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Summary.—The purpose of this report is to provide experimental results for comparison with theoretical analyses of stress diffusion problems. The structures considered consist of plane reinforced sheet which has been assumed not to buckle. Symmetrical loads are applied to the edge booms connected to the sheet by continuous no-slip joints. Attention is concentrated on the stress distribution near the ends of the parallel strips of plate.

An outline of the existing theoretical work which is applicable to this type of problem is given. The stringer-sheet theory, the only one capable of dealing adequately with unreinforced sheet, is compared with the photoelastic results.

It is shown that the stringer-sheet theory overestimates the peak shear stresses near the corners of the strips and consequently also the rate of diffusion of load from boom to sheet. It is also shown that the experimental shear stresses are in reasonable agreement with those predicted by a more exact plane-stress theory. This theory predicts that the peak shear stress in the plate is $2/\pi$ times the direct stress in the boom at the end of the panel. However, with the type of joint considered here, the maximum shear stress is likely to be much higher than the value given by this prediction.

Some attention is also given to transverse end stiffeners and it would seem that these normally have very little effect on the shear stresses.

The photoelastic models were made from the Allylstrene plastic called C.R.39. The no-slip joints were obtained by gluing the stiffeners to the plates.

1. *Introduction.*—The problems of stress analysis known as stress diffusion and shear lag‡ have been confronting aeronautical engineers for about fifteen years. The papers published during this period on these problems have been mainly theoretical, and indeed very few confirmatory experiments appear to have been carried out. Until recently it has been the practice to allow the thin plate skin of aircraft to buckle between the stringers at loads below the ultimate; in some cases buckling occurred at working loads. For this reason and also because the theory is thereby simplified, it was usually assumed that the plates take no direct load and merely transfer load from one stringer to the next by shear. Skin buckling is now no longer considered permissible or desirable, owing firstly to the need for more accurate aerofoil shapes, and secondly, to the investigations which have shown that under current conditions lighter structures are possible if buckling is postponed until ultimate failure occurs. Hence with the use of heavier plates and lighter stiffeners, the above assumption will no longer lead to sufficiently accurate solutions of these problems.

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* Carried out at Bristol University and communicated by Prof. A. G. Pugsley.

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‡ The recent British Standard Glossary[1] defines stress diffusion as: 'Variation along the length of a structure of transverse distribution of stress due to direct loads applied along its length', and shear lag as: 'The type of diffusion in which the lag of longitudinal displacement of one part of a transverse section relative to that of another results entirely from shear loading applied along lines parallel to the length of the structure.'

In the United States the term 'shear lag' often embraces both these meanings.

Owing to the complex nature of the mathematics involved, all the theoretical papers on stress diffusion and shear lag are concerned with two-dimensional elasticity and for this reason the stresses are assumed constant over any section of stringer or boom and through the thickness of the plate. Due to the difficulty in solving the differential equation of plane stress, further simplifying assumptions are also made in the majority of instances. For this reason the theories may be classified into three groups. The first group, which uses what is known as the finite-stringer method, assumes that the plate has only shear stiffness and that it therefore takes no direct load. The second group assumes that the strain in the plate in the transverse direction is suppressed entirely by many rigid ribs and by this means the problem is reduced to solving Laplace's equation, which is simpler than the equation of plane stress. If in addition it is assumed that the stringers are replaced by a uniform sheet, which is the limit of a large number of small stringers, then the method is known as the stringer-sheet solution. Thirdly, there exist a few semi-empirical solutions. The following brief review of the published papers is given so that the assumptions and difficulties may be more fully understood.

1.1. *Finite-Stringer Method.*—The first noteworthy paper using the finite-stringer method is that by Cox, Smith and Conway[2] (1937) who derived the basic set of differential equations

$$P_r = EA_r \frac{du_r}{dx} \qquad (1)$$

$$\frac{dP_r}{dx} = \frac{Gt_r}{b_r}(u_r - u_{r-1}) - \frac{Gt_{r+1}}{b_{r+1}}(u_{r+1} - u_r) \qquad (2)$$

(*see* Fig. 1).

assuming a plate with shear stiffness only and where u_r is the displacement of the r^{th} stiffener in the x–direction.

In the general case a set of $n - 1$ second-order equations is obtained where n is the number of stringers, and thus as this number increases, so does the labour involved in solving the problem. To make the problem more tractable, constant-stress stringers were considered and by this means the differential equations may be integrated independently leaving a set of linear simultaneous equations to be solved. For both constant-stress and uniform stringers, Duncan[3] (1938) and Smith[4] (1938) give solutions of the above equations in a few cases where both structure and applied load are symmetrical about the longitudinal centre-line. They were able to show that the stresses are diffused exponentially along the length and therefore with typical dimensions the load rapidly becomes uniformly distributed across the panel. Cox[5] (1938) shows that the amount of calculation necessary is much reduced if a few of the stringers are such that the direct stress is constant along their length.

The finite-stringer theory was further refined by Hadji-Argyris and Cox[6] (1944) for the type of panel shown in Fig. 2. The results for a parallel panel were found by a limiting process. They found it possible to calculate the average stringer stresses and the shear stress at the edge without the need to solve determinantal equations. It is also shown that the solution for the original stringer-sheet assumptions can be obtained from that already derived by taking the limit as the number of stringers tends to infinity. Further simplifications are indicated by Hadji-Argyris[7] (1944) and examples are given in both papers.

When the load and structure are such that bending occurs, the shear strain can no longer be taken as $\partial u/\partial y$ but becomes $\partial u/\partial y + \partial v/\partial x$. (Differential coefficients are used here for brevity.) Williams and Fine[8] (1940) show that by introducing $\partial v/\partial x$, where v is a function of x only, and the additional equation relating the bending moment at any section to the applied moment, the problem is often still soluble. Where structure and load are symmetrical this latter equation is automatically satisfied by taking $v = 0$.

As the relatively heavy plate of modern aircraft structures is now often important in resisting the direct applied load, this method is becoming less useful. In any event it cannot hope to give accurate peak shear stress values when the plate remains unbuckled. The stringer-sheet method, which is more easily applied to panels with large numbers of stringers, is more likely to be of use with modern structures.

1.2. *Stringer-sheet Method.*—The stringer-sheet method was first proposed in a paper by Williams, Starkey and Taylor[9] (1939). In its original form the assumption was again made that the plate takes no direct stress, but this was not done by Hildebrand[10] (1943) and Goodey[11] (1946) in later work. The approximations are that the strain in the transverse direction is entirely suppressed by infinitely many rigid ribs and the stringers are replaced by a 'stringer sheet'. Thus for direct stresses the sheet thickness is effectively $t + t_x$ where t_x is the area of added stringers per unit width of plate. Only the sheet thickness t is available to resist the shear loads. Using well-known notation the relations set out below follow: the stress in the longitudinal direction is $\sigma_x = E \, . \, \partial u/\partial x$; the shear stress is $\tau = G(\partial u/\partial y + \partial v/\partial x)$ where v is a function of x only, and the equation of equilibrium for constant t and t_x is

$$\frac{\partial \tau}{\partial y} + k_x \frac{\partial \sigma_x}{\partial x} = 0 \text{ where } k_x = 1 + \frac{t_x}{t} \, .$$

In usual elastic theory a second equation is required, but because of the rigid ribs which supply the forces in the direction necessary for equilibrium, it is trivial here. Substituting for τ and σ_x, the following form of Laplace's equation is obtained:

$$\frac{\partial^2 u}{\partial y^2} + k^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{(3)}$$

where $k^2 = 2(1 + \nu)k_x$ and ν is Poisson's ratio. The displacement v, which can only affect the shear stresses, is determined by equating the total transverse shear force on any section to the applied shear.

Except in a few instances, reasonably simple solutions of equation (3) satisfying all the boundary conditions are not known. Williams, Starkey and Taylor[9] (1939) give the solution of some problems; for example when the stress is zero at one end and the displacement zero at the other, the solution is obtained in terms of a single Fourier series. Mansfield[12] (1947) has shown that where the panel is bounded by constant area booms and the load is applied at the ends, satisfaction of all boundary conditions is easily obtained with the aid of generalized Fourier series*. In a similar way, if solutions exist satisfying the conditions along the long edges they may be adapted to any conditions at the ends.

All the existing solutions indicate that with practical panels the diffusion is almost complete within a length equal to one or two panel breadths. In other words, the stress distribution at distances greater than this from an end is independent of the boundary conditions at that end. Consequently, in a number of papers semi-infinite strips have been considered, as it is easier to satisfy one boundary condition at a time. To reduce the amount of work required the end conditions considered are usually either that the direct stress or the displacement is zero. Goodey[11] (1946) and Hildebrand[10] (1943) both solve a number of problems of this type. Goodey uses Fourier integrals which are capable of dealing with complicated loading along the sides but not with variations in panel dimensions. Some of his results are perhaps more simply deduced with generalized Fourier series. Some of the problems which have been solved are shown in Fig. 3.

Mansfield[13] (1947) has discussed the importance of the bending stiffness of an end rib for the problem of Fig. 3a. With the rib built into the edge stringers, considerable reduction in the peak shear stresses is achieved. Ribs in practice will usually have a small bending rigidity compared with that of the sheet in its own plane so that the main effect, as is shown in the paper, is the reduction of the peak shear stresses at points XX in Fig. 3a. Infinite values are predicted by the stringer sheet method at these points when no rib is employed. Because of the local character of these peak shear stresses, Mansfield[14,15] (1947, 1948) has also considered the stress distribution in semi-infinite sheets. Using plane stress theory he has found the stresses in plates loaded as shown in Fig. 4.

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* $\sum_{n=n_1}^{\infty} \alpha_n \binom{\sin}{\cos} nx$, where the n are the roots of certain transcendental equations and not necessarily multiples of π, is a generalized Fourier series and the α_n can be chosen so that the series represents a given function over a certain interval. The terms of the series are orthogonal over this interval.

It can be shown that the singularity in stress which occurs at the points XX of a panel similar to that of Fig. 3a but without an end rib is precisely the same as those occurring at the points XX of the problem illustrated in Fig. 5. The shear stress at such a point is dependent upon the direction from which the point is approached but is always finite (*e.g.*, *see* Timoshenko[16]). Physically, however, the theoretical solution breaks down as it predicts a rotation of the corner through 90 deg as indicated in Fig. 6. The solution for the displacement u under the load is given by Timoshenko and from this the following result is easily obtained.

$$\frac{\partial u}{\partial y} = \frac{2q}{\pi E} \log_e \frac{1 - x/2b}{x/2b}$$

where q is the load per unit area applied to the boundary. It is seen that $\partial u/\partial y$ is infinite at $x = 0$ and $x = 2b$. The importance of this theoretical rotation of the corner can be estimated by finding the value of x for which $\partial u/\partial y$ is $0 \cdot 1$ say. With small x and $\partial u/\partial y = 0 \cdot 1$,

$$\log_e (x/2b) = - 0 \cdot 1 \pi E/2q \text{ approximately.}$$

With usual structural materials $x/2b$ will almost certainly be less than 10^{-8}. Thus the theory is still likely to hold quite close to the corner. From this approach to the problem it would appear that close to the corner, the shear stress will be of the order $2/\pi$ times the direct stress in the booms. The stringer-sheet solution predicts very much higher values for the peak shear stress and for this reason the results of Mansfield[12,14] (1947) are of doubtful value, as are also the quantitative results of Mansfield[17] (1949). Here the effect on the peak shear stress of slip in the media attaching the edge stiffeners to the plate for panels of the type shown in Fig. 3a is discussed. The slip is taken directly proportional to the shear transmitted, an assumption that is probably rarely fulfilled in practice.

The accuracy of the stringer-sheet approximation has been open to doubt for some considerable time. One source of error which is not generally recognized is due to the assumption that $\sigma_x = E \,.\, \partial u/\partial x$.

For the plate the correct expression is $\sigma_x = \dfrac{E}{1 - \nu^2} \dfrac{\partial u}{\partial x}$ when the transverse strain is zero. However, the condition of zero transverse strain is impossible to realize in practice and further attention to this point is unwarranted. Fine[18] (1941) solved ' exactly ' the problem shown in Fig. 7 for the purpose of comparison with this approximate method. The solution given by Fine implicitly assumes that the transverse strain is zero at the ' free ' end but with the length of panel considered the effect on the stresses at the built-in end should be negligible. Very good agreement between the two theories is obtained. Goodey[11] (1946) also solves the plane-stress equations for a problem of the type shown in Fig. 3a. The results suggest that in this problem agreement between the exact and the stringer-sheet theory is dependent on the assumption of an end rib rigid in the transverse direction, *i.e.*, on the assumption that the transverse strain at the end is zero. In one case Goodey modifies the plane-stress equations to take into account many ribs and stiffeners by the stringer-sheet technique. Thus apart from the problem discussed by Goodey where the stringer-sheet method predicts infinite shear stresses, the stringer-sheet method appears to give good agreement with the more accurate plane-stress solutions.

1.3 *Other Approximations.*—For panels where a number of stiffeners are attached to the plate Kuhn and Chiarito[19] (1942) suggest that for purposes of design a substitute panel with only edge and one central stiffeners be discussed. With the aid of both stringer-sheet and finite-stringer methods, suitable dimensions for the substitute panels and semi-empirical rules for obtaining direct-stress distributions are given. With the subsequent development of the two former methods, these further approximations do not seem to be warranted. The direct stresses in a panel having tapered stringers are given and good agreement is obtained with the proposed method. The method is further simplified by Kuhn and Peterson[20] (1948) and an attempt is made to deal with finite transverse stiffness of panels, but the method developed is very approximate. Some test results are given but the agreement obtained is probably fortuitous as the panels were riveted and no allowance is made for this fact in the theory. Some tests have also been carried out by Allen[21] (1948) but the same objection applies to this paper also.

1.4. *Remarks.*—Thus it is seen that the accuracy of existing approximate theories has never been adequately tested. The experimental work described here was carried out with the aim of providing more definite information with regard to the accuracy of the stringer-sheet theory. Most attention has been given to the type of diffusion problem shown in Fig. 3a, as it was felt that the discrepancy was likely to be greatest in this case. This type of problem occurs in aircraft structures at cut-outs in the wing surfaces and sometimes at the wing roots. Usually shear forces will also be applied along the edge stiffeners but the stress distribution due to these shears can be treated as a separate problem providing that the structure remains elastic.

A photo-elastic method was used as symmetry of loading is visually apparent and shear stresses are very easy to determine. Probably the method also enables peak shear stresses to be evaluated with more accuracy than with the use of strain gauges. The models tested were made to compare with theoretical solutions rather than as replicas of actual structures. No attempt was made to reproduce the effect of rivet slip as no existing theory claims to predict rivet loads.

2. *Photo-Elastic Technique.*—It was decided that the best and most convenient means of obtaining experimental evidence on stress diffusion was by the use of photo-elasticity. The method also enables the effect of transverse stiffeners to be evaluated. Perhaps the most important reason for this choice was that continuous joints can be readily achieved and thus better agreement with the theoretical assumptions can be obtained. It might be argued that this type of connection is rare in practice, but the characteristics of riveted joints, the common connection, are still uncertain and therefore without building full-scale structures, no accurate knowledge can be gained of their effect. The shear stresses are also easily obtained. The photo-elastic polariscope and loading rig at Bristol University has a four-inch diameter field and this governed to some extent the size of the models constructed. The apparatus used was designed and built at Bristol.

2.1. *Model Construction.*—Of the several available photo-elastic materials it was decided that the Allylstrene type plastic called C.R.39 was the most suitable for these experiments. As it is made in sheets with polished surfaces no polishing need be done and therefore compared with other materials much labour was saved in the preparation of models. Plates of the material are readily glued together and this property enabled stiffening members to be easily attached to the plates.

A jig saw was used for the rough cutting of the plastic sheets and a vertical milling machine was used for most of the final shaping. The ends of the plates had to be filed and consequently were not as well finished as the sides. This was partly due to the brittle nature of C.R.39. The photograph of Fig. 8 shows the most complicated model made, but most of the models had the same general form. As it was desired to obtain a uniform stress distribution in the booms where they protrude beyond the end of the plate, an effort was made to locate the loading holes as near the axis of the booms as possible. By locating the sides of the stringers from pins fitted in the loading holes, the error in the position of these holes is probably less than 0·003 in. Even with this small error the maximum bending stress is 7 per cent of the average direct stress in the stiffener of width 0·25 in. which was usually used. The desired width of plates and stiffeners and also parallel edges were readily obtained with the vertical miller if the members were of reasonable width. With stiffeners appreciably under 0·25 in. in width, 0·001 to 0·002 in. error was usual owing to the lack of rigidity even though a steel backing piece was used whilst milling. However as the thickness of the plastic varied by as much as 0·005 in. this error was relatively unimportant.

The prepared pieces of the models were glued together using the special glue obtainable which was required to be baked at 80 deg C and under pressure. It was found most convenient to sandwich the model between two glass plates and to apply pressure by means of about a pound of weights on the top plate. This permitted the model to be seen and it could thus be easily determined whether or not the stiffeners had been disturbed. The models were baked for approximately sixteen hours at about 75 to 80 deg C and allowed to cool slowly for about three hours after removal of the top glass plate. The slow cooling was an attempt at annealing the model but was not over-successful due to the rather large stresses (possibly 'time-edge' stress), introduced by the baking. Before testing, the free end of the plate was filed square and by this means the worst time-edge effects were removed. Unfortunately these time-edge stresses prevented stress measurements being made for the stiffeners.

The dimensions of the models tested are given in Figs. 9 to 13. The length of each model was chosen so that the direct stress at its centre was very nearly uniform and usually two cases were covered by each by making the ends different. The numbers given in the sketches are for convenience of reference. For all the models a tree system was used to apply the load. Knife-edges were not used at the hinges, as, if the load was unsymmetrical, it was immediately obvious from the photo-elastic pattern. If such a condition did exist, the fault could usually be rectified by applying transverse forces by hand whilst the model was lightly loaded.

2.2. *Calibration of the Photo-elastic Material.*—Tension specimens approximately $\frac{1}{4}$-in. wide were used to find the photo-elastic sensitivity of the C.R.39 sheets. As the plastic became rather yellow when baked, some of the specimens were baked with the models but no significant change was found in the stress-optical properties. The mean value obtained for the optical sensitivity of sheet nominally $\frac{1}{8}$-in. thick was 97·6 lb/in. width/fringe. This is a little lower than the value of 100 quoted by Jessop and Harris[22] (1949). The material starts to become non-linear at about the appearance of the third fringe with $\frac{1}{8}$-in. sheet and this corresponds approximately to the elastic limit of the material.

2.3. *Interpretation of the Photo-elastic Results.*—The photo-elastic results have only been used to determine the shear stresses adjacent to the booms. This is partly due to necessity and partly because these stresses, and the sheet efficiencies which have been derived from them, are the most interesting results obtainable from the experiments. The difficulty of finding the direct stresses in the sheet along the centre-line of the model is due to the zero fringe or near-zero fringe difference occurring on the centre-line of all the models. Integration methods for finding the separate stresses are quite inaccurate when such a point lies on the line of integration. In the case of the stresses adjacent to the booms, the integration methods depend on the state of stress at the edge of this sheet. As this is mathematically a singular point, the nature of the stress distribution at this point cannot be found by experiment with any certainty. The only feasible method of separating the direct stresses is by measurement of the variation in thickness of the plate under load, but it was felt that this method involved too much labour for the limited amount of further knowledge obtainable.

No information was obtainable from the stringers and booms as considerable stress was frozen in them and in any case the stresses here could not be considered constant through the thickness. The stress in the plate adjacent to the boom is also not constant through the thickness but the photo-elastic method should give fairly accurate values for the average shear stress through the thickness of the plate.

The maximum shear stress at any point in the plate is proportional to the fringe order at that point and as the direction of the principle stresses is given by the isoclinic pattern, the shear stress across any plane is easily determined. The principle-stress difference which is equal to twice the maximum shear stress is given by

$$\frac{\text{diff. principal stresses}}{\sigma_0} = \frac{97 \cdot 6nA}{Pt}$$

where σ_0 is the direct stress in the end of the boom of area A due to a load P, n is the fringe order, 97·6 the calibration constant of the material and t the plate thickness. In some cases, very few fringes were photographed and a more accurate plot of fringe order was obtained by noting the position of a fringe at various loads. The sheet thickness used was an average value as the plates varied in thickness by about 5 per cent in distances of the order of a few inches.

To deduce the sheet efficiency from the curve of shear stress along the stiffener is merely a matter of graphical integration. For the case where the sheet is unstiffened, it is easily shown that the sheet efficiency at any point distant x' from the end of the panel is

$$\eta = \frac{\text{load in sheet}}{\text{load at infinity}} = \frac{A + bt}{A} \int_0^{x'} \frac{\tau}{\sigma_0} d\left(\frac{x}{b}\right)$$

where b is the half-width of the plate. If the plate has stringers attached, a similar expression can readily be obtained. Where the end of the model is provided with an end rib, integration was begun at the centre of the model where it is assumed that the sheet efficiency is unity. That this is not far from the truth is clear from the photographs (Figs. 14 to 19) of the fringe patterns which indicate that the stress distribution across the centre of the models is very nearly uniform.

3. *Discussion of Results*.—In practice cut-outs in stiffened panels will usually be provided with boundary stiffeners. However, considerable attention has been given to the panel without an end stiffener depicted in Fig. 3a for the following reasons. Firstly, the shear stresses at the corners of the cut-out of this panel will give an upper limit for the plate shear stress since the effect of a practical end stiffener will be to reduce the shear strain at these points. It is assumed that the boundary stiffener is always attached to the boom as otherwise it might be possible for heavy loads to be transferred from boom to stiffener *via* the sheet. Also the remarks apply to structures with continuous connecting media ; in the case of rivets they can only apply in an average sense. Secondly, end stiffeners are unlikely to transfer such load from the booms since in general these end stiffeners will have a small bending stiffness compared with that of the sheet in its own plane. For this reason the sheet efficiency in carrying direct load will only be slightly underestimated by assuming no stiffener. Finally this is by far the easiest panel to analyse theoretically and therefore experiments on such panels are more suitable for purposes of comparison with theoretical results.

All the published theoretical solutions which can be applied to unbuckled sheet have assumed that there is no transverse strain in the sheet and all have predicted infinite shear stresses at the corners of panels with no stiffener at the end. These solutions use the stringer-sheet assumptions and are given by Hildebrand[10] (1943), Goodey[11] (1946) and others. For this reason the validity of the basic assumption is open to doubt with this type of panel. The following comparisons between the theoretical results and the experimental results obtained here are made for this reason.

3.1. *Load Taken by Sheet and Stringers*.—(*a*) *Panels with Booms and Plates only*.—To enable a comparison to be made between the theoretical and experimental results for the load taken by sheet and stringers, the concept of sheet efficiency η has been introduced. The sheet efficiency at any cross-section is defined as the load carried by the sheet and stringers divided by the load taken by that section if the complete section were stressed uniformly. Thus at the free end the efficiency is zero and at an infinite distance from that end it is unity.

In Figs. 20 and 21 the edge shear stress and sheet efficiency obtained from the stringer-sheet solution (for example Goodey[11], 1946) and the experimental results for model I are shown. For this model the ratio of the area of one boom to half the plate cross-sectional area, λ, is unity. It is seen that the stringer-sheet solution is in considerable error and over-estimates the plate efficiency. This is due to the very high shear stresses near the end of the panel predicted by theory. The shear has been plotted as a fraction of the stress σ_0. The experimental values are somewhat in error, as the sheet efficiency, obtained by integrating the shear stress, is greater than unity, the upper limit. For the case considered here, the maximum value of η is $1 \cdot 04$, and for some of the models the error is as great as 10 per cent. Since the errors for both ends of each model seemed to be about the same, this error is assumed to be systematic.

The stringer-sheet solution and the experimental values from models I, IIIa and IVa for the plate efficiency are given in Figs. 22 and 23. The variation of η with the panel width appears to be fairly accurately estimated by the approximate theory although actual values are over-estimated. The experimental values are corrected for the assumed systematic errors.

(*b*) *Panels with Stringers.*—The model No. V was provided with one central stringer and a comparison between the stringer-sheet solution of Goodey[11] (1946) and the experimental results is made in Figs. 24 and 25. Also included in the figures are the results of an analysis using the same assumptions as Goodey but with the central stringer considered as a discrete member instead of considering it uniformly spread across the width of the panel. It is seen that both solutions over-estimate the sheet efficiency but that the latter analysis is nearer the truth.

As the number of stringers increases, so the stringer-sheet assumptions might be expected to approach nearer the truth since the plate itself takes a smaller proportion of the direct load. With a large number of stringers the only feasible solution applicable to unbuckled plate is the stringer-sheet theory and the results from this, according to Goodey[11] (1946), are compared with the experimental curves of Figs. 26 and 27 for the shear stresses and sheet efficiency obtained for model VI. This model has five longitudinal members in addition to the booms. It is seen that the agreement in the case of the sheet efficiency is no better than that previously obtained in Figs. 21 to 25 and that for small x the theoretical shear stress is much too high. Model VI was designed to represent a stiffened panel more or less typical of aircraft practice. The booms are perhaps a little heavy but constructional difficulties prevented a smaller size being used.

(*c*) *Panels with Transverse End Stiffeners.*—A satisfactory determination of the effect of the bending stiffness of a transverse end stiffener on the stress distribution of the panel of Fig. 3a is difficult. One analysis has been done by Mansfield[12] (1947) using stringer-sheet assumption but a plane-stress solution, even if theoretically possible, would almost certainly involve an immense amount of calculation. A solution on this latter basis appears to be the only one which might yield results of reasonable accuracy for the type of problem considered here. From a consideration of the relative stiffnesses of sheet and end stiffener it is clear that practical stiffeners will only transmit a small fraction of the load and this is found to be so by Mansfield.

In Fig. 28 is shown plotted the results from model I, IIa and IIb for the edge shear stresses, which have been corrected for the systematic errors which were commented on above. In the case of model IIb, which has built-in stiffeners, it would seem that about 10 per cent of the load is transferred by the stiffener. The effect of the end stiffeners is seen to be quite small but the relative positions of the curves are as expected. For small x model IIa gives the highest stresses and IIb the smallest; for large x model I gives the highest and IIb the smallest stresses.

A comparison of the stiffened and free ends of the models III, IV, V and VI is given in Fig. 29. No correction for systematic errors was necessary in this instance as the compared curves came from the same model. The end stiffeners have very little effect on the shear stresses except for model IV. For this model, owing to its small width, the bending stiffness of the end rib is quite large compared with that likely in practice. In every case the maximum measurable shear stress at the stiffened end was a little less than the corresponding value at the free end. The end stiffener prevented measurements being made closer than $\frac{1}{8}$ in. from the end of the model.

3.2. *Shear-Stress Concentration.*—The type of singularity predicted by plane-stress theory at the points XX of Fig. 3a is indicated above in the Introduction. Unlike stringer-sheet theories, the shear stress obtained by approaching the corner from the direction along the booms is finite and is equal to $2/\pi$ times the direct stress in the boom at that point. This value of $2/\pi$ is independent of panel width and is consequently applicable to any panel with an unstiffened end. It is also pointed out above that this solution predicts a rotation of the corner through a right-angle and in practice this solution must break down in a very local region near the corner.

A sketch of this corner displacement is given in Fig. 6. Further sources of error in predictions of stresses at the corner, common to all theories, are the assumptions that the booms have no bending stiffness, and that the direct stress is constant over the boom cross-sectional area. For these reasons a departure from the value of $2/\pi$ can be expected.

It is most improbable that the stress concentration at the corners will be seriously affected by the addition of longitudinal stringers to the plate and this is borne out by the experiments which have been done. That this corner distribution is relatively independent of panel configuration is demonstrated by the photographs of Figs. 14 to 16 which show the fringe patterns obtained with constant stiffener end stress except for Fig. 16 where σ_0 is about 20 per cent higher. The following values for the shear stress and principal stress difference were found by extrapolation.

Model No.	I	IIIa	IVa	Va	VIa
$\frac{\pi}{2}\frac{\tau_0}{\sigma_0} = \frac{\text{observed shear stress}}{\text{theoretical stress}}$	0·93	0·90	0·80	0·88	1·07
$\frac{\text{Principal stress diff.}}{\text{Theoretical difference}}$	1·00	1·02	0·93	0·96	1·08

It is seen that the agreement is quite good. However, the accuracy of extrapolating where the stress is rising steeply is open to doubt. Also by the averaging effect of the photo-elastic method over a small region, an underestimate of the stresses is likely to be obtained by experiment. The averaging effect of photo-elasticity is due to the fact that a cone of light passes through every point of the model. This effect may just be appreciable with the lens system used and is consistent with the results which are low for the narrowest model IVa and high for the wider models IIIa and VIa. By plotting the results in different ways an endeavour was made to find out whether or not the stresses become very large. The most profitable graph is shown in Fig. 30 where the shear stress is plotted against $\log_e (x/b)$. The experimental points are for the fringes adjacent to the boom obtained from the photographs. Quite considerable scatter is obtained and it was not affected to any great extent by plotting principal stress differences nor by using $x/b\lambda$ instead of x/b. The former ratio indicates actual distances from the corner since all the models considered for this particular plot had the same size booms. With this logarithmic plot the stringer-sheet solution gives approximately a straight line for small values of x/b. About all that can be concluded from this graph is that the shear stress at the corner is likely to be greater than $2/\pi$ but x/b will be less than $0 \cdot 02$ for these larger values. It is also seen that the stringer-sheet solution for $\lambda = A/bt = 1$ badly over-estimates the shear stress for small x when the slope of the graph is independent of λ.

In any event both the theoretical solutions and the experimental results are of doubtful value at points so close to the corner. The theoretical solutions, with their assumptions of idealized booms and a model of two dimensions, cannot hope to give accurate predictions of the stresses at the corners. At these points the structure is essentially three-dimensional and thus the peak stresses are not obtainable from two-dimensional theory, which at most can only give average values through the thickness of the sheet. Similarly, the photo-elastic results can only give average values although these average values may be somewhat more accurate than the theoretical estimates. However, both methods should give reasonably accurate results at distances equal to one or two times the plate thickness from the boom, providing that this is continuously attached to the plate. There is also the point that it is practically impossible to produce a right-angle re-entrant corner. With the models discussed here the corners were produced with an ordinary six-inch file and consequently a small irregular fillet is left in the corners. This fillet may alleviate the peak stresses.

4. *Conclusions.*—The actual state of stress at the corner of the panels discussed here, where the sheet remains plane, is largely dependent on the geometry of the structure at that point, and cannot be predicted by approximate theories. The results of both theory and the experiments described above indicate that high values of shear stress may occur very close to the edge of the sheet. However it is futile to try and estimate these high values which have been shown to occur within a distance equal to 0·01 of the width of the panel for the reason that they will be quite dependent on the local details of any particular panel. It is in fact a stress-concentration problem and if a true right-angle corner were practicable, a theoretically infinite stress might result. Such a stress means that local yielding would occur no matter how small the load applied to the structure. Such action would only be possible with a continuous no-slip boom-sheet connection. With riveted joints the discussion of these very high shears is pointless as they only occur over such a short distance, in fact over only a fraction of the usual rivet pitch. Rivets have stress concentration problems of their own, but this subject is beyond the scope of the present work.

At a distance equal to 0·01 of the width of the panel from the end of the plate, the shear stress is shown here to be about $2/\pi$ times the stiffener end stress, the value predicted by plane-stress theory. Thus, except for very wide panels, the load on the first rivet of a riveted boom to sheet connection is likely to be about $2td\sigma_0/\pi$ where t is the plate thickness, d is the rivet pitch and σ_0 the boom end stress. This is on the assumption that the rivets are a non-slip connecting media. If slip occurs between boom and plate, this value is much reduced. With a practical joint such slip is bound to occur and therefore this estimated rivet load can only be treated as an upper limit.

For this type of panel the stringer-sheet theory, which assumes zero transverse strain in the plate, has been shown above to overestimate the shear stress and consequently the sheet efficiency. This is no doubt due to the fact that this theory requires large transverse forces to suppress the transverse strain at the end of the panel. The error can be as much as 40 per cent in the case of a panel with unstiffened sheet.

The effect of a transverse end stiffener is shown to usually be small owing to its small bending resistance compared with that of the plate in its own plane. The maximum load transmitted to the sheet *via* such a stiffener is unlikely to be more than 15 per cent of the applied load with usual stiffener sizes. Similarly the peak shear stresses adjacent to the booms are unlikely to be markedly reduced.

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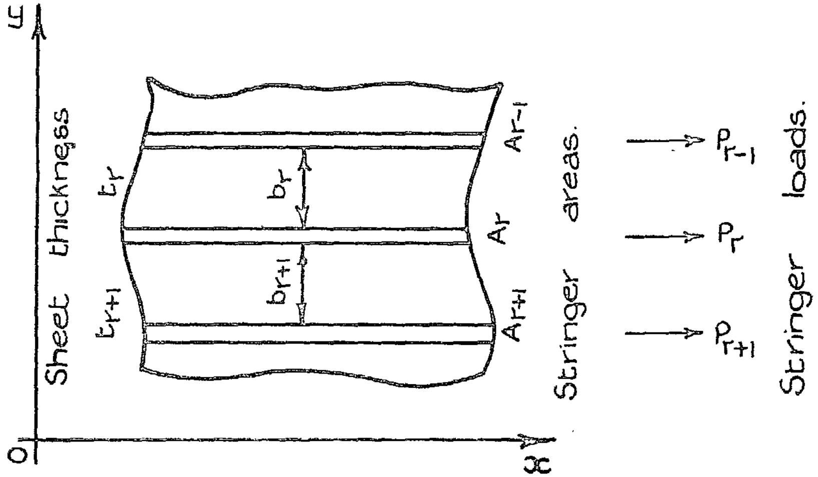


FIG. 1. Notation for finite stringer theory.

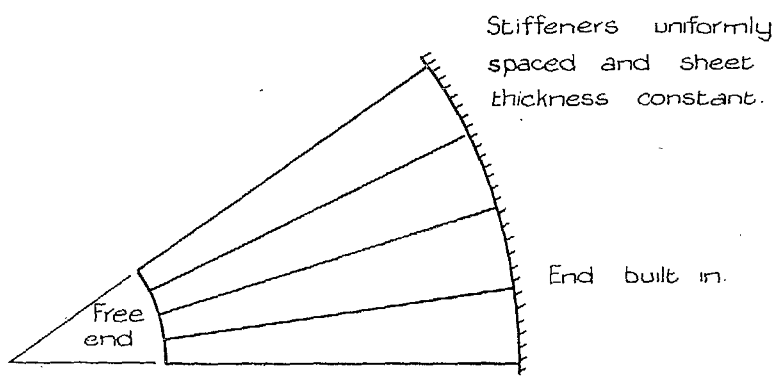


FIG. 2. Structure considered by Hadji-Argyris and Cox.

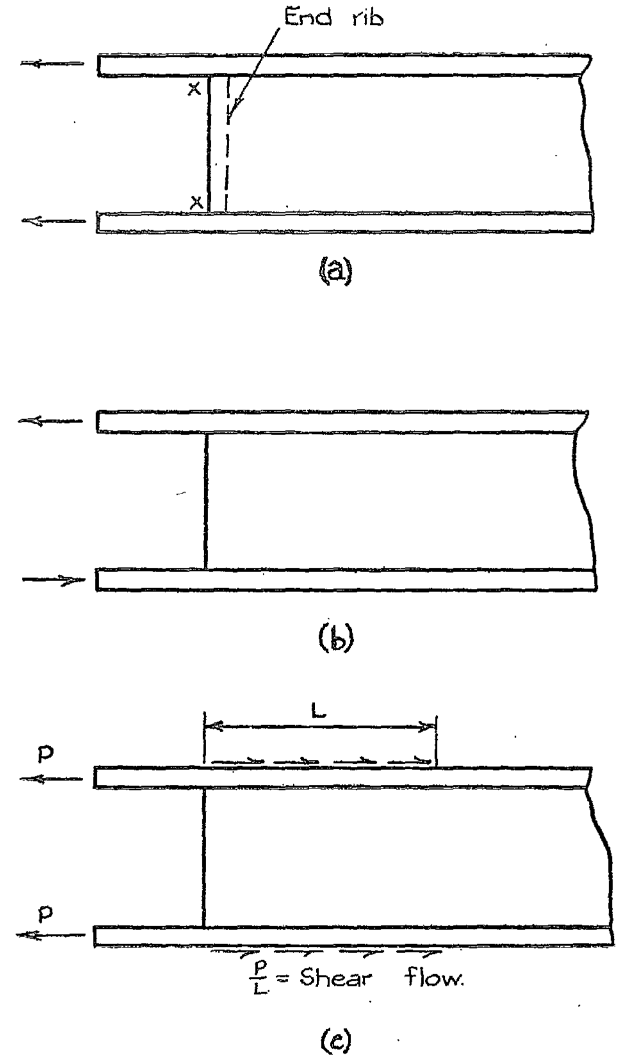
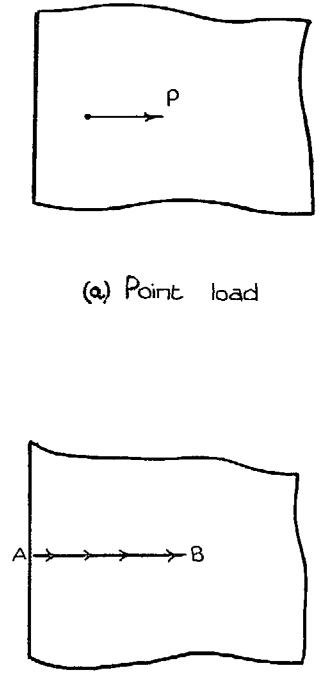


FIG. 3. Problems solved by stringer-sheet method. Booms and plate are uniform along the length.



(a) Point load

(b) Uniform or linearly varying shear along AB.

Fig. 4. Load applied to semi-infinite plates.

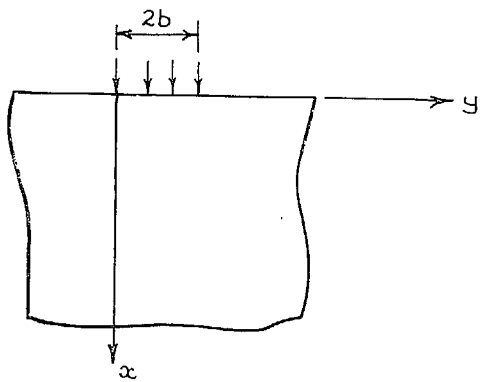


Fig. 5. Uniform load applied to a semi-finite plate.

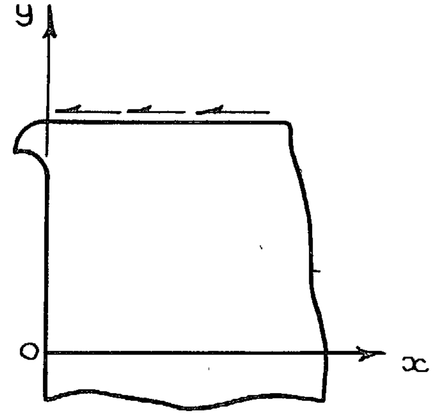


Fig. 6. Theoretical rotation of the corner of the plate.

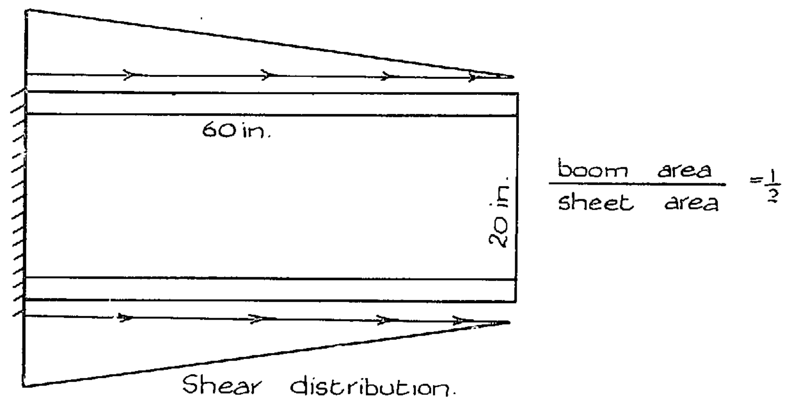


Fig. 7. Structure considered by Fine.

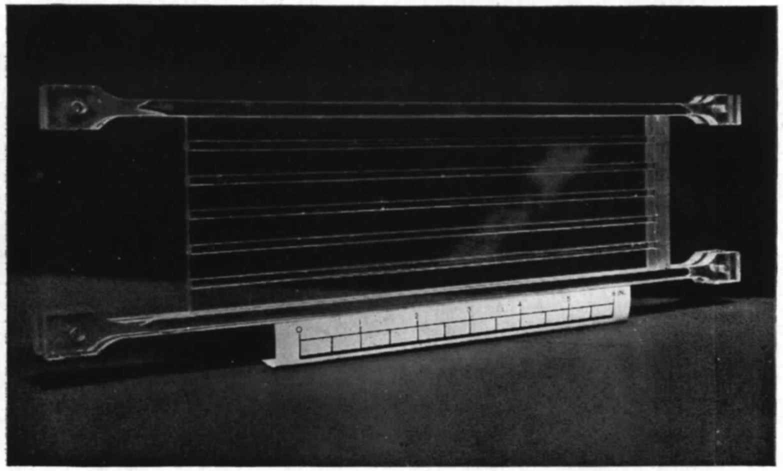


FIG. 8. Model No. VI.

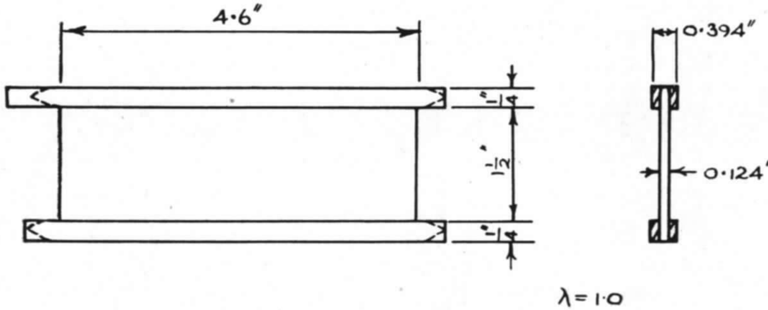


FIG. 9. Model No. I.

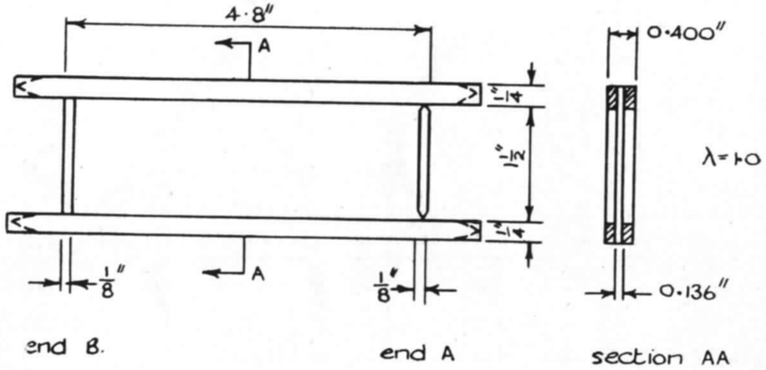
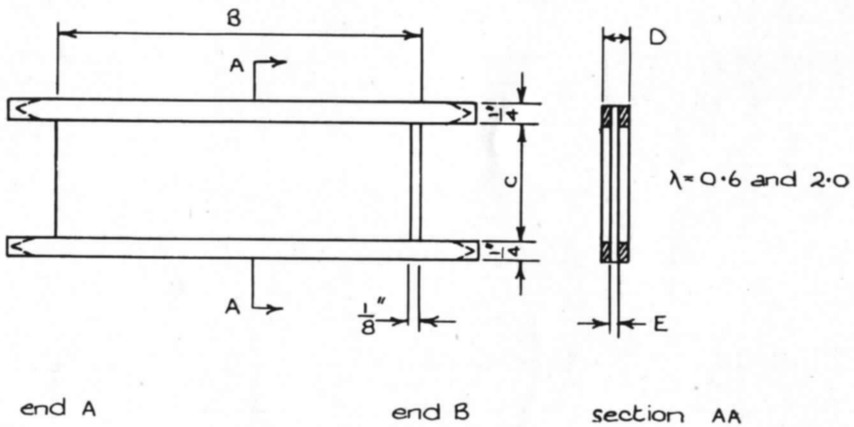


FIG. 10. Model No. II.



Model No.	III	IV
B	7·4"	4·4"
C	2·5"	0·75"
D	0·393"	0·395"
E	0·124"	0·127"

FIG. 11. Models Nos. III and IV.

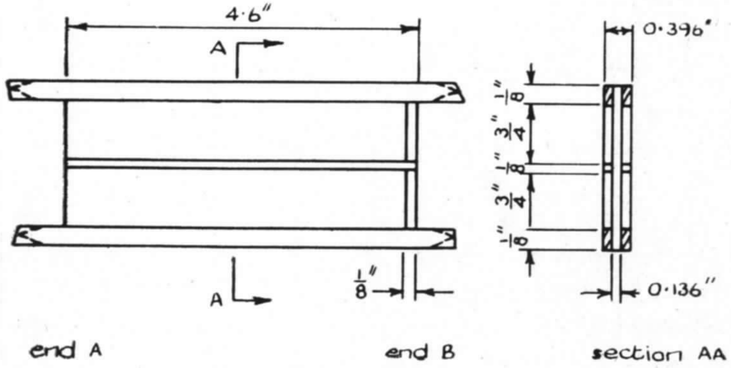


FIG. 12. Model No. V.

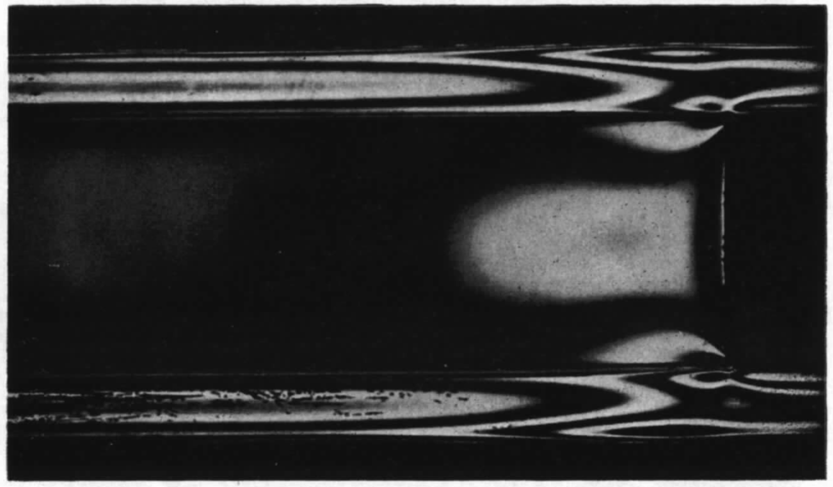


FIG. 14. Fringe pattern of model No. IVa.

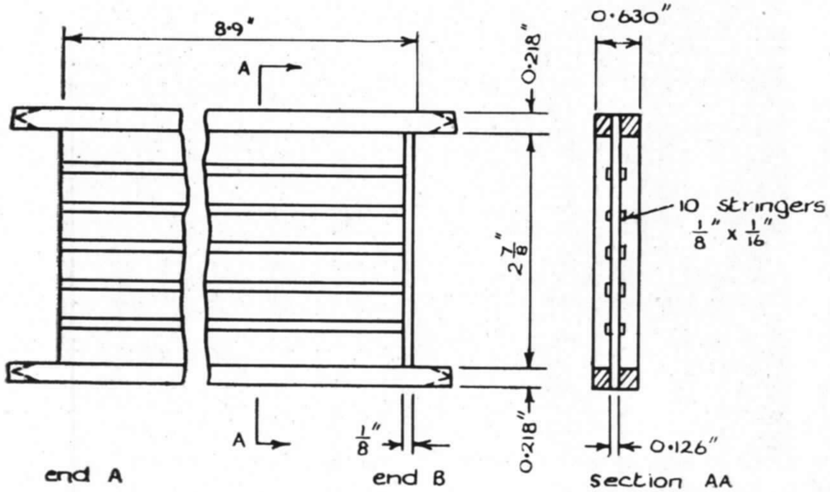


FIG. 13. Model No. VI.

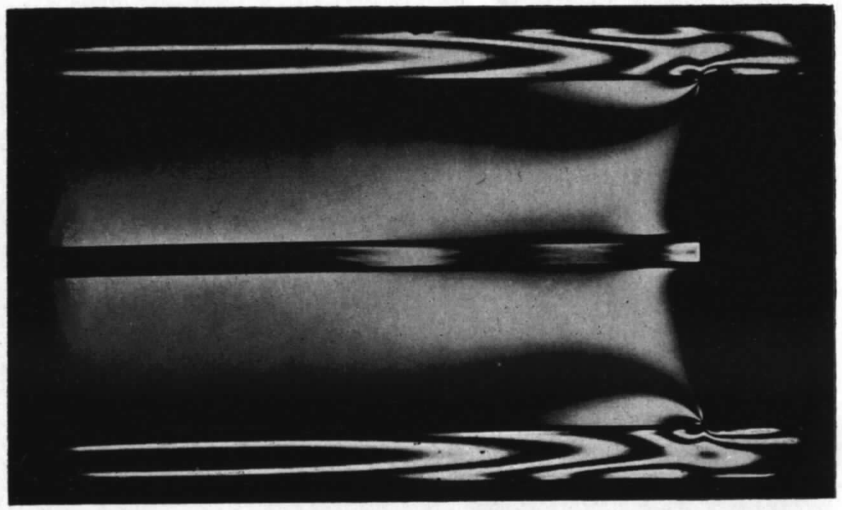


FIG. 15. Fringe pattern of model No. Va.

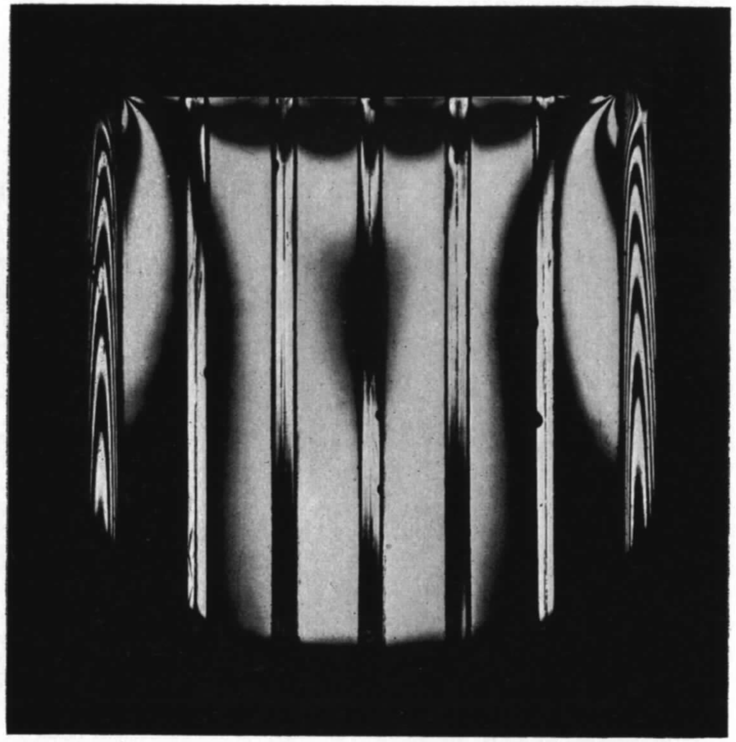


FIG. 16. Fringe pattern of model No. VIa.

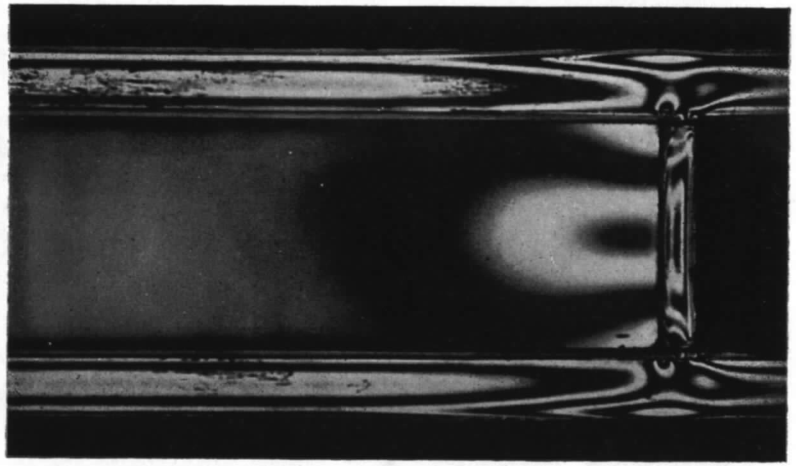


FIG. 17. Fringe pattern of model No. IVb.

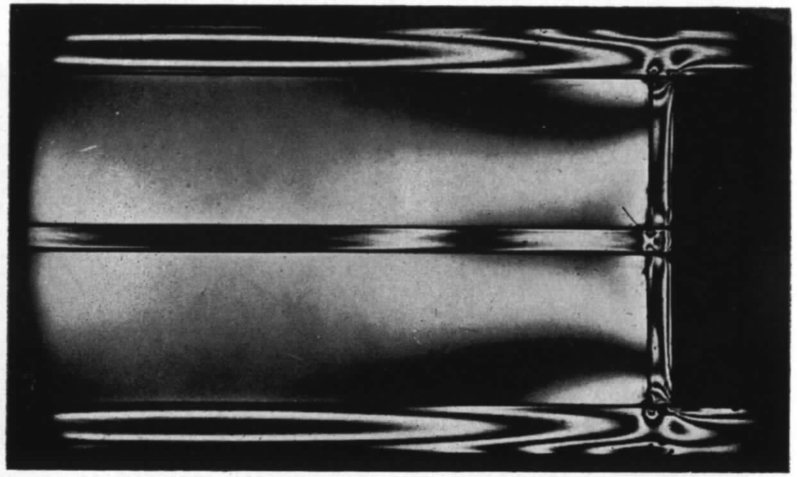


FIG. 18. Fringe pattern of model No. Vb.

FIG. 19. Fringe pattern of model No. VIb.

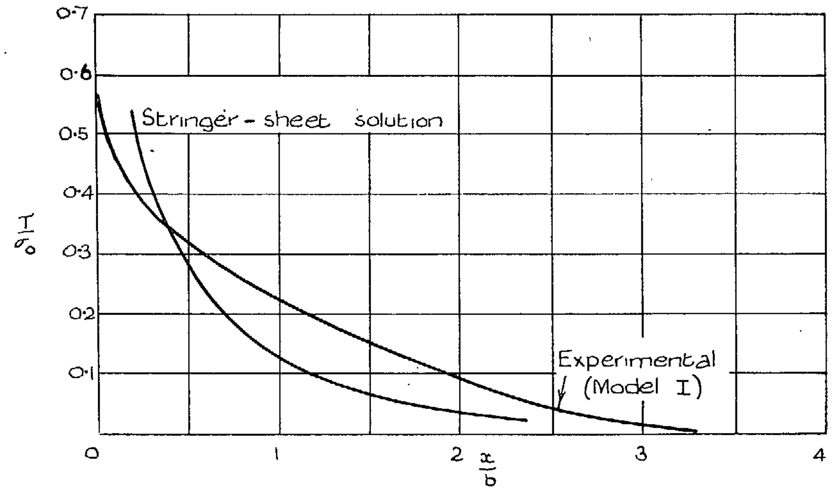


FIG. 20. Edge shear stress. $\lambda = 1 \cdot 0$.

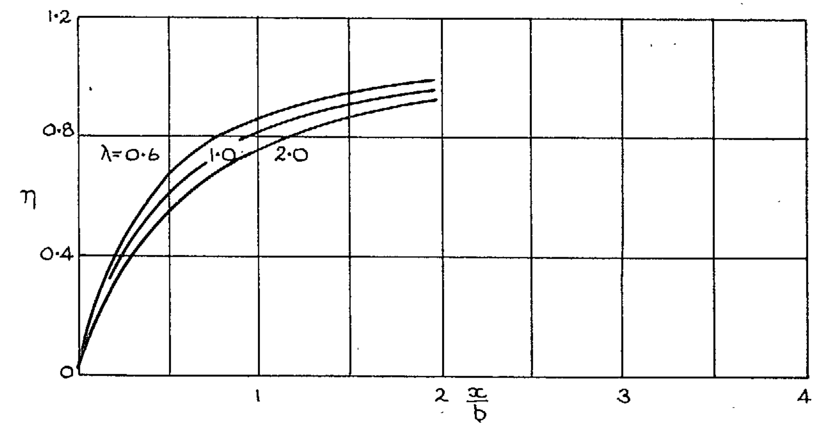


FIG. 22. Theoretical sheet efficiency.

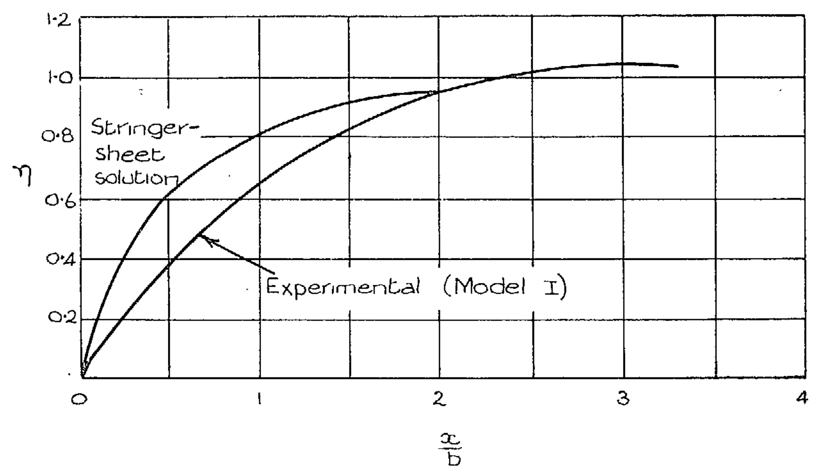


FIG. 21. Sheet efficiency. $\lambda = 1 \cdot 0$.

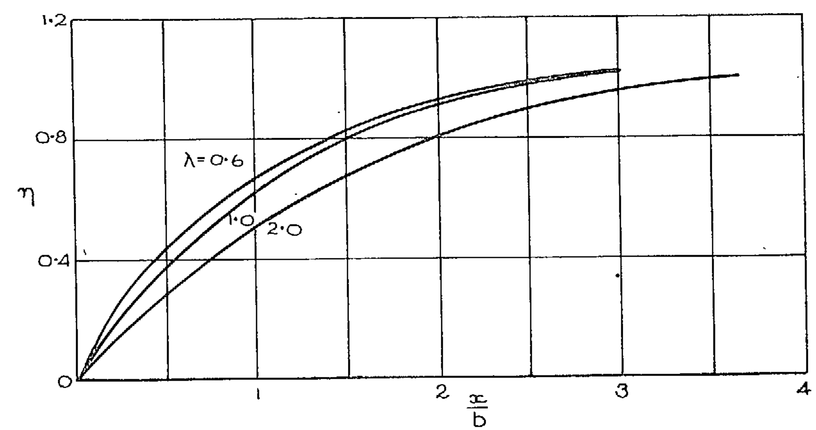


FIG. 23. Experimental sheet efficiency.

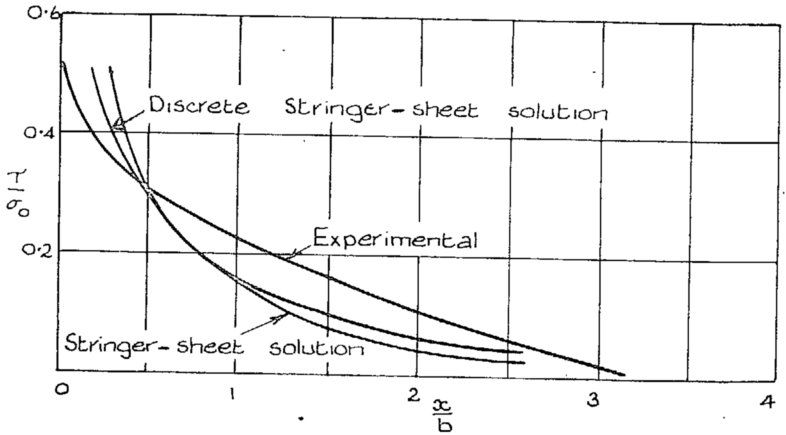


FIG. 24. Edge shear stress of model No. V.

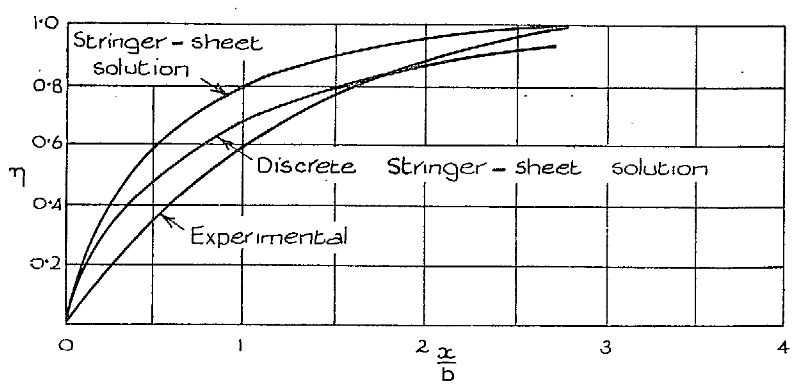


FIG. 25. Sheet efficiency of model No. V.

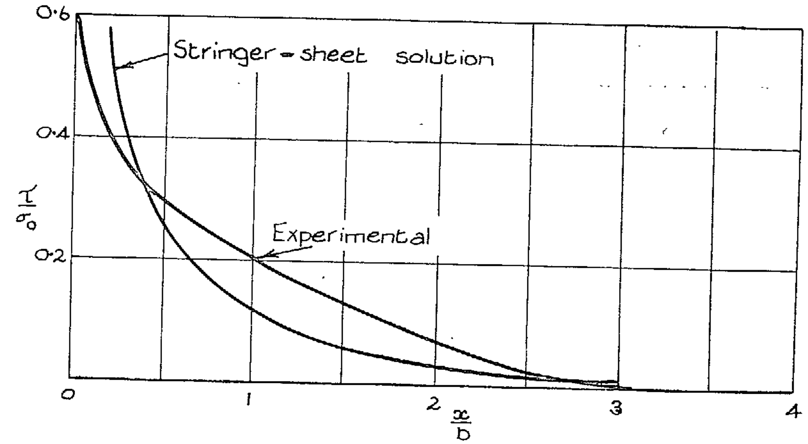


FIG. 26. Edge shear stress of model No. VI.

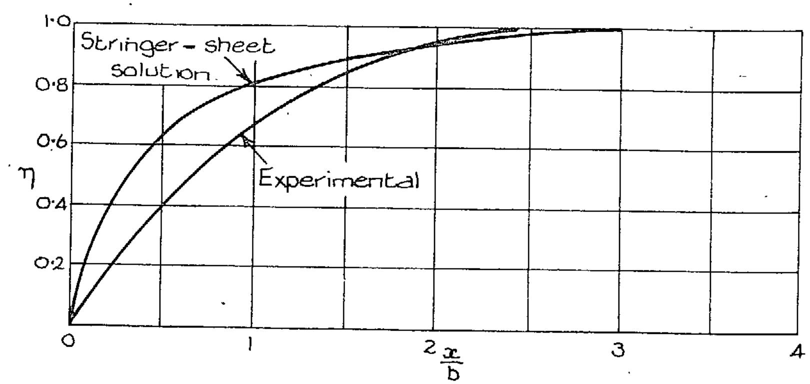


FIG. 27. Sheet efficiency of model No. VI.

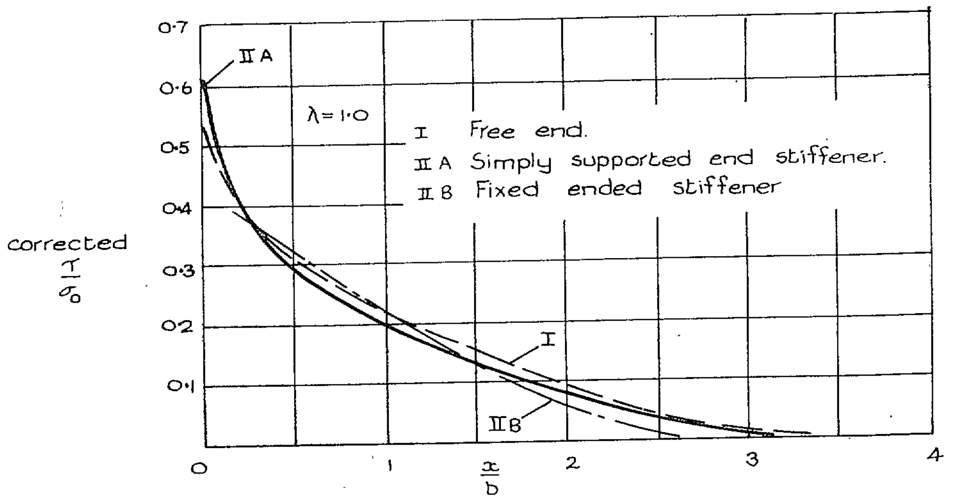


FIG. 28. Effect of stiffeners on edge shear stress.

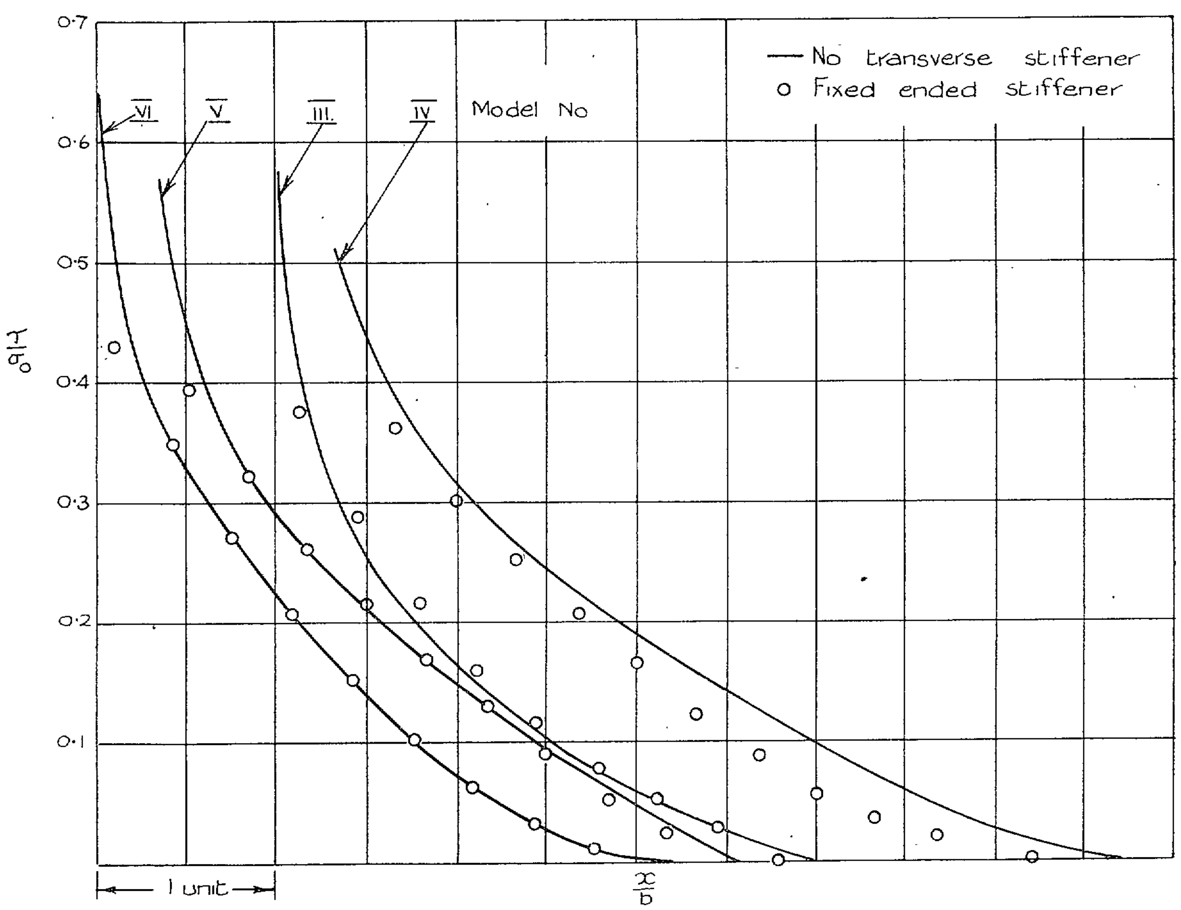


FIG. 29. Effect of stiffeners on edge shear stresses.

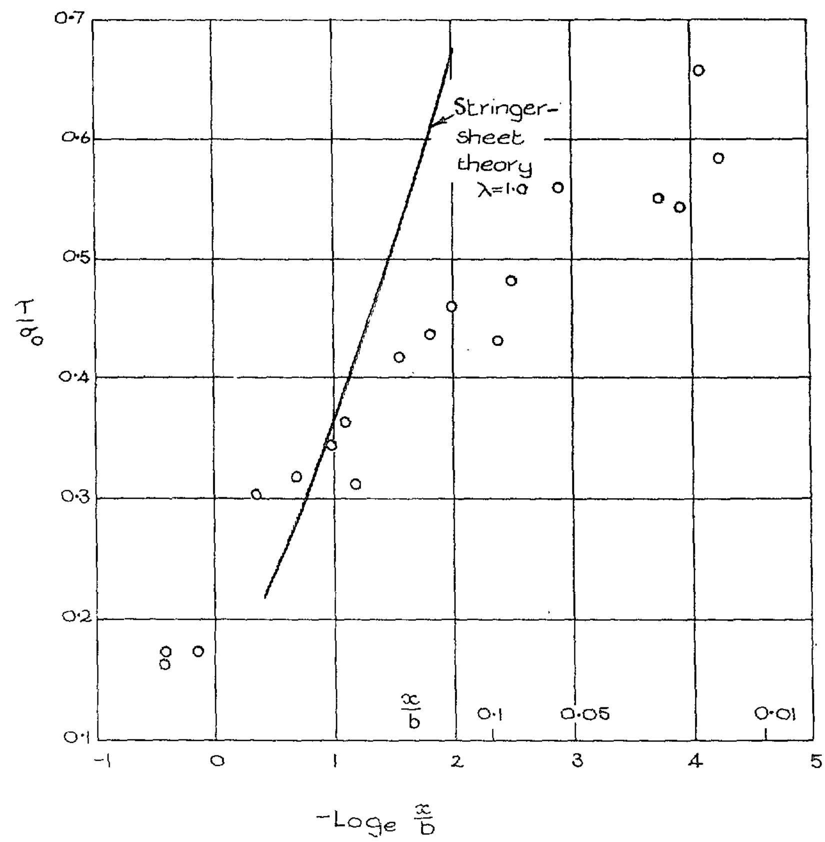


FIG. 30. Logarithmic plot of shear stress.

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