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Calculation of the Effect of Slipstream on Lift and Induced Drag

By

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PART I

Wing of Infinite Span

Summary.—The lift distribution along a wing of infinite span with a central jet of higher velocity is calculated by standard methods of aerofoil theory for several values of (jet velocity/stream velocity) and of (jet diameter/wing chord). The lift increment and the induced drag are determined and the application of the results to practical cases is discussed.

1. *Introduction.*—The effect of slipstream on induced drag is of interest in connection with the reduction of drag of aircraft cruising at fairly high lift coefficients, and the present calculations have been made to get some idea of the magnitude of this effect. The general problem of the wing of finite span with a jet passing over it is complicated and the present calculations deal only with the infinite-span wing. The results have as far as possible been given in a form which enables them to be applied in practice to wings of finite span to estimate the induced-drag increment due to slipstream. But it is desirable to obtain the correct solution for the finite span wing for comparison; Dr. K. Mitchell has made considerable progress with this problem and has kindly lent us some auxiliary tables for use in the present calculations.

2. *Mathematical Analysis.*—The wing is of infinite span and has a constant chord c and constant incidence α along the span. The quarter-chord line of the wing is a diameter of the contracted slipstream, which is a circular jet of velocity v and radius R. The stream velocity outside this jet is denoted by V.

* Three reports:

R.A.E. Report No. Aero. 2083A, A.R.C. 9480, October, 1945 (Part I).

R.A.E. Report No. Aero. 2167, A.R.C. 10,256, November, 1946 (Part II).

R.A.E. Report No. Aero. 2083B, A.R.C. 9262, October, 1945 (Part III)

are collected here together under one cover for convenience. Each report is separate as the notations and treatments are necessarily different.

Lifting-line theory is assumed to be valid, so that the circulation round the wing is given by:—

$$\left.\begin{aligned} 2k &= m\,c\,(v\alpha - w)\,, \quad \text{for } |\,x\,| < R\,, \\ 2K &= m\,c\,(V\alpha - W)\,, \text{for } |\,x\,| > R\,, \end{aligned}\right\} \qquad \ldots \ldots \ldots \ldots \ldots \quad (1)$$

where x denotes distance along the span measured from the centre of the jet,

k, K denote the circulations round the wing at any station inside and outside the slipstream respectively,

w, W denote the downwashes at points inside and outside the slipstream respectively,

and m is the lift-curve slope of the aerofoil section.

We shall now calculate the downwash at a point P (x) on the wing due to a semi-infinite trailing vortex* of strength Γ, which starts from the point Q (x') on the wing and extends downstream. This is an image problem which has been solved by Karman and Burgers[1] and by Koning[2]. Four cases have to be distinguished as shown in Fig. 1; the results are as follows:—

Case I. P and Q both inside the slipstream.

The downwash at P is that due to a vortex of strength Γ at Q (x') plus that due to a vortex of strength $\lambda_2\Gamma$ at the image point Q′ (R^2/x'), where $\lambda_2 = (v^2 - V^2)/(v^2 + V^2)$. Hence the downwash velocity is :—

$$w = \frac{\Gamma}{4\pi}\left[\frac{1}{x' - x} + \frac{\lambda_2}{R^2/(x' - x)}\right]$$

Case II. P inside slipstream, Q outside slipstream.

The downwash at P (x) is that due to a vortex of strength $\lambda_1\Gamma$ at Q (x'), where $\lambda_1 = 2vV/(v^2 + V^2)$. Hence the downwash velocity is:

$$w = \frac{\Gamma}{4\pi}\left[\frac{\lambda_1}{x' - x}\right]$$

Case III. P outside slipstream, Q inside slipstream.

The downwash at P (x) is due to a vortex of strength $\lambda_1\Gamma$ at Q (x') together with a vortex at the origin. The latter can be ignored since we are always concerned with pairs of vortices of opposite sign, and for any such pair the images at the origin cancel. Hence the downwash velocity is:—

$$W = \frac{\Gamma}{4\pi}\left[\frac{\lambda_1}{x' - x}\right]$$

Case IV. P and Q both outside the slipstream.

The downwash at P (x) is that due to a vortex of strength Γ at Q (x') together with a vortex of strength $-\lambda_2\Gamma$ at the image point Q′ (R^2/x') and a vortex at the origin. As before, the vortex at the origin can be ignored and the downwash velocity is:—

$$W = \frac{\Gamma}{4\pi}\left[\frac{1}{x' - x} - \frac{\lambda_2}{(R^2/x') - x}\right]$$

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* The counter-clockwise direction is taken as positive.

The strength of the trailing vortex system at any point Q (x') is equal to $-\,k'(x')$ per unit length inside the slipstream and to $-\,K'(x')$ per unit length outside the slipstream, where the dash denotes differentiation, and the total downwash is obtained by integration along the wing span. For a point inside the slipstream we obtain

$$4\pi w(x) = -\int_{-R}^{R} k'(x')\left[\frac{1}{x'-x} + \frac{\lambda_2}{R^2/x' - x}\right] dx' - \left[\int_{-\infty}^{-R} + \int_{R}^{\infty}\right] K'(x')\left[\frac{\lambda_1}{x'-x}\right] dx' . \qquad (2)$$

In the first integral we put

$$x = Ry , \qquad x' = Ry',$$

and in the second integral

$$x = Ry , \qquad x' = R/Y' .$$

We also put

$$\left.\begin{aligned} k(y) &= 4\pi R\, v\, \alpha f(y) \\ K(Y) &= 4\pi R\, V\, \alpha\, F(Y) \end{aligned}\right\} . \qquad (3)$$

With these substitutions (2) becomes

$$\frac{w(y)}{v\alpha} = -\int_{-1}^{1} f'(y')\left[\frac{1}{y'-y} + \frac{\lambda_2 y'}{1-yy'}\right] dy' + \frac{\lambda_1 V}{v}\int_{-1}^{1} F'(Y')\,\frac{Y'dY'}{1-yY'} \qquad (4)$$

For a point outside the slipstream the downwash is given by

$$4\pi W(x) = -\int_{-R}^{R} k'(x')\left[\frac{\lambda_1}{x'-x}\right] dx' - \left[\int_{-\infty}^{-R} + \int_{+R}^{\infty}\right] K'(x')\left[\frac{1}{x'-x} - \frac{\lambda_2}{R^2/x' - x}\right] dx'. \quad (5)$$

We make the substitutions (3) and in the first integral put

$$x = R/Y , \qquad x' = Ry' ,$$

and in the second integral

$$x = R/Y , \qquad x' = R/Y' .$$

These lead to the formula

$$\frac{W(Y)}{V\alpha} = \frac{\lambda_1 v}{V}\int_{-1}^{1} f'(y')\,\frac{Ydy'}{1-Yy'} - \int_{-1}^{1} F'(Y')\left[\frac{YY'}{Y'-Y} - \frac{\lambda_2 Y}{1-YY'}\right] dY' . \qquad (6)$$

We now assume that the circulation functions $f(y)$ and $F(Y)$ can be represented by series expansions of the form :—

$$f(y) = a + b\,[2\log 2 - (1-y)\log(1-y) - (1+y)\log(1+y)] - \sum_{0}^{\infty} c_n \frac{(1-y^{2n+2})}{2n+2} \qquad (7a)$$

$$F(Y) = A + B\,[2\log 2 - (1-Y)\log(1-Y) - (1+Y)\log(1+Y)] - \sum_{0}^{\infty} C_n \frac{(1-Y^{2n+2})}{2n+2} . \qquad (7b)$$

The form of these expansions is suggested by the solution of the problem of the wing spanning a free jet[3]. With these expressions

$$f'(y) = b\log\left(\frac{1-y}{1+y}\right) + \sum_{0}^{\infty} c_n y^{2n+1}$$

$$F'(Y) = B\log\left(\frac{1-Y}{1+Y}\right) + \sum_{0}^{\infty} C_n Y^{2n+1} .$$

Substitution in (4) and (6) gives

$$\frac{w}{v\alpha} = -\int_{-1}^{1}\left[b\log\frac{1-y'}{1+y'} + \sum_{0}^{\infty} c_n y'^{2n+1}\right]\left[\frac{1}{y'-y} + \frac{\lambda_2 y'}{1-yy'}\right]dy'$$

$$+\frac{\lambda_1 V}{v}\int_{-1}^{1}\left[B\log\left(\frac{1-Y'}{1+Y'}\right) + \sum_{0}^{\infty} C_n Y'^{2n+1}\right]\frac{Y'dY'}{1-yY'} \quad \ldots \quad \ldots \quad \ldots \quad (8a)$$

$$\frac{W}{V\alpha} = \frac{\lambda_1 v}{V}\int_{-1}^{1}\left[b\log\frac{1-y'}{1+y'} + \sum_{0}^{\infty} c_n y'^{2n+1}\right]\frac{Ydy'}{1-Yy'}$$

$$-\int_{-1}^{1}\left[B\log\frac{1-Y'}{1+Y'} + \sum_{0}^{\infty} C_n Y'^{2n+1}\right]\left[\frac{YY'}{Y'-Y} - \frac{\lambda_2 Y}{1-YY'}\right]dY' . \quad \ldots \quad (8b)$$

To evaluate the integrals we note that[3]

$$\int_{-1}^{1}\log\left(\frac{1-y'}{1+y'}\right)\frac{dy'}{y'-y} = -\frac{\pi^2}{2} + \tfrac{1}{2}\log^2\left(\frac{1+y}{1-y}\right)$$

$$\int_{-1}^{1}\log\left(\frac{1-y'}{1+y'}\right)\frac{y'dy'}{1-yy'} = -\frac{1}{2y^2}\log^2\left(\frac{1+y}{1-y}\right).$$

With these results and the relations

$$\frac{yy'}{y'-y} = y^2\left(\frac{1}{y'-y}\right) + y, \qquad \frac{y}{1-yy'} = y + y^2\left(\frac{y'}{1-yy'}\right)$$

we obtain

$$\int_{-1}^{1}\log\left(\frac{1-y'}{1+y'}\right)\frac{yy'dy'}{y'-y} = y^2\left[-\frac{\pi^2}{2} + \tfrac{1}{2}\log^2\left(\frac{1+y}{1-y}\right)\right]$$

$$\int_{-1}^{1}\log\left(\frac{1-y,}{1+y'}\right)\frac{y\,dy'}{1-yy'} = -\tfrac{1}{2}\log^2\left(\frac{1+y}{1-y}\right).$$

To determine the other integrals we write, as in Ref. 3,

$$J_n(y) = \int_{-1}^{1}\frac{y'^{2n+1}dy'}{y'-y}, \qquad K_n(y) = \int_{-1}^{1}\frac{y'^{2n+2}dy'}{1-yy'}.$$

Also

$$\int_{-1}^{1}\frac{y'^{2n+1}y\,dy'}{1-yy'} = y^2\int_{-1}^{1}y'^{2n+1}\left(\frac{1}{y} + \frac{y'}{1-yy'}\right)dy' = y^2 K_n(y),$$

$$\int_{-1}^{1}\frac{y'^{2n+2}y\,dy'}{y'-y} = y^2\int_{-1}^{1}y'^{2n+1}\left(\frac{1}{y'-y} + \frac{1}{y}\right)dy' = y^2 J_n(y).$$

With these values (8a) and (8b) become

$$\frac{w}{v\alpha} = b\left[\frac{\pi^2}{2} - \tfrac{1}{2}\left(1-\frac{\lambda_2}{y^2}\right)\log^2\left(\frac{1+y}{1-y}\right)\right] - \sum_{0}^{\infty} c_n\left[J_n(y) + \lambda_2 K_n(y)\right]$$

$$-\frac{\lambda_1 V}{v}\left[\frac{B}{2y^2}\log^2\left(\frac{1+y}{1-y}\right) - \sum_{0}^{\infty} C_n K_n(y)\right]; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9a)$$

$$\frac{W}{V\alpha} = \frac{\lambda_1 v}{V}\left[-\frac{b}{2}\log^2\left(\frac{1+Y}{1-Y}\right) + \sum_{0}^{\infty} c_n Y^2 K_n(Y)\right]$$

$$-B\left[-\frac{\pi^2 Y^2}{2} + \tfrac{1}{2}(Y^2+\lambda_2)\log^2\left(\frac{1+Y}{1-Y}\right)\right] - \sum_{0}^{\infty} C_n\left[Y^2 J_n(Y) - \lambda_2 Y^2 K_n(Y)\right], \quad (9b)$$

We must ensure that the downwash does not become infinite at the slipstream boundary, which is given by $y = 1 = Y$. Putting $y = 1 - \eta$ in (9a) and $Y = 1 - \eta$ in (9b), where η is a small quantity, and equating the coefficients of $\log^2 \eta$ to zero gives

$$-\frac{b}{2}(1 - \lambda_2) - \frac{\lambda_1 V}{v} \cdot \frac{B}{2} = 0\,,$$

$$-\frac{\lambda_1 v}{V} \cdot \frac{b}{2} - \frac{B}{2}(1 + \lambda_2) = 0\,.$$

With $\lambda_1 = \frac{2v\,V}{v^2 + V^2}$, $\lambda_2 = \frac{v^2 - V^2}{v^2 + V^2}$, these two equations are consistent and give $B = -\,b$. With this value of B (9a) and (9b) become

$$\frac{w(y)}{v\alpha} = \frac{b}{2}\left[\pi^2 - \left(\frac{1 - y^2}{y^2}\right)\log^2\left(\frac{1 + y}{1 - y}\right)\right] - \sum_0^\infty c_n \left[J_n(y) + K_n(y)\right]$$
$$+ \frac{\lambda_1 V}{v}\sum_0^\infty (c_n + C_n)\, K_n(y)\,, \qquad \ldots \qquad (10a)$$

$$\frac{W(Y)}{V\alpha} = -\frac{b}{2}\left[\pi^2 Y^2 + (1 - Y^2)\log^2\left(\frac{1 + Y}{1 - Y}\right)\right] - Y^2\sum_0^\infty C_n \left[J_n(Y) + K_n(Y)\right]$$
$$+ \frac{\lambda_1 v}{V}\, Y^2 \sum_0^\infty (c_n + C_n)\, K_n(Y)\,, \qquad \ldots \qquad (10b)$$

where use has been made of the relations,

$$1 - \lambda_2 = \frac{\lambda_1 V}{v}\,, \qquad 1 + \lambda_2 = \frac{\lambda_1 v}{V}\,.$$

As y and Y tend to unity $(J_n + K_n)$ tends to zero and K_n tends to infinity. For $y = 1 - \eta$, where η is small,

$$-\,K_n(y) = \log(\eta/2) + 2\left[\frac{1}{2n + 1} + \frac{1}{2n - 1} + \ldots\ldots + \tfrac{1}{5} + \tfrac{1}{3} + 1\right],$$

and hence

$$-\sum_0^\infty (c_n + C_n)\, K_n(y) = \left[\log(\eta/2) + 2\right]\sum_0^\infty (c_n + C_n) + 2\sum_0^\infty \left(\frac{c_n + C_n}{2n + 1}\right).$$

In order that the downwash shall be finite at the edges of the jet the relation

$$\sum_0^\infty (c_n + C_n) = 0 \qquad \ldots \qquad (11)$$

must therefore be satisfied. When this holds we have the result

$$-\sum_0^\infty (c_n + C_n)\, K_n(1) = 2\sum_0^\infty \left(\frac{c_n + C_n}{2n + 1}\right). \qquad \ldots \qquad (12)$$

We are now ready to substitute in the basic equations (1) from equation (3), and we get

$$\left.\begin{aligned} \frac{w(y)}{v\alpha} &= 1 - \mu f(y) \\ \frac{W(Y)}{V\alpha} &= 1 - \mu F(Y) \end{aligned}\right\} \qquad \ldots \qquad (13)$$

where $\mu = \dfrac{8\pi R}{mc}$, and w, W are given by (10) and f, F are given by (7).

3. *Boundary conditions.*—At large distances from the slipstream the circulation tends to a constant value and the downwash tends to zero. The downwash at infinity is obtained by putting Y equal to zero in (10b) and it will be seen that a zero value is obtained.

At the edge of the slipstream the first condition to be satisfied is the continuity of the lift: this is a consequence of the requirement of the continuity of pressure. Hence

$$kv = KV \text{ for } y = 1 = Y ,$$

or

$$v^2 f(1) = V^2 F(1) ,$$

whence, from (7),

$$v^2 a = V^2 A . \qquad (14)$$

We see that the circulation and the downwash are discontinuous at the boundary.

Some of the coefficients have been related to ensure that the downwash is finite at the edge of the slipstream. The further condition that (13) is satisfied for $y = 1 = Y$ will also be imposed; this gives

$$\frac{\pi^2}{2} b - \frac{2\lambda_1 V}{v} \sum_1^\infty \left(\frac{c_n + C_n}{2n + 1}\right) = 1 - \mu a , \qquad (15)$$

$$-\frac{\pi^2}{2} b - \frac{2\lambda_1 v}{V} \sum_0^\infty \left(\frac{c_n + C_n}{2n + 1}\right) = 1 - \mu A , \qquad (16)$$

making use of (12).

From (15) and (16) we obtain

$$b = \frac{2}{\pi^2}\left(\frac{v^2 - V^2}{v^2 + V^2}\right) = \frac{2}{\pi^2}\lambda_2 , \qquad (17)$$

$$\mu a v^2 = \mu A V^2 = \frac{2v^2V^2}{v^2 + V^2}\left[1 + 2\sum_0^\infty \left(\frac{c_n + C_n}{2n + 1}\right)\right] . \qquad (18)$$

Substituting from (10), (17) and (18), equations (13) become

$$\lambda_2 \left[1 - \left(\frac{1 - y^2}{\pi^2 y^2}\right) \log^2 \left(\frac{1 + y}{1 - y}\right)\right] - \sum_0^\infty c_n \left[J_n(y) + K_n(y)\right]$$
$$+ \frac{\lambda_1 V}{v} \sum_0^\infty (c_n + C_n) K_n(y)$$
$$= 1 - \mu \left[a + \frac{2\lambda_2}{\pi^2}\left\{2 \log 2 - (1 - y) \log (1 - y) - (1 + y) \log (1 + y)\right\}\right.$$
$$\left. - \sum_0^\infty \frac{c_n(1 - y^{2n+2})}{2n + 2}\right] \qquad (19)$$

and

$$-\lambda_2 \left[Y^2 + \left(\frac{1 - Y^2}{\pi^2}\right) \log^2 \left(\frac{1 + y}{1 - y}\right)\right] - Y^2 \sum_0^\infty c_n \left[J_n(Y) + K_n(Y)\right]$$
$$+ \frac{\lambda_1 v}{V} Y^2 \sum_0^\infty (c_n + C_n) K_n(Y)$$
$$= 1 - \mu \left[A - \frac{2\lambda_2}{\pi^2}\left\{2 \log 2 - (1 - Y) \log (1 - Y) - (1 + Y) \log (1 + Y)\right\}\right.$$
$$\left. - \sum_0^\infty \frac{C_n (1 - Y^{2n+2})}{2n + 2}\right] . \qquad (20)$$

The quantities a and A are given by (18); also (11) must be satisfied.

4. *Range of Solutions.*—The above equations have been solved approximately by retaining only the coefficients c_0, c_1, c_2, C_0, C_1, C_2, C_3, the higher coefficients being put equal to zero. These coefficients were determined by satisfying equation (19) at $y = 0$, $0 \cdot 6$ and $0 \cdot 9$, and equation (20) at $Y = 0$, $0 \cdot 6$, and $0 \cdot 9$; in addition, the equations are satisfied for $y = 1$ and $Y = 1$. The values of μ covered were $1 \cdot 0$, $2 \cdot 5$ and $5 \cdot 0$, corresponding approximately to aspect ratios of the part of the wing in the slipstream of $0 \cdot 5$, $1 \cdot 25$ and $2 \cdot 5$. The values of v/V considered were $1 \cdot 2$, $2 \cdot 0$ and ∞ ($V = 0$).

The accuracy to which (19) and (20) were satisfied along the wing was determined by calculating the values of the functions appearing at a large number of points. In all cases except one the errors were equivalent to changes of wing incidence everywhere less than $1\frac{1}{2}$ per cent.; for $\mu = 1 \cdot 0$, $v/V = 2 \cdot 0$, the error amounted to an incidence variation of 5 per cent. over a small part of the span, but no reason for the larger error in this case has been discovered.

5. *Results of calculations.*—The values of the circulation functions $f(y)$ and $F(Y)$ are given in Table 1. The spanwise distributions of lift, circulation, and downwash are shown in Figs. 2, 3, 4, in which the values of the lift and circulation given by simple strip theory are also shown. To show the relation between the present calculations and those of Koning[2], the lift variation along the span, plotted as a proportion of the lift increment given by simple strip theory, is shown in Fig. 5 for $\mu = 1$. It will be seen that Koning's calculations fit in well with those of the present report.

The total lift increment due to the jet, denoted by ΔL, is given by

$$\Delta L = \int_{-R}^{R} \rho\,(vk - \tfrac{1}{2}cV^2 m\alpha)\,dx + \left[\int_{-\infty}^{-R} + \int_{R}^{\infty}\right] \rho\,(VK - \tfrac{1}{2}cV^2 m\alpha)\,dx$$

$$= \tfrac{1}{2}\rho V^2 cR\,m\alpha \left[\int_{-1}^{1} \left(\frac{v^2}{V^2}\mu f - 1\right) dy + \int_{-1}^{1} \frac{(\mu F - 1)}{Y^2}\,dY\right].$$

The lift increment ΔL_0, which would be obtained on a simple strip theory basis, is given by:

$$\Delta L_0 = \tfrac{1}{2}\rho\,(v^2 - V^2)\,2cR\,m\alpha$$

The ratio $\Delta L/\Delta L_0$ was determined for all the cases and the results are given in Table 2 and Fig. 6. In drawing Fig. 6, use has been made of Koning's result that, for slipstream velocities which differ only slightly from the stream velocity, the lift increment is equal to that given by simple strip theory, so that $\Delta L/\Delta L_0 \rightarrow 1$ as $v \rightarrow V$.

The induced drag ΔD_i is defined by the relation:—

$$\Delta D_i = \int_{-R}^{R} \rho\, w\, k\, dx + \left[\int_{-\infty}^{-R} + \int_{R}^{\infty}\right] \rho\, W\, K\, dx$$

$$= 4\pi R^2\, \rho v^2\, \alpha^2 \left[\int_{-1}^{1} \frac{w}{v\alpha} f(y)\, dy + \frac{V^2}{v^2}\int_{-1}^{1} \frac{W}{V\alpha} \cdot \frac{F(Y)}{Y^2}\, dY\right]$$

$$= 4\pi R^2\, \rho v^2\, \alpha^2 \left[\int_{-1}^{1} (1 - \mu f)\, f\, dy + \frac{V^2}{v^2}\int_{-1}^{1} \frac{(1 - \mu F)\, F}{Y^2}\, dY\right],$$

making use of (13). The integrals were evaluated numerically and the values of $\Delta D_i/4\pi R^2 \rho v^2 \alpha^2$ are given in Table 3. Fig. 7 shows the values of $(\Delta D_i)T/(\Delta L)^2$, where T is the jet thrust, defined as $T = \rho v\,(v - V)\,\pi R^2$. It should be noted that, in drawing the curve in Fig. 7, no distinction is made between the results for different values of μ. The induced drag is the difference between drag inside the jet and thrust outside it; these forces tend to equality as the jet and stream velocities tend to equality. Consequently a very high standard of numerical accuracy is required to determine the induced drag correctly. A sufficiently high standard has not been attained in the present calculations to do more than draw a mean curve through the points in Fig. 7.

6. *Discussion.*—The calculation of the lift distribution of a wing of finite span with slipstream present is difficult. We are therefore faced with the problem of applying the calculations for the infinite wing as far as possible to the practical case of the finite wing.

We note at the outset that a close agreement between theory and experiment for the lift increment due to slipstream is not normally to be expected. The principal reasons for this are:—

(1) The jet is not sharply defined, due to mixing at the edges, also it contains periodic components and rotation for ordinary propellers;

(2) The aspect ratio of the part of the wing in the slipstream is too small for the assumptions of lifting line theory to be really valid.

(3) In many cases there is an inclination between the propeller axis and the wind direction, which introduces an inclination into the jet.

But the value of the above investigation does not wholly depend on the accuracy to which the lift increment can be predicted. For a given lift increment, which can be determined by measurement or from generalised data (R. & M. 1788[4]), the distribution of this increment for contra-rotating propellers will be similar in shape to the distribution given by calculation. In particular the induced drag parameter $\Delta D_i \, . \, T/(\Delta L)^2$ will probably be nearly correct, and have the same value for wings of finite span as for wings of infinite span.

If the calculated value of the induced drag has a significant influence on the performance of an aircraft, then it is worth while to consider whether this induced drag can be eliminated. For the case of the wing of infinite span, which has been considered in detail above, the induced drag can be reduced to zero by adjusting the incidence of the part of the wing in the slipstream, so that the loading is constant along the wing; there are then no trailing vortices and the induced drag is identically zero.

NOTATION

ρ	density of fluid
c	chord of aerofoil
α	aerofoil incidence measured from zero lift direction
m	slope of lift curve of aerofoil section
R	radius of jet representing slipstream
x	distance along span measured from centre of jet
V	stream velocity outside jet
v	velocity inside slipstream jet in contracted condition
W	downwash velocity at wing outside jet
w	downwash velocity at wing inside jet.
K	circulation at section of wing outside jet
k	circulation at section of wing inside jet
ΔL	lift increment due to slipstream
ΔL_0	lift increment due to slipstream on simple strip theory
T	jet thrust, equal to $\rho v\ (v - \mathrm{V})\ \pi R^2$
ΔD_i	induced drag
y	x/R for $x < R$
Y	R/x for $x > R$
λ_1	$\dfrac{2vV}{v^2 + V^2}$
λ_2	$\dfrac{v^2 - V^2}{v^2 + V^2}$
μ	$\dfrac{8\pi R}{mc}$
$f(y)$, $F(Y)$	defined by equation (7).

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No.	Author	Title, etc.
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2	Koning	Influence of the Propeller on other parts of the Airplane Structure. Aerodynamic Theory (Durand). Vol. 4, Div. M. Julius Springer. Berlin. 1935.
3	Squire	The Lift and Drag of a Rectangular Wing spanning a Free Jet of Circular Section. *Phil. Mag.*, Vol. 27, 1939, p. 229.
4	Smelt and Davies	Estimation of Increase in Lift due to Slipstream. R. & M. 1788. 1937.

TABLE 1

Circulation Functions

(a) $V/v = 0$

μ	1·0		2·5		5·0	
y, Y	$f(y)$	$F(y)$	$f(y)$	$F(Y)$	$f(y)$	$F(Y)$
0	0·1708	—	0·1378	—	0·1038	—
0·2	0·1667	—	0·1349	—	0·1021	—
0·4	0·1536	—	0·1256	—	0·0963	—
0·6	0·1295	—	0·1068	—	0·08455	—
0·8	0·0886	—	0·0762	—	0·0623	—
1·0	0	—	0	—	0	—

(b) $V/v = 0·5$

μ	1·0		2·5		5·0	
y, Y	$f(y)$	$F(Y)$	$f(y)$	$F(Y)$	$f(y)$	$F(Y)$
0	0·4180	1·0	0·2237	0·4000	0·1375	0·2000
0·2	0·4152	1·0186	0·2219	0·4084	0·1364	0·2035
0·4	0·4066	1·0642	0·2158	0·4298	0·1326	0·2127
0·6	0·3907	1·1159	0·2044	0·4571	0·1253	0·2259
0·8	0·3640	1·1643	0·1841	0·4895	0·1113	0·2461
1·0	0·3074	1·2294	0·1362	0·5446	0·0725	0·2901

(c) $V/v = 0·833$

μ	1·0		2·5		5·0	
y, Y	$f(y)$	$F(Y)$	$f(y)$	$F(Y)$	$f(y)$	$F(Y)$
0	0·7872	1·0	0·3394	0·4000	0·1794	0·2000
0·2	0·7863	1·0068	0·3388	0·4029	0·1790	0·2012
0·4	0·7833	1·0233	0·3368	0·4102	0·1778	0·2042
0·6	0·7779	1·0419	0·3330	0·4194	0·1755	0·2085
0·8	0·7689	1·0587	0·3265	0·4302	0·1710	0·2150
1·0	0·7503	1·0805	0·3111	0·4480	0·1589	0·2288

TABLE 2

Values of $\Delta L/\Delta L_0$

V/v	μ		
	1·0	2·5	5·0
0	0·1257	0·2628	0·4081
0·5	0·3062	0·4880	0·6225
0·833	0·5685	0·7861	0·8863

TABLE 3

Values of $\Delta D_i/4\pi R^2 \rho v^2 \alpha^2$

V/v	μ		
	1·0	2·5	5·0
0	0·1087	0·1846	0·2231
0·5	0·1397	0·1254	0·1172
0·833	0·0797	0·0329	0·0151

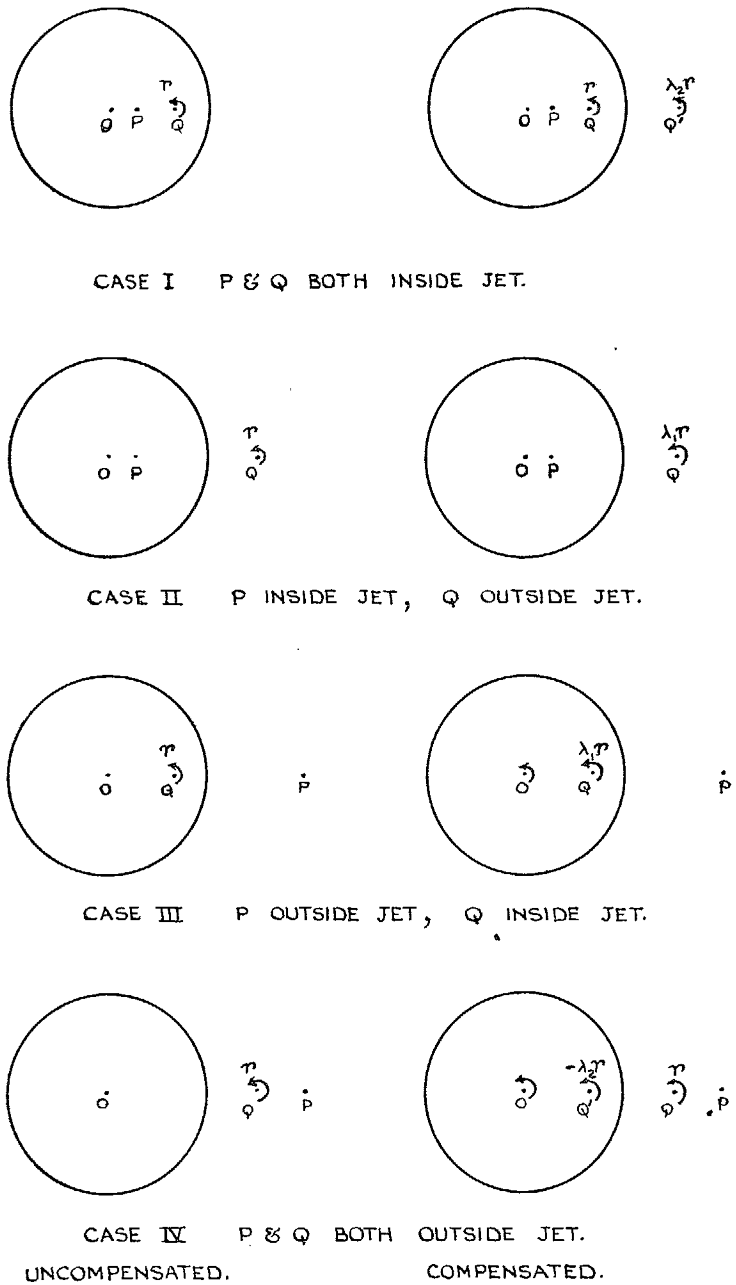

FIG. 1. Effect at a Point P of a Vortex at Q.

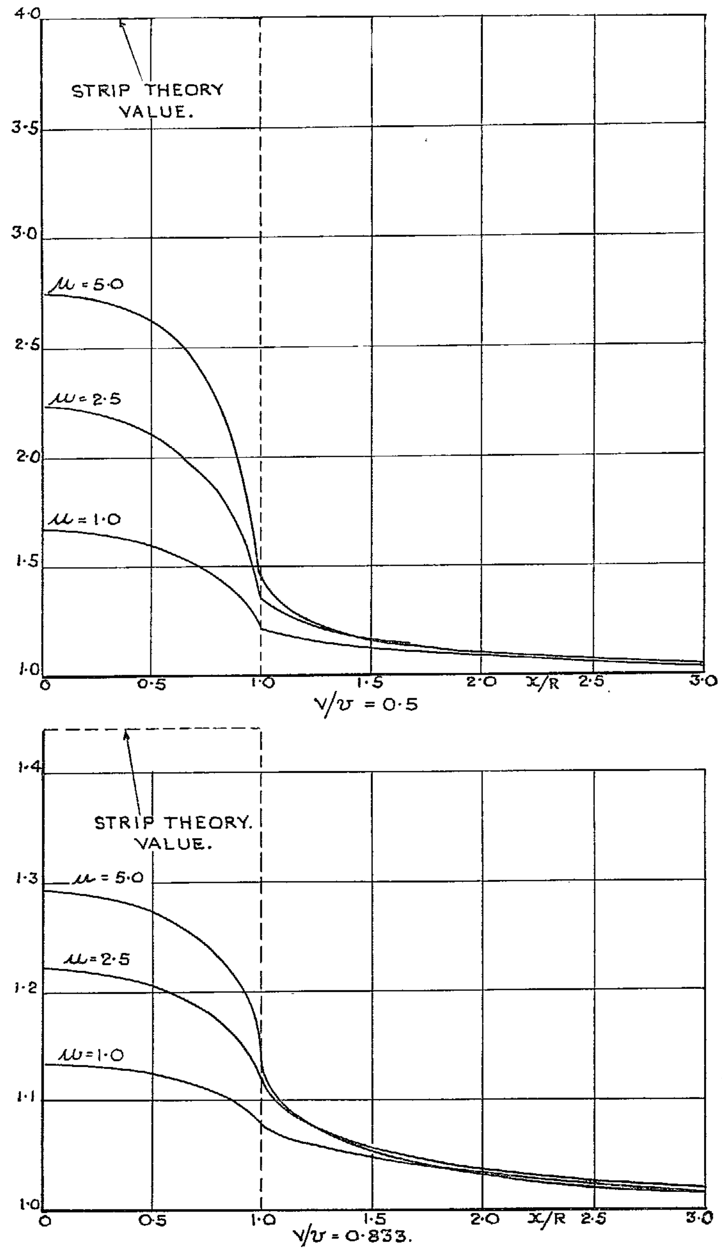

FIG. 2. Lift Distribution along Span.

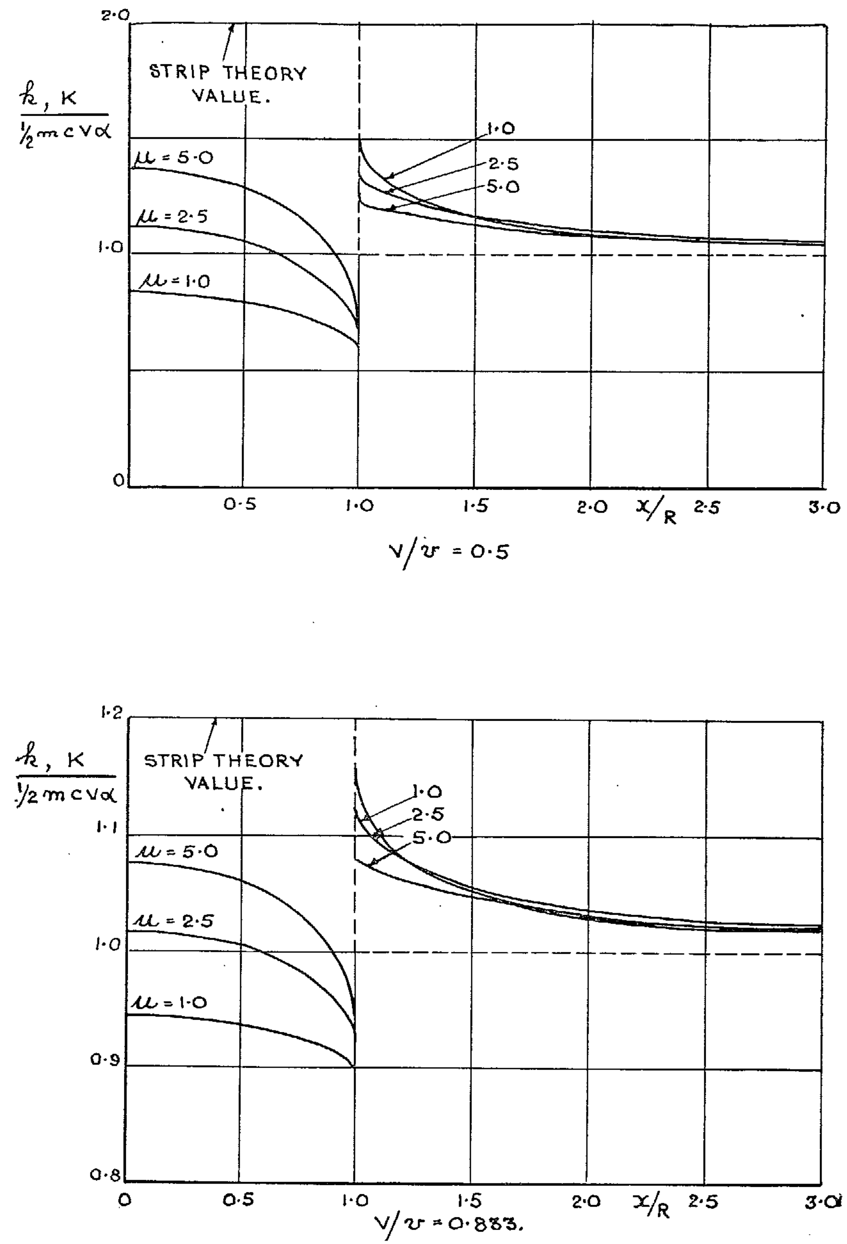

FIG. 3. Circulation Distribution along Span.

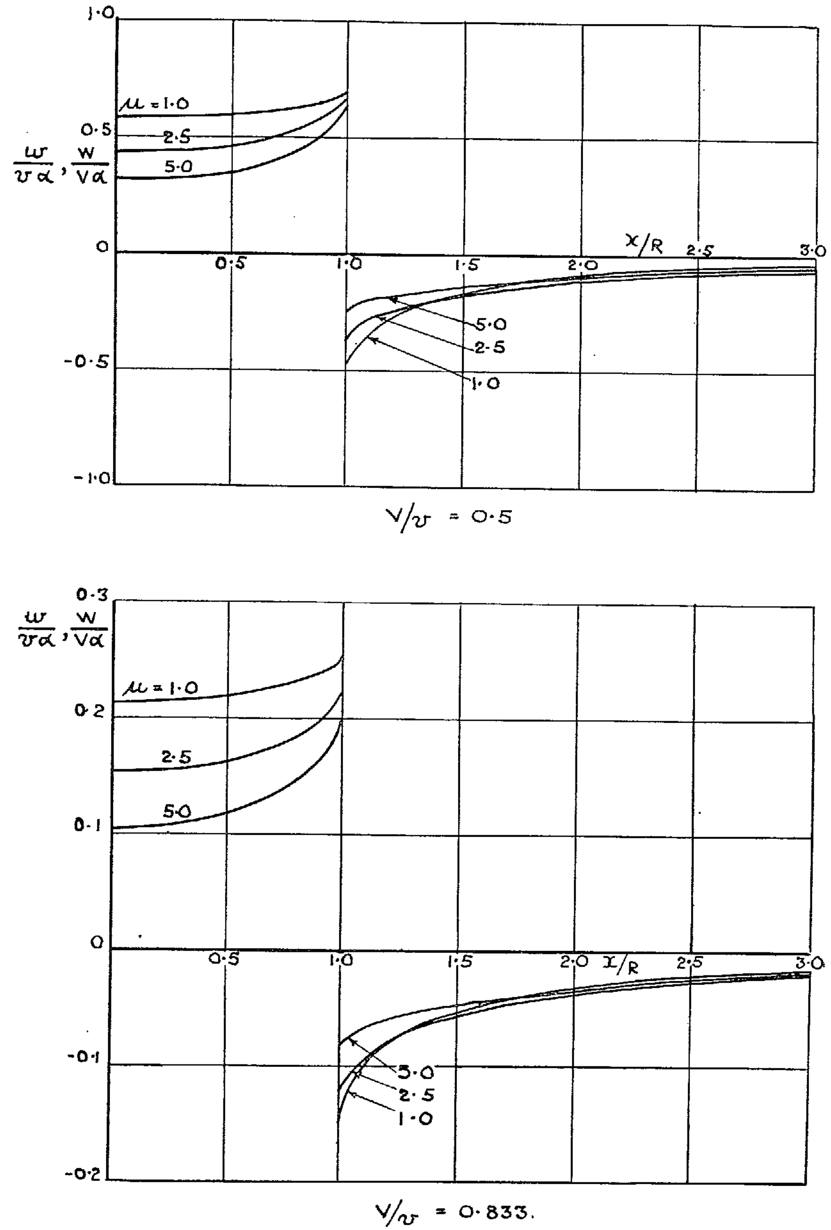

FIG. 4. Downwash Distribution along Span.

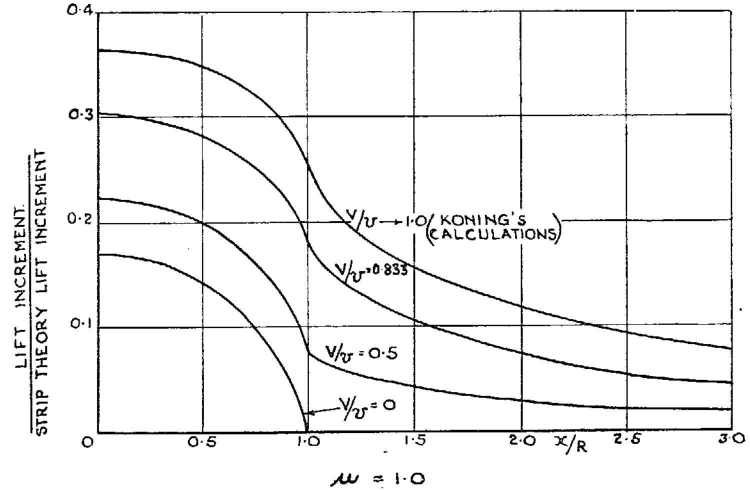

FIG. 5. Lift Increment grading along the Span.

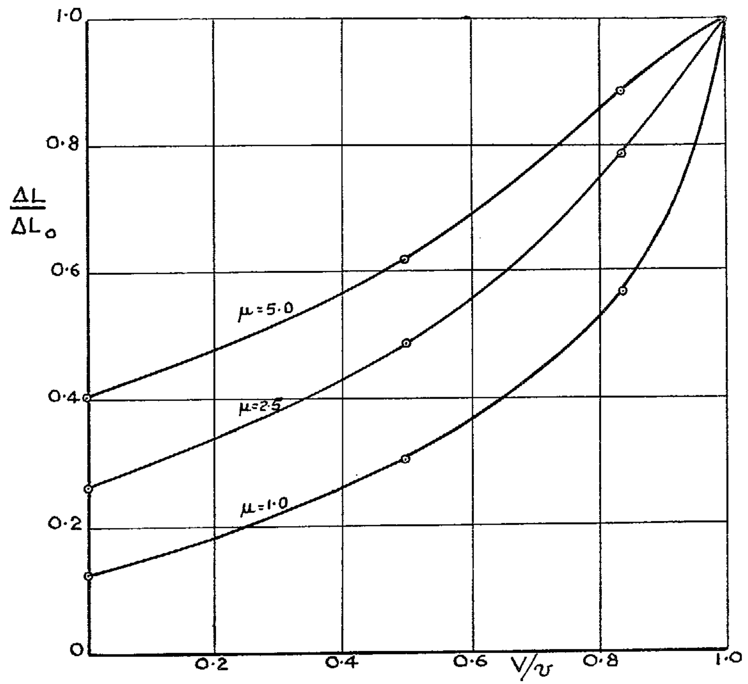

FIG. 6. Lift Increment Due to Slipstream.

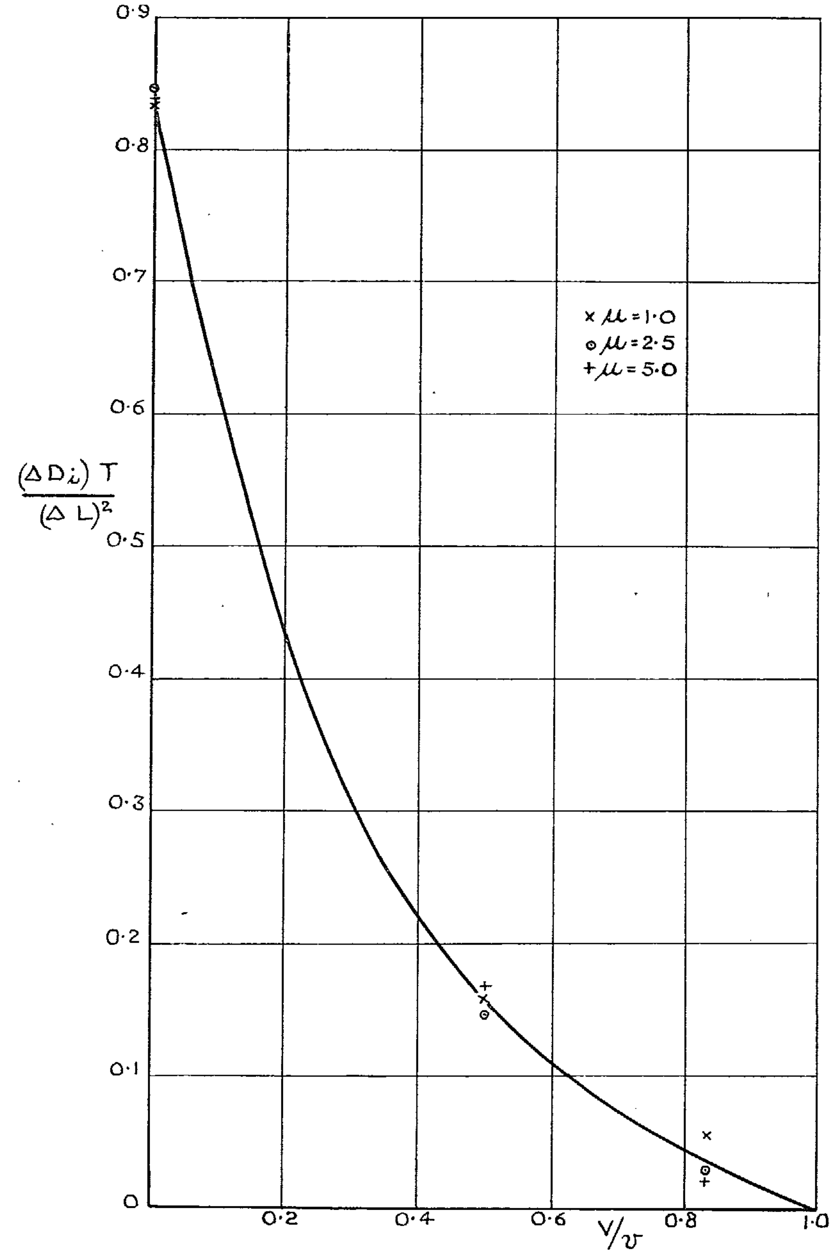

FIG. 7. Induced Drag Due to Slipstream.

PART II

Slipstream Rotation

Summary.—Calculations are made by lifting-line theory of the distribution of circulation along a wing of infinite span which is located in a rotating slipstream. The wing torque and induced thrust are evaluated. The numerical values of the wing torque are not entirely reliable because of the approximations introduced into the analysis. The induced thrust power is calculated to be between 30 per cent. and 40 per cent. of the rotational power input to the propeller.

---

1. *Introduction.*—This is the second part of an investigation of the effect of slipstream on lift and induced drag and deals with the effect of slipstream rotation. We make the following assumptions:—

(1) The wing is of infinite span and has a constant chord and incidence.

(2) Lifting-line theory may be applied although the ratio of propeller diameter to wing chord is not large in practice.

(3) The axis of the cylindrical slipstream is parallel to the direction of the undisturbed stream and passes through the quarter-chord point of the wing.

(4) The axial velocity in the slipstream is equal to the velocity in the free stream.

(5) Ahead of the wing the slipstream rotates as a rigid cylinder.

(6) The effect of rotation is equivalent to a change in the stream direction at the wing, giving rise to an effectively linear variation of wing incidence inside the slipstream.

2. *General Analysis.*—The above assumptions make the problem, as far as the circulation distribution along the wing span is concerned, equivalent to the problem of a wing of infinite span with a linear variation of incidence within the slipstream boundary and constant incidence outside the boundary. Since a constant increase of incidence along the whole span can be made without affecting the slipstream rotation effect, it may be considered that the wing has zero incidence outside the slipstream. Hence the wing incidence α is given by

$$\left.\begin{aligned} \alpha(x) &= \omega y/V, && \text{for } -R < x < R\,, \\ \alpha(x) &= 0\,, && \text{for } |\,x\,| > R\,, \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1)$$

where x is measured along the span from the centre of the jet, R is the jet radius and ω is the angular velocity of rotation in the slipstream.

To determine the corresponding circulation distribution we start with Prandtl's result[1] for a sinusoidal variation of incidence, which gives for the incidence distribution

$$\alpha = \alpha_0 \sin \mu x\,,$$

the corresponding circulation distribution

$$K = \left(\frac{4V\,\alpha_0}{\mu + \dfrac{8}{mc}}\right) \sin \mu x\,,$$

where V is the stream velocity, c is the aerofoil chord, m is the lift curve slope in two dimensional flow, and α_0 is a constant. The downwash w is related to the circulation and incidence by the equation

$$\alpha - \frac{w}{V} = \frac{2K}{mcV}, \qquad \ldots \qquad (2)$$

and hence

$$\frac{w}{V} = \frac{\mu K}{4V} = \frac{\mu \alpha_0}{\mu + (8/mc)} \sin \mu x .$$

We generalise the above result by use of Fourier's double integral theorem. For any incidence distribution $\alpha(x)$, which is an odd function of x, we have the formula

$$\alpha(x) = \frac{2}{\pi} \int_0^\infty \sin \mu x \, d\mu \int_0^\infty \alpha(t) \sin \mu t \, dt ,$$

and, making use of the above results for a simple sinusoidal variation, we can write down the value of the circulation corresponding to the arbitrary incidence distribution $\alpha(x)$ to be

$$K(x) = \frac{2}{\pi} \int_0^\infty \frac{4V}{\mu + (8/mc)} \sin \mu x \, d\mu \int_0^\infty \alpha(t) \sin \mu t \, dt .$$

With the specified linear incidence distribution inside the rotating slipstream, given by (1), we get

$$\int_0^\infty \alpha(t) \sin \mu t \, dt = \frac{\omega}{V} \int_0^\infty t \sin \mu t \, dt$$

$$= \frac{\omega}{V\mu^2} \Big[\sin \mu R - \mu R \cos \mu R \Big] .$$

Hence the circulation distribution corresponding to (1) is

$$K(x) = \frac{8\omega}{\pi} \int_0^\infty \frac{\sin \mu x \, [\sin \mu R - \mu R \cos \mu R]}{\mu^2 \, [\mu + (8/mc)]} \, d\mu .$$

Putting $x/R = y$, $\mu R = \lambda$, and $\frac{8R}{mc} = a$, this becomes

$$\frac{2K(y)}{mc\omega R} = \frac{2a}{\pi} \int_0^\infty \frac{(\sin \lambda - \lambda \cos \lambda)}{\lambda^2 \, (\lambda + a)} \sin (\lambda y) \, d\lambda , \qquad \ldots \qquad (3)$$

corresponding to the incidence distribution

$$\left. \begin{aligned} \alpha(y) &= \frac{\omega R}{V} y \quad \text{for } -1 < y < 1 \\ \alpha(y) &= 0 \quad \text{for } |y| > 1 \end{aligned} \right\} \qquad \ldots \qquad (4)$$

The integral in (3) can be expressed in terms of the sine and cosine integrals, which have been tabulated by Jahnke and Emde.

Calculations of the distributions of circulation along the span have been made for $a = 0{\cdot}5$, $1{\cdot}0$, $1{\cdot}5$, and $2{\cdot}0$, corresponding to the following ratios of jet diameter to wing chord:

$$2R/c = 0{\cdot}785,\ 1{\cdot}57,\ 2{\cdot}35,\ 3{\cdot}14,$$

taking the section lift-curve slope m to be equal to 2π. The results of the calculations are shown in Fig. 1, which give $2K/mc\ \omega R$ as a function of y/R for the various values of a or $2R/c$.

3. *Torque and Thrust on the Wing.*—The anti-symmetric loading on the wing induced by the slipstream rotation produces a torque which opposes the propeller torque. The magnitude of the total wing torque is

$$Q = \int_{-\infty}^{\infty} (\rho V K)\, x\, dx \,,$$

since the lift per unit span is equal to ρVK. It can be shown that the wing torque is equal to the torque calculated by strip theory, ignoring all downwash effects. The latter torque is given by

$$Q_0 = \int_{-R}^{R} \tfrac{1}{2}\rho V^2\, mc \left(\frac{\omega x}{V}\right) x\, dx = \tfrac{1}{3}\, \rho\, mc\, R^3 \omega V \,.$$

Further the propeller torque Q_1 is equal to the rate of increase of angular momentum in the slipstream so that

$$Q_1 = \frac{\pi}{2}\, \rho R^4\, \omega V \,.$$

Hence, since $Q = Q_0$,

$$\frac{Q}{Q_1} = \frac{Q_0}{Q_1} = \frac{2}{3\pi}\,\frac{mc}{R} = \frac{16}{3\pi a} \,.$$

This leads to the surprising result that the torque on the wing is greater than the propeller torque for $2R/c < 8/3$, i.e. for wing chords greater than 0·375 of the slipstream diameter. The theory is however only reliable for values of the wing chord which are less than this amount.

In addition to the torque on the wing there is an induced thrust, because the stream inside the slipstream is inclined to the direction of motion of the wing. Inside the jet the lift vector on the wing is inclined backwards from the local stream direction but forwards relative to the general stream direction; outside the jet the lift vector is inclined forwards because of the upwash there. The induced thrust T is given by the formula

$$T = \int_{-\infty}^{\infty} (\rho V K) \left(\alpha - \frac{w}{V}\right) dx \,,$$

since the lift vector is inclined backwards at the angle w/V relative to the local stream direction, the local stream direction being inclined forwards at the angle α relative to the general stream direction. Substituting from (2) and making use of the symmetry of the jet we obtain

$$T = \rho mc\, R^3\, \omega^2 \int_0^{\infty} \left(\frac{2K}{mc\, \omega R}\right)^2 dy \,.$$

It is convenient to relate the induced thrust to the power expended in generating the rotation of the slipstream. The presence of the wing has the effect of recovering a part of this power input. We shall therefore express the induced thrust power as a proportion of the rotational power input.* This power input P_1 is given by

$$P_1 = \tfrac{1}{2}\omega Q_1 = \frac{\pi}{4}\, \rho R^4\, \omega^2\, V \,.$$

Hence

$$\frac{TV}{P_1} = \frac{32}{\pi a} \int_0^{\infty} \left(\frac{2K}{mc\, \omega R}\right)^2 dx \,,$$

where, as before, $a = 8R/mc$. The values of this quantity for the various cases are given in Table 1. It will be seen that the efficiency of energy recovery reaches a maximum value of 37 per cent.

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* The rotational power input P_1 is normally about 3 per cent. of the total power input of a propeller.

4. *Discussion.*—If the chord is less than one-third of the jet diameter and the angular velocity in the jet is small there is good reason for supposing that the method given above is fairly reliable. Unfortunately the practical range corresponds more to wing chords about equal to the jet diameter and in this region lifting line theory ceases to be an acceptable approximation. The justification for working out examples in this region is that they may give some rough guidance until lifting surface theory can be applied, if this is ever done.

A further weakness in the theory is the assumption that for the purpose of calculating the lift distribution, the rotation in the slipstream is equivalent to a twist in the wing. This assumption will fail if the distortion of the slipstream boundary becomes appreciable and if the wing chord is too large a proportion of the jet diameter. Here again we can hardly expect the theory to be fully reliable for wing chords greater than one-third of the jet diameter.

A comparison with *ad hoc* experimental data has not been made. A reliable comparison would require a programme of tests specially arranged to determine the relevant quantities.

REFERENCE

No.	*Author*	*Title, etc.*
1	Prandtl and Betz	Vier Abhandlungen zur Hydrodynamik und Aerodynamik, p.29. Kaiser Wilhelm Institute, Göttingen. 1927.

NOTATION

R	radius of slipstream
y	distance along wing span measured from the centre of the slipstream.
V	stream velocity
w	downwash velocity
ω	angular velocity in slipstream
c	wing chord
α	wing incidence to local stream direction
m	Section lift curve slope, taken to be equal to 2π in the calculations
K	circulation along the wing.
x	y/R
a	$8R/mc$
Q	torque on wing
Q_0	torque on wing calculated by strip theory
Q_1	propeller torque
T	thrust on wing
P_1	rotational power input to propeller

TABLE 1

Induced Thrust Power on the Wing as a Fraction of the Rotational Power Input by the Propeller

a	0·5	1·0	1·5	2·0
$2R/c$	0·785	1·57	2·35	3·14
TV/P_1	0·280	0·350	0·369	0·367

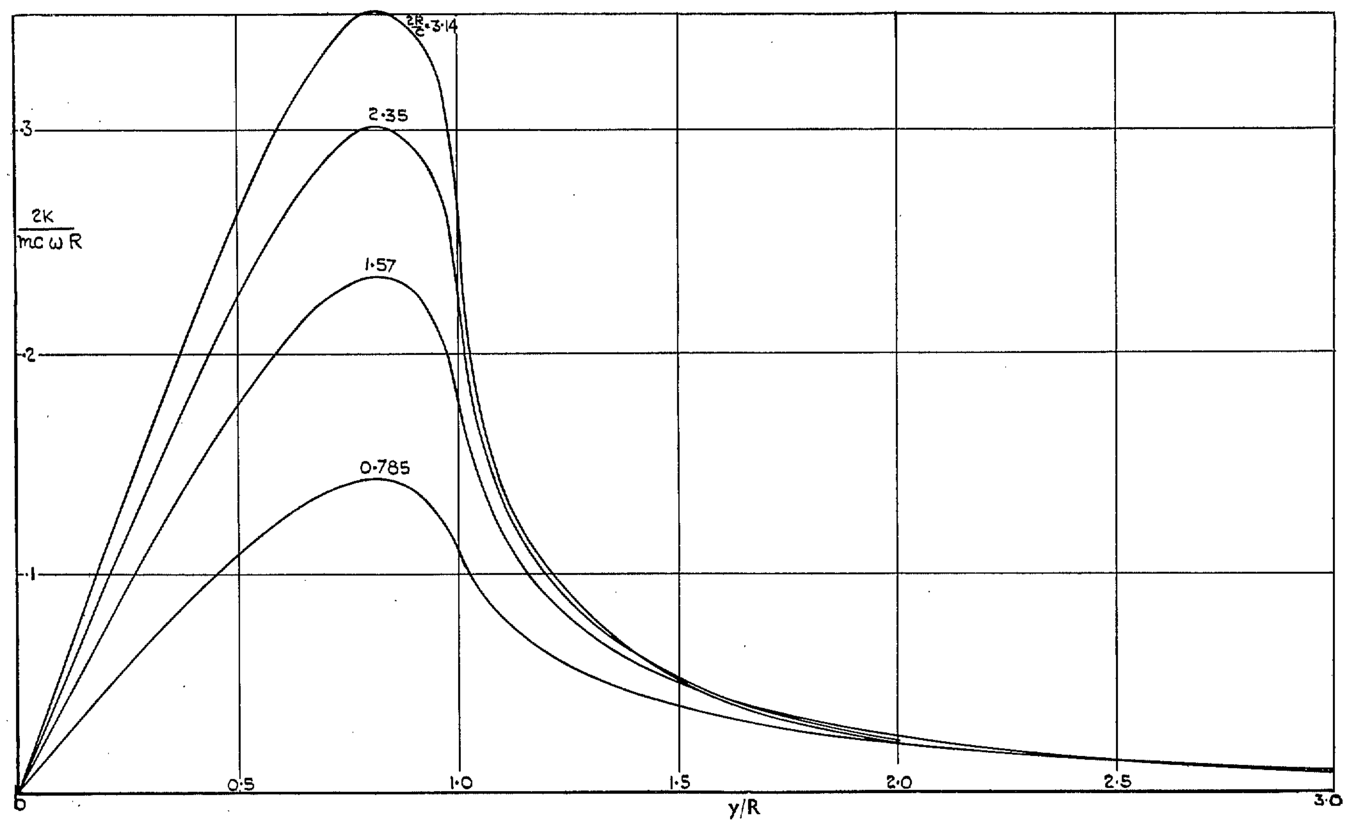

FIG. 1. Circulation Distribution along the Span.

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PART III

Minimum Induced Drag

Summary.—A proof is given that minimum induced drag is obtained for constant downwash angle along the wing, and the calculation of the span loading of a wing in this condition is considered. Examples of wings with a central jet and with a pair of jets are given. It is concluded that in practical cases the span loading in the minimum drag condition is nearly elliptic and that the minimum induced drag is nearly the same as for a wing with elliptic loading in a uniform stream.

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1. *Introduction*.—Calculations of the increment in lift and of the induced drag for a single slipstream on an infinite wing were given in Part I. It was there pointed out that the induced drag for an infinite wing could be eliminated by reducing the incidence of the part of the wing in the slipstream, so that the lift should be uniformly distributed along the wing span. This suggests (1) that a similar technique might be used to reduce the induced drag due to slipstream for a wing of finite span and (2) that the span loading giving minimum induced drag with slipstream should be investigated. The latter investigation is described in this part of the report; in section 2 the condition for minimum induced drag is determined and in sections 3 and 4 some span loadings satisfying this condition are calculated.

The conclusion of the present investigation is that the minimum induced drag in practical cases is obtained by adjusting the incidence to give elliptic loading along the span and that the induced drag is then equal to the minimum induced drag with slipstream absent.

2. *Condition for Minimum Induced Drag*.—It has been shown by Kármán and Tsien[1] that, for a wing in a stream of non-uniform velocity, *minimum induced drag is obtained if the downwash angle is constant along the span*. A simple alternative proof of this result, following the procedure used by Betz[2] for the problem of the propeller with minimum energy loss, will now be given:

Since the induced drag is related to the energy in the wake far downstream, the lifting elements can be moved along the wind direction without change in total drag provided that their incidences are adjusted to maintain the lift of the elements unchanged; this is Munk's stagger theorem. If a small element giving a lift δL is added some way downstream the induced velocity at the wing is negligible, and the increase in induced drag is equal to $\varepsilon \delta L$, where ε is the downwash angle at the station where the element is placed; as usual the downwash far downstream is equal to twice the downwash at the wing. If two elements with lifts δL_1, δL_2, are added to a lifting system the total lift will be unchanged, provided that

$$\delta L_1 + \delta L_2 = 0 .$$

For a wing in the minimum drag condition a small change in the loading distribution at constant total lift produces no change in the induced drag; the condition for this is

$$\varepsilon_1 \delta L_1 + \varepsilon_2 \delta L_2 = 0 ,$$

where ε_1 and ε_2 are the downwash angles due to the wing field far downstream from the locations of the two lifting elements. These two equations are only compatible for arbitrary lifting elements if the downwash angle is constant along the wing span, both far downstream and at the wing itself. This is the required result.

The flow round a wing in the minimum induced drag condition is the same as if the wake were made into a rigid sheet; this is, of course, also true for a wing in a stream of uniform velocity. Hence, both with and without slipstream, the lift distribution for minimum induced drag on a monoplane wing is obtained by solving the two-dimensional potential problem of an infinite rigid plate in a stream which is normal to the plate. At the cylindrical boundaries of the slipstreams, the following conditions must be satisfied:[3]

$$\left.\begin{aligned} v\phi &= V\Phi\,, && \text{(continuity of pressure)} \\ \frac{1}{v}\frac{\partial\phi}{\partial n} &= \frac{1}{V}\frac{\partial\Phi}{\partial n}\,, && \text{(conservation of mass)} \end{aligned}\right\} \qquad \text{(1)}$$

where v and V are the axial velocities inside and outside the slipstream respectively, ϕ and Φ are the velocity potentials inside and outside the slipstream respectively, and n is measured normal to the boundary of the slipstream.

Consider now a monoplane with a single central slipstream or with a pair of slipstreams as shown in Fig. 1. Take axes as shown in the figure. Let ε be the (constant) downwash angle along the span at infinity downstream, so that our potential problem is the determination of the flow of a stream which has a velocity εV at infinity and which is normal to the span of the wing. The conditions to be satisfied on the boundary of the slipstream are given in equations (1).

Now the gradient of circulation along the wing span is related to the velocity potential as follows:

$$\frac{dK}{dx} = 2\frac{\partial\Phi}{\partial x}\,, \text{ (outside)}, \qquad \text{and } \frac{dk}{dx} = 2\frac{\partial\phi}{\partial x}\,, \text{ (inside)},$$

where K, k are the circulations outside and inside the slipstream respectively. We now introduce second harmonic functions f, F which are related to the velocity potential by the formulae

$$\varepsilon VF = \Phi\,, \text{ (outside)}, \qquad \text{and } \varepsilon Vf = \frac{v}{V}\phi\,, \text{ (inside)},$$

so that we have, along the wing span,

$$\frac{dK}{dx} = 2\varepsilon V\frac{\partial F}{\partial x}\,, \text{ (outside)} \qquad \text{and } \frac{dk}{dx} = 2\varepsilon\frac{V^2}{v}\frac{\partial f}{\partial x}\,, \text{ (inside)}.$$

Hence

$$K = 2\varepsilon VF\,, \qquad k = 2\varepsilon\frac{V^2}{v}f\,,$$

if it is arranged that F vanishes at the wing tips, where the circulation vanishes, and we note that k and f must vanish together.

On the jet boundary equations (1) are replaced by

$$f = F \text{ and } \frac{1}{v^2}\frac{\partial f}{\partial n} = \frac{1}{V^2}\frac{\partial F}{\partial n}\,.$$

At infinity we shall have

$$F = y\,.$$

The wing lift is given by

$$L = \int_{\text{span}} \rho\,(vk, VK)\,dx\,.$$

Substituting for k and K in terms of f and F, this becomes

$$L = 2\rho\varepsilon V^2\int_{\text{span}} (f, F)\,dx\,. \qquad \text{(2)}$$

The induced drag for constant downwash angle ε at infinity is given by

$$D_i = \tfrac{1}{2}\varepsilon L\,, \qquad \qquad (3)$$

since the downwash angle at the wing is one-half the downwash angle at infinity. Eliminating ε between (2) and (3) we obtain

$$D_i = \frac{L^2}{4\rho V^2 \int_{\text{span}} (f, F)\, dx}\,. \qquad \qquad (4)$$

The problem is thus reduced to the determination of the functions f and F and the evaluation of the integral in (4). The distribution of lift along the span is the same as the variation of f and F along the span. If no slipstream is present minimum induced drag is obtained with elliptic loading along the span; it will be found convenient to compare the loading and induced drag with slipstream present with the standard case of elliptic loading.

3. *Single Central Slipstream.*—For convenience we shall take the semi-span to be unity, which can be done without loss of generality. If we write

$$f = a_0 + \sum_1^\infty a_n \left(\frac{r}{R}\right)^{2n} \cos 2n\theta\,,$$

$$F = a_0 + \sum_1^\infty a_n \left[\left(\frac{1+p}{2}\right)\left(\frac{r}{R}\right)^{2n} + \left(\frac{1-p}{2}\right)\left(\frac{R}{r}\right)^{2n}\right] \cos 2n\theta\,,$$

where r, θ are measured as shown in Fig. 1,

R is the jet radius,

a_0, a_n are constants,

and p stands for V^2/v^2,

then f, F are harmonic functions which satisfy the boundary conditions at the jet boundary. For

$$\left.\begin{aligned} F &= a_0 + \sum_1^\infty a_n \cos 2n\theta = f\,, \\ \frac{\partial F}{\partial r} &= 2np \sum_1^\infty a_n \cos 2n\theta = p\frac{\partial f}{\partial r}\,. \end{aligned}\right\} \text{ on } r = R\,.$$

If f and F are the real parts of complex potentials h, H, then

$$h = a_0 + \sum_1^\infty a_n (z/R)^{2n}\,, \qquad \qquad (5)$$

$$H = a_0 + \sum_1^\infty a_n \left[\left(\frac{1+p}{2}\right)\left(\frac{z}{R}\right)^{2n} + \left(\frac{1-p}{2}\right)\left(\frac{R}{z}\right)^{2n}\right] \qquad \qquad (6)$$

It is also possible to express the external potential H in another form; we must have F tending to y at infinity, further H must be real on the y axis and on the x axis between $(+1, 0)$ and $(-1, 0)$, and purely imaginary on the x axis outside this interval. Consideration of these conditions and of the solution with slipstream absent shows that H must be of the form

$$H = (1 - z^2)^{1/2}\,(1 - B_1 z^{-2} - B_2 z^{-4} + \ldots)\,,$$

where $B_1, B_2, \ldots$ are constants. For $|z| < 1$ we can expand this expression in the form

$$
\begin{aligned}
H &= (1 - \tfrac{1}{2}z^2 - \tfrac{1}{8}z^4 - \tfrac{1}{16}z^6 + \ldots)(1 - B_1 z^{-2} - B_2 z^{-4} + \ldots) \\
&= 1 + \frac{B_1}{2} + \frac{B_2}{8} + \frac{B_3}{16} - \ldots \\
&\quad + z^2 \left[-\tfrac{1}{2} + \frac{B_1}{8} + \frac{B_2}{16} + \ldots \right] + z^{-2} \left[-B_1 + \frac{B_2}{2} + \frac{B_3}{8} + \ldots \right] \\
&\quad + z^4 \left[-\tfrac{1}{8} + \frac{B_1}{16} + \ldots \right] + z^{-4} \left[-B_2 + \frac{B_3}{2} \ldots \right] \\
&\quad + z^6 \left[-\tfrac{1}{16} + \ldots \right] + z^{-6} \left[-B_3 + \ldots \right] \\
&\quad + \ldots
\end{aligned}
\tag{7}
$$

Equating coefficients in the expansions (6) and (7) we obtain

$$a_0 = 1 + \frac{B_1}{2} + \frac{B_2}{8} + \frac{B_3}{16} - \ldots - ,$$

$$a_1 \left(\frac{1+p}{2} \right) R^{-2} = -\tfrac{1}{2} + \frac{B_1}{8} + \frac{B_2}{16} + \ldots ; \qquad a_1 \left(\frac{1-p}{2} \right) R^2 = -B_1 + \frac{B_2}{2} + \frac{B_3}{8} - \ldots ,$$

$$a_2 \left(\frac{1+p}{2} \right) R^{-4} = -\tfrac{1}{8} + \frac{B_1}{16} + \ldots , \qquad a_2 \left(\frac{1-p}{2} \right) R^4 = -B_2 + \frac{B_3}{2} \ldots ,$$

$$a_3 \left(\frac{1+p}{2} \right) R^{-6} = -\tfrac{1}{16} + \ldots , \qquad a_3 \left(\frac{1-p}{2} \right) R^6 = -B_3 + \ldots ,$$

Next a_1, a_2, a_3 are eliminated from these equations and B_1, B_2, B_3 evaluated by successive approximation as series expansions in R. Retaining terms up to R^{12} only we get

$$
\begin{aligned}
B_1 &= \tfrac{1}{2} q R^4 + \tfrac{1}{16} q R^8 (1 - q) + \tfrac{1}{128} q R^{12} (1 - q)(3 - q) , \\
B_2 &= \tfrac{1}{8} q R^8 + \tfrac{1}{32} q R^{12} (1 - q) , \\
B_3 &= \tfrac{1}{16} q R^{12} ,
\end{aligned}
$$

where $$q = \frac{1-p}{1+p} = \frac{v^2 - V^2}{v^2 + V^2} .$$

With these values of B_1, B_2, B_3 we can calculate the values of a_0, a_1, a_2, a_3 to the same order of accuracy and obtain

$$
\begin{aligned}
a_0 &= 1 + \tfrac{1}{4} q R^4 + \tfrac{1}{64} q R^8 (3 - 2q) + \tfrac{1}{256} q R^{12} (5 - 5q + q^2) , \\
a_1 &= -\frac{R^2}{(1+p)} \left[1 - \tfrac{1}{8} q R^4 - \tfrac{1}{64} q R^8 (2 - q) \right] , \\
a_2 &= -\frac{R^4}{4(1+p)} \left[1 - \tfrac{1}{4} q R^4 - \tfrac{1}{128} q R^8 (9 - 4q) \right] , \\
a_3 &= -\frac{R^6}{8(1+p)} \left[1 - \tfrac{5}{16} q R^4 \right] .
\end{aligned}
$$

As an example we shall calculate the span loading giving minimum induced drag for a wing for which the slipstream diameter is one-half the span and the velocity in the jet is twice the stream velocity. We therefore put

$$R = 0\cdot5\,, \qquad v/V = 2\cdot0\,, \qquad p = \frac{V^2}{v^2} = \tfrac{1}{4}\,, \qquad q = \frac{1-p}{1+p} = 0\cdot6\,,$$

and obtain

$$a_0 = 1\cdot0095\,, \qquad a_1 = -\,0\cdot1991. \qquad a_2 = -\,0\cdot0125\,, \qquad a_3 = -\,0\cdot0016$$

$$B_1 = 0\cdot01886\,, \qquad B_2 = 0\cdot00029\,, \qquad B_3 = 0\cdot00001\,.$$

The calculated span-loading distribution is given in Fig. 2 and compared with the elliptic distribution. It will be seen that the differences are not large, in spite of the extreme values of radius and velocity ratio which have been used. The induced drag is 2·7 per cent. greater than for the case of elliptic loading with slipstream absent.

It may be concluded from the above that minimum induced drag for a single slipstream is obtained in all practical cases by making the span loading elliptic, and that the induced drag is then nearly the same as for elliptic loading with slipstream absent.

4. *Pair of Symmetrically Placed Jets.*—We again take the semi-span to be unity and adopt notation as shown in Fig. 1. Then the functions f and F are given by

$$f = a_0 + \sum_1^\infty a^n \left(\frac{r}{R}\right)^n \cos n\theta\,,$$

$$F = a_0 + \sum_1^\infty a_n \left[\left(\frac{1+p}{2}\right)\left(\frac{r}{R}\right)^n + \left(\frac{1-p}{2}\right)\left(\frac{R}{r}\right)^n\right] \cos n\theta\,,$$

and the associated complex potentials are given by

$$h = a_0 + \sum_1^\infty a_n \left(\frac{z_1}{R}\right)^n,$$

$$H = a_0 + \sum_1^\infty a_n \left[\left(\frac{1+p}{2}\right)\left(\frac{z_1}{R}\right)^n + \left(\frac{1-p}{2}\right)\left(\frac{R}{z_1}\right)^n\right],$$

where

$$z_1 = z - b\,, \qquad p = V^2/v^2\,,$$

and b is the distance of the jet centres from the mid-point of the wing. As before we can express H as

$$H = (1 - z^2)^{1/2}\left[1 - \frac{A_1}{z^2 - b^2} - \frac{A_1}{(z^2 - b^2)^2} \cdot \cdot \cdot\right],$$

where A_1 and A_2 are real constants. This form satisfies the symmetry conditions and the conditions at infinity.

The procedure for the determination of the constants follows the same lines as before, but it is more complicated and will not be set out here. The results obtained for two examples were:—

Case I

$$b = 0\cdot3\,, \qquad R = 0\cdot2\,, \qquad V/v = 0\cdot5;$$

$$A_1 = 0\cdot00536\,, \qquad A_2 = 0\cdot000042\,,$$

$$a_0 = 0\cdot976\,, \qquad a_1 = -\,0\cdot104, \qquad a_2 = -\,0\cdot037$$

Case II

$$b = 0\cdot5\,, \qquad R = 0\cdot2\,, \qquad V/v = 0\cdot5;$$

$$A_1 = 0\cdot01738\,, \qquad A_2 = 0\cdot00085\,,$$

$$a_0 = 0\cdot888\,. \qquad a_1 = -\,0\cdot184\,, \qquad a_2 = -\,0\cdot049$$

The corresponding calculated span loadings are given in Fig. 3 and 4. It will be seen that the loadings do not differ greatly from elliptic, although the ratio of jet velocity to stream velocity, which has been taken equal to 2·0 in the examples, is high. For the first example with the slipstreams nearer towards the centre of the wing (Fig. 3) the induced drag is 1·5 per cent. less than for a plain wing with elliptic loading, and for the second example (Fig. 4) the induced drag is 2·5 per cent. less than for this standard case.

It may be concluded also for the case of a pair of jets that minimum induced drag is obtained in all practical cases by making the span loading elliptic, and that the induced drag is then practically the same as for the standard case of a wing with elliptic loading in a uniform stream.

NOTATION

ρ	density of fluid
x	distance along span measured from centre of wing
y	distance from wing measured normal to wing span
z	$z + iy$
r, θ	polar coordinates measured from centre of jet
b	distance of centre of jets from centre of wing
z_1	$z - b$
R	radius of jet
v	velocity inside jet
V	velocity of main stream
p	V^2/v^2
q	$\dfrac{1 - p}{1 + p}$
ϕ	velocity potential inside jet
Φ	velocity potential outside jet
k	circulation at section of wing inside jet
K	circulation at section of wing outside jet
ε	downwash angle far downstream
L	wing lift
D_i	induced drag

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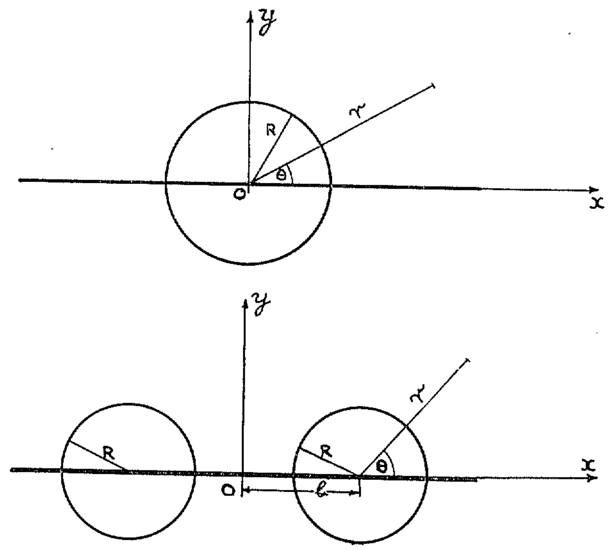

FIG. 1. Notation for Slipstream Calculations.

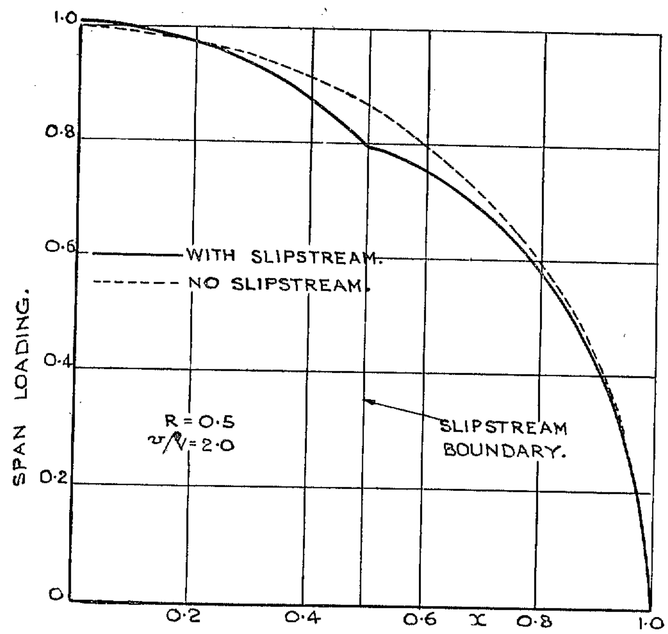

FIG. 2. Lift Distribution for a Single Central Jet.

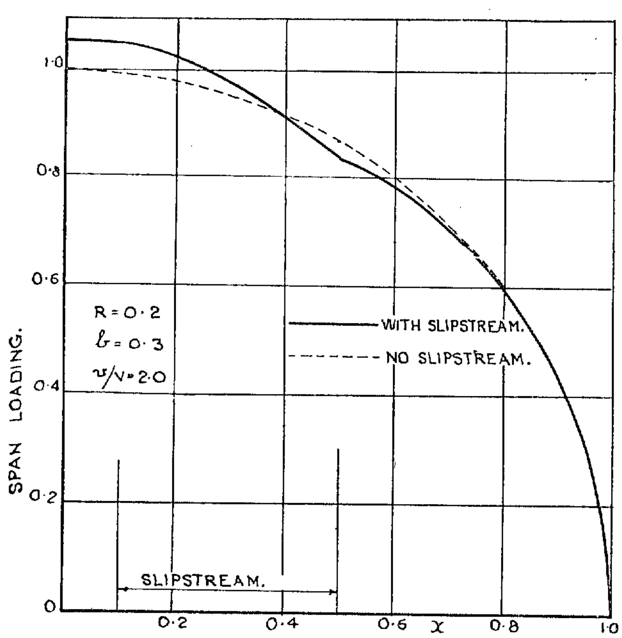

FIG. 3. Lift Distribution for a Pair of Jets. Case I.

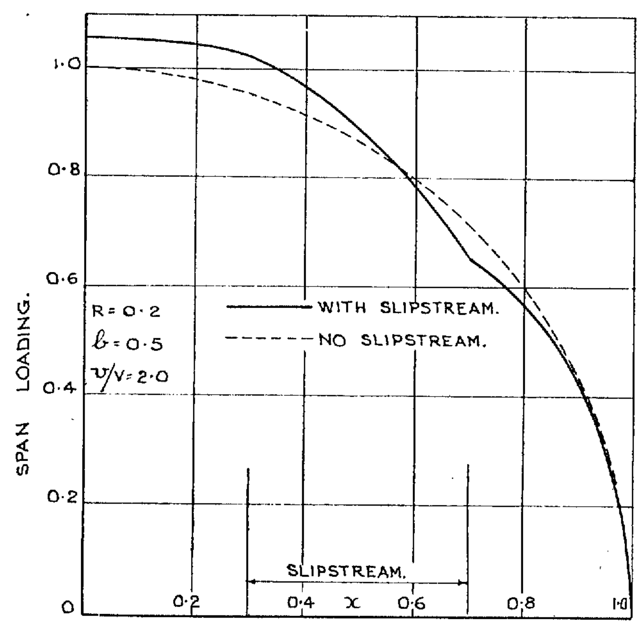

FIG. 4. Lift Distribution for a Pair of Jets. Case II.

(89049) Wt. 13/806 K 5 6/50 Hw.

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