R. & M. No. 2903
(13,667)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

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LONDON: HER MAJESTY'S STATIONERY OFFICE
1955
PRICE 4*s* 6*d* NET

Theoretical Requirements of Tunnel Experiments for Determining Stability Derivatives in Oscillatory Longitudinal Disturbances

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COMMUNICATED BY THE PRINCIPAL DIRECTOR OF SCIENTIFIC RESEARCH (AIR), MINISTRY OF SUPPLY

*Reports and Memoranda No. 2903**

June, 1950

Summary.—The need of using stability derivatives of unsteady (oscillatory) motion is explained, and requirements of tunnel experiments for determining them are established. These are based on simple theoretical considerations, valid for both incompressible and compressible (sub- and supersonic) flow. Two alternative schemes of experimental tests are critically examined. Non-dimensional derivatives are defined and applied in modified stability equations, referred to fixed or moving systems of co-ordinate axes.

1. *Introduction.*—Recent experience with some prototypes of modern aircraft (both tailed and tailless) has revealed a definite loss of damping in the short-period longitudinal oscillation at high subsonic speeds. This may become so serious that it may jeopardize the highest performance which would otherwise be attainable.

Until recently, the damping of the short-period oscillation has always been more than adequate, and these oscillations almost disappeared within a few seconds after an initial disturbance, leaving only the phugoid oscillation to persist. With our uncertainty about the stability derivatives, it was difficult to predict the damping of the phugoid oscillation with any accuracy, but although this damping was always small and sometimes negative, the motion was of such long period that it could always be controlled by the pilot, and the uncertainty about the value of the damping was not considered important. Uncertainty about the damping of the short-period oscillation is, however, much more serious, for if this oscillation becomes unstable it may reach a dangerous amplitude in a few seconds. Under these conditions it becomes of paramount importance that we should increase our knowledge of the stability derivatives at both low and high Mach numbers, and particularly those which affect the damping of the short period motion.

In the past our study of aircraft dynamics has been based on the theory of 'quasi-steady' derivatives, *i.e.*, we have assumed that the forces and moments, which arise from the aircraft motion at any instant, depend only on the attitude and velocities at the same instant as if they were constant. This has led to a substantial simplification both of the theory and of experimental technique, and to the derivatives dependent only on geometrical data. For instance, the important derivative m_q has been calculated as due to a rotation at constant rate, as in a steady circular flight, and determined experimentally by whirling-arm tests rather than by oscillatory experiments in wind tunnel. The forces and moments, however, depend not only on the instantaneous values of the variables but on the complete history of the motion. To deal with the

* R.A.E. Tech. Note Aero. 2059, received 8th January, 1951.

equations of motion in such a way as to allow for an arbitrary past history is both difficult and laborious, but if we are concerned mainly with poorly damped oscillations, it may be adequate to consider the effects which arise in simple harmonic motion. In this motion the past history is completely defined by the instantaneous values of the displacements and velocities, and the frequency. We can therefore obtain derivatives which will be analogous to our 'quasi-steady' derivatives but will now be dependent on frequency as well as on the geometrical data.

The failure of the quasi-steady treatment of the problem may be expected to increase with the frequency of the motion. The derivatives of simple harmonic motion have been used therefore for many years in flutter work where the reduced frequency $\omega = nc/V$ (n = angular frequency, c = mean chord, V = forward speed) is high, of the order unity. In a disturbed motion of an aircraft much lower reduced frequencies occur, thus about $0 \cdot 01$ or less in phugoid motion, and about $0 \cdot 1$ in so-called 'short-period' oscillations which have now become troublesome. It now seems necessary to use the harmonic treatment in this low frequency régime.

It may be mentioned that flutter calculations have dealt practically with two-dimensional derivatives only, and the theory of those has been well known for quite a long time.

Some indications that aerodynamic derivatives (in particular the 'damping' ones) of unsteady motion might be needed in the investigations of flight disturbances, even with their low frequencies, resulted already from the two-dimensional theory of unsteady flow. Glauert[1] has shown that it is precisely at low frequencies that the rotary derivative may change sign so as to lead to increasing rotary oscillations, though this would occur at rather unrealistic positions of the axis (in front of quarter-chord point). The three-dimensional 'unsteady' theory is still in its infancy, but there are indications already[2, 4] that it may lead to results considerably different from those of the two-dimensional one, especially to reduced or negative rotary damping in wider regions for various designs, in particular highly swept-back and tailless. Finally, the occurrence of negative damping may be strongly enhanced by compressibility at high Mach numbers[2, 5].

It is seen that stability derivatives must be considered, for every geometrical shape and arrangement, as functions of reduced frequency and Mach number. Their theoretical estimates should be based on the theory of unsteady flow, and the experimental determination on simple harmonic oscillatory tests in wind tunnels, of low reduced frequency (up to $0 \cdot 2$ at the highest as far as can be foreseen).

As the theory of unsteady flow in three dimensions can be expected to develop only gradually, and will always need an independent confirmation, the experimental tests are absolutely indispensable at the present stage. The experimental technique will probably be able to furnish the necessary data sooner than the theory. It is very important that the general scheme of tests be planned in a way to provide the users with such data as will be needed by them. The purpose of the present paper is to summarise the requirements, but not to meddle with the experimental technique itself.

The elementary theory of dynamic stability shows that the short period damping factor is roughly proportional to the linear combination of derivatives:

$$B = -z_w - \frac{m_{\dot{\vartheta}}}{i_B} = -z_w - \frac{m_q + m_{\dot{w}}}{i_B} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1.1)$$

(*see* Appendix, A.15). It may seem that only the derivatives appearing in this expression are needed, and particular attention should be paid to $m_{\dot{\vartheta}}$ which is most likely to change sign. However, a complete study of dynamic stability requires a greater number of derivatives. The longitudinal disturbance may be treated as a combination of a vertical (heaving) oscillation (w) and of a rotary oscillation (ϑ) about the c.g. axis. Each of these oscillations gives rise to force and moment increments, so that four effects will appear, and all ought to be known. If the oscillations are simple harmonic, the effects will also be simple harmonic functions of time, normally not in phase with the originating oscillation; each of the four effects may be split into two components, one in phase with the oscillation, and another 90 deg out of phase. We therefore obtain the total of 8 derivatives which, in the notation most likely to be used in connection with tunnel experiments, will be styled:

$$z_w,\ z_{\dot{w}},\ m_w,\ m_{\dot{w}} \quad \text{and} \quad z_\vartheta,\ z_{\dot{\vartheta}},\ m_\vartheta,\ m_{\dot{\vartheta}}. \qquad \ldots \qquad (1.2)$$

Of those, only the first and last one appear in (1.1), but all others are needed for determining both frequency and damping of the aircraft oscillations. As, however, all derivatives depend on reduced frequency, it is clear that limiting the investigation to two selected ones would deprive it of much of its value. It might be argued that the chief aim is merely to examine the important critical case of zero total damping ($B = 0$) within the anticipated range of frequency, which could be roughly delimitated according to flight experience. However, it is possible that the unsteady derivatives at high Mach numbers may produce unexpected frequency values and even aperiodic modes. Even if this proves not to be so, there is everything to be said for starting with an experimental scheme which will determine all the derivatives for any axis of rotation, as functions of mean incidence, Mach number and relative frequency. This may lead to safe approximations which would simplify the subsequent scheme of tests.

Section 2 of this paper gives the definitions of fundamental derivatives and the relationships between their values corresponding to various c.g. positions. In sections 3 and 4, two most natural schemes A and B of experimental tests are suggested and critically examined, both involving only rotary oscillations and avoiding the more troublesome heaving ones. The scheme A limits the number of axes of oscillation to two, but requires recording both resultant moments and forces; the scheme B requires three axes, but only moments to be recorded. It is shown that the scheme A is to be greatly preferred theoretically, but it is expected that it will be also more welcome from the point of view of experimental technique. In section 5 the derivatives are written (as they always should be in summarising experimental results) in the standard non-dimensional form.

As commonly done at present in all investigations of oscillatory phenomena, complex variables are used throughout, so that the derivatives, eight in number, are combined into four 'complex derivatives', thus, *e.g.*, m_ϑ and $m_{\dot{\vartheta}}$ are represented by one complex quantity, and similarly for other derivatives. This practically halves the number of equations and formulae, but is also convenient for presenting the experimental results which are always recorded as sine curves, with their amplitudes and phase angles determining exactly the complex derivatives. Splitting up the final numerical results into real and imaginary components for substituting into dynamical equations is, of course, an extremely simple procedure.

The Appendix gives the equations of small motion of the combined heaving and rotary motion (colloquially the short-period oscillation) as a disturbance from straight flight. These are referred first to the same space axes as are used in the analysis of the model tests, and then to the more usual wind axes fixed in the body. The derivatives in both systems are furnished completely by the model tests. It is hoped that this discussion will clear up the confusion that sometimes arises between workers in the different systems.

It should be noticed finally that the analysis, which gives in effect no more than a framework for expressing the aerodynamic forces in a prescribed motion, is based on first principles of kinematics and dynamics only, and therefore applies to the whole range of Mach number.

2. *Definition of Derivatives in a Fixed System of Co-ordinates, and Fundamental Relationships.*—A longitudinal disturbance in the motion of an aircraft (supposed to fly horizontally with the velocity V) may be considered to consist of a translatory vertical (heaving) motion, coinciding with that of the c.g., and of a pitching rotation about the c.g.; the translatory horizontal disturbance will be neglected. The vertical motion is conveniently defined by its velocity w (positive downwards, Fig. 1) as a function of time. Similarly, the rotary motion is defined by the angle of pitch ϑ (positive clockwise in Figs. 2 and 3) as a function of time. The problem consists in determining vertical forces, and moments about c.g., resulting from the disturbance. As only small disturbances are to be taken into account, the forces and moments in the resultant motion

can be determined by superposition of the two component motions. Therefore, all stability derivatives can be defined by examining the two component motions separately. We shall assume that both are simple harmonic oscillations of the same angular frequency $n = 2\pi f$, but generally out of phase with each other. The two component motions will therefore be defined, in complex notation, by the equations:

$$w = w^* \, e^{int}, \quad \vartheta = \vartheta^* \, e^{i(nt+\beta)} \qquad (2.1)$$

where w^* and ϑ^* are amplitudes of w and ϑ, respectively, and β is the angle of phase difference. Differentiating (2.1) with respect to time, we obtain:

$$\dot{w} = inw^* \, e^{int} = inw; \quad \dot{\vartheta} = in\vartheta^* \, e^{i(nt+\beta)} = in\vartheta \qquad (2.2)$$

Considering the *vertical (heaving) oscillation* first (Fig. 1), the resultant force Z, and moment $M^{(o)}$ about the origin of co-ordinates O, must both be simple harmonic functions of time, of frequency n, generally out of phase with either w or $\dot{w}$. Such functions may be determined either theoretically or experimentally, and they can always be represented as sums of two terms, in phase with w and $\dot{w}$, respectively†:

$$Z = Z_w w + Z_{\dot{w}} \dot{w} = (Z_w + inZ_{\dot{w}})w = \bar{Z}_w w \,, \qquad (2.3)$$

$$M^{(o)} = M_w^{(o)} w + M_{\dot{w}}^{(o)} \dot{w} = (M_w^{(o)} + inM_{\dot{w}}^{(o)})w = \bar{M}_w^{(o)} w \,, \qquad (2.4)$$

where it has been found convenient to introduce complex derivatives $\bar{Z}_w = Z_w + inZ_{\dot{w}}$ and $\bar{M}_w^{(o)} = M_w^{(o)} + inM_{\dot{w}}^{(o)}$. The force Z and its derivatives obviously do not depend on the position of the origin O in this case; the moment, however, does, and if it is required to determine the moment about any alternative point A, at a distance x from O, this will be given by:

$$M = M^{(o)} - xZ = (\bar{M}_w^{(o)} - x\bar{Z}_w)w = \bar{M}_w w \,, \qquad (2.5)$$

so that the complex moment derivative (with respect to A) $\bar{M}_w = M_w + inM_{\dot{w}}$ becomes:

$$\underline{\bar{M}_w = \bar{M}_w^{(o)} - \bar{Z}_w x} \,. \qquad (2.6)$$

Let us now consider the *rotary oscillation*, first *about the origin O* (Fig. 2). The resultant force Z and moment M_o (about O) are again simple harmonic functions of time, of frequency n, generally out of phase with both ϑ and $\dot{\vartheta}$, and can be represented as sums of two terms, in phase with ϑ and $\dot{\vartheta}$, respectively:

$$Z = Z_{\vartheta,o} \vartheta + Z_{\dot{\vartheta},o} \dot{\vartheta} = (Z_{\vartheta,o} + inZ_{\dot{\vartheta},o})\vartheta = \bar{Z}_{\vartheta,o} \vartheta \,, \qquad (2.7)$$

$$M_o = M_{\vartheta,o} \vartheta + M_{\dot{\vartheta},o} \dot{\vartheta} = (M_{\vartheta,o} + inM_{\dot{\vartheta},o})\vartheta = \bar{M}_{\vartheta,o} \vartheta \,, \qquad (2.8)$$

where, again, complex derivatives $\bar{Z}_{\vartheta,o}$ and $\bar{M}_{\vartheta,o}$ have been introduced.

Next, let us consider a *rotary oscillation about an arbitrary point* A (Fig. 3)‡. Such an oscillation may be obtained by superposition of a rotary oscillation about O and a vertical oscillation whose velocity w is permanently determined by the relationship:

$$w = -x\dot{\vartheta} = -inx\vartheta, \qquad (2.9)$$

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† If Z has been found experimentally as a harmonic function out of phase with the motion, which can be expressed in the form:

$$Z = Z^* \, e^{i(nt+\delta)}, \text{ [written instead of } Z = Z^* \cos (nt + \delta)] \qquad (2.3a)$$

Z^* being the amplitude and δ phase difference angle relative to w, then, comparing (2.3) and (2.3a), we obtain:

$$Z_w = \frac{Z^*}{w^*} \cos \delta, \quad Z_{\dot{w}} = \frac{Z^*}{nw^*} \sin \delta, \quad \bar{Z}_w = \frac{Z^*}{w^*} e^{i\delta} \,. \qquad (2.3b)$$

An analogous procedure applies to $\bar{M}_w$, and similarly to $\bar{Z}_\vartheta$ and $\bar{M}_\vartheta$, as defined below.

‡ For simplicity, Figs. 1 to 3 have been drawn as if the equilibrium position were that of zero incidence, but the theory applies in the more general case.

so that it differs in phase by 180 deg from ϑ, and by 270 deg from $\dot{\vartheta}$. The *resultant force* Z will now be obtained by adding the expressions (2.7) and (2.3), while taking (2.9) into account:

$$Z = Z_{\vartheta,o}\vartheta - Z_w x\dot{\vartheta} = (\bar{Z}_{\vartheta,o} - inZ_w x)\vartheta, \quad \ldots \quad (2.10)$$

and hence the complex derivative $\bar{Z}_\vartheta = Z_\vartheta + inZ_{\dot{\vartheta}}$ will become:

$$\underline{\bar{Z}_\vartheta = \bar{Z}_{\vartheta,o} - in\bar{Z}_w x}\,. \quad \ldots \quad (2.11)$$

Similarly, we obtain the *resultant moment about O*, by adding (2.8) and (2.4), while taking (2.9) into account:

$$M^{(o)} = \bar{M}_{\vartheta,o}\vartheta - \bar{M}_w{}^{(o)}x\dot{\vartheta} = (\bar{M}_{\vartheta,o} - in\bar{M}_w{}^{(o)}x)\vartheta\,, \quad \ldots \quad (2.12)$$

so that the complex derivative $\bar{M}_\vartheta{}^{(o)} = M_\vartheta{}^{(o)} + inM_{\dot{\vartheta}}{}^{(o)}$ becomes:

$$\bar{M}_\vartheta{}^{(o)} = \bar{M}_{\vartheta,o} - in\bar{M}_w{}^{(o)}x\,. \quad \ldots \quad (2.13)$$

In this case, the *resultant moment about the centre* A *of rotary oscillations* is what really matters, and this will be:

$$M = M^{(o)} - Zx = (\bar{M}_{\vartheta,o} - in\bar{M}_w{}^{(o)}x - \bar{Z}_{\vartheta,o}x + in\bar{Z}_w x^2)\vartheta\,, \quad \ldots \quad (2.14)$$

and the complex derivative $\bar{M}_\vartheta = M_\vartheta + inM_{\dot{\vartheta}}$ finally becomes:

$$\underline{\bar{M}_\vartheta = \bar{M}_{\vartheta,o} - (\bar{Z}_{\vartheta,o} + in\bar{M}_w{}^{(o)})x + in\bar{Z}_w x^2}\,. \quad \ldots \quad (2.15)$$

It is seen that, when considering a disturbed motion of an aircraft, with its c.g. at an arbitrary point A, we shall need four complex derivatives $\bar{Z}_w$, $\bar{M}_w$, $\bar{Z}_\vartheta$ and $\bar{M}_\vartheta$ (which really represent eight real derivatives). Of these, $\bar{Z}_w$ does not depend on x; $\bar{M}_w$ and $\bar{Z}_\vartheta$ are linear functions of x (equations 2.6 and 2.11); and $\bar{M}_\vartheta$ is a quadratic function of x (equation 2.15). Supposing that the values of derivatives $\bar{Z}_w$, $\bar{M}_w{}^{(o)}$, $\bar{Z}_{\vartheta,o}$ and $\bar{M}_{\vartheta,o}$ (related to the origin O) have been determined theoretically or experimentally, we can easily calculate the corresponding values for an arbitrary point A.

If the derivatives are to be determined experimentally in a wind tunnel, one difficulty arises: it is comparatively simple to arrange experiments with rotary oscillations (and to record variable forces and moments during such oscillations), but it would be much more difficult to do this for heaving oscillations. However, if experiments with *rotary oscillations about more than one axis* are made, and a sufficient number of force and/or moment records obtained, it will be possible, by using equations (2.6), (2.11) and (2.15), to determine all derivatives, either for the origin O, or for an arbitrary c.g. position. If only some particular derivatives are needed, a certain reduction of the number of measurements may be expected. Two schemes of this kind are examined in the next sections.

3. *Scheme A: Oscillatory Tests About Two Axes, Forces and Moments Recorded.*—For every particular aircraft model, there are four independent unknown complex derivatives. To determine all of them we require four independent measurements, each recording one sine curve, or rather its amplitude and phase, which count as one complex experimental datum. The simplest way to achieve that is to arrange for oscillatory tests about two convenient axes, sufficiently distant from each other, say A_1 and A_2 (abscissae x_1 and x_2), and to record Z and M in each case, thus obtaining four complex derivatives $\bar{Z}_{\vartheta,1}$, $\bar{Z}_{\vartheta,2}$, $\bar{M}_{\vartheta,1}$ and $\bar{M}_{\vartheta,2}$. Applying the equations (2.11) and (2.15) twice, for A_1 and A_2, we obtain the following system of four linear equations with four unknowns $\bar{Z}_w$, $\bar{M}_w{}^{(o)}$, $\bar{Z}_{\vartheta,o}$ and $\bar{M}_{\vartheta,o}$:

$$\left.\begin{aligned}
&\bar{Z}_{\vartheta,o} - in\bar{Z}_w x_1 = \bar{Z}_{\vartheta,1}\\
&\bar{Z}_{\vartheta,o} - in\bar{Z}_w x_2 = \bar{Z}_{\vartheta,2}\\
&\bar{M}_{\vartheta,o} - (\bar{Z}_{\vartheta,o} + in\bar{M}_w{}^{(o)})x_1 + in\bar{Z}_w x_1{}^2 = \bar{M}_{\vartheta,1}\\
&\bar{M}_{\vartheta,o} - (\bar{Z}_{\vartheta,o} + in\bar{M}_w{}^{(o)})x_2 + in\bar{Z}_w x_2{}^2 = \bar{M}_{\vartheta,2}
\end{aligned}\right\}, \quad \ldots \quad (3.1)$$

the solution of which is:

$$\left.\begin{aligned}\bar{Z}_{\vartheta,o} &= \frac{x_2\bar{Z}_{\vartheta,1} - x_1\bar{Z}_{\vartheta,2}}{x_2 - x_1}\,; \quad in\bar{Z}_w = \frac{\bar{Z}_{\vartheta,1} - \bar{Z}_{\vartheta,2}}{x_2 - x_1} \\ \bar{M}_{\vartheta,o} &= \frac{(x_2\bar{M}_{\vartheta,1} - x_1\bar{M}_{\vartheta,2}) + x_1x_2(\bar{Z}_{\vartheta,1} - \bar{Z}_{\vartheta,2})}{x_2 - x_1} \\ in\bar{M}_w^{(o)} &= \frac{(\bar{M}_{\vartheta,1} - \bar{M}_{\vartheta,2}) + (x_1\bar{Z}_{\vartheta,1} - x_2\bar{Z}_{\vartheta,2})}{x_2 - x_1}\end{aligned}\right\} \quad \ldots \quad (3.2)$$

Substituting (3.2) into (2.6), (2.11) and (2.15), we obtain *all derivatives in terms of those originally measured, for arbitrary x* :

$$\left.\begin{aligned}in\bar{Z}_w &= \frac{\bar{Z}_{\vartheta,1} - \bar{Z}_{\vartheta,2}}{x_2 - x_1} \\ \bar{Z}_\vartheta &= \bar{Z}_{\vartheta,1}\,\frac{x_2 - x}{x_2 - x_1} + \bar{Z}_{\vartheta,2}\,\frac{x - x_1}{x_2 - x_1} \\ in\bar{M}_w &= \frac{\bar{M}_{\vartheta,1} - \bar{M}_{\vartheta,2} - \bar{Z}_{\vartheta,2}(x_2 - x) - \bar{Z}_{\vartheta,1}(x - x_1)}{x_2 - x_1} \\ \bar{M}_\vartheta &= \frac{\bar{M}_{\vartheta,1}(x_2 - x) + \bar{M}_{\vartheta,2}(x - x_1) + (\bar{Z}_{\vartheta,1} - \bar{Z}_{\vartheta,2})(x - x_1)(x - x_2)}{x_2 - x_1}\end{aligned}\right\} \quad \ldots \quad (3.3)$$

The formulae (3.3) would become somewhat simpler (though less symmetrical) if the origin of co-ordinates O were chosen so as to coincide with one of the oscillation centres A_1 or A_2. However, it will be more convenient to have them written as above, because the c.g. position (x) will usually be measured from the leading edge of the standard mean chord, or of the root chord, and it will be rather unusual to have one of the experimental oscillation centres exactly at either of these positions.

The four complex formulae (3.3) are equivalent to eight formulae in real terms, which are obtained by simply separating the real and imaginary parts in (3.3). For example, the first formula may be written:

$$in(Z_w + inZ_{\dot{w}}) = \frac{(Z_{\vartheta,1} + inZ_{\dot{\vartheta},1}) - (Z_{\vartheta,2} + inZ_{\dot{\vartheta},2})}{x_2 - x_1}\,,$$

and hence:

$$Z_w = \frac{Z_{\dot{\vartheta},1} - Z_{\dot{\vartheta},2}}{x_2 - x_1}\,, \quad n^2Z_{\dot{w}} = \frac{Z_{\vartheta,2} - Z_{\vartheta,1}}{x_2 - x_1}\,, \quad \ldots \quad (3.4)$$

and the remaining formulae are split in a similar way.

4. *Scheme B: Oscillatory Test About Three Axes, Only Moments Recorded.*—From the point of view of the experimental technique, it might be convenient to measure only resultant moments about the axes of oscillation, while avoiding the troublesome procedure of measuring the resultant forces. The question arises, what can be achieved by such partial measurements. If only oscillations about two axes were considered, we should have merely the last two of equations (3.1) at our disposal, and none of the four unknowns could be determined. If, however, three axes are used, then, applying the equation (2.15) three times, we get the following system of three linear equations:

$$\left.\begin{aligned}\bar{M}_{\vartheta,o} - (\bar{Z}_{\vartheta,o} + in\bar{M}_w^{(o)})x_1 + in\bar{Z}_wx_1^2 &= \bar{M}_{\vartheta,1} \\ \bar{M}_{\vartheta,o} - (\bar{Z}_{\vartheta,o} + in\bar{M}_w^{(o)})x_2 + in\bar{Z}_wx_2^2 &= \bar{M}_{\vartheta,2} \\ \bar{M}_{\vartheta,o} - (\bar{Z}_{\vartheta,o} + in\bar{M}_w^{(o)})x_3 + in\bar{Z}_wx_3^2 &= \bar{M}_{\vartheta,3}\end{aligned}\right\}, \quad \ldots \quad (4.1)$$

which may be thought to contain three unknowns $\bar{M}_{\vartheta,o}$, $\bar{Z}_w$ and $(\bar{Z}_{\vartheta,o} + in\bar{M}_w^{(o)})$. These are easily determined:

$$\left.\begin{aligned}
in\bar{Z}_w &= \frac{\bar{M}_{\vartheta,1}}{(x_1 - x_2)(x_1 - x_3)} + \frac{\bar{M}_{\vartheta,2}}{(x_2 - x_1)(x_2 - x_3)} + \frac{\bar{M}_{\vartheta,3}}{(x_3 - x_1)(x_3 - x_2)} \\
\bar{M}_{\vartheta,o} &= \bar{M}_{\vartheta,1}\frac{x_2 x_3}{(x_1 - x_2)(x_1 - x_3)} + \bar{M}_{\vartheta,2}\frac{x_1 x_3}{(x_2 - x_1)(x_2 - x_3)} + \bar{M}_{\vartheta,3}\frac{x_1 x_2}{(x_3 - x_1)(x_3 - x_2)} \\
\bar{Z}_{\vartheta,o} + in\bar{M}_w^{(o)} &= \bar{M}_{\vartheta,1}\frac{x_2 + x_3}{(x_1 - x_2)(x_1 - x_3)} + \bar{M}_{\vartheta,2}\frac{x_1 + x_3}{(x_2 - x_1)(x_2 - x_3)} + \bar{M}_{\vartheta,3}\frac{x_1 + x_2}{(x_3 - x_1)(x_3 - x_2)}
\end{aligned}\right\} \quad (4.2)$$

and, substituting into (2.6), (2.11) and (2.15), we obtain, *for arbitrary* x:

$$\left.\begin{aligned}
in\bar{Z}_w &= \frac{\bar{M}_{\vartheta,1}}{(x_1 - x_2)(x_1 - x_3)} + \frac{\bar{M}_{\vartheta,2}}{(x_2 - x_1)(x_2 - x_3)} + \frac{\bar{M}_{\vartheta,3}}{(x_3 - x_1)(x_3 - x_2)} \\
\bar{Z}_\vartheta + in\bar{M}_w &= \bar{M}_{\vartheta,1}\frac{x_2 + x_3 - 2x}{(x_1 - x_2)(x_1 - x_3)} + \bar{M}_{\vartheta,2}\frac{x_1 + x_3 - 2x}{(x_2 - x_1)(x_2 - x_3)} + \bar{M}_{\vartheta,3}\frac{x_1 + x_2 - 2x}{(x_3 - x_1)(x_3 - x_2)} \\
\bar{M}_\vartheta &= \bar{M}_{\vartheta,1}\frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + \bar{M}_{\vartheta,2}\frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + \bar{M}_{\vartheta,3}\frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}
\end{aligned}\right\}. \quad (4.3)$$

It is seen that the two complex derivatives $\bar{Z}_w = Z_w + inZ_{\dot{w}}$ and $\bar{M}_\vartheta = M_\vartheta + inM_{\dot{\vartheta}}$ can be determined for any x. As to the remaining derivatives, only their linear combination:

$$\bar{Z}_\vartheta + in\bar{M}_w = (Z_\vartheta - n^2 M_{\dot{w}}) + in(Z_{\dot{\vartheta}} + M_w) \quad (4.4)$$

can be found. It would obviously not help at all to use four axes of oscillation: an additional equation of the same kind as in (4.1) would be redundant, while it would not enable us to find $\bar{Z}_\vartheta$ and $\bar{M}_w$ separately. Therefore, methods consisting in measuring only moments about the axes of oscillation do not provide a full solution of the problem, *i.e.*, do not furnish the values of all derivatives.

Nevertheless, the experimental scheme B can be at least partly useful in some cases:

(*a*) The damping of short-period oscillations depends primarily on two derivatives, Z_w and $M_{\dot{\vartheta}}$, both determinable in this scheme. If, therefore, we are merely interested in the condition of zero damping, it may seem sufficient to know these two particular derivatives. The matter is not so simple, however. One must keep in mind that all derivatives are functions of Mach number and of reduced frequency. If Z_w and $M_{\dot{\vartheta}}$ are determined as such functions, it will be possible to find the condition of zero damping, *e.g.*, the critical value of Mach number for any assumed reduced frequency. However, the true frequency, in any given conditions of flight, depends itself on all other derivatives. Hence, the data furnished by the scheme B are insufficient, though quite valuable.

(*b*) If only small reduced frequencies are taken into consideration, then two additional relationships between the unknown (real) derivatives hold approximately, *viz.*,

$$M_\vartheta \simeq VM_w, \quad Z_\vartheta \simeq VZ_w. \quad (4.5)$$

These relationships are obviously and exactly true in the case of steady motions, and then they mean simply that, for a constant ϑ, the resultant force and moment are the same as for a constant $w = V\vartheta$. For oscillatory motions, the relationships are only approximately true, because the effects of apparent mass and of periodically varying vortex wake differ in the cases of rotary and plunging oscillations. However, it can be shown easily, at least in the two-dimensional case, that the differences are small of the order higher than one*, in reduced frequency. If, therefore,

* In the two-dimensional case, the order of magnitude of the differences is that of $\omega^2 \ln \omega$, where ω is reduced frequency; in the three-dimensional case, we may expect[2] differences of the order ω^2.

an accuracy of the first order is deemed sufficient, then the relationships (4.5) may be used as a makeshift, and then all derivatives are determinable in the scheme B. The procedure is obviously not quite satisfactory: the experimental method should be free from all inexact or doubtful theoretical assumptions, for which it should provide the ultimate check.

The final conclusion is that the scheme B may be of some limited use, but the scheme A should be greatly preferred.

5. *Non-dimensional derivatives.*—It is essential to present the experimental results in non-dimensional notation, so that data obtained in different laboratories for different models, wind speeds, and frequencies, could be easily compared and supplement each other. The *non-dimensional independent variables*, against which the stability derivatives will be plotted, are:

Non-dimensional abscissa of the axis: $h = x/c$, ... (5.1)

and reduced frequency: $\omega = nc/V = 2\pi fc/V$. .. .. (5.2)

The Mach number V/a will be an important additonal parameter, but it will not appear in our equations. It should be noticed that the definition of the reduced frequency is based on the length c (mean chord) according to the British custom, and not on the semi-chord, as in German and most American publications. Our definition leads to simpler formulae in what follows. The difference must be remembered when comparing results from various sources.

The following *non-dimensional stability derivatives* will be introduced, equivalent or analogous to the standard expressions of Bryant and Gates[6]:

$$\left.\begin{array}{ll} z_w = Z_w/\rho SV & z_{\dot w} = Z_{\dot w}/\rho Sc \\ m_w = M_w/\rho SVc & m_{\dot w} = M_{\dot w}/\rho Sc^2 \\ z_\vartheta = Z_\vartheta/\rho SV^2 & z_{\dot\vartheta} = Z_{\dot\vartheta}/\rho SVc \\ m_\vartheta = M_\vartheta/\rho SV^2c & m_{\dot\vartheta} = M_{\dot\vartheta}/\rho SVc^2 \end{array}\right\} \quad \text{.. .. .. ..} \quad (5.3)$$

It will be noticed that the standard mean chord c has been used throughout as length of reference, instead of l (tail arm). This is because the technique will be often applied to tailless models, and it also leads to a desirable simplification of the formulae*.

The non-dimensional complex derivatives will be obtained as follows:

$$\bar z_w = z_w + i\omega z_{\dot w} = (Z_w + inZ_{\dot w})/\rho SV = \bar Z_w/\rho SV,$$

and similarly:

$$\left.\begin{array}{l} \bar m_w = m_w + i\omega m_{\dot w} = \bar M_w/\rho SVc \\ \bar z_\vartheta = z_\vartheta + i\omega z_{\dot\vartheta} = \bar Z_\vartheta/\rho SV^2 \\ \bar m_\vartheta = m_\vartheta + i\omega m_{\dot\vartheta} = \bar M_\vartheta/\rho SV^2c \end{array}\right\} \quad \text{.. .. .. ..} \quad (5.4)$$

The *fundamental relationships* (2.6), (2.11) and (2.15) for stability derivatives, corresponding to varying positions of the axis, will assume the following form in non-dimensional notation:

$$\left.\begin{array}{l} \bar m_w = \bar m_w^{(o)} - \bar z_w h, \quad \bar z_\vartheta = \bar z_{\vartheta,o} - i\omega \bar z_w h, \\ \bar m_\vartheta = \bar m_{\vartheta,o} - (\bar z_{\vartheta,o} + i\omega \bar m_w^{(o)})h + i\omega \bar z_w h^2 \end{array}\right\} \quad \text{.. .. .. ..} \quad (5.5)$$

* Another difference will be found in the definition of $m_{\dot w}$ which, according to Ref. 6, should be defined as $M_{\dot w}g/Wc$ (W being the weight of the aircraft) which would be μ times smaller than in (5.3), μ denoting the relative density. Our definition is more consistent with those of other derivatives.

These three complex formulae are equivalent to the following six formulae in real terms:

$$\left.\begin{array}{ll} m_w = m_w^{(o)} - z_{\dot{w}} h & m_{\dot{w}} = m_{\dot{w}}^{(o)} - z_{\dot{w}} h \\ z_\vartheta = z_{\vartheta,o} + \omega^2 z_{\dot{w}} h & z_{\dot{\vartheta}} = z_{\dot{\vartheta},o} - z_w h \\ m_\vartheta = m_{\vartheta,o} - (z_{\vartheta,o} - \omega^2 m_{\dot{w}}^{(o)}) h - \omega^2 z_{\dot{w}} h^2 & m_{\dot{\vartheta}} = m_{\dot{\vartheta},o} - (z_{\dot{\vartheta},o} + m_w^{(o)}) h + z_w h^2 \end{array}\right\} . \quad (5.5a)$$

We shall now transform into non-dimensional notation the formulae of sections 3 and 4, giving the derivatives for arbitrary position of the axis in terms of those originally measured, for the schemes A and B.

Scheme A (formulae 3.3):

$$\left.\begin{aligned} i\omega \bar{z}_w &= \frac{\bar{z}_{\vartheta,1} - \bar{z}_{\vartheta,2}}{h_2 - h_1} \\ \bar{z}_\vartheta &= \bar{z}_{\vartheta,1} \frac{h_2 - h}{h_2 - h_1} + \bar{z}_{\vartheta,2} \frac{h - h_1}{h_2 - h_1} \\ i\omega \bar{m}_w &= \frac{\bar{m}_{\vartheta,1} - \bar{m}_{\vartheta,2} - \bar{z}_{\vartheta,2}(h_2 - h) - \bar{z}_{\vartheta,1}(h - h_1)}{h_2 - h_1} \\ \bar{m}_\vartheta &= \frac{\bar{m}_{\vartheta,1}(h_2 - h) + \bar{m}_{\vartheta,2}(h - h_1) + (\bar{z}_{\vartheta,1} - \bar{z}_{\vartheta,2})(h - h_1)(h - h_2)}{h_2 - h_1} \end{aligned}\right\} \ldots \quad (5.6)$$

Scheme B (formulae 4.3):

$$\left.\begin{aligned} i\omega \bar{z}_w &= \frac{\bar{m}_{\vartheta,1}}{(h_1 - h_2)(h_1 - h_3)} + \frac{\bar{m}_{\vartheta,2}}{(h_2 - h_1)(h_2 - h_3)} + \frac{\bar{m}_{\vartheta,3}}{(h_3 - h_1)(h_3 - h_2)} \\ \bar{z}_\vartheta + i\omega \bar{m}_w &= \bar{m}_{\vartheta,1} \frac{h_2 + h_3 - 2h}{(h_1 - h_2)(h_1 - h_3)} + \bar{m}_{\vartheta,2} \frac{h_1 + h_3 - 2h}{(h_2 - h_1)(h_2 - h_3)} + \bar{m}_{\vartheta,3} \frac{h_1 + h_2 - 2h}{(h_3 - h_1)(h_3 - h_2)} \\ \bar{m}_\vartheta &= \bar{m}_{\vartheta,1} \frac{(h - h_2)(h - h_3)}{(h_1 - h_2)(h_1 - h_3)} + \bar{m}_{\vartheta,2} \frac{(h - h_1)(h - h_3)}{(h_2 - h_1)(h_2 - h_3)} + \bar{m}_{\vartheta,3} \frac{(h - h_1)(h - h_2)}{(h_3 - h_1)(h_3 - h_2)} \end{aligned}\right\} . \quad (5.7)$$

It must be kept in mind that each of the complex formulae (5.6) and (5.7) represents two formulae in real terms, obtainable by separating the real and imaginary terms in each formula.

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APPENDIX

Equations of Dynamic Stability for Level Flight with Derivatives of Oscillatory Motion

In the theory of dynamic stability, the equations are usually set up in the system of moving axes x, z, the x-axis being fixed in the moving aircraft and directed along the undisturbed flight velocity V, thus horizontally in undisturbed level flight (Fig. 4), and the z-axis pointing vertically downward in such a flight*. Hence, the moving axes coincide, in undisturbed flight, with those x_1, z_1, fixed in space, which are always horizontal and vertical respectively.

During a disturbance, the attitude of the aircraft in space varies and is determined, at any time, by the small angle of pitch θ (positive clockwise in Fig. 5). The moving axes x, z rotate with the aircraft through the same angle θ, while x_1, z_1 remain unaltered. The velocity of the c.g. varies throughout the disturbance so that, in addition to V, there may be small incremental velocity components u and w along x and z-axes. We neglect the component u (thus assuming a disturbance at constant forward speed, and eliminating the phugoid motion). Therefore, the small variable components of the disturbance are only θ and w, and the resultant linear velocity:

$$V_r = \sqrt{(V^2 + w^2)} \simeq V, \quad \ldots \quad \text{(A.1)}$$

the small terms of the second order being neglected. The same disturbance can be referred to the fixed axes x_1, z_1, the resultant velocity being now resolved along them, so that the horizontal component will become:

$$V_h = V \cos\theta \simeq V, \quad \ldots \quad \text{(A.2)}$$

and the vertical component:

$$w_1 = w\cos\theta - V\sin\theta \simeq w - V\theta, \quad \ldots \quad \text{(A.3)}$$

the higher-order terms being again neglected. It is seen that the horizontal component may be considered as constant, and the entire disturbance represented, in line with the 'experimental' method used in the main text, as a combination of vertical (heaving) oscillation with small variable velocity w, and rotary oscillation with the small variable angle ϑ, always equal to θ:

$$\vartheta = \theta. \quad \ldots \quad \text{(A.4)}$$

It is clear now that the equation of motion can be written either in the moving system (with w, θ as unknown functions of time), or in the fixed system (with w_1, ϑ as unknown functions of time), and the transformation needed for passing from one system to the other one will be simply (A.3) and (A.4).

We shall start by writing the *equations in the fixed system*, so as to use the 'experimental' aerodynamic derivatives of the main text, and to transform them afterwards to the more usual moving system. The equations are:

$$\left.\begin{aligned} \frac{W}{g}\cdot\frac{dw_1}{dt} &= Z_\vartheta\vartheta + Z_{\dot\vartheta}\frac{d\vartheta}{dt} + Z_w w_1 + Z_{\dot w}\frac{dw_1}{dt} \\ \frac{Wk_B{}^2}{g}\frac{d^2\vartheta}{dt^2} &= M_\vartheta\vartheta + M_{\dot\vartheta}\frac{d\vartheta}{dt} + M_w w_1 + M_{\dot w}\frac{dw_1}{dt} \end{aligned}\right\} \quad \ldots \quad \text{(A.5)}$$

To write them in non-dimensional terms, we introduce three auxiliary constants:

$$\left.\begin{aligned} &\text{unit of aerodynamic time} && \hat{t} = W/g\rho SV \\ &\text{relative density of the aircraft} && \mu = W/g\rho Sc \\ &\text{moment of inertia ratio} && i_B = k_B{}^2/c^2 \end{aligned}\right\} \quad \ldots \quad \text{(A.6)}$$

* In Figs. 1 and 3, referring to the main text of this Report, the abscissa x was conveniently measured positive backwards: this does not lead to any inconsistency.

further, non-dimensional values of time, and vertical velocity component:

$$\tau = t/\hat{t} \quad \text{and} \quad \hat{w}_1 = w_1/V \;, \quad \ldots \qquad \text{(A.7)}$$

and finally, non-dimensional derivatives, as defined by (5.3). The equations (A.5), after multiplying the first one by

$$g\hat{t}/WV = 1/\rho SV^2 = \hat{t}/\rho SVc\mu \;,$$

and the second one by

$$g\hat{t}^2/Wk_B{}^2 = \mu/\rho SV^2 ci_B = \hat{t}/\rho SVc^2 i_B,$$

become:

$$\left.\begin{aligned} \frac{d\hat{w}_1}{d\tau} &= z_\vartheta \vartheta + \frac{z_{\dot\vartheta}}{\mu}\cdot\frac{d\vartheta}{d\tau} + z_w \hat{w}_1 + \frac{z_{\dot w}}{\mu}\cdot\frac{d\hat{w}_1}{d\tau} \\ \frac{d^2\vartheta}{d\tau^2} &= \frac{\mu m_\vartheta}{i_B}\vartheta + \frac{m_{\dot\vartheta}}{i_B}\frac{d\vartheta}{d\tau} + \frac{\mu m_w}{i_B}\hat{w}_1 + \frac{m_{\dot w}}{i_B}\frac{d\hat{w}_1}{d\tau} \end{aligned}\right\} \ldots \qquad \text{(A.8)}$$

We now obtain the *equations in the moving system*, by replacing w_1 by (A.1) and ϑ by θ in (A.5):

$$\left.\begin{aligned} \frac{W}{g}\left(\frac{dw}{dt} - V\frac{d\theta}{dt}\right) &= (Z_\vartheta - VZ_w)\theta + (Z_{\dot\vartheta} - VZ_{\dot w})\frac{d\theta}{dt} + Z_w w + Z_{\dot w}\dot{w} \\ \frac{Wk_B{}^2}{g}\cdot\frac{d^2\theta}{dt^2} &= (M_\vartheta - VM_w)\theta + (M_{\dot\vartheta} - VM_{\dot w})\frac{d\theta}{dt} + M_w w + M_{\dot w}\dot{w} \end{aligned}\right\} \ldots \qquad \text{(A.9)}$$

It is seen that w-derivatives (Z_w, $Z_{\dot w}$, M_w, $M_{\dot w}$) are exactly the same in both systems, referring as well to w_1 and w. Contrariwise, the θ-derivatives differ from ϑ-derivatives, and may be written:

$$\left.\begin{aligned} Z_\theta &= Z_\vartheta - VZ_w, \qquad Z_{\dot\theta} = Z_{\dot\vartheta} - VZ_{\dot w} \\ M_\theta &= M_\vartheta - VM_w, \qquad M_{\dot\theta} = M_{\dot\vartheta} - VM_{\dot w} \end{aligned}\right\} \ldots \qquad \text{(A.10)}$$

The new equations can also be immediately re-written in non-dimensional form, by transforming exactly as before, while introducing new non-dimensional θ-derivatives, analogous to ϑ-derivatives in (5.3):

$$\left.\begin{aligned} z_\theta &= Z_\theta/\rho SV^2 = z_\vartheta - z_w \\ z_q = z_{\dot\theta} &= Z_{\dot\theta}/\rho SVc = z_{\dot\vartheta} - z_{\dot w} \\ m_\theta &= M_\theta/\rho SV^2 c = m_\vartheta - m_w \\ m_q = m_{\dot\theta} &= M_{\dot\theta}/\rho SVc^2 = m_{\dot\vartheta} - m_{\dot w} \end{aligned}\right\} \ldots \qquad \text{(A.11)}$$

and putting $\hat{q} = q\hat{t} = d\theta/d\tau$ (non-dimensional angular velocity). The equations (A.9) then become:

$$\left.\begin{aligned} \frac{d\hat{w}}{d\tau} &= z_\theta\theta + \left(1 + \frac{z_q}{\mu}\right)\hat{q} + z_w\hat{w} + \frac{z_{\dot w}}{\mu}\frac{d\hat{w}}{d\tau} \\ \frac{d\hat{q}}{d\tau} &= \frac{\mu m_\theta}{i_B}\theta + \frac{m_q}{i_B}\hat{q} + \frac{\mu m_w}{i_B}\hat{w} + \frac{m_{\dot w}}{i_B}\frac{d\hat{w}}{d\tau} \end{aligned}\right\} \ldots \qquad \text{(A.12)}$$

Equations (A.12) contain a few more terms than usually written hitherto, *viz.*, the terms with derivatives z_θ, m_θ, z_q and $z_{\dot{w}}$. The two last ones have been normally omitted as supposedly very small compared with μ (the latter, particularly for fast aircraft, being of the order of 100); this will apply in all cases we have in view, and therefore the derivatives z_q and $z_{\dot{w}}$ are of very little importance, and the corresponding terms may be usually neglected—this also applies to terms with $z_{\dot{\vartheta}}$ and $z_{\dot{w}}$ in (A.8)*. As to the terms with derivatives z_θ and m_θ, they are of particular interest. On the basis of the 'quasi-steady' theory, the relationships (4.5) would apply, therefore (A.10) and (A.11) would give $z_\theta = z_\vartheta - z_w = 0$ and $m_\theta = m_\vartheta - m_w = 0$, and both terms would disappear, the system of differential equations becoming of the second order in $\hat{q}$ and $\hat{w}$, instead of third order in θ and $\hat{w}$. The condition $z_\theta = m_\theta = 0$ has been always tacitly implied, this meaning with reference to the moving axis exactly the same as (4.5) in fixed axes. When having to deal with aerodynamics of unsteady (oscillatory) motion, we may at most expect z_θ and m_θ to be small of second order in ω. The latter fact does not warrant the neglect of these terms in (A.12), without a thorough investigation and numerical analysis. The simplification involved is not particularly valuable, as the primary aim of the stability equations is to establish the conditions of zero damping, and these will be simple enough even without neglecting the z_θ, m_θ terms. The determinantal equation for stability roots, corresponding to the full system (A.12) is:

$$\Delta(\lambda) = \begin{vmatrix} \left(1 - \frac{z_{\dot{w}}}{\mu}\right)\lambda - z_w & -\left(1 + \frac{z_q}{\mu}\right)\lambda - z_\theta \\ -\frac{m_{\dot{w}}}{i_B}\lambda - \frac{\mu m_w}{i_B} & \lambda^2 - \frac{m_q}{i_B}\lambda - \frac{\mu m_\theta}{i_B} \end{vmatrix} = A\lambda^3 + B\lambda^2 + C\lambda + D = 0, \quad \text{(A.13)}$$

where:

$$\left.\begin{aligned} A &= 1 - \frac{z_{\dot{w}}}{\mu} \\ B &= -z_w - \left(1 - \frac{z_{\dot{w}}}{\mu}\right)\frac{m_q}{i_B} - \left(1 + \frac{z_q}{\mu}\right)\frac{m_{\dot{w}}}{i_B} \\ C &= \frac{z_w m_q - z_\theta m_{\dot{w}}}{i_B} - \left(1 - \frac{z_{\dot{w}}}{\mu}\right)\frac{\mu m_\theta}{i_B} - \left(1 + \frac{z_q}{\mu}\right)\frac{\mu m_w}{i_B} \\ D &= \frac{\mu}{i_B}\,(z_w m_\theta - z_\theta m_w) \end{aligned}\right\} \quad \text{(A.14)}$$

while, assuming $z_\theta = m_\theta = 0$ (and neglecting $z_{\dot{w}}/\mu$ and z_q/μ, as proper for consistency reasons), we obtain:

$$\left.\begin{aligned} A &\simeq 1 \\ B &\simeq -z_w - \frac{m_q + m_{\dot{w}}}{i_B} = -z_w - \frac{m_{\dot{\vartheta}}}{i_B} \\ C &\simeq \frac{z_w m_q - \mu m_w}{i_B} = \frac{z_w(m_{\dot{\vartheta}} - m_{\dot{w}}) - \mu m_w}{i_B} \\ D &\simeq 0 \end{aligned}\right\} \quad \text{(A.15)}$$

---

* The derivatives $z_{\dot{w}}$ and $z_{\dot{\vartheta}}$, although negligible in dynamic equations, are of great importance for determining $m_{\dot{w}}$ and $m_{\dot{\vartheta}}$ for varying c.g. positions, *see* (5.5a).

The condition of zero damping in the last simpler case (quadratic equation) is:

$$B \simeq -z_w - \frac{m_q + m_{\dot{w}}}{i_B} = 0, \quad \ldots \quad \text{(A.16)}$$

while, taking the full cubic ((A.13) with coefficients (A.14)) into account, it becomes:

$$B - D/C = 0. \quad \ldots \quad \text{(A.17)}$$

As a thorough experimental investigation must provide all derivatives, there should be no difficulties with using the more exact condition.

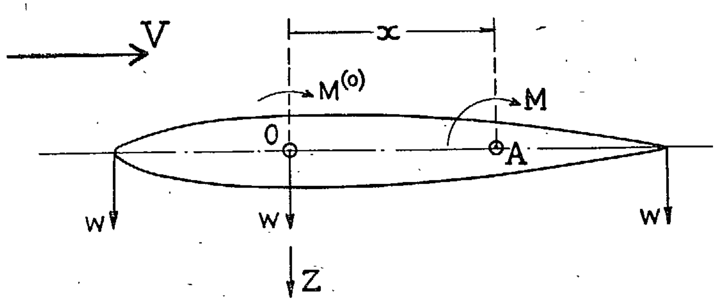

FIG. 1. Model in heaving oscillation.

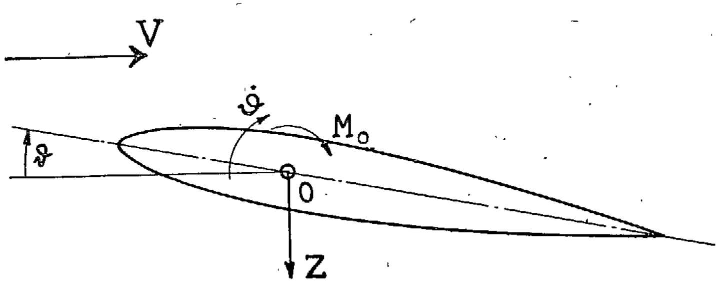

FIG. 2. Model in rotary oscillation about origin.

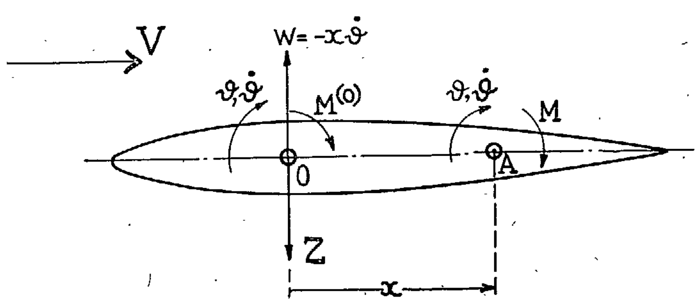

FIG. 3. Model in rotary oscillation about arbitrary axis.

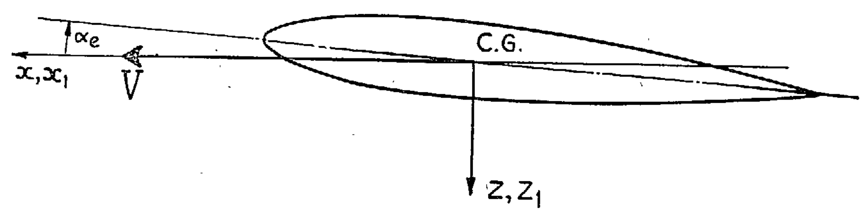

FIG. 4. Aircraft and co-ordinate systems in equilibrium conditions.

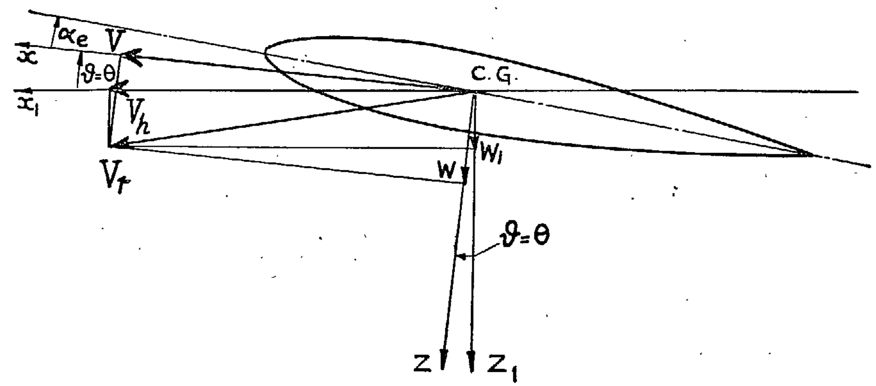

FIG. 5. Aircraft and co-ordinate systems in disturbed flight.

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