frac{\partial C_M}{\partial C_T} + \frac{\partial C_M}{\partial q} \frac{dq}{dC_T}\right]$$

where

$$\frac{dq}{dC_T} = \frac{V_e}{R} \frac{1}{2\mu_1} = \frac{\rho S V_e}{2m} .$$

As in the case of the static margin, the rotor speed is assumed to be constant. The derivatives for the conditions in the pull-out are determined in the Appendix for a rotor with blades of constant chord and angle ; in particular,

$$\frac{\partial a_1}{\partial q} = -\frac{16}{\gamma \Omega}$$

where $\gamma = \rho a c R^4 / I_b$, the blade inertia number, $I_b$ being the blade moment of inertia about the flapping hinge.

Also $C_J' = 0$ and $\partial \alpha_T / \partial q = L / V_e$, $\alpha_T$ being the tailplane incidence.

Thus differentiating (4) and eliminating $B_1$, the controls-fixed manoeuvre margin is found to be

$$H_m = -a_1 \frac{(1 + \eta_B) a_T \bar{V}_T}{C_T} - C_T \frac{h_e}{R} \left[\frac{\partial a_1}{\partial C_T} - \frac{8\rho S V}{\gamma \Omega m}\right] + \frac{k}{R}\left(1 - \frac{h}{h_\eta}\right) - \frac{h}{R}\left[C_H' - \frac{C_H}{C_T} \frac{h}{h_\eta}\right]$$

$$+ \frac{C_{F0}}{C_T} \frac{h}{h_\eta} + a_T \bar{V}_T \left[(\alpha_p' - \varepsilon') + \frac{h}{h_\eta} \frac{\alpha_p - \varepsilon}{C_T} + \frac{\rho S L}{2m}\right] . \qquad (8)$$

If there are no fuselage or tailplane pitching moments this reduces to,

$$H_m = -C_T \frac{h_e}{R} \left[\frac{\partial a_1}{\partial C_T} - \frac{8\rho S V}{\gamma \Omega m}\right] + \frac{k}{R}\left[1 - \frac{h}{h_e}\right] - \frac{h}{R}\left[C_H' - \frac{C_H h}{C_T . h_e}\right] .$$

From the Appendix, for blades of constant chord and length,

$$\frac{\partial a_1}{\partial C_T} = \frac{4\mu^3}{a\sigma} \quad ; \quad C_H' = -\mu\left[\theta(1 + \tfrac{1}{3}\mu^2) - \frac{4C_T\mu^2}{a\sigma}\right],$$

where $a$ is the slope of blade lift-coefficient curve.

4.3. *Manoeuvre Margin in Terms of Control Changes to Trim.*—For the steady manoeuvre state, the change of pitching moment with controls fixed due to variation in the thrust and the rate of pitch must be balanced by a pitching moment due to control application, including possibly changes in both $B_1$ and $\theta$. Thus

$$H_m = -\left(\frac{dC_M}{dC_T}\right)_{B_1, \theta} = \frac{\partial C_M}{\partial B_1}\left(\frac{dB_1}{dC_T}\right)_{C_M=0} + \frac{\partial C_M}{\partial \theta}\left(\frac{d\theta}{dC_T}\right)_{C_M=0}$$

and so from (4) neglecting variation in the downwash $\varepsilon$,

$$H_m = -C_T\frac{h_\eta}{R}\left(\frac{dB_1}{dC_T}\right)_{C_M=0} + \left[\frac{\partial a_1}{\partial \theta}C_T\frac{h_e}{R} + \frac{\partial C_H}{\partial \theta}\frac{h}{R} - a_T\bar{V}_T\frac{\partial \alpha_p}{\partial \theta}\right]\left(\frac{d\theta}{dC_T}\right)_{C_M=0}.$$

From the Appendix, for blades of constant chord and angle,

$$\frac{\partial a_1}{\partial \theta} = \frac{4}{3}\frac{\mu}{1+\mu^2} \quad ; \quad \frac{\partial C_H}{\partial \theta} = \frac{\sigma a}{3\mu}\left[\tfrac{2}{3}\theta(1 + \tfrac{3}{2}\mu^2) - \frac{3C_T\mu^2}{a\sigma}\right]$$

$$\frac{\partial \alpha_p}{\partial \theta} = -\frac{2}{3\mu(1 - \frac{3}{2}\mu^2)}.$$

It will be noted that all parts of $\partial C_M/\partial\theta$ are positive except part of the second term ; this is less however than the first term, for practical values of $\mu$, so that $\partial C_M/\partial\theta$ is positive.

The manoeuvre margin can therefore be determined from flight measurements of cyclic and collective pitch values to trim for a range of steady accelerations in pull-outs at the same speed. It also follows since $C_T = (1 + n)C_W$ that

$$H_m = -(1+n)\frac{h_\eta}{R}\left(\frac{dB_1}{dn}\right)_{C_M=0} + \left[\frac{\partial a_1}{\partial \theta}(1+n)\frac{h_e}{R} + \frac{\partial C_H}{\partial \theta}\frac{h}{R} - \frac{a_T\bar{V}_T}{C_W}\frac{\partial \alpha_p}{\partial \theta}\right]\left(\frac{d\theta}{dn}\right)_{C_M=0}.$$

This relation gives the rate of change of longitudinal cyclic pitch and collective pitch applications at constant speed in terms of the manoeuvre margin.

It will be seen that at constant collective pitch, a positive $H_m$ results in a backwards control movement (decreasing $B_1$) for positive normal acceleration ; alternatively (since $\partial C_M/\partial\theta > 0$) an increase in collective pitch is required if the cyclic pitch is constant. A study is required of the common type of pull-out manoeuvre in which both cyclic and collective pitch control changes are made by the pilot, to determine effective and acceptable combinations of control displacements. As already noted in connection with the static margin, however, the steady flying qualities of the helicopter depend more on the characteristics for constant collective pitch. The stick-fixed characteristics of the helicopter are of a similar nature whether the manoeuvre is initiated by use of one control only or by an equivalent combination of both, and for simplicity, further analysis, much of which is concerned with the growth of acceleration in a pull-out, is confined to the case of motion for constant collective pitch. The relation of the manoeuvre margin to the rate of change of trim with acceleration at constant collective pitch then takes the simpler form

$$H_m = -\left[(1+n)\frac{h}{R} + \frac{C_J}{C_W}\frac{e}{R} + (1+\eta_B)\frac{a_T\bar{V}_T}{C_W}\right]\left(\frac{dB_1}{dn}\right)_{C_M=0}.$$

In pull-out tests to determine the manoeuvre margin, it is possible for the rate of growth of acceleration to be so slow that the steady acceleration corresponding to a given control displacement is difficult to achieve. It may be easier to determine the trim changes for the steady acceleration in a turn, and the manoeuvre margin can be determined by comparing stick positions in the turn and in straight flight at the same speed.

The difference between the stick position in the turn and in steady straight pull-out with the same normal acceleration arises from the difference in the rate of pitching in the two states. To eliminate some of the secondary pitching-moment effects from the main and tail rotors it is convenient to take mean values for $B_1$ from turns in near-level flight, made in both directions. If the flight path deviates markedly from level flight it may be necessary to make allowance for inertia couples (Ref. 3).

In a steady pull-out with normal acceleration $ng$, $q = ng/V$.

In a turn with normal acceleration $ng$, $q = \frac{g}{V}\left(n + 1 - \frac{1}{n+1}\right)$.

It follows from the pitching-moment equation (4), using the values of $\partial a_1/\partial q$ and $\partial \alpha_T/\partial q$ from section 4.2, that

$$(B_1)_{\text{pull-out}} = (B_1)_{\text{turn}} + \left[\frac{h_e}{h_\eta}\frac{16g}{\gamma \Omega V}\frac{n}{n+1} + \frac{a_T \bar{V}_T}{C_W}\frac{R}{h_\eta}\frac{Lg}{V^2}\frac{n}{(n+1)^2}\right].$$

Thus the manoeuvre margin may be determined from the trim curves in turns from the relationship,

$$H_m = -(1+n)\frac{h_\eta}{R}\left[\left(\frac{dB_1}{dn}\right)_{\text{turn}} + \left\{\frac{h_e}{h_\eta}\frac{16g}{\gamma \Omega V}\frac{1}{(n+1)^2} + \frac{a_T \bar{V}_T}{C_W}\frac{R}{h_\eta}\frac{Lg}{V^2}\frac{(1-n)}{(1+n)^3}\right\}\right].$$

4.4. *Growth of Acceleration in the Pull-out.*—A positive value for the manoeuvre margin ensures that the helicopter is not subject to a rapid divergence and that a stable control movement is required to develop a steady acceleration. It is important also, however, to have some knowledge of the growth of the acceleration to ensure that sufficient time is allowed to reach the steady acceleration in pull-out tests. In addition, there is evidence (Ref. 1) that pilots consider that the general flying characteristics of a helicopter depend to a marked extent on the shape of the acceleration-time curve in a manoeuvre, and the fuller significance of the manoeuvre margin can be judged by the influence the value of $H_m$ has on the way in which the growth of acceleration occurs.

From the equations of motion at constant speed, it follows as in section 4.1, assuming $w$, $q$ proportional to $e^{\lambda t}$ (using $t$ in preference to $\tau$ for a particular application) that the motion with controls fixed depends on the roots of the equation,

$$\lambda'^2 + B'\lambda' + C' = 0 \qquad \text{(9)}$$

where

$$\left.\begin{aligned} B' &= -\left(\frac{M_q}{I} + \frac{Z_w}{m}\right) \\ C' &= \left(\frac{Z_w}{m}\frac{M_q}{I} - \frac{M_w V_e}{I}\right) \end{aligned}\right\} \qquad \text{(10)}$$

In studying the effect of $H_m$ on the motion it is useful to note that

$$\begin{aligned} C' &= -\frac{1}{I}\frac{dM(\alpha, q)}{d\alpha} \\ &= \frac{RT_a}{I} H_m . \end{aligned}$$

The steady acceleration in the ultimate state of the motion can be determined in terms of $H_m$ as in section 4.3; putting $\delta C_T = n_s C_w$ and $\delta B_1 = B_S$ where $n_s g$ is the steady increment in the acceleration and $B_S$ is the fixed control displacement, $n_s$ is given, for constant collective pitch, by

$$n_s = \frac{1}{C_W}\frac{\partial C_M}{\partial B_1}\frac{B_S}{H_m} . \qquad (11)$$

The significance of the analysis in section 2 of the $M_{B_1}$ term in the equations of motion can be seen from an alternative derivation of this relation (11), from the equations of motion in the steady state for which $\dot{w} = \dot{q} = 0$. Eliminating $w$ from the equations it is found that

$$n_s = \frac{V_e q}{g} = \frac{Z_a}{mg}\frac{(M_a + M_{B_1})}{(dM/d\alpha)_{B_1}} B_S .$$

Now from (1),

$$M_{B_1} = (M_{B_1})_{a_p} - M_a$$
$$= (M_{B_1})_{C_T} - M_a \text{ at constant speed,}$$

and, as before,

$$n_s = \frac{1}{C_W}\frac{\partial C_M}{\partial B_1}\frac{B_S}{H_m} .$$

The derivative $\partial C_M/\partial B_1$ depends on $C_T$, and a mean value is taken corresponding to the mean acceleration $n_m g$, so that,

$$\frac{n_s}{B_S} = -\frac{h_m}{R}\frac{1}{H_m}$$

where

$$\frac{h_m}{R} = (1 + n_m)\frac{h}{R} + \frac{C_J}{C_W}\frac{e}{R} + \frac{a_T \bar{V}_T}{C_W}(1 + \eta_B). \qquad (12)$$

For a positive value of the manoeuvre margin, the roots of the quadratic (9) may be both real and negative, or may both consist of negative real and imaginary parts. For real roots $B'^2 - 4C' > 0$, so that

$$0 < H_m < \frac{I}{4RT_a} B'^2 .$$

If the roots are $-b'/2 \pm \xi$, where $\xi = \frac{1}{2}(B'^2 - 4C')^{1/2}$, the growth of the acceleration, from an initial steady condition in which $w = q = 0$, following the instantaneous displacement of the control plane by the angle $B_s$, is given by the relation,

$$\frac{n}{B_S} = \frac{T_a e^{-B't/2}}{WV_e\xi}\left[\frac{Tg}{W}\sinh \xi t + V_e\left(\frac{W}{T_a}\frac{h_m}{RH_m} - 1\right)\left(\frac{B'}{2}\sinh \xi t + \cosh \xi t\right)\right] - \frac{h_m}{RH_m} .$$

When $H_m > \frac{I}{4RT} B'^2$, the roots are of the form $- B' \pm i\zeta$, where $\zeta = \frac{1}{2}(4C' - B'^2)^{1/2}$. The growth of acceleration is given by

$$\frac{n}{B_S} = \frac{T_a e^{-B't/2}}{WV_e\zeta}\left[\frac{T_a g}{W}\sin \zeta t + V_e\left(\frac{W}{T_a}\frac{h_m}{RH_m} - 1\right)\left(\frac{B'}{2}\sin \zeta t + \zeta \cos \zeta t\right)\right] - \frac{h_m}{RH_m} .$$

For $H_m \leqslant 0$, the motion is divergent and does not attain a steady state. When $H_m = 0$, the growth of the acceleration is given by,

$$\frac{n}{B_S} = \frac{T_a}{WB'}\left[\left(\frac{Wh_m}{B'I} + \frac{Tg}{WV_e}\right)(1 - e^{-B't}) - \frac{Wh_m}{I} t\right] - \frac{T_a}{W} .$$

Estimates of the acceleration have been made, using these relations, for a single-rotor helicopter, for three values of $B'$ covering a range of aerodynamic derivatives and speed conditions, and a range of values of $H_m$. The variation in $H_m$ may be assumed to arise mainly through changes in $M_\alpha$, which does not affect $B'$; for simplicity $h_m/R$ is assumed to be constant, though this might normally vary with $H_m$ and $M_\alpha$. The characteristics assumed for the helicopter are :

$$W = 5{,}000 \text{ lb.} ; \quad I = 7{,}000 \text{ slug-ft}^2 ; \quad \Omega = 28 \cdot 3 \text{ radn/sec} ; \quad \gamma = 9 \cdot 35 ;$$

$$R = 24 \text{ ft} ; \quad h_m/R = 0 \cdot 155 ; \quad gT_a/W = 120 \text{ ft/sec}^2$$

Curves of $n/B_s$ against time are plotted in Fig. 2, for values of $H_m$ from 0 to $0 \cdot 015$ for the case of $B' = 1 \cdot 2$, and in Fig. 3 for selected values of $H_m$ for $B' = 0 \cdot 8$ and $2 \cdot 0$. There is in all cases an instantaneous build-up of acceleration to $- T_a B_s/m$ corresponding to the assumed immediate increase in thrust when control is applied. This is followed by a slight decrease in $n$, before the main trend of the growth of acceleration becomes apparent. For $H_m = 0$, the motion is a divergence and this would be more rapid for $H_m < 0$. For $0 < H_m < IB'^2/4RT_a$, the motion is steadily asymptotic, after the initial stage, to the ultimate acceleration value, while for $H_m > IB'^2/4RT_a$, the curve achieves a maximum value in a time decreasing as $H_m$ is increased.

The early decrease in acceleration is clearly an undesirable feature, and as stated in the N.A.C.A. 'anticipation requirement' (Ref. 1), the slope of the curves should remain positive until the maximum acceleration is attained. The fall-off in acceleration is due to rapid damping of the initial acceleration and the damping can only be reduced at the expense of the characteristics later in the motion. The effect may be less marked, as suggested in Ref. 5, with a linked rotor and tailplane control system, because due to the additional control power, a smaller rotor tilt is required for a given acceleration ; there is in consequence a smaller initial acceleration and the reduction is relatively smaller compared to the final value. In terms of the above analysis, if $n = n_0$ at $t = 0$, it can be shown that (providing $H_m > 0$),

$$\frac{n_0}{n_s} = \frac{T_a}{W} \frac{R}{h_m} H_m.$$

It is apparent from (12) that the addition of the $\eta_B$ term for the linked tailplane can have a considerable effect on $h_m$ and if, as appears possible from (8), $H_m$ is not greatly affected by the $\eta_B$ terms, $n_0/n_s$ will be appreciably reduced. In practice there would of course be a less marked fall-off in acceleration than estimated because the control action and thrust increase would not be instantaneous in the manner assumed in the analysis.

The shape of the acceleration-time curves following the reduction stage shows that the ultimate acceleration is approached relatively slowly for $H_m < 0 \cdot 01$ ; it would be difficult in such cases to establish the steady state in pull-out tests and the manoeuvre margin should be determined from trim changes in turns. An assessment can be made of the shape of the curves in terms of the N.A.C.A. divergence requirement that the shape of the curve should become concave downwards within 2 seconds, which is equivalent to the condition that $d^2n/dt^2$ should become negative before $t = 2 \cdot 0$. The variation of $d^2n/dt^2$ with $H_m$ at $t = 2 \cdot 0$ for the set of curves in Fig. 2 is shown in Fig. 4. The value of $H_m$ to meet the requirement varies with $B'$, the largest value being $H_m = 0 \cdot 0085$ for $B' = 0 \cdot 8$. N.A.C.A. specify that the requirement is to be met at higher speeds, which correspond to lower values of $B'$, since the second (and normally dominant) part in the form given in (10) is equal to $T_a/mV_e$.

These results suggest that an assessment of the manoeuvre stability equivalent to the N.A.C.A. test may be obtained from the simpler criterion that the manoeuvre margin should be greater than a small positive value. Further evidence on this point will be obtained from flight tests now being made on a number of helicopters, to establish their general stability and control characteristics for correlation with pilots' impressions of the handling qualities. An analysis is also to be made of American test data (Ref. 1) for a helicopter without and with a tailplane, either fixed or linked to the rotor control.

5. *Dynamic Stability.*—Since the theory of the longitudinal motion of a helicopter has been presented in a similar form to that for a fixed-wing aircraft, many of the general arguments used in, for example, Ref. 3 to show the significance of the static and manoeuvre margins with reference to the dynamic stability, can also be applied to the helicopter. It is important, however, to remember the approximate assumptions in the development of the theory for a helicopter for both static and manoeuvre margins ; these included the assumption of near-level flight ($\chi$ small) and that the rotor thrust can be equated to the lift force normal to the flight path ($\alpha_p$ small). The theory can therefore be expected to be less accurate at lower speeds and the following discussion is to be taken to apply only to flight states in which the approximations are valid.

It appears, from the form of the aerodynamic derivatives in the coefficient $B$ in the complete quartic (2) governing the dynamic stability motion, that $B$ is large and positive for a helicopter (as well as for a fixed wing aircraft). The coefficient $D$, which is approximately equal to $\frac{1}{2}\mu_1 m_u C_T/i_E$, is small, while $E$ and $C$ are approximately proportional to the static and manoeuvre margins respectively. The conditions for a stable dynamic motion are that all the coefficients and the Routhian discriminant should be positive. Following the argument in Ref. 3, it may be said that if all the coefficients are positive except $E$, there is a divergence which is relatively slow because it depends on speed changes which take time to develop. On a helicopter however if $E$ is negative it is probably because of a large positive value of $\partial C_M/\partial\alpha$, so that $C$ also may be negative and the resultant divergence is likely to be rapid because it is a pitching divergence which can occur without change of speed.

It is also possible, if $\partial C_M/\partial V$ is large, for a positive static margin or $E$ to be combined with a very small or negative manoeuvre margin or $C$ . In this case there may be an unstable oscillation associated with a negative value of the Routhian discriminant (3). Neglecting the small term in $D^2$, the discriminant condition shows instability to occur when $(CD - BE) \leqslant 0$ and $\mathscr{R}$ may therefore be negative while $C$ and $D$ are positive, if $E$ is still positive. If $C$ is negative there will be a more unstable oscillation or a rapid divergence (combined with a slower one, since there must be two positive roots when $E > 0$). A negative manoeuvre margin is therefore a definite indication of serious instability.

These qualititative arguments provide a useful indication of the general significance of the values of $K_n$ and $H_m$ for the dynamic stability motion. More definite information is desirable on the relative magnitudes of the coefficients, and of the effect of, for example, a fixed or controlled tailplane, and detailed estimates of the motion for representative helicopter types are to be made.

6. *Discussion and Conclusions.*—Summarising the course of the analysis, in the first place, the theory of the longitudinal motion of a single rotor helicopter in forward flight has been presented in the standard form for fixed-wing aircraft. It has been shown possible to establish for the helicopter, forms of the stick-fixed static and manoeuvre margins.

The static margin $K_n$ is proportional to the coefficient $E$ in the stability quartic and the helicopter is statically stable if $K_n > 0$ ; to the pilot a positive value of $K_n$ means that (for constant collective pitch) a forward stick displacement (proportional to $K_n$) is required to hold an increased speed and conversely.

The manoeuvre theory is based on an approximate analysis of the motion in a pull-out, in which changes in speed and in the normal component of gravity are neglected. The manoeuvre margin, $H_m$, is approximately proportional to the coefficient $C$ in the stability quartic ; a rapid divergence is unlikely to occur in a disturbance if $H_m > 0$ and $H_m$ is therefore some measure of the stability following control application for a pull-out. In addition, to the pilot, a positive value of $H_m$ ensures that a stable (backwards) stick displacement proportional to $H_m$ is required (for constant collective pitch) to produce positive normal acceleration.

Theoretical relations have been derived in a general form for a helicopter with a tailplane, for both $K_n$ and $H_m$, and relations are given for determining them in terms of measured control changes to trim; for constant collective pitch and rotor speed, $K_n$ is related to the change of stick position with speed, and $H_m$ to the change of stick position with steady acceleration at the same speed.

Estimates made of the growth of acceleration in a pull-out for a range of helicopter characteristics have confirmed the motion to be divergent for $H_m \leqslant 0$. Also the ultimate steady acceleration is attained only slowly for possible small values of $H_m$; it appears preferable therefore to rely on the trim changes in steady turns for determination of $H_m$.

The estimated acceleration-time curves have been considered in relation to the N.A.C.A. 'divergence requirement' (essentially that $d^2n/dt^2$ should become negative in less than 2 seconds) and it has been found that the requirement is met in all the cases considered for a small value of $H_m$. This suggests that the manoeuvre margin might replace the N.A.C.A. test as a near-equivalent and simpler alternative measure of the stability in a pull-out.

The importance of the manoeuvre margin is confirmed by general consideration of its significance in relation to the dynamic stability of the helicopter. A negative static margin is undesirable because of unstable trim changes with speed, but will lead only to a relatively slow divergence, whereas a large divergence in pitch can be developed only when $H_m \leqslant 0$.

Flight tests are being made on a number of helicopters to establish the stability and control characteristics, including the stick-fixed static and manoeuvre margins, and to determine how the quantitative assessments compare with the pilots' views of desirable longitudinal handling qualities. An analysis is to be made also of American test data (Ref. 1) for a helicopter with and without a tailplane, either fixed or linked to the rotor control. Consideration is being given to the definition of criteria for satisfactory handling characteristics in hovering and low speed flight.

Extension of the theory is required for stick-free longitudinal stability. It appears that a similar method of analysis can be used if the characteristics of the rotor forces on the control plane are known; attention is being given to the analysis of these forces. Extension of the theory is required also for rotor wing combinations and for tandem rotor helicopters.

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## LIST OF SYMBOLS

| | | |
|---|---|---|
| $a$ | | Slope of blade lift-coefficient curve |
| $a_1$ | | Backwards tilt of rotor tip-path plane relative to control plane |
| $a_T$ | | Slope of tailplane lift-coefficient curve |
| $b$ | | Number of rotor blades |
| $B$ | | Coefficient of $\lambda^3$ in the stability quartic, equation (2) |
| $B'$ | | Coefficient of $\lambda'$ in the pull-out quadratic, equation (9) |
| $B_s$ | | Control-plane displacement during pull-out |
| $B_1$ | | Forward tilt of control plane relative to rotor hub plane |
| $B_{1d}$ | | Displacement of control plane from equilibrium value, $B_{1e}$ |
| $B_{1e}$ | | Equilibrium value of $B_1$ |
| $c$ | | Blade chord |
| $C$ | | Coefficient of $\lambda^2$ in the stability quartic, equation (2) |
| $C'$ | | Term independent of $\lambda'$ in the pull-out quadratic, equation (9) |
| $C_B$ | | Fuselage body pitching-moment coefficient |
| $C_D$ | | Mean blade profile-drag coefficient |
| $C_{F_0}$ | | Fuselage and tailplane pitching-moment coefficient for $B_1 = 0$ |
| $C_H$ | $=$ | $H/\frac{1}{2}\rho V^2 S$ |
| $C_J$ | $=$ | $bW_b/\rho SR\mu^2$ |
| $C_M$ | $=$ | $M/\frac{1}{2}\rho V^2 SR$ |
| $C_T$ | $=$ | $T/\frac{1}{2}\rho V^2 S$ |
| $C_W$ | $=$ | $W/\frac{1}{2}\rho V^2 S$ |
| $D$ | | Coefficient of $\lambda$ in stability quartic, equation (2) |
| $e$ | | Flapping hinge offset |
| $E$ | | Term independent of $\lambda$ in stability quartic, equation (2) |
| $f$ | | Trimmer or bias spring setting |
| $g$ | | Acceleration due to gravity |
| $h$ | | Distance of c.g. below rotor hub |
| $h_e$ | $=$ | $h + \dfrac{C_J}{C_T} e$ |
| $h_m$ | $=$ | $(1 + n_m)h + \dfrac{C_J}{C_W} e + \dfrac{a_T \bar{V}_T R}{C_W} (1 + \eta_B)$ |
| $h_\eta$ | $=$ | $h_e + \dfrac{a_T \bar{V}_T R}{C_T} (1 + \eta_B)$ |
| $H$ | | Transverse force on rotor |
| $H_m$ | | Manoeuvre margin, stick fixed |
| $i_E$ | | $I/mR^2$ |

| | | |
|---|---|---|
| $I$ | | Helicopter pitching moment of inertia |
| $I_b$ | | Blade moment of inertia about the flapping hinge |
| $J$ | $=$ | $W_b\Omega^2R/g$, the centrifugal force on a blade |
| $K$ | | Distance of c.g. forward of the rotor hub axis |
| $K_n$ | | Static margin, stick fixed |
| $L$ | | Distance of tailplane from centre of gravity |
| $m$ | | Helicopter mass |
| $m_u$, $m_w$, etc. | $=$ | $M_u/\rho SRV$, $M_w/\rho SRV$, etc. |
| $M_u$, $M_w$, etc. | $=$ | $\partial M/\partial u$, $\partial M/\partial w$, etc. |
| $M$ | | Pitching moment of complete aircraft about centre of gravity |
| $M_F$ | | Pitching moment of fuselage and tailplane about centre of gravity |
| $ng$ | | Normal acceleration |
| $n_0g$ | | Normal acceleration at $t = 0$ following control displacement |
| $n_mg$ | | Mean normal acceleration during growth of acceleration |
| $n_sg$ | | Ultimate steady increment in acceleration |
| $q$ | | Angular velocity about $y$-axis |
| $R$ | | Rotor radius |
| $\mathscr{R}$ | | Routhian discriminant |
| $S$ | | Rotor disc area, $\pi R^2$ |
| $S_T$ | | Tailplane area |
| $t$ | | Time |
| $T$ | | Rotor thrust |
| $T_a$ | | $\partial T/\partial \alpha$ |
| $u$ | | Velocity increment along $x$-axis in disturbed flight |
| $v$ | | Rotor induced velocity, assumed constant over rotor disc |
| $V$ | | Flight speed |
| $V_e$ | | Flight speed in equilibrium condition |
| $\bar{V}_T$ | | Tail volume ratio, $S_TL/SR$ |
| $w$ | | Velocity increment along $z$-axis in disturbed flight |
| $W$ | | Helicopter weight |
| $W_b$ | | Weight of one blade |
| $x$ | | Axis fixed in aircraft in disturbed motion, in direction of flight in equilibrium condition |
| $X$ | | Force along $x$-axis |
| $X_{B_1}$, $X_q$, etc. | $=$ | $\partial X/\partial B_1$, $\partial X/\partial q$, etc. |

| | | |
|---|---|---|
| $y$ | | Axis to starboard, perpendicular to plane of symmetry |
| $z$ | | Axis fixed in aircraft in disturbed motion, normal to direction of flight in equilibrium condition |
| $Z$ | | Force along $z$-axis |
| $z_u$, $z_w$, etc. | = | $Z_u/\rho SV$, $Z_w/\rho SV$, etc. |
| $Z_{B_1}$, $Z_q$, etc. | = | $\partial Z/\partial B_1$, $\partial Z/\partial q$, etc. |
| $\alpha$ | | Angle of incidence of rotor hub plane to the flight path |
| $\alpha_e$ | | Equilibrium value of $\alpha$ |
| $\alpha_p$ | | Control plane incidence to the flight path |
| $\alpha_{pe}$ | | Equilibrium value of $\alpha_p$ |
| $\alpha_T$ | | Tailplane incidence to the flight path |
| $\gamma$ | | Blade inertia number, $\rho acR^4/I_b$ |
| $\varepsilon$ | | Angle of downwash at the tailplane |
| $\zeta$ | = | $\frac{1}{2}(4C' - B'^2)^{1/2}$ |
| $\eta_B$ | = | $\partial\eta_T/\partial B_1$ |
| $\eta_0$ | | Tailplane setting for $B_1 = 0$ |
| $\eta_T$ | | Tailplane setting to fuselage |
| $\theta$ | | Blade collective pitch angle |
| $\lambda$ | | Root of stability quartic, equation (2) |
| $\lambda'$ | | Root of pull-out quadratic, equation (9) |
| $\mu$ | = | $V \cos \alpha_p/\Omega R$ |
| $\mu_1$ | = | $m/\rho SR$ |
| $\nu$ | = | $(v - V \sin \alpha_p)/\Omega R$ |
| $\xi$ | = | $\frac{1}{2}(B'^2 - 4C')^{1/2}$ |
| $\rho$ | | Air density |
| $\sigma$ | | Rotor solidity, $bc/\pi R$ |
| $\tau$ | | Aerodynamic time, $t\rho SV_e/m$ |
| $\chi$ | | Flight path angle to horizontal, positive on climb |
| $\chi_e$ | | Equilibrium value of $\chi$ |
| $\Omega$ | | Rotor speed |

Superscripts :

| | | |
|---|---|---|
| $(')$ | $\equiv$ | $d/dC_T$ |
| $(\dot{})$ | $\equiv$ | $d/dt$ |

# APPENDIX

## *Stability Functions and Derivatives*

1. *General.*—The theory of the longitudinal motion is developed with the aerodynamic functions and derivatives in a general form, and it can be applied to particular configurations using appropriate values for these quantities. As an illustration, approximate values are given here for a simple rotor system with blades of constant chord and constant angle along their length. The formulae can be applied also to tapered and twisted blades if the chord and blade angle are assumed to be the mean equivalent values for untapered and untwisted blades.

2. *Stability Functions.*—The quantities in the stability analysis from which values are required (for given values of $C_T$ and $\mu$), are the blade flapping angle $a_1$, the control-plane incidence $\alpha_p$, the transverse force coefficient $C_H$, the blade centrifugal force coefficient $C_J$, and the downwash angle at the tailplane $\varepsilon$. An analysis has been made of the flight of a helicopter in R. & M. 1730 (Ref. 6) assuming uniform induced flow over the rotor and neglecting tip and root losses; it follows from this work, assuming that excessive blade stalling does not occur over the retreating blades, and neglecting terms of $\mu^4$ and over, that

$$\frac{T}{a\sigma\rho S\,\Omega^2R^2} = 4\left[\frac{\frac{2}{3}\theta(1-\mu^2)-\nu(1-\frac{1}{2}\mu^2)}{1+\frac{2}{3}\mu^2}\right]$$

where

$$\nu = \frac{v - V\sin(\alpha_p + a_1)}{\Omega R,}$$ $v$ being the mean induced velocity at the rotor.

Thus, for $V > 0$,

$$C_T = \frac{T}{\frac{1}{2}\rho V^2 S} = \frac{a\sigma}{2\mu^2}\,\frac{\frac{2}{3}\theta(1-\mu^2)-\nu(1-\frac{1}{2}\mu^2)}{1+\frac{3}{2}\mu^2}\,. \qquad \text{(A.1)}$$

Also from Ref. 6,

$$a_1 = \frac{8}{3}\mu\,\frac{\theta - \frac{3}{4}\nu}{1+\frac{3}{2}\mu^2}\,. \qquad \text{(A.2)}$$

Eliminating $\nu$ from (A.1) and (A.2)

$$a_1 = \frac{4}{3}\left(\frac{\mu}{1+\mu^2}\right)\left[\theta + \frac{3C_T\mu^2}{a\sigma}\right]. \qquad \text{(A.3)}$$

From the momentum theory formula for the thrust

$$\frac{v}{V} = \frac{C_T}{4} \text{ approximately.}$$

Thus, with $\mu = V/\Omega R$ for small rotor-plane incidence,

$$\nu = \mu[\tfrac{1}{4}C_T - \sin(\alpha_p + a_1)]$$

and from (A.1)

$$\sin(\alpha_p + a_1) = \frac{C_T}{4a\sigma}\left[a\sigma + \frac{8\mu}{(1-2\mu^2)} - \frac{2\theta}{3\mu(1+\frac{1}{2}\mu^2)}\right]. \qquad \text{(A.4)}$$

This relation determines $\alpha_p$ as a function of $C_T$, $\mu$ and $\theta$ since $a_1$ is given as a function of these quantities in (A.3). Similarly, since

$$C_H = \frac{H}{\frac{1}{2}\rho V^2 S} = \frac{\sigma}{2\mu}\left[C_D + a\nu\left\{\frac{\nu}{(1+\frac{3}{2}\mu^2)} - \frac{\theta}{3}\,\frac{1-\frac{9}{2}\mu^2}{1+\frac{3}{2}\mu^2}\right\}\right],$$

eliminating $v$

$$C_H = \frac{\sigma}{2\mu}\left[C_D + a(1 + \tfrac{3}{2}\mu^2)\left\{\tfrac{2}{3}\theta - \frac{2\mu^2 C^T}{a\sigma}\right\}\left\{\tfrac{1}{3}\theta - \frac{2\mu^2 C_T}{a\sigma}\right\}\right] \quad \ldots \quad \ldots \quad (A.5)$$

Generally $\mu^4$ terms and higher orders have been neglected in the above formulae except where simpler derivatives result from retaining them. For accuracy at high tip speed ratios, it may however be necessary to retain $\mu^4$ terms throughout.

The rotor inertia moment is $\frac{1}{2}bJe(B_1 - a_1)$ where $J$ is the centrifugal force on a blade. For a blade of constant chord of weight $W_b$,

$$J = \frac{W_b \Omega^2 R}{g} \text{ and } C_J = \frac{bW_b}{\rho S R \mu^2} \cdot \quad \ldots \quad (A.6)$$

Little quantitative information is available on the downwash conditions in the vicinity of the tail on a helicopter. An approximate indication of the downwash angle is given by the mean value at the rotor, so that

$$\varepsilon = \frac{v}{V} = \frac{C_T}{4} \cdot \quad \ldots \quad (A.7)$$

3. *Stability Derivatives.*—Throughout the analysis of the longitudinal motion, the rotor speed is assumed to be constant and the stability derivatives are determined for this condition ; blade and control circuit distortion and unsteady aerodynamic effects are neglected.

*Static margin derivatives.*—In the static margin analysis, $B_1$, $\theta$, and $\Omega$ are constant and $C_T . \frac{1}{2}\rho V^2 S = W$. Thus from (A.3)

$$a_1' = \frac{da_1}{dC_T} = -\frac{a_1}{2C_T} \cdot \frac{1 - \mu^2}{1 + \mu^2} \cdot \quad \ldots \quad (A.8)$$

Also from (A.4) and (A.8)

$$\alpha_p' = -a_1' + \tfrac{1}{4} + \frac{\mu}{a\sigma(1 + 2\mu^2)} - \frac{\theta}{3\mu C_T}\frac{1}{(1 - \frac{1}{2}\mu^2)}$$

$$= \tfrac{1}{4} + \frac{\mu}{a\sigma} - \frac{\theta}{3\mu C_T} \cdot \frac{1}{(1 + \frac{3}{2}\mu^2)} \cdot$$

From (A.5)

$$C_H' = \left[\frac{C_H}{2C_T}(1 - \tfrac{3}{2}\mu^2) + \frac{3\mu\sigma C_D}{4C_T}\right]\frac{1}{(1 + \frac{3}{2}\mu^2)} \cdot$$

From (A.6) and (A.7)

$$C_J' = \frac{C_J}{C_T}\,; \quad \varepsilon' = \tfrac{1}{4} \,.$$

4. *Manoeuvre Margin Derivatives.*—In the manoeuvre margin analysis, $B_1$, $\theta$, $\Omega$ and $V$ are constant ; the condition for a steady circle at constant speed is,

$$\frac{dq}{dC_T} = \frac{V_e}{R}\frac{1}{2\mu_1} = \frac{\rho S V_e}{2m} \cdot$$

In this case

$$a_1' = \frac{da_1}{dC_T} = \frac{\partial a_1}{\partial C_T} + \frac{\partial a_1}{\partial q}\frac{dq}{dC_T} \cdot$$

From (A.3)

$$\frac{\partial a_1}{\partial C_T} = \frac{4\mu^3}{a\sigma} .$$

It has been shown by analysis of blade flapping motions (Ref. 7) that the approximate lag of the rotor behind the shaft due to a rate of pitch $q$, is $16q/\gamma\Omega$. In addition $a_1$ is affected by the linear velocity, $hq$, resulting from the fact that the pitching occurs about the helicopter centre of gravity ; this effect however is small and is neglected here. Thus

$$\frac{\partial a_1}{\partial q} = -\frac{16}{\gamma\Omega} \text{ approximately.}$$

From (A.4),

$$\alpha_p{}' = \frac{d\alpha_p}{dC_T} = \frac{\partial \alpha_p}{\partial C_T} - \frac{\partial a_1}{\partial q}\frac{dq}{dC_T}$$

where

$$\frac{\partial \alpha_p}{\partial C_T} = \frac{1}{4} + \frac{2\mu}{a\sigma} .$$

From (A.5),

$$C_H{}' = -\mu\left[\theta(1 + \tfrac{1}{3}\mu^2) - \frac{4C_T\mu^2}{a\sigma}\right] .$$

From Ref. 3

$$\frac{\partial \alpha_T}{\partial q} = \frac{L}{V_e} .$$

From (A.6) and (A.7), $C_J{}' = 0$ and $\varepsilon' = \frac{1}{4}$.

In the analysis to determine the manoeuvre margin in terms of the control changes to trim, the derivatives are required with respect to $\theta$ of $a_1$, $\alpha_p$ and $C_H$ for given $C_T$ and $\mu$. From the relations in section 2 of this Appendix

$$\frac{\partial a_1}{\partial \theta} = \frac{4}{3}\frac{\mu}{1 + \mu^2}$$

$$\frac{\partial \alpha_p}{\partial \theta} = -\frac{2}{3\mu(1 - \frac{3}{2}\mu^2)}$$

$$\frac{\partial C_H}{\partial \theta} = \frac{\sigma a}{3\mu}\left[\tfrac{2}{3}\theta(1 + \tfrac{3}{2}\mu^2) - \frac{3C_T\mu^2}{a\sigma}\right] .$$

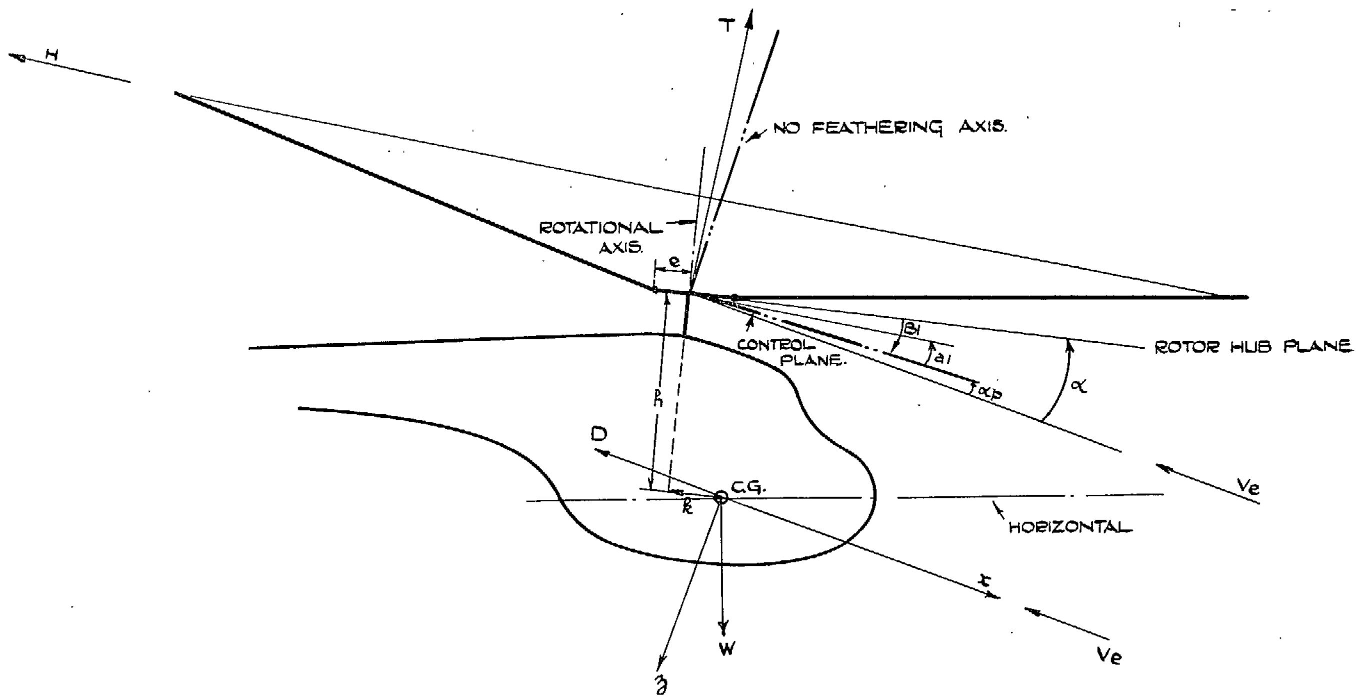


FIG. 1. Helicopter layout in pitching plane.

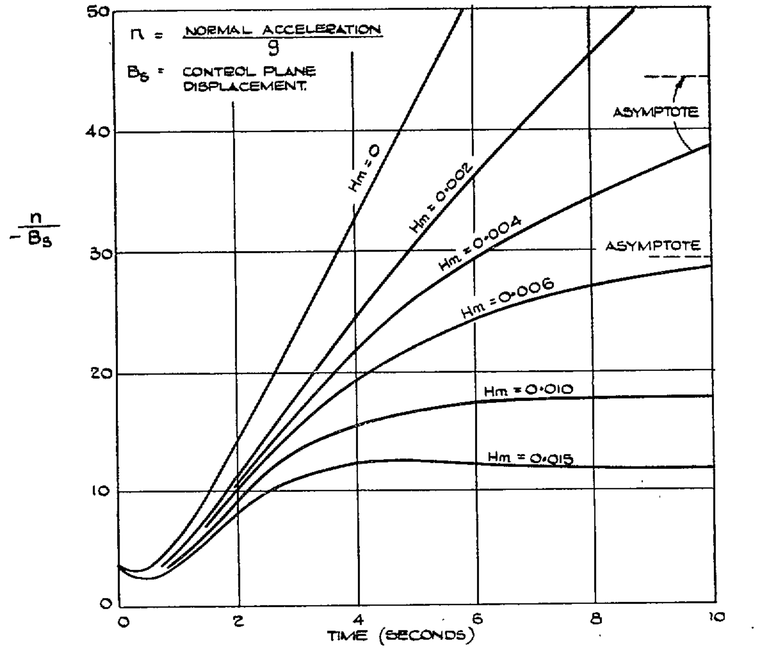


FIG. 2. Effect of manoeuvre margin on growth of acceleration in a pull-out for $B' = 1 \cdot 4$.

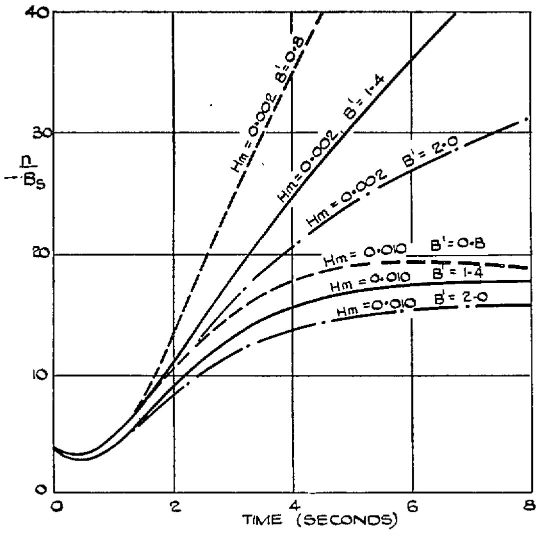


FIG. 3. Growth of acceleration in a pull-out for different values of $B'$ and $H_m$.

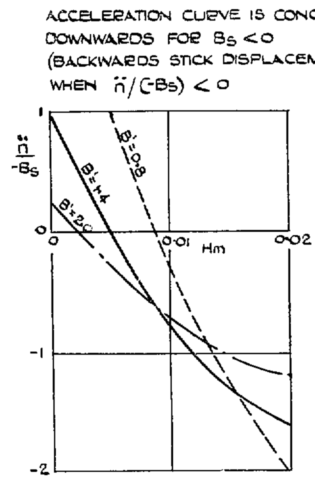


FIG. 4. Variation of $d^2n/dt^2$ at $t = 2 \cdot 0$ sec with $H_m$ for different values of $B'$.

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