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E. H. Mansfield, B.A.
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Summary.—In many problems relating to the stressing of thin-walled cylinders, and in particular those concerned with the stresses set up in a cylinder under torsion when one section is restrained against warping, it has been commonly assumed that sections have their shape retained by closely spaced stiff ribs. Justification for this assumption is that, for certain types of loading, the ribs of most practical structures do little work in maintaining the section shape (and the analysis is considerably simplified).

In this report the effect of discrete, flexible ribs has been investigated and the results have been incorporated in a number of graphs which show the effect of rib-flexibility in a long thin-walled cylinder of arbitrary shape under end constraint. Some of the results of these investigations are, as would be expected, of a negative character, in that they show that for certain types of end conditions (roughly, those in which the predominating self-equilibrating loads act parallel to the cylinder axis) the effect of rib-flexibility is negligible. But rib-flexibility is of paramount importance when self-equilibrating shear-distorting forces are applied to a cylinder—such as occur at a wing cut-out or near an overhanging engine—and this report makes the stress distribution in such a case readily determinable.

It is shown that the complete stress die-away pattern depends, apart from the boundary conditions, on two non-dimensional parameters. These parameters are functions of the type of end constraint as well as of the structure dimensions and elastic constants. Expressions are given for determining these parameters when the cylinder shape and loading are arbitrary.

The simplified case of a four-boom cylinder of rectangular section under torque is treated separately in a second appendix.

The solution is strictly true for a four-boom cylinder or when the self-equilibrating end-load system is orthogonal (eigenload)[5]; but as minimum-energy methods are used in the analysis, the results are believed to be substantially correct for a smoothly varying end-load system applied to a cylinder of arbitrary shape.

1. *Introduction.*—When a thin-walled cylinder undergoes torsion, or bending with shear, there will be in general an axial warping of all sections; however, if one section is restrained against warping then a self-equilibrating stress system will exist at, and in the neighbourhood of, the restrained section. It is convenient to regard the stresses in such a cylinder as being made up of two parts, the one due to the primary applied loads and the other due to the self-equilibrating system. This system can be regarded as that which will just liquidate the warping at the constrained section.

* R.A.E. Report Structures 6, received 8th November, 1947.

In calculating the magnitude and rate of decay of these self-equilibrating stresses it has been commonly assumed that the cylinder sections have their shape retained by closely spaced stiff ribs. This assumption has the advantage of considerably simplifying the analysis and it is usually sufficiently accurate in problems in which the ribs do little work in retaining the cylinder shape and may therefore be regarded as infinitely stiff. However, in such problems as the stress determination near large cut-outs, etc., this assumption is not valid.

It will be noticed that by considering the ribs to be flexible we have introduced a second degree of freedom for the cylinder displacements, and hence the stresses will have two independent modes. Here, therefore, we consider the stresses set up by applying to one end of a long cylinder, stiffened by uniformly spaced flexible ribs, two self-equilibrating sets of stresses. The first of these corresponds to the case where the root section is subjected to axial warping alone and no shear distortion ; the second to the case where there is only shear-distortion of the root section (*i.e.*, distortion in its own plane) but no axial warping. Any end condition may be obtained by a linear combination of these two fundamental systems.

Mention has been made earlier of the application of this work to the ' torsion problem ' and, although the main part of the analysis here is concerned with the more general problem of applying any smoothly varying self-equilibrating system to one end of a long cylinder of arbitrary shape, the particular application to the torsion of a cylinder of rectangular section is dealt with in Appendix II.

2. *Assumptions and Method of Solution.*—The fundamental assumption made is that the *distribution* of direct stress round a section is constant along the cylinder. This is strictly true[5] only in the case of a four-boom cylinder or when the stress distribution is ' orthogonal ' (as it is for example, in the building-in effect of a tube of rectangular section under torque). The method developed here cannot therefore be applied to find the effect of rib-flexibility on the stresses in a cylinder to one end of which is applied a ' four-point loading ', and other problems where the stress diffusion has a discontinuous character. But, as stated in the summary, the method used here being a minimum-energy one, the stress die-away pattern will be substantially correct—being the best possible subject to the given assumptions. The second main assumption that the direct stress varies linearly between ribs is then in agreement with previous investigations[3].

In addition to these assumptions the following are also made : the material is isotropic and the stress-strain relations are linear ; the structure is integral and constant along its length ; stringers, if present, may be spread into a stringer sheet ; the peripheral stresses are negligible in comparison with the longitudinal and shear stresses. The ribs are assumed to have no stiffness normal to their plane. Reference is made throughout to an ' equivalent shear-rib thickness ' : for a rib of framework construction this is understood to mean the thickness of a simple shear rib (*i.e.*, a uniform sheet) which has the same stiffness as the actual rib.

The method of solution is an energy one. The direct-stress distribution round the section at the root is one of the data of the problem and, by introducing a representative decay factor (because of the two possible modes of stress distribution there will be *four* decay factors altogether, two increasing and two corresponding ones decreasing), the variation along the cylinder is also known. From considerations of equilibrium the shear stresses in the cylinder walls and the rib stresses are therefore known in terms of the unknown decay factor. The total strain energy of the system is now evaluated and Castigliano's Energy Theorem used to get the quartic equation which determines the decay factors.

For the particular case of end constraint of a cylinder of rectangular section under torsion it is of interest to note that the decay factors obtained by this method are in complete agreement with those obtained by solving the difference equation which results from considering compatibility of displacements and stresses.

3. *Description of Results.*—In Appendix I an equation for determining the decay factors is derived; these factors are shown to be functions of two non-dimensional parameters, one, β, a measure of the rib pitch and the other, γ, a 'generalised rib stiffness'. Precise definitions of β and γ for cylinders of arbitrary shape are given in equation (40). For the particular case of a rectangular box under torsion β and γ are given by equations (92) and (93).

3.1. *Influence of Rib-Flexibility when one Section is Restrained against Shear Distortion.*—In Appendix IA the particular problem of the effect of uniformly spaced flexible ribs on the stresses due to a self-equilibrating system applied to a long thin-walled cylinder, when one section—the root section—is restrained against shear-distortion, is considered in detail. This condition of zero shear-distortion may be regarded as produced by a stiff root rib.

Figs. 1 to 4 show the effect of β and γ on the direct stresses and shear stresses at the root. Previous theory, in which cross-sections are assumed to retain their shape, corresponds to the particular case when β is zero and γ infinite; it is seen therefore from Figs. 1 to 4 that the functions plotted there, i.e., ψ_w, ϕ_w and Λ_w, may be regarded as factors by which the stresses obtained by such theory hereafter called elementary theory, must be multiplied to give the correct stresses. Over the current practical range of β and γ (roughly, $\beta < 1, \gamma > 2$) the factor for the direct stresses lies between $0 \cdot 9$ and $1 \cdot 0$ and for the shear stresses between $0 \cdot 6$ and $1 \cdot 0$.

Fig. 5 shows the maximal value of the self-equilibrating shear-distorting forces applied to a rib in terms of the forces exerted by the stiff root rib; this ratio is less than $0 \cdot 5$ over the practical range of β and γ. The most highly stressed rib may be determined from Fig. 6: it is usually the first or second.

The ratio of the direct stress at the first rib to that at the root has been plotted for various β and γ in Fig. 8; it will be seen that rib-flexibility is relatively unimportant and this ratio differs little from that given by elementary theory (shown by broken line). A more marked dependence on rib-flexibility is shown in Fig. 9 where the ratio $\frac{\text{magnitude of self-equilibrating shear stresses in first bay}}{\text{magnitude of self-equilibrating shear stresses in root bay}}$ has been plotted.

The direct stress distribution along the cylinder when each rib is infinitely stiff has been plotted for various β in Fig. 10. It is of interest to note that the area under each curve is independent of β and it follows that the direct stresses at rib stations are lower than those given by elementary theory but are higher between ribs.

For a full explanation of the curves in Figs. 7 and 11 reference should be made to A.6.3 where the concept of a 'rib medium' is introduced.

The functions λ_1, λ_2 of Figs. 12 and 13 are for use in determining the stresses along the cylinder; they apply to this particular end condition and when there is zero axial warping at the root.

3.2. *Influence of Rib-Flexibility when one Section is Restrained against Axial Warping.*—In Appendix IB the particular problem of the effect of uniformly spaced flexible ribs on the stresses due to a self-equilibrating system applied to a long thin-walled cylinder, when one section—the root section—is restrained against axial warping, is considered in detail. This condition of zero warping may be regarded as produced by a self-equilibrating shear-distorting system applied at a section in the middle of a long cylinder; it will be seen from symmetry that this root section will not experience axial warping.

From a physical point of view it would be expected that the degree of rib-flexibility and spacing would have a marked effect upon the stresses and their rates of die-away. This is abundantly clear from Figs. 14 to 18 where numerous families of curves, analogous to the previous ones, have been drawn.

That part of the notation which is relevant to Figs. 1 to 18 is reproduced in Note II.

3.3. *Application to a Cylinder of Rectangular Section under Torsion.*—This has been treated in detail in Appendix II where expressions for the standard decay factor, k and the non-dimensional rib spacing and stiffness parameters β and γ have been given ; these expressions are given in A.9.2 and are more manageable than the general ones derived in Appendix I.

A complete numerical example demonstrating the practical application of this report is given at the end of Appendix II.

An equivalent γ for a cylinder with walls of sandwich construction is found in A.12 where a simple formula is given in equation (101).

3.4. *Subsidiary Problems.*—Numerous particular cases of the general problem have been investigated in Appendices IA and IB, *e.g.*, the concept of a rib medium, etc. : and the effect of a rib medium in combination with discrete ribs is discussed in A.6.4.

4. *Conclusions.*—When one end of a thin-walled cylinder is subjected to a smoothly varying self-equilibrating system, it has been general to assume that cross-sections have their shape retained by closely spaced stiff ribs. In this report the effect of flexible and discrete ribs has been investigated, and the stess at any point in the cylinder is shown to be that given by elementary theory multiplied by an appropriate function of two non-dimensional parameters. Expressions are given for these parameters, and the functions referred to above have been plotted in Figs. 1 to 18.

It is shown that the effect of rib-flexibility upon the stresses in a cylinder when one section is restrained against shear-distortion is small (producing errors, over the current practical range, of less than 10 per cent in the direct stresses and less than 40 per cent in the shear stresses, and these on the conservative side).

The corresponding problem when one section undergoes shear-distortion but is prevented from axial warping has been investigated and the stresses are shown to be markedly dependent upon rib-flexibility. Such a condition of rib distortion may occur near large cut-outs in a cylinder under torsion or where a sudden torque is applied in a ' non-Batho ' manner, *e.g.*, an up-and-down torque produced by an over-hanging engine in an aircraft wing.

An example demonstrating the practical application of this report has been given at the end of Appendix II.

5. LIST OF SYMBOLS

Notation of Appendix I

Symbol		Description	first introduced in equation :—
E, G		Elastic moduli	
z		Distance along cylinder measured from a rib	(1), (2)
s		Distance around section measured from a fixed point	(3)
u		Axial displacement, assumed to be of the form $u_s(s)\ u_z(z)$	(1)
$\bar{u}$	=	$\bar{u}(s)$ which determines the direct-stress distribution round a section	(1)
f		Constant of proportionality such that	
$f\bar{u}$		Direct stress (axial stress)	(1)
n		Number of rib or bay (*see* Diagram 1)	(1)
r		Stress decay factor	(1)

Notation of Appendix I—continued

			first introduced in equation :—
p		Rib pitch (assumed constant)	(2)
t		Skin thickness of cylinder walls	(3)
t_s		Stringer-sheet thickness	(3)
q		Shear stress in cylinder walls	(3)
Q	=	qt = shear flux	(5)
Q_0		Value of Q at $s = 0$	(5)
F_j		Section area of jth flange	(6)
$\Delta_j Q$		.. ..	(7)
P		Length of perpendicular from Oz to tangent to point on section	(11)
A		Total area of cylinder section $= \frac{1}{2}\oint P\,ds$ (area of equivalent shear rib)	(13) (19)
A_m		Area of section bounded by radii from Oz to $s = s_1$, $s = s_m$	(13)
R		.. ..	(14)
S		.. ..	(16)
H		Rib boundary shear flux	(17)
S', H'		Values of S, H at $s = s'$	(19)
t_{rib}		Thickness of equivalent shear rib	(19)
N		.. ..	(19)
W_R		Strain energy stored in rib	(19)
W_F		Strain energy stored in flanges	(21)
W_S		Strain energy stored in stringer-sheet	(24)
W_Q		Strain energy stored in cylinder walls (due to shear)	(25)
W		Total strain energy	(27)
B_1, B_2, B_3		.. ..	(29), (30)
e_0		.. ..	(35)
k		'Standard decay factor'	(40), (70)
β		'Non-dimensional rib pitch measure'	(40)
γ		'Generalised rib stiffness'	(40), (76) *et seq.*
r_1	=	$r_1(\beta, \gamma)$ } values of r whose moduli are less than unity	
r_2	=	$r_2(\beta, \gamma)$ }	(42)

Additional notation in Appendices IA *and* IB

f_1, f_2	Values of f corresponding to r_1, r_2	(43)
ξ_1, ξ_2	.. ..	(52)
ψ_w	.. ..	(53)
Δ_w	.. ..	(54)
λ_1, λ_2	.. ..	(56), (57)
J	.. ..	(59)

Additional notation in Appendices IA *and* IB*—continued*

Δ_w	(59)
ϕ_w	(60)
Δ_{wQ}	(61)
Ξ	(64)
z_m	(77)
ϱ	(79) *et supra*
ϕ_D	(80)
μ	(80)
Δ_D	(82) *et supra*
Δ_{DD}	(86)

The notation used in Appendix II is given in section A.1.9.
The notation relevant to the Figures only is given in Note II.
Throughout all Greek letters are non-dimensional.

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APPENDIX I

Derivation of Equation for Determining the Decay Factors

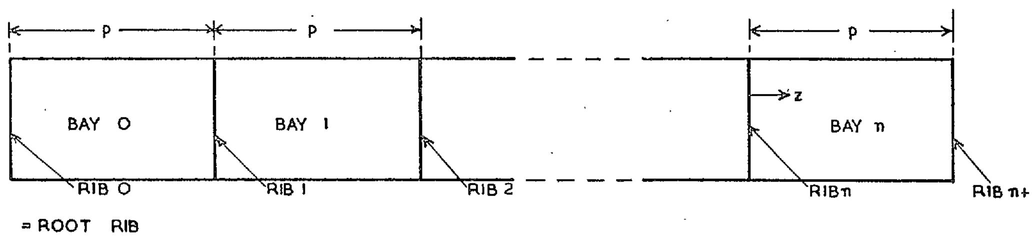

Diagram 1. Elevation of cylinder showing rib and bay notation.

A.1.1. *Direct Stresses.*—We have assumed the self-equilibrating direct-stress system applied at the root to be proportional to a given axial warping $\bar{u}$, a function of s only, to vary linearly between consecutive ribs and to decay as a geometric progression at rib positions. We can thus write the direct stress at the nth rib in the form :

$$E\frac{\partial u_n}{\partial z} = f\bar{u}r^n, \qquad (1)$$

where f is a constant of proportionality and r a representative decay factor*. Thence the direct stress in the nth bay is given by

$$E\frac{\partial u}{\partial z} = f\bar{u}r^n\left\{1 + \frac{z}{p}(r-1)\right\}, \qquad (2)$$

where z is zero at the nth rib.

A.1.2. *Shear Stresses.*—For longitudinal equilibrium of direct and shear-stresses in the region between flanges, we have

$$\frac{\partial}{\partial z}\left(Et_s\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial s}(qt) = 0, \qquad (3)$$

which on substitution from (2) gives

$$\frac{f\bar{u}t_s}{p}r^n(r-1) + \frac{\partial}{\partial s}(qt) = 0. \qquad (4)$$

Hence, apart from changes in shear flux $qt = Q$, due to flange loads,

$$Q = Q_0 + \frac{fr^n(r-1)}{p}\int_0^s t_s\bar{u}\,ds, \qquad (5)$$

where Q_0 is the value of Q at $s = 0$.

Changes in shear flux due to the presence of flanges will now be considered. The flange load in the jth flange

$$= F_jf\bar{u}_jr^n\left\{1 + \frac{z}{p}(r-1)\right\} \qquad (6)$$

and so

$$F_jf\bar{u}_j\frac{r^n(r-1)}{p} + \Delta_jQ = 0, \qquad (7)$$

where Δ_jQ is the increase in Q due to load in F_j.

Hence Q due to flanges alone $= \sum_0^s \Delta_jQ$, $\qquad (8)$

where $\sum_0^s$ denotes summation over the interval $(0, s)$,

$$= \frac{fr^n(1-r)}{p}\sum_0^s F_j\bar{u}_j\,. \qquad (9)$$

* By writing r^n in the form $\exp(n \log r)$ we can regard the decay as exponential.

Equations (5) and (9) give the total shear flux

$$Q = Q_0 + \frac{fr^n(1-r)}{p}\left\{\int_0^s t_s\bar{u}\, ds + \sum_0^s F_j\bar{u}_j\right\}. \quad \text{.. .. .. ..} \quad (10)$$

Since the system is self-equilibrating the torque must vanish; this condition determines Q_0. Taking moments about Oz gives

$$\oint QP\, ds = 0, \quad \text{.. .. ..} \quad (11)$$

i.e.

$$\oint P\left[Q_0 + \frac{fr^n(1-r)}{p}\left\{\int_0^s t_s\bar{u}\, ds + \sum_0^s F_j\bar{u}_j\right\}\right] ds = 0 \quad \text{.. ..} \quad (12)$$

and, writing,

$$\Sigma\int_{s_{m-1}}^{s_m} P\left(\sum_0^s F_j\bar{u}_j\right) ds \text{ for } \oint\left(P\sum_0^s F_j\bar{u}_j\right) ds, \quad \text{.. .. ..} \quad (12)$$

becomes

$$2AQ_0 = -\frac{fr^n(1-r)}{p}\left[\oint\left\{P\int_0^s t_s\bar{u}\, ds\right\}ds + 2\Sigma\left\{(A_m - A_{m-1})\sum_0^{s_m} F_j\bar{u}_j\right\}\right] \quad \text{..} \quad (13)$$

or

$$Q_0 = \frac{fRr^n(1-r)}{p}, \text{ say,} \quad \text{.. .. ..} \quad (14)$$

where R depends only on $\bar{u}$ and the geometry of the cylinder.

Consequently from (10) and (14), and introducing the suffix $_n$,

$$Q_n = \frac{fr^n(1-r)}{p}\, R + \int_0^s t_s\bar{u}\, ds + \sum_0^s F_j\bar{u}_j \quad \text{.. .. ..} \quad (15)$$

$$= \frac{fSr^n(1-r)}{p}, \text{ say,} \quad \text{.. .. ..} \quad (16)$$

where S depends only on $\bar{u}$ and the geometry of the cylinder.

A.1.3. *Limitation on Possible Forms for S.*—It will be seen from the above that for a given smoothly varying $\bar{u}$ we can find the corresponding shear-stress distribution given by S; that the converse is not necessarily true is evident from the following: differentiating (16) we have in the region between flanges,

$$\frac{\partial S}{\partial s} = t_s\bar{u}. \quad \text{.. .. ..} \quad (16a)$$

Now, because of the continuous character of $\bar{u}$, it will be seen that (16a) determines $\bar{u}$ *all round* the section. Substituting the value of $\bar{u}$ given by (16a) in (16) it will be seen that unless there are jumps at the flange positions in S of magnitude $(F/t_s)(\partial S/\partial s)$ there is no corresponding smoothly varying form for $\bar{u}$.

A.1.4. *Rib Stresses.*—

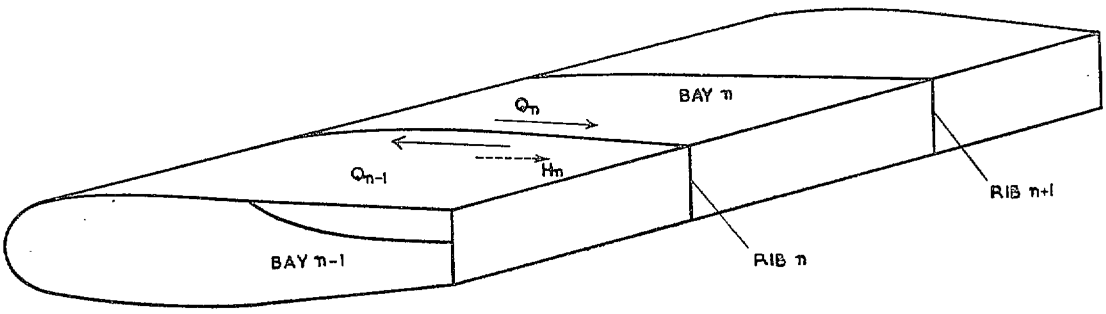

Diagram 2. Shear flux along boundary of rib.

Consider the forces per unit length that act at a point s on the boundary of the nth rib. For equilibrium we have :

$$H_n + Q_n - Q_{n-1} = 0, \qquad \ldots \qquad (17)$$

where H is the ' rib-boundary shear flux '.

Using (16) we can write (17) in the form

$$H_n = \frac{fSr^{n-1}}{p}(1 - r)^2. \qquad \ldots \qquad (18)$$

It will be seen from the above equation that the distribution of H round the rib periphery is proportional to S and is therefore similar to the distribution of Q.

A.2. *Strain Energy.*—The total strain energy will now be found and Castigliano's Energy Theorem applied to obtain an equation for the decay factor r.

A.2.1. *Energy Stored in the Ribs.*—As has been stated above the distribution of shear flux round the rib periphery is proportional to S, i.e., if we know the numerical value of H at any point, say at $s = s'$, then, provided that value ($= H'$, say) is not identically zero, the shear flux at all other points on the rib periphery is known.

Now we have assumed that the stress-strain relations are linear and hence the strain energy in a rib will be proportional to the square of the shear stresses. These shear stresses are proportional to H' and so we may write :

$$\text{strain energy in rib } n \equiv W_{R,n}$$

$$\propto (H_n')^2$$

$$\equiv \frac{NA(H_n')^2}{2Gt_{\text{rib}}}, \text{ say,} \qquad \ldots \qquad (19)$$

where N is a non-dimensional coefficient dependent on the geometry and elastic properties of the rib and the mode of connection between the rib and the cylinder walls. The form of equation (19) has been chosen so that $N = 1$ when the ribs are rectangular (in a four-boom cylinder of doubly-symmetrical rectangular section) and of uniform thickness and when the applied $\bar{u}$ is that obtained when the cylinder undergoes pure torsion.

Using (18), (19) becomes

$$W_{R,n} = \frac{f^2(S')^2NAr^{2n-2}(1 - r)^4}{2p^2Gt_{\text{rib}}}. \qquad \ldots \qquad (20)$$

A.2.2. *Energy Due to Direct Stresses.*—The strain energy in the flanges in bay n due to direct stress is

$$W_{F,n} = \frac{1}{2E} \Sigma \int_0^p F_j \left(E \frac{\partial u}{\partial z} \right)^2 dz \quad \ldots \quad (21)$$

$$= \frac{f^2 r^{2n}}{2E} \Sigma F_j \bar{u}_j^2 \int_0^p \left\{ 1 + \frac{z}{p}(r-1) \right\}^2 dz, \text{ from (2),} \quad \ldots \quad (22)$$

$$= \frac{f^2 p}{6E} r^{2n}(1 + r + r^2) \, \Sigma F_j \bar{u}_j^2 , \quad \ldots \quad (23)$$

the summation being taken over all values of j.

Similarly the strain energy due to direct stress in the skin and stringers in bay n is

$$W_{S,n} = \frac{f^2 p}{6E} r^{2n}(1 + r + r^2) \oint t_s \bar{u}^2 \, ds \; . \quad \ldots \quad (24)$$

A.2.3. *Energy Due to Shear Stress in the Cylinder Walls.*—Due to the shear flux Q, the strain energy in bay n is

$$W_{Q,n} = \frac{p}{2G} \oint \frac{Q^2}{t} \, ds \quad \ldots \quad (25)$$

$$= \frac{f^2 r^{2n}(1-r)^2}{2pG} \oint \frac{S^2}{t} \, ds, \text{ from (16)} \; . \quad \ldots \quad (26)$$

A.2.4. *Total Strain Energy.*—The total strain energy stored in bay n and rib n will be

$$W_n = W_{F,n} + W_{S,n} + W_{Q,n} + W_{R,n} , \quad \ldots \quad (27)$$

and the total strain energy for a long cylinder is

$$W = \sum_{n=0}^{\infty} W_n \; . \quad \ldots \quad (28)$$

Now $\sum_{n=0}^{\infty} r^{2n} = 1/(1-r^2)$, $|r| < 1$, and hence from (27) and (28) :

$$W = f^2 \left\{ \frac{pB_1(1 + r + r^2)}{6(1 - r^2)} + \frac{B_2(1-r)}{2p(1+r)} + \frac{B_3(1-r)^3}{p^3 2r^2(1+r)} \right\} . \quad \ldots \quad (29)$$

where

$$\left. \begin{aligned} B_1 &= \frac{1}{E} \left\{ \Sigma F_j \bar{u}_j^2 + \oint t_s \bar{u}^2 \, ds \right\} \\ B_2 &= \frac{1}{G} \oint \frac{S^2}{t} \, ds \\ B_3 &= \frac{(S')^2 NAp}{Gt_{\text{rib}}} \end{aligned} \right\} \quad \ldots \quad (30)$$

A.3. *External Forces Applied at the Root.*—In order to apply an energy theorem, by means of which we can evaluate the decay factor r, we must first find out what are the externally applied forces and the displacements through which they move.

The externally applied direct-stress system is one of the data of the problem and was assumed to be given by

$$\left[E\frac{\partial u}{\partial z}\right]_{\text{root}} = f\bar{u}, \qquad (31)$$

which is *a priori* independent of r, and it will be seen later from equation (37) that the corresponding axial displacements need not be found.

Consider now the external self-equilibrating shear loads that are applied at the root and the displacements through which they move.

The externally applied shear flux is given by

$$H_0 + Q_0 \; (= Q_{-1})$$

$$= \frac{fS(1-r)}{pr}, \text{ from (16).} \qquad (32)$$

To find the displacements through which Q_{-1} moves we find the corresponding root-rib peripheral movement—which is necessarily the same.

At the root rib we have from (18)

$$H_0 = \frac{fS(1-r)^2}{pr} \qquad (33)$$

and the strain energy in the root rib is

$$W_{R,0} = \frac{NAH_0^2}{2Gt_{\text{rib}}}; \qquad (34)$$

hence the generalised displacement of the root-rib periphery will be

$$e_0, \text{ say, } = \frac{\partial W_{R,0}}{\partial H_0} \qquad (35)$$

$$= \frac{fSNA(1-r)^2}{prGt_{\text{rib}}}, \text{ from (33), (34).} \qquad (36)$$

From continuity of displacements e_0 will be the displacement of the externally applied shear flux Q_{-1}.

A.3.1. We can now apply Castigliano's Theorem, in the form: 'When a structure obeying Hooke's Law is in equilibrium under a system of external forces the correct stress distribution is such that for a small change in that stress distribution the change in the strain energy stored in the structure is equal to the work done by the changes in the external forces moving through their equilibrium displacements.' (37)

This theorem may therefore be written:

$$\frac{\partial W}{\partial r} = \left\{\frac{\partial Q_{-1}}{\partial r}\right\} e_0 ; \qquad (38)$$

the direct stresses at the root are, as stated in (31), independent of the parameter r and so contribute nothing towards (38).

Substituting from (29), (34) and (36) in (38) and dividing by f^2 gives

$$\frac{\partial}{\partial r}\left\{\frac{pB_1(1+r+r^2)}{6(1-r^2)}+\frac{B_2(1-r)}{2p(1+r)}+\frac{B_3(1-r)^3}{2p^3r^2(1+r)}\right\}=-\frac{B_3(1-r)^2}{p^3r^3},$$

i.e.,

$$p^4B_1r(r^2+4r+1)-6p^2B_2r(1-r)^2+6B_3(1-r)^4=0. \quad \ldots \quad (39)$$

A.3.2. We now introduce k, β and γ, defined as follows :—

$$\left.\begin{aligned} k &= +\sqrt{\frac{B_1}{B_2}} \\ \beta &= kp \text{ (non-dimensional)} \\ \text{and } \gamma &= \frac{B_2^2}{B_1B_3}=\frac{B_2}{k^2B_3} \text{ (non-dimensional)} \end{aligned}\right\} \quad \ldots \quad (40)$$

From the definition of the B's, given in equation (30), it will be seen that k is independent of the rib-pitch and flexibility and that β is independent of the rib-flexibility and proportional to the rib-pitch. All are independent of the magnitude of $\bar{u}$.

With this notation the equation for r may be written

$$\begin{aligned} r^4 - r^3\{4+\gamma\beta^2(1-\beta^2/6)\} + r^2\{6+2\gamma\beta^2(1+\beta^2/3)\} \\ - r\{4+\gamma\beta^2(1-\beta^2/6)\}+1=0, \end{aligned} \quad \ldots \quad (41)$$

a reciprocal equation of which two roots are therefore less than unity and two greater.

We shall call the roots :

$$r_1,\ 1/r_1,\ r_2,\ 1/r_2$$

where, say,

$$|r_{1,2}|<1\ . \quad \ldots \quad (42)$$

A.4. *Determination of f*.—In A.1.1 it was assumed that the direct stresses at the nth rib were given by

$$E\frac{\partial u_n}{\partial z}=f\bar{u}r^n\ , \quad \ldots \quad (1\ bis)$$

but from (42) it will be seen that there are four different values of r, i.e., there are four possible modes of axial stress distribution and the complete form of (1) will be*

$$E\frac{\partial u_n}{\partial z}=\bar{u}(f_1r_1^n+f_2r_2^n+f_3r_1^{-n}+f_4r_2^{-n})\ . \quad \ldots \quad (43)$$

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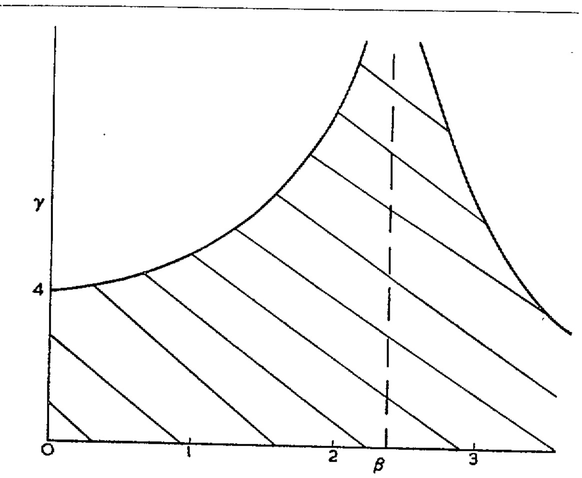
* The f's and r's are not necessarily real. The accompanying diagram indicates the region (shaded) in the $\beta-\gamma$ domain in which the f's and r's are complex.



For a cylinder of finite length the four unknown f's in (43) are determined from the four end conditions, *viz.*, the magnitude of the direct stresses and the rib distortion at each end of the cylinder.

In the following work we assume the cylinder is long enough for it to be considered semi-infinite (*i.e.*, conditions at the root end do not appreciably affect those at the far end), in which case f_3 and f_4 in equation (43) are zero.

In Appendix IA the particular problem in which there is no shear-distortion of the root rib is considered and in Appendix IB that in which there is distortion at the root but zero axial warp. All systems of self-equilibrating load applied at the root may be obtained by a suitable combination of these two systems. In both cases the root direct stress and rate of stress die-away, etc., are found for varying values of the parameters β and γ.

APPENDIX IA

Particular Case of Zero Shear-distortion at the Root

A.5.1. In Appendix I the equation for determining the decay factors r was found—*see* equations (41) and (42). The roots of this equation whose moduli are less than unity are

$$\left.\begin{aligned} r_1 &= r_1(\beta, \gamma) \\ \text{and} \quad r_2 &= r_2(\beta, \gamma) \end{aligned}\right\} \qquad (44)$$

and the direct stresses at the nth rib in a long cylinder are given by

$$E\frac{\partial u_n}{\partial z} = \bar{u}(f_1 r_1^{\,n} + f_2 r_2^{\,n}) \qquad (45)$$

with a linear variation between ribs.

A.5.2. *Determination of f_1 and f_2.*—We shall now find the f's for the case when there is no shear-distortion of the root rib and when the axial warp at the root is $-\bar{u}$. (Chosen negative so that if $\bar{u}$ is a warp in the unconstrained state of the cylinder—under torsion, say—then for complete building-in at the root we should have $u_0 + \bar{u} = 0$, i.e., $u_0 = -\bar{u}$.)

From (2) we have, leaving out the suffices $_1$ and $_2$ and considering the general term,

$$\frac{\partial u}{\partial z} = \frac{f\bar{u}r^n}{E}\left\{1 + \frac{z(r-1)}{p}\right\}, \qquad (2\ bis)$$

which on integrating over bay n gives

$$\begin{aligned} u_{n+1} - u_n &= \int_0^p \frac{\partial u}{\partial z}\,dz \\ &= \frac{f\bar{u}pr^n(1+r)}{2E}\,; \end{aligned} \qquad (46)$$

therefore $-u_0 = u_\infty - u_0$, since u_∞ is zero,

$$\begin{aligned} &= \sum_0^\infty (u_{n+1} - u_n) \\ &= \frac{f\bar{u}p}{2E}\left(\frac{1+r}{1-r}\right) \text{ from equation (46)} \qquad (47) \\ &= \bar{u} \text{ for the particular case under consideration.} \qquad (48) \end{aligned}$$

Hence, introducing the suffices $_1$ and $_2$ and rearranging, we get

$$f_1\left(\frac{1+r_1}{1-r_1}\right)+f_2\left(\frac{1+r_2}{1-r_2}\right)=\frac{2E}{p}\,. \qquad \ldots \qquad (49)$$

The condition of zero shear-distortion at the root implies that the root rib is unstressed, *i.e.*,

$$H_0 = 0\,. \qquad \ldots \qquad (50)$$

Hence from (18) :

$$\frac{f_1(1-r_1)^2}{r_1}+\frac{f_2(1-r_2)^2}{r_2}=0\,. \qquad \ldots \qquad (51)$$

This condition of zero shear-distortion at the root can be regarded as produced by an infinitely stiff root rib even though flexible ribs have been considered in deriving the decay factors.

Equations (49) and (51) determine f_1 and f_2, which are of the form

$$\left.\begin{array}{l} f_{1,2} = (E/p)\,\xi_{1,2}\,(r_1, r_2), \text{ say,} \\ \text{where from (44)} \\ r = r(\beta, \gamma)\,. \end{array}\right\} \qquad \ldots \qquad (52)$$

A.5.3. *Direct Stress at the Root.*—The direct stress at the root is

$$\bar{u}(f_1+f_2) = \frac{E\bar{u}}{p}(\xi_1+\xi_2) \text{ from (52)}$$

$$= E\bar{u}k\,\frac{(\xi_1+\xi_2)}{\beta}$$

$$= E\bar{u}k\psi_w, \text{ say,} \qquad \ldots \qquad (53)$$

where ψ_w is non-dimensional, being a function of β and γ only. The suffix w has been introduced as the end conditions of the self-equilibrating system which we are applying to the structure are such that there is axial *warping* at the root section and no shear-distortion. In Appendix IB we consider the effect of a self-equilibrating system which produces shear *distortion* and no axial warping of the root section.

Two ψ_w–β–γ families of curves have been drawn in Figs. 1 and 2, one showing lines of constant β, the other lines of constant γ. It will be seen from these that over the current practical range $(0{\cdot}2 < \beta < 1)$ it is sufficient to assume $\beta = 0$; this assumption is shown in A.6.3 to be the same as assuming the ribs to be spread out into a homogeneous ‘ rib medium ’.

A.5.4. *Direct Stress along the Cylinder.*—Owing to the existence of two decay factors there is no simple ‘ exponential ’ expression for the direct stress along the cylinder ; there is however a simple recurrence relation for the stresses at the nth rib.

The ratio $\dfrac{\text{magnitude of self-equilibrating axial stresses at first rib}}{\text{magnitude of self-equilibrating axial stresses at root rib}}$

$$= \frac{f_1r_1+f_2r_2}{f_1+f_2}$$

$$\equiv \Delta_w, \text{ say,} \qquad \ldots \qquad (54)$$

where $\Delta_w = \Delta_w(\beta, \gamma)$. The Δ_w–β–γ family of curves has been drawn in Fig. 8.

In general we have

$$\frac{\text{self-equilibrating axial stresses at } n\text{th rib}}{\text{self-equilibrating axial stresses at root rib}} = \frac{f_1 r_1^n + f_2 r_2^n}{f_1 + f_2} \equiv \Delta_{w,n}, \text{ say,} \quad \ldots \quad (55)$$

$$= \lambda_1 \Delta_{w,n-1} - \lambda_2 \Delta_{w,n-2} \quad \ldots \quad (56)$$

by virtue of (55) and where

$$\lambda_1 = r_1 + r_2 = \lambda_1(\beta, \gamma)$$

and

$$\lambda_2 = r_1 r_2 = \lambda_2(\beta, \gamma) \quad \ldots \quad (57)$$

Figs. 12 and 13 show the λ_1–β–γ and λ_2–β–γ families of curves.

$\Delta_{w,0}$ which occurs in (56) when $n = 2$, is, of course, unity. We have for convenience written $\Delta_{w,1} = \Delta_w$ as there is no risk of ambiguity.

A.5.5. *Externally Applied Shears.*—In order to keep the root section free from shear distortion there must be applied to the cylinder, either externally or by a stiff rib, a system of shears which will be determined by equation (16) with $n = -1$ (or zero, by virtue of the particular end conditions under consideration).

We therefore have :

$$Q_{-1} = \frac{S}{p}\left\{\frac{f_1(1 - r_1)}{r_1} + \frac{f_2(1 - r_2)}{r_2}\right\} ; \quad \ldots \quad (58)$$

using equation (52) we may write this in the form

$$J \equiv Q_{-1} = k^2 E S \Delta_w \ , \quad \ldots \quad (59)$$

where Δ_w is a function of β and γ only. The Δ_w–β–γ family of curves has been drawn in Fig. 4.

Alternatively we can express J in terms of the direct stresses at the root. We then have the relation :

$$(\text{direct stress at root}) \div \left(\frac{J\,\bar{u}}{kS}\right) = \psi_w \div \Delta_w$$

$$= \phi_w \quad \ldots \quad (60)$$

Fig. 3 shows the ϕ_w–β–γ family of curves.

A.5.6. *Shear Stresses along the Cylinder.*—The self-equilibrating shear flux in bay O $= Q_0$*

$$= Q_{-1} - H_0 \text{ from (17)}$$

$$= J, \text{ since } H_0 \text{ is zero.} \quad \ldots \quad (61)$$

The ratio $Q_1 : Q_0$ will be denoted by Δ_{wQ} (this is similar to the notation used in A.5.4).

The Δ_{wQ}–β–γ family of curves is shown in Fig. 9. The shear flux in the nth bay satisfies the recurrence relation

$$\Delta_{wQ,n} = \lambda_1 \Delta_{wQ,n-1} - \lambda_2 \Delta_{wQ,n-2} \ , \quad \ldots \quad (63)$$

where $\Delta_{wQ,n} = Q_n : Q_0$.

* This must not be confused with the Q_0 of equation (5).

A.5.7. *Rib Stresses.*—The root rib undergoes no shear-distortion and is therefore unstressed. The ribs at the far end of the cylinder are unstressed—from the principle of Saint-Venant. The self-equilibrating rib shear flux H is however not zero near the root but increases towards a maximal value and then decreases to zero as indicated in Diagram 3 below ;

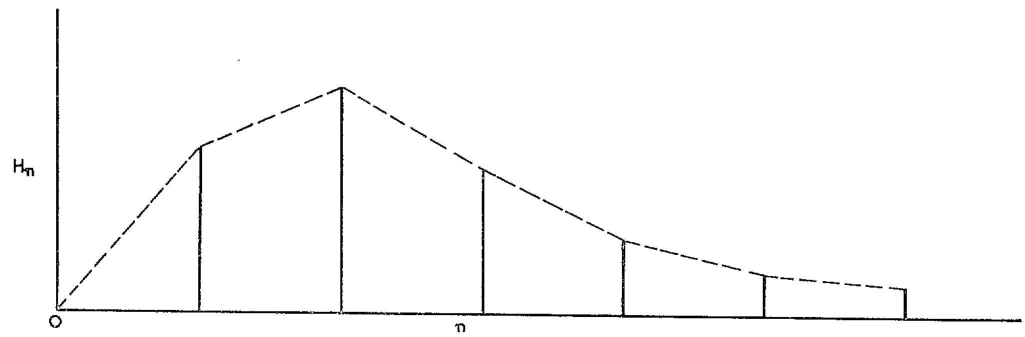


Diagram 3. Variation of rib distortion ($\propto H$) along the cylinder.

The maximal value of H can be expressed in the form

$$H_{\max} = \Xi J, \qquad (64)$$

where J is the externally applied self-equilibrating shear flux (*see* A.5.5), and $\Xi = \Xi(\beta, \gamma)$; this has been plotted in Fig. 5.

The position of the most highly stressed rib is also a function of β and γ only. Fig. 6 gives the appropriate value of n for given values of β and γ ; e.g., if $\beta = 0 \cdot 4$, $\gamma = 4$, it follows that the 2nd rib is the most highly stressed.

A.6. *Some Particular Cases and a Physical Interpretation of k, β and γ.*—The results obtained in this Appendix will now be adapted to the following special cases :

(*a*) the ribs are infinitely stiff—the rib-pitch being finite
(*b*) the rib-pitch tends to zero—ribs of finite stiffness
(*c*) the ribs are considered to be spread out into a uniform elastic rib-medium of effective thickness t_{rib}/p per unit length.

A.6.1. *Case* (*a*).—Since the ribs are infinitely stiff we have, in effect, $t_{\text{rib}} = \infty$ and hence from equation (30) $B_3 = 0$. It follows from equation (40) that $\gamma = \infty$ which means that the equation for the decay factor r takes the form

$$r^2 - 2\left(\frac{1 + \beta^2/3}{1 - \beta^2/6}\right) r + 1 = 0 \; . \qquad (65)$$

The direct stress at the root is

$$\frac{2E\bar{u}}{p}\left(\frac{1 - r_1}{1 + r_1}\right) \qquad (66)$$

where $|r_1| < 1$. This reduces to

$$E\bar{u}k\left(1 + \frac{\beta^2}{12}\right)^{-1/2} , \qquad (67)$$

which is in agreement with the results obtained in Ref. 4 where a doubly symmetrical rectangular box under torsion was considered. The variation of direct stress along the cylinder for various values of β is shown in Fig. 10.

A.6.2. *Case* (*b*).—Since the rib-pitch tends to zero it is reasonable to assume that the variation of direct stress from one rib to the next will be very small. We accordingly search for a solution in the form

$$r = 1 + c_1 p + c_2 p^2 + O(p^3) \ , \qquad \qquad (68)$$

where the c's are constants. Observing that as $p \to 0$, $\gamma \to \infty$ and so the equation for r is a particular case of equation (65). Substituting (68) in (65) gives $c_1^2 p^2 = \beta^2$,

i.e.,
$$c_1 = \pm k \ , \qquad \qquad (69)$$

terms of higher order in p being neglected.

The expression for the direct stress at the nth rib becomes :

$$E\frac{\partial u}{\partial z} = \lim_{p \to 0} \left\{ f_1 \bar{u}(1 + kp)^n + f_2 \bar{u}(1 - kp)^n \right\}$$

$$= \lim_{p \to 0} \left\{ f_1 \bar{u}(1 + kp)^{z/p} + f_2 \bar{u}(1 - kp)^{z/p} \right\} ,$$

where z is now measured only from the root,

$$= \bar{u}(f_1 \, e^{kz} + f_2 \, e^{-kz}) \ . \qquad \qquad (70)$$

Thus k has been identified with the decay factor obtained when cross-sections are assumed to retain their shape (since $\gamma = \infty$). For this reason it is convenient to call k the 'standard decay factor'. This k, given by equation (40), is in agreement with that obtained by other writers in considering cylinders under torsion[1,2].

For a long cylinder where there is complete building-in at the root (70) becomes

$$E\frac{\partial u}{\partial z} = E\bar{u}k \, e^{-kz} \qquad \qquad (71)$$

$\beta (= kp)$ may be regarded as a non-dimensional measure of the rib-pitch.

It is of interest to note that to retain the shape of cross-section it is necessary and sufficient that $p = 0$ and $\gamma = \infty$, i.e., $\lim_{p \to 0} (t_{\text{rib}}/p) = \infty$. Previous writers have assumed that $p = 0$, and, in effect, $t_{\text{rib}} = \infty$. Provided the ribs are infinitely close together they could, in theory, even have zero stiffness, *e.g.*, if t_{rib} behaves like $\sqrt{p}$ as $p \to 0$.

A.6.3. *Case* (*c*).—The concept of a stringer-sheet, in which the stringers have been spread out into a uniform sheet with equivalent uni-directional properties, is well known and this concept may likewise be extended to the discrete ribs of the present problem. In this case the individual ribs are represented by a uniform elastic rib-medium of thickness t_{rib}/p per unit length. This means that γ is unchanged. The properties of this medium are such that it distorts in a manner similar to that of the individual ribs and that there is no axial interaction.

As in Case (*b*) the direct stresses will decay exponentially and accordingly we put $r = 1 + c_1 p + O(p^2)$ so that as $p \to 0$ the solution takes the form

$$E\frac{\partial u}{\partial z} = f\bar{u} \exp(c_1 z) \ . \qquad \qquad (72)$$

The resulting equation for c_1 is the quadratic in c_1^2

$$c_1^4 - \gamma k^2 c_1^2 + \gamma k^4 = 0 \ . \qquad \qquad (73)$$

Let the roots of this equation, corresponding to decaying stresses only, be $c_1 = -k_1$ and $-k_2$. The condition of complete axial building-in at the root is

$$\frac{f_1}{k_1} + \frac{f_2}{k_2} = E \, , \qquad (74)$$

and for zero shear-distortion at the root

$$k_1^2 f_1 + k_2^2 f_2 = 0 \, . \qquad (75)$$

The direct stress at the root is now

$$\bar{u}(f_1 + f_2) = E\bar{u}k \left\{ \frac{\gamma^{1/4}(2 + \gamma^{1/2})^{1/2}}{1 + \gamma^{1/2}} \right\} . \qquad (76)$$

This function of γ is of course a particular case of the ψ_w of equation (53). It will be seen from Figs. 1 and 2 that over the current practical range of β, $0 \cdot 2 < \beta < 1$, this function of γ gives a good representation of ψ_w, i.e., for the particular problem under consideration the assumption that the effect of rib-flexibility may be estimated sufficiently accurately by spreading the ribs out into a uniform elastic rib medium is justified. Because of this fact and that $\gamma \propto t_{\text{rib}}/p$ it is convenient to call γ the ' generalised rib stiffness '.

There are practical cases where an equivalent rib-medium does actually exist. For example, consider a cylinder whose walls are of a sandwich construction. Any shear distortion of the section will be resisted by the bending stiffness of the walls and we have in effect an equivalent rib-medium. The fictitious value of t_{rib}/p which we must use for calculating γ is that which, when subjected to a set of self-equilibrating shears per unit length (proportional to S), will store the same amount of strain energy as the actual structure. An example is given in A.12.

A.6.3.1. *Direct stresses away from the root.*—The direct stresses along the cylinder have been plotted, in non-dimensional form, for various values of γ in Fig. 11. It will be seen that over the current practical range (approximately: $1 < \gamma < 32$) the effect of rib-medium flexibility is small. One reason for this is that at the root, where rib distortions would produce most effect on the direct stresses, we have prevented any distortion.

A.6.3.2. *Rib distortion along the cylinder.*—The rib-medium distortion is proportional to $(e^{-k_1 z} - e^{-k_2 z})$ and is therefore of the form indicated in Diagram 4 below.

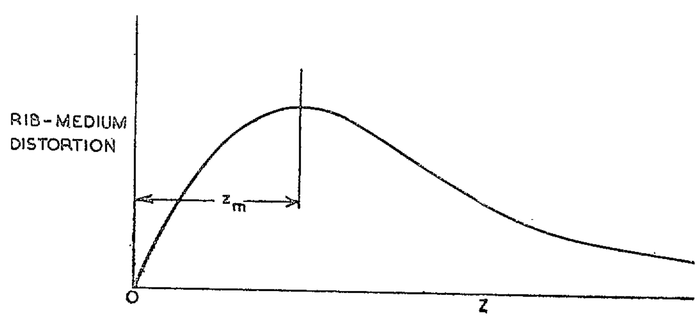


Diagram 4. Rib-medium distortion.

If z_m is the value of z at which the distortion is a maximum then we have the relation :

$$\left.\begin{aligned} kz_m &= \frac{\text{arg cosh } (\gamma^{1/2}/2)}{\gamma^{1/4}(\gamma^{1/2} - 2)^{1/2}} \text{ for } \gamma > 4 \\ \text{and} \quad & \frac{\text{arc cos } (\gamma^{1/2}/2)}{\gamma^{1/4}(2 - \gamma^{1/2})^{1/2}} \text{ for } \gamma < 4 \end{aligned}\right\} \quad \ldots \quad (77)$$

These have been plotted in Fig. 7.

If it were desirable to keep the distortion of the section everywhere as small as possible then an extra rib in the structure at the section $z = z_m$, where the distortion would otherwise be very large, would be most effective.

A.6.4. *Combination of Rib Medium with Discrete Ribs.*—With a slight modification of the analysis in Appendix I we can derive expressions for the stresses at the rib stations and the variation, now non-linear, between ribs. These are of academic interest only as the effect of discrete flexible ribs upon the various stresses is, in most cases, a second-order one, and there is no justification for investigating accurately the combination of two second-order effects (since this would introduce third-order effects).

It is probable that in such a case (combination of rib-medium with discrete ribs) the following device will give the stresses, etc., with sufficient accuracy :

let suffix $_m$ refer to the rib-medium and suffix $_r$ to the discrete ribs and let $F(\beta, \gamma)$ be a stress or bending moment, say,

then : $$F \simeq \left(\frac{\gamma_r}{\gamma_r + \gamma_m}\right) F(\beta_r, \gamma_r + \gamma_m) + \left(\frac{\gamma_m}{\gamma_r + \gamma_m}\right) F(0, \gamma_r + \gamma_m).$$

APPENDIX IB

Particular Case of Zero Axial Warp at the Root

A.7. In Appendix I the equation for determining the decay factors r was found, *see* Equations (41) and (42). The roots of this equation whose moduli are less than unity are

$$\left.\begin{aligned} r_1 &= r_1(\beta, \gamma) \\ \text{and} \quad r_2 &= r_2(\beta, \gamma), \end{aligned}\right\} \quad \ldots \quad (44 \textit{ bis})$$

and the direct stresses at the nth rib in a long cylinder are given by

$$\frac{E\partial u_n}{\partial z} = \bar{u}(f_1 r_1^n + f_2 r_2^n) \quad \ldots \quad (45 \textit{ bis})$$

with a linear variation between ribs.

A.7.1. *Expressions for Determining f_1 and f_2.*—In Appendix IA the special case of zero shear-distortion at the root was considered ; in this Appendix the similar problem of zero axial warp, but non-zero shear-distortion, at the root is considered. All systems of self-equilibrating load applied at the root may be obtained by suitable combination of these two systems.

To fix ideas we consider a long cylinder, as represented in Diagrams 5 and 6, to one rib of which is applied a self-equilibrating shear system $2J$ proportional to S. From symmetry it will be seen that there will be no axial warping of the section at this (root) rib. For slightly increased generality we take the stiffness of this 'root rib' to be 2ϱ times the stiffness of each of the other ribs; the γ in the following work refers to the rest of the structure.

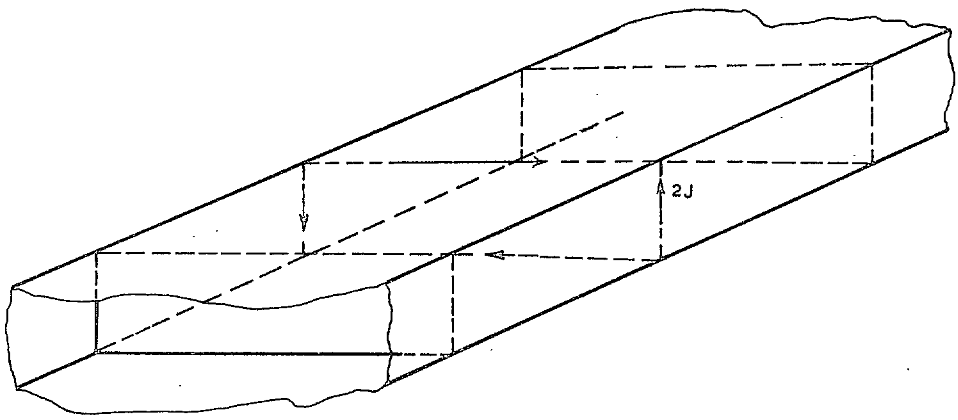


Diagram 5. Self-equilibriating shear system.

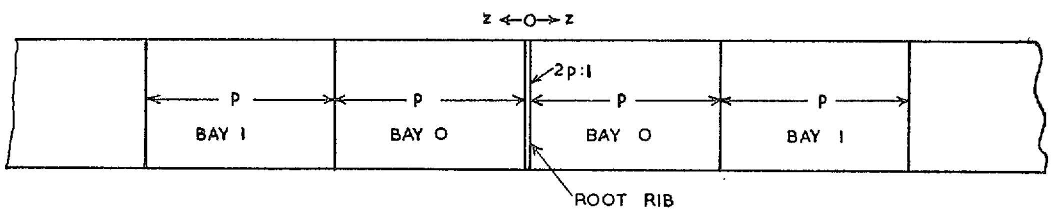


Diagram 6. Section of cylinder showing ribs.

The condition of zero axial warp at the root is, from equation (47),

$$f_1\left(\frac{1+r_1}{1-r_1}\right)+f_2\left(\frac{1+r_2}{1-r_2}\right)\gamma = 0. \quad \ldots \quad (78)$$

The equation of shear equilibrium is evidently

$$2J = 2\varrho H_0 + 2Q_0\,, \quad \ldots \quad (79)$$

which, on substituting from equations (16) and (18), gives another relation between f_1 and f_2.

A.7.2. *Direct Stress at the Root.*—Solving equations (78) and (79) for f_1 and f_2, we obtain the following expression for the direct stress at the root:

$$\frac{E\,\partial u_0}{\partial z} = \bar{u}(f_1+f_2)$$

$$= \left(\frac{J\bar{u}}{Sk}\right)\frac{\phi_D}{(2\varrho-1)\mu+1}\,, \quad \ldots \quad (80)$$

where ϕ_D and μ are functions of β and γ only. These have been plotted in Figs. 14 and 18. It will be observed that the magnitude of $\bar{u}$ in (80) is immaterial as it is proportional to that of S, If $2\varrho = 1$, *i.e.*, all ribs are equally stiff, equation (80) reduces to

$$E\frac{\partial u_0}{\partial z} = \left(\frac{J\bar{u}}{Sk}\right)\phi_D \quad \ldots \quad (81)$$

and it will be seen that ϕ_D is analogous to ϕ_W of equation (60).

A.7.3. *Direct Stress along the Cylinder.*—The ratio :

$\frac{\text{magnitude of self-equilibriating axial stresses at first rib}}{\text{magnitude of self-equilibriating axial stresses at root rib}}$ has been denoted by Δ_D and the Δ_D–β–γ family of curves has been drawn in Fig. 16 ; and using the notation :

$$\frac{\text{self-equilibriating axial stresses at } n\text{th rib}}{\text{self-equilibriating axial stresses at root rib}} = \Delta_{D,n} \quad \ldots \quad (82)$$

we have the recurrence relation, similar to that of equations (56) and (63),

$$\Delta_{D,n} = \lambda_1\Delta_{D,n-1} - \lambda_2\Delta_{D,n-2} . \quad \ldots \quad (83)$$

This enables the direct stress to be found at any point along the cylinder.

A.7.4. *Rib Loads.*—That part of the applied self-equilibrating shear flux $2J$ which is taken by the root rib is given by

$$H_0 = \frac{4\mu\varrho J}{1 + \mu(2\varrho - 1)} , \quad \ldots \quad (84)$$

which, when all the ribs are equally stiff, reduces to

$$H_0 = 2\mu J. \quad \ldots \quad (85)$$

The shear flux taken by each adjacent rib is

$$H_1 = \left(\frac{H_0}{2\varrho}\right)\Delta_{DD} , \quad \ldots \quad (86)$$

where $\Delta_{DD} = \Delta_{DD}(\beta, \gamma)$ and is shown in Fig. 17.

Using the notation :

$$H_n = \left(\frac{H_0}{2\varrho}\right)\Delta_{DD,n} , \quad \ldots \quad (87)$$

we have the recurrence relation

$$\Delta_{DD,n} = \lambda_1\Delta_{DD,n-1} - \lambda_2\Delta_{DD,n-2} . \quad \ldots \quad (88)$$

A.8. *Rib Medium.*—When such a self-equilibrating shear system is applied to a cylinder with a rib-medium—such as that mentioned in A.6.3—the direct stress at the root is given by

$$E\frac{\partial u_0}{\partial z} = \left\{\frac{J\bar{u}}{Sk}\right\}\left\{\frac{1}{\gamma^{1/4}(2 + \gamma^{1/2})^{1/2}}\right\} . \quad \ldots \quad (89)$$

This function of γ is analogous to that of equation (76) and is a particular case of ϕ_D.

The direct-stress distribution along the cylinder has been plotted (in non-dimensional form) for various values of γ in Fig. 15.

APPENDIX II

Application to a Rectangular Box under Torsion

A.9. We consider now the particular problem of the effect of self-equilibrating constraints applied to a cylinder of rectangular section under torsion. The cylinder is stiffened by uniformly spaced flexible ribs. Expressions are given for k, β, γ, etc., which are more manageable than the general ones derived in Appendix I.

The notation—given below—is similar to that of R. & M. 1761[2].

A.9.1. *Notation*

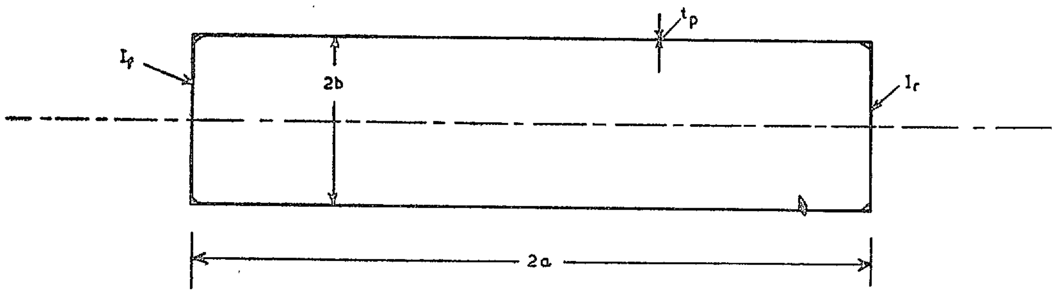


Diagram 7. Section of box.

E, G	Elastic moduli
$2a$	Distance between front and rear spars
$2b$	Distance between top and bottom panels
I	Moment of inertia of spar section about its centre-line
t	Skin thickness
A_p	Area of section of top panel capable of taking direct stress
=	Area of section of bottom panel capable of taking direct stress
p	Rib pitch
t_{rib}	Thickness of equivalent shear rib
M	Bending moment in spar or panel
T	Applied torque

suffices $_f$, $_r$ and $_p$ refer to front spar, rear spar and top or bottom panel,
suffix $_0$ refers to the root section.

A.9.2. *Expressions for k, β and γ.*—Introducing τ, I_x, C_1, C_2 and C_3 defined by the relations :

$$\left.\begin{aligned} \tau &= \frac{1 + b^2 A_p/I_f}{1 + b^2 A_p/I_r} \\ 1/I_x &= 1/I_f + \tau/I_r \\ C_1 &= 1/at_f + 1/at_r + 2/bt_p \\ C_2 &= 1/at_f + 1/at_r - 2/bt_p \\ C_3 &= (1 + \tau)a^2 I_x + b^2 I_p \end{aligned}\right\} , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (90)$$

we have the following expression for k^2

$$k^2 = \frac{16abG}{C_1 C_3 E} \quad \ldots \quad \ldots \quad \ldots \quad (91)$$

and hence

$$\left.\begin{aligned}\beta &= 4p\sqrt{\left(\frac{abG}{C_1C_3E}\right)} \\ &= pk\ .\end{aligned}\right\} \quad \ldots \quad (92)$$

The equation for γ corresponding to equation (40) is

$$\gamma = \frac{C_1 t_{\text{rib}}}{2k^2 p} \quad \ldots \quad (93)$$

A.9.3. *Direct and Shear Stresses at the Root.* (Constant torque case).—We shall now find the self-equilibrating shear loads and bending moments at the root due to complete building-in at that section when a constant torque is applied to the far end of the box. This end condition corresponds to that of Appendix IA.

In the unconstrained state there will be shear stresses distributed according to the Batho-Bredt torsion theory, *i.e.*, a constant shear-flux round the section, and no direct stresses. There will also be in general an axial warping of all sections and if this warp is given by $\bar{u}$ then there must be applied to the box at the root a warp of $-\bar{u}$* in order that there will be complete building-in. (For partial building-in we impose a warp of $-K\bar{u}$, etc.) We shall concern ourselves with the stresses due to this self-equilibrating system as was done in Appendix IA. It is clear that since $\bar{u}$ will be proportional to the applied torque T we can express the stresses at the root directly in terms of T.

It readily follows that

$$\left.\begin{aligned} M_{f0} &= -T\left(\frac{EkC_2I_x}{16Gb}\right)\psi_w \\ \text{and} \quad M_{r0} &= \tau M_{f0} \end{aligned}\right\} \quad \ldots \quad (94)$$

This equation corresponds to equation (53) where the stresses at the root were considered—bending moments being, of course, meaningless for a section of arbitrary shape.

The self-equilibrating shear-flux system is of the form indicated in Diagram 8 below, but we can considerably simplify our working is we consider the *total* shear load in the spars and panels (S_s and S_p) and not the *distribution* along the spar webs and panels.

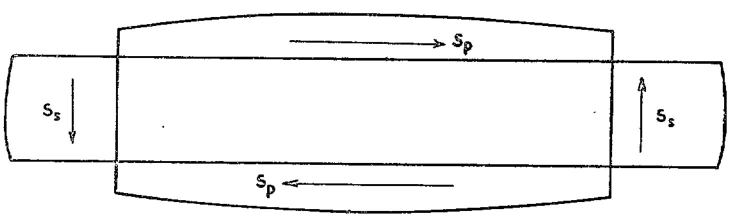


Diagram 8. Distribution of self-equilibrating shear system.

With the notation introduced above we have the equilibrium relation

$$2bS_p = 2aS_s\ , \quad \ldots \quad (95)$$

and the equation corresponding to equation (60) of Appendix IA is

$$2aS_s = \left(\frac{kC_3}{2aI_x}\right)\frac{M_{f0}}{\phi_w}\ ; \quad \ldots \quad (96)$$

---

*In order to ensure that the resulting system is self-equilibrating it will generally be necessary to add to $\bar{u}$ a linear warp of the form $A + Bx + Cy$.

and that corresponding to equation (59) :

$$4aS_s = 2aS_s + 2bS_p = \left(\frac{C_2}{C_1}\right) TA_w \quad . \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (97)$$

It will be remembered that these shear loads are in addition to the Batho-Bredt torsion loads. If these are designated by S^1 we have the following relation :

$$4aS_s^1 = 2aS_s^1 - 2bS_p^1 = -T \quad . \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (98)$$

A.9.4. *Stresses along the Cylinder.*—The direct and shear stresses along the cylinder may be found by using the recurrence relations given in equations (55) and (63).

A.10. *Varying Torque.*—When the torque is not constant but is a function of z the $\bar{u}$ at the root may be calculated by standard methods (*e.g.*, Ref. 2) with sufficient accuracy and the present method used for determining the direct stresses due to building-in ; these stresses will be additional to those due to the rate of change of torque. If the torque varies considerably over any one bay then the stressing problem may need further investigation.

A.11. *Direct and Shear Stresses at the Root when that Section undergoes Shear-distortion only.*—Proceeding on lines similar to those of Appendix IB and using the notation of the previous sections we obtain the following relation between the bending moments at the 'root' and the applied shear system $2J$:

$$M_{f0} = \left(\frac{4a^2 I_x}{kC_3}\right) \frac{S_s \phi_D}{(2\varrho - 1)\mu + 1} , \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (99)$$

where J is as in Diagram 8. As in section A.9.3 we have the simple relation between the front and rear spar bending moments :

$$M_{r0} = \tau M_{f0} \quad . \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (100)$$

The application of the rest of Appendix IB is quite straight-forward and requires no explanation.

A.12. *Rib-Medium Representation of Cylinder with Walls of Sandwich Construction.*—It was mentioned in section A.6.3 that in cases where the cylinder walls were of a sandwich construction any shear-distortion of the section will be resisted by the bending stiffness of the walls and we have in effect an equivalent rib medium. This means that β is zero and we can use the results of equations (72) to (76).

We shall now give an expression for γ when the section of the box is as in Diagram 9 below.

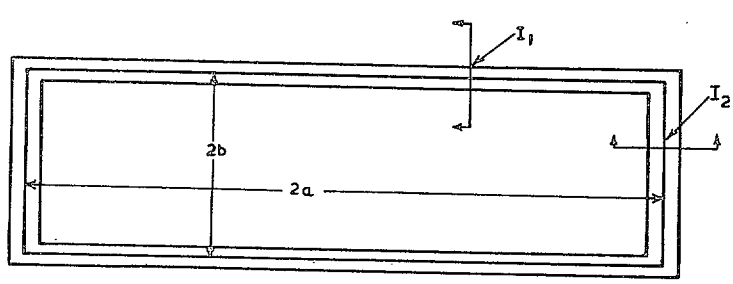


Diagram 9. Section of cylinder with walls of sandwich construction.

We introduce the following additional notation :

I_1 moment of inertia of unit axial length of top panel section
= ,, ,, ,, ,, ,, ,, ,, ,, bottom panel section
I_2 ,, ,, ,, ,, ,, ,, ,, ,, front spar web section
= ,, ,, ,, ,, ,, ,, ,, ,, rear spar web section.

By considering the strain-energy stored in the structure when self-equilibrating loads of the type shown in Diagram 8 are applied and comparing the strain energy with that which would be stored in a 'pure shear rib' (*i.e.*, a rib of constant skin thickness) we arrive at this equation for γ :—

$$\gamma = \frac{3EC_1}{2Gk^2ab}\left(\frac{a}{I_1} + \frac{b}{I_2}\right)^{-1} . \qquad (101)$$

EXAMPLE

A long thin-walled cylinder of doubly symmetrical rectangular section is resisting a constant torque applied at one end of the cylinder. The other end of the cylinder is rigidly built-in, and the top and bottom panels and the ribs are removed over a length L from this encastréd end. The rest of the cylinder is stiffened by uniformly spaced ribs, the junction rib being ϱ times the stiffness of each of the remainder. The problem is to determine how the spar bending moments throughout, and at the junction rib in particular, are influenced by rib-flexibility and spacing. An expression is also derived for the load taken by the junction rib. The results are also given in Figs. 19 to 22.

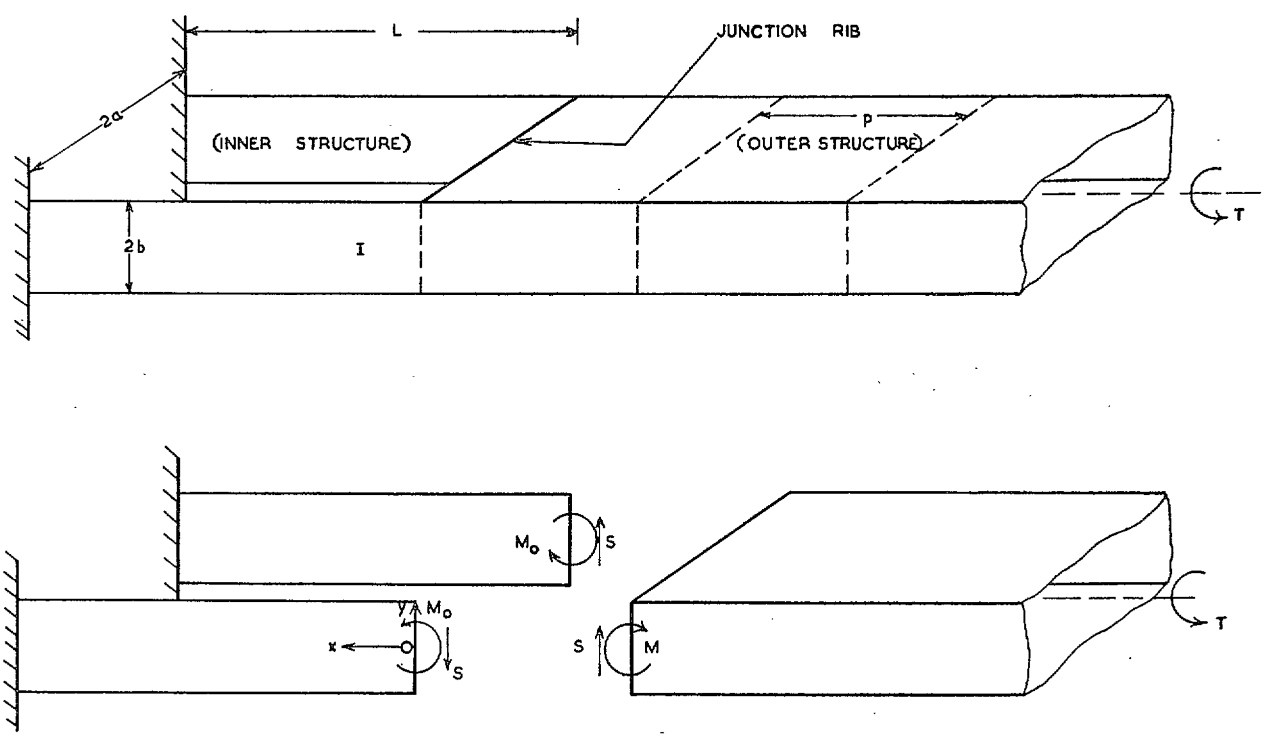


Diagram 10. Torque applied to rectangular box with cut-out.

T	Applied torque (constant throughout)
I	Moment of inertia of front and rear spar
L	Distance along cylinder (from encastréd end to the junction rib) in which the spars alone are present to resist torsion
$2a$, $2b$	Width and depth of cylinder section
S	Shear load in each spar (as in Diagram 10)
$M(x)$	Bending moment in spars between encastréd end and junction rib
M_0	Value of $M(x)$ at junction rib
M	Bending moment in spars just outboard of junction rib
E, G	Elastic moduli
x, y	Co-ordinates measured from junction rib (*see* Diagram 10)
k	Standard decay factor } as in Appendix II.
J, ψ_W, ϕ_W, Λ_W, ϕ_D	} as in Appendix II.
C_1, C_2, ϱ, β, γ, μ	} as in Appendix II.

Other symbols introduced where necessary.

We proceed to find relations between the spar slopes and spar bending-moments in the inner and outer structures at the junction rib section. Compatibility of slope and moment at that section then determines each.

Inner Structure.—The torque is resisted solely by the spars and hence

$$2aS = T. \qquad \text{(i)}$$

Also, $$M(x) = EI\frac{d^2y}{dx^2} = M_0 - Sx . \qquad \text{(ii)}$$

Integrating (ii) gives

$$EI\frac{dy}{dx} = M_0(x - L) - \frac{S}{2}(x^2 - L^2) , \qquad \text{(iii)}$$

it being assumed that the spars are rigidly encastréd at $x = L$. At the junction rib the spar slope (due to bending) given by (iii) must be the slope of the spars in the outer structure at that section ; and we have :—

$$\left(\frac{dy}{dx}\right)_0 = \frac{L}{2EI}(SL - 2M_0) . \qquad \text{(iv)}$$

Outer Structure.—Bending Moments.—The bending moment at the junction rib section in the front and rear spars of the outer structure will be slightly less than M_0 because of the contribution in bending stiffness of the top and bottom panels. We can write however

$$M = M_0/K \qquad \text{(v)}$$

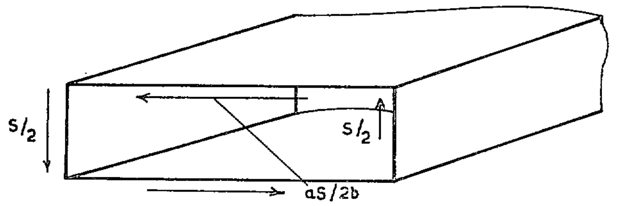
where K is a constant depending on the section dimensions of the outer structure.

If A_p is the area of section of the top and of the bottom panel capable of taking direct stress the bending moment taken by these panels will be $b^2A_p/2I$ of the moment in the adjacent spars, a result obtained by equating the stresses in the panels and in the spars at their common points.

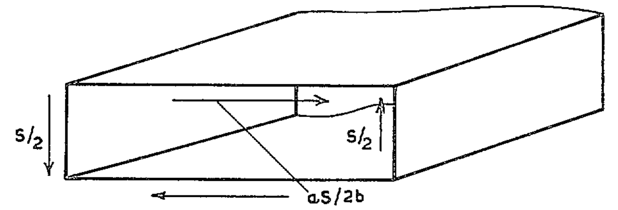
The value for K above is accordingly

$$1 + b^2A_p/2I . \qquad \text{(va)}$$

Shears.—We regard the up-and-down shears to be made up of two systems one of the type :—



which represents a Batho system, and the other :



which is a self-equilibrating 'J' system.

The Batho system will produce a slope of the front spar (equal and opposite to that in the rear spar) given by

$$\left(\frac{dy}{dx}\right)_0 = -\frac{TC_2}{32Gb} \qquad \text{(vi)}$$

Equation (vi) may be deduced from first principles or indirectly from equation (94) when closely spaced stiff ribs are considered ; the bending moment that would be caused by building-in would then decay as exp($-$ kx) and may be integrated to give this slope.

The self-equilibrating shear and differential bending system may be split up into two further systems, as was done in Appendices IA and IB, one in which axial warping only of the section at the junction rib is permitted and the other in which shear-distortion of the section and no axial warping is permitted. Distinguishing these two systems by using suffices W and D we can therefore write.

$$\left.\begin{aligned} M &= M_W + M_D \\ \text{and} \quad S/2 &= S_W + S_D \end{aligned}\right\} \qquad \text{(vii)}$$

And from equations (96) and (99) :—

$$\left.\begin{aligned} M_W &= C\phi_W S_W \ , \\ M_D &= C\phi_D' S_D \end{aligned}\right\} \qquad \text{(viii)}$$

where $\quad C = -\ 2a^2I/kC_3 \qquad \text{(ix)}$

and ϕ_D' is the modified ϕ_D, to take account of different flexibility of the junction rib, and is given by

$$\phi_D' = \frac{\phi_D}{(2\varrho - 1)\mu + 1} \qquad \text{(x)}$$

Of the four unknowns M_W, M_D, S_W and S_D, the first alone produces a slope in the spars due to differential bending. Accordingly we solve (vii) and (viii) for M_W :—

$$M_W = \frac{M - C\phi_D'S/2}{1 - \phi_D'/\phi_W} \qquad \text{(xi)}$$

The slope in the spars due to M_W is proportional to the axial warping due to M_W and is given by

$$\left(\frac{dy}{dx}\right)_0 = M_W/EkI\psi_W \quad \text{(xii)}$$

The total differential spar slope is given by (vi) and (xii) and must be the spar slope (due to bending only) of the inner structure at the junction rib : given by (iv). Equating these spar slopes gives :—

$$\frac{L}{2EI}(SL - 2M_0) = \frac{M - C\phi_D'S/2}{EkI\psi_W(1 - \phi_D'/\phi_W)} - \frac{TC_2}{32Gb} \quad \text{(xiii)}$$

and hence

$$M = \frac{TL^2/4a + (C_2 - C_1/g_1)TEI/32Gb}{LK + 1/kg_2} \quad \text{(xiv)}$$

where

$$\left.\begin{aligned} g_1 &= \psi_W(1/\phi_D' - 1/\phi_W) \\ g_2 &= \phi_D' g_1 \end{aligned}\right\} \quad \text{(xv)}$$

and

It is worth noting that for closely spaced stiff ribs g_1 becomes infinite and g_2 becomes unity. We accordingly expect the bending moment at the junction rib to be very small when

$$L^2/4a + C_2EI/32Gb = 0 ;$$

which on substituting for C_2 and simplifying gives

$$L \simeq \sqrt{\{(\text{single boom area})\ a/tp)\}} \quad \text{(xvi)}$$

Die-away of Bending Moments along the Cylinder.—The bending moments elsewhere in the outer structure may be found accurately as follows. M_W is calculated from equation (xi) and thence M_D by subtraction from M. The bending moment varies linearly from the junction rib to the first rib where it has the value

$$\left.\begin{aligned} M_1 &= \Delta_W M_W + \Delta_D M_D \\ &= \Delta_1 M, \text{ say ;} \end{aligned}\right\} \quad \text{(xvii)}$$

and, in general, at the nth rib

$$M_n = \Delta_n M \ , \quad \text{(xviii)}$$

where the Δ's are found from the recurrence relationship

$$\Delta_n = \lambda_1 \Delta_{n-1} - \lambda_2 \Delta_{n-2} \ . \quad \text{(xix)}$$

Δ_W, Δ_D, λ_1 and λ_2 are taken from the figures at the end of the report.

Considerable errors may be made if the bending moments are assumed to die away as exp $-(kx)$, as will be seen from Figs. 19 and 21 which show graphically the results of the numerical example below.

Load taken by the junction rib.—The load taken by the junction rib is given directly by equation (84), and we have

$$S_{JR} = \frac{2\mu\varrho S_D}{1 + \mu(2\varrho - 1)} \ , \quad \text{(xx)}$$

and S_D, found from (vii) and (viii), $= \dfrac{\phi_W T/4a - M/C}{\phi_W - \phi_D'}$. (xxi)

Numerical example.—Let us assume that

$$2a = 30 \text{ in.}$$
$$2b = 10 \text{ in.}$$
$$t_s = 0 \cdot 1 \text{ in.}$$
$$t_p = 0 \cdot 04 \text{ in.}$$
$$A_p = 1 \cdot 5 \text{ in.}^2$$

So that $I_p = 110 \text{ in.}^4$

boom area $= 0 \cdot 44 \text{ in.}^2$ each

So that $I = 30 \text{ in.}^4$

$$G/E = 0 \cdot 4$$
$$p = 6 \text{ in.}$$
$$t_{\text{rib}} = 0 \cdot 02 \text{ in. } (0 \cdot 03 \text{ in. sheet with lightening holes})$$
$$L = 8 \text{ in. [Case } (a)], \; 24 \text{ in. [Case } (b)].$$

Substituting in equations (90), (91), (92) and (93) we have

$$C_1 = \frac{2}{15 \times 0 \cdot 1} + \frac{2}{5 \times 0 \cdot 04} = 11 \cdot 3 \; ,$$

$$C_2 = \frac{2}{15 \times 0 \cdot 1} - \frac{2}{5 \times 0 \cdot 04} = - 8 \cdot 67 \; ,$$

$$C_3 = a^2 I + b^2 I_p = 6750 + 2750 = 9500.$$

Therefore $k^2 = \dfrac{16 \times 15 \times 5 \times 0 \cdot 4}{11 \cdot 3 \times 9500} = 0 \cdot 00446$

and $k = 0 \cdot 0668$

and $\beta = 6k = 0 \cdot 40$

and $\gamma = \dfrac{11 \cdot 3 \times 0 \cdot 02}{0 \cdot 00446 \times 12} = 4 \cdot 2$

K (*see* equation (va)) $= 1 + \dfrac{25 \times 1 \cdot 5}{60} = 1 \cdot 62$

and C (*see* equation (ix)) $= \dfrac{- 2 \times 15^2 \times 30}{0 \cdot 0668 \times 9500} = - 21 \cdot 3$.

These values of β and γ give

$$\psi_W = 0 \cdot 94, \qquad \phi_W = 1 \cdot 47, \qquad \phi_D = 0 \cdot 318, \qquad \mu = 0 \cdot 392.$$

These were obtained directly from the Figures at the end of the report.

For Case (*a*) in which $L = 8$ in. we have from equation (xiv) :

$$\frac{M}{T} = \frac{8^2/60 + (- 8 \cdot 67 - 11 \cdot 3/g_1)30/32 \times 0 \cdot 4 \times 5}{8 \times 1 \cdot 62 + 1/0 \cdot 0668 g_2}$$

$$= - \frac{(3 + 5 \cdot 33/g_1)}{13 + 15/g_2} \qquad \text{(xxii)}$$

If $\underline{\varrho = 1}$:

$$\phi_D{}' = \frac{0 \cdot 318}{1 + 0 \cdot 392} \text{ from equation (x)}$$
$$= 0 \cdot 228,$$

and hence

$$g_1 = 0\cdot94(1/0\cdot228 - 1/1\cdot47) = 3\cdot49$$

and

$$g_2 = 0\cdot228 \times 3\cdot49 \qquad = 0\cdot796$$

and therefore from (xxii)

$$M/T = -\underline{0\cdot142} .$$

If $\underline{\varrho = 4}$:

$$\phi_D' = \frac{0\cdot318}{1 + 2\cdot74} = 0\cdot0850 ,$$

so that $\quad g_1 = 10\cdot4, \qquad g_2 = 0\cdot885$

and therefore

$$M/T = \underline{-0\cdot117}$$

For closely spaced stiff ribs throughout we have $M/T = -0\cdot107$.

If $L = 24$ in. the equation corresponding to (xxii) becomes

$$\frac{M}{T} = \frac{5\cdot54 - 5\cdot33/g_1}{39 + 15/g_2} \qquad \text{(xxiii)}$$

and if $\varrho = 1$

$$M/T = \underline{+0\cdot07} ,$$

if $\varrho = 4$

$$M/T = \underline{+0\cdot09} ,$$

and for closely spaced stiff ribs throughout $M/T = \underline{+0\cdot103}$.

The load taken by the junction rib (expressed in terms of the vertical shears applied by it to the spars) is given by (xxi), and we can write this in the form

$$S_{JR}/T = \left[\frac{0\cdot784\,\varrho}{1 + 0\cdot392(2\varrho - 1)}\right]\left[\frac{1\cdot47/60 + (M/T)/21\cdot3}{1\cdot47 - \phi_D'}\right] . \qquad \text{(xxiv)}$$

If $L = 8$ in. and $\varrho = 1$:

$$S_{JR}/T = \frac{0\cdot784}{1\cdot392}\left[\frac{0\cdot0246 - 0\cdot142/21\cdot3}{1\cdot47 - 0\cdot228}\right]$$

$$= \underline{0\cdot0081} ,$$

c.f. $1/4a = 0\cdot0167$ if we assume the junction rib alone puts the up-and-down torque into a Batho distribution.

If $\varrho = 4$:

$$S_{JR}/T = \underline{0\cdot0115} .$$

Similarly for $L = 24$ in. we have:

if $\varrho = 1$

$$S_{JR}/T = \frac{0\cdot784}{1\cdot392}\left[\frac{0\cdot0245 + 0\cdot07/21\cdot3}{1\cdot47 - 0\cdot228}\right]$$

$$= \underline{0\cdot0125} .$$

and if $\varrho = 4$:

$$S_{JR}/T = \underline{0\cdot0174} .$$

All these numerical results are shown in graph form in Figs. 19 to 22.

NOTE I

Summary of Formulae for Various Functions of β, γ

λ_1 and λ_2 calculated from :—

$$\frac{1}{\lambda_2}(1 + \lambda_1 + \lambda_2)^2 = 16 + \gamma\beta^2(4 + \beta^2/3) ,$$

$$\frac{1}{\lambda_2}(1 - \lambda_1 + \lambda_2)^2 = \gamma\beta^4 ,$$

and

$$\phi_w = \frac{\beta(1 - \lambda_2)}{1 - \lambda_1 + \lambda_2} = \frac{1 - \lambda_2}{\beta\sqrt{(\gamma\lambda_2)}} ,$$

$$\psi_W = \frac{2(1 - \lambda_2)(1 - \lambda_1 + \lambda_2)}{\beta\{1 + \lambda_1 - \lambda_2(6 - \lambda_1 - \lambda_2)\}} ,$$

$$\Delta_W = \frac{2\lambda_2\gamma\beta^2}{1 + \lambda_1 - \lambda_2(6 - \lambda_1 - \lambda_2)} ,$$

$$\Delta_W = \frac{\lambda_1 - 2\lambda_2}{1 - \lambda_2} ,$$

$$\Delta_{WQ} = \lambda_1 - \lambda_2 ,$$

$$\phi_D = \frac{4\beta\lambda_2}{(1 - \lambda_2)(1 + \lambda_1 + \lambda_2)} ,$$

$$\Delta_D = \frac{\lambda_1 + \lambda_2 - 1}{2} ,$$

$$\Delta_{DD} = \frac{\lambda_2(4 - 3\lambda_1 - 4\lambda_2 + \lambda_1\lambda_2 + \lambda_1^2)}{1 + \lambda_1 - \lambda_2(6 - \lambda_1 - \lambda_2)} ,$$

$$\mu = \frac{2(1 + \lambda_1 - 3\lambda_2)}{(1 - \lambda_2)(1 + \lambda_1 + \lambda_2)} - 1 .$$

NOTE II

On the curves of Figs. 1 *to* 18

Accuracy of the Graphs.—The values of λ_1 and λ_2 ($= r_1 + r_2$ and r_1r_2) have been calculated to six significant figures for some fifty pairs of values of β and γ over the range $0 < \beta < 3$, $1 < \gamma < 64$. The various functions of β and γ in the Figures have been calculated to three places of decimals using the formulae tabulated in Note II and the values of λ_1, λ_2 mentioned above. In most cases lines of constant γ have been drawn and in such cases there is a change of scale at $\beta = 1$; this has been done to include the effect of comparatively large values of $\beta (\beta \leqslant 4)$ without altering the accuracy of the curves over the practical range. In Figs. 2, 6 and 7 a scale for γ over the range $0 < \gamma < \infty$ is given ; this has been obtained from a linear scale by a transformation of the form $\gamma = 4x/(1 - x)$.

Notation relevant to Figs. 1 *to* 18

E		Young's Modulus
k		'Standard decay factor', *see* equations (40), (30), (16) or (91)
β		Non-dimensional rib-pitch measure
	=	$k \times$ rib-pitch
γ		'Generalised rib stiffness', *see* equations (40), (30), (16) or (93)
$\bar{u}$		A warping, round a section of the cylinder, which determines the distribution of the self-equilibrating direct stress around a section: *see* A.1.1
S		A function of the cylinder dimensions which is proportional to the magnitude of $\bar{u}$ and determines the distribution of self-equilibrating shear-flux across a section, *see* equation (16)
J		Self-equilibrating shear-flux applied at root to semi-infinite cylinder; it is proportional to S.

Those Figures which appertain to the particular end condition investigated in Appendix IA, *i.e.*, axial warping but zero shear-distortion at the root, have the sign Ⓦ displayed at the head of the page: those which refer to the complementary case of shear-distortion but zero warp have the sign Ⓓ.

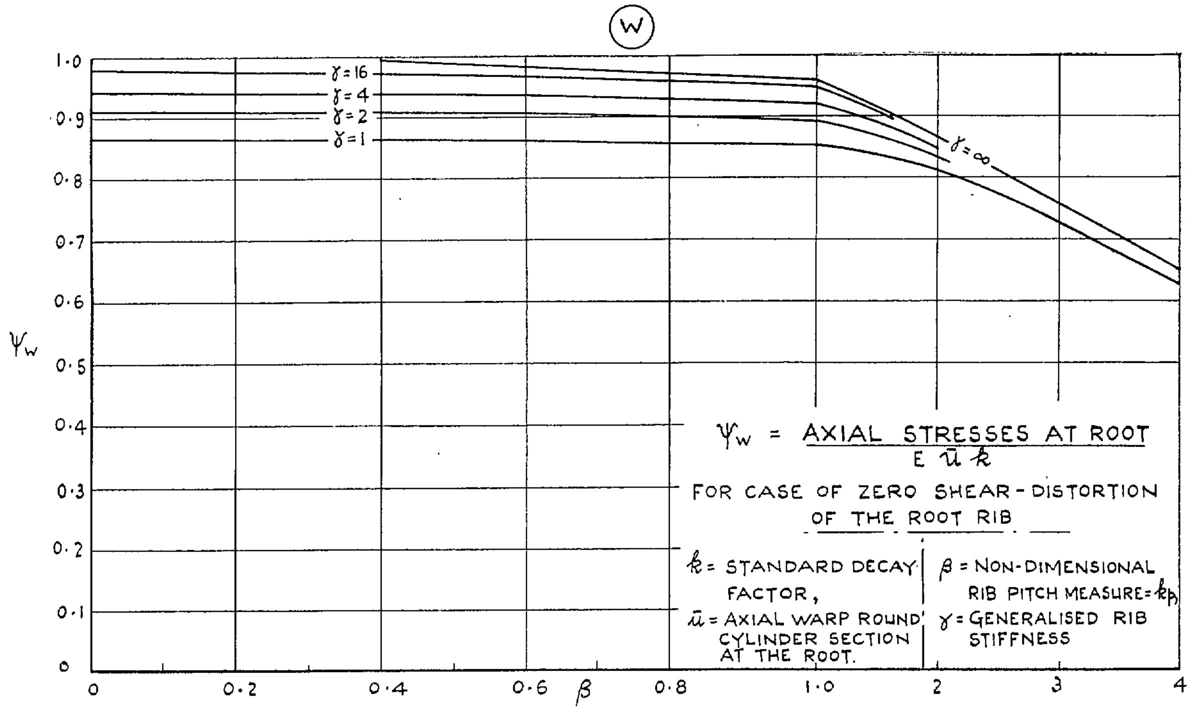


FIG. 1. Axial stresses at root due to axial warp at the root.

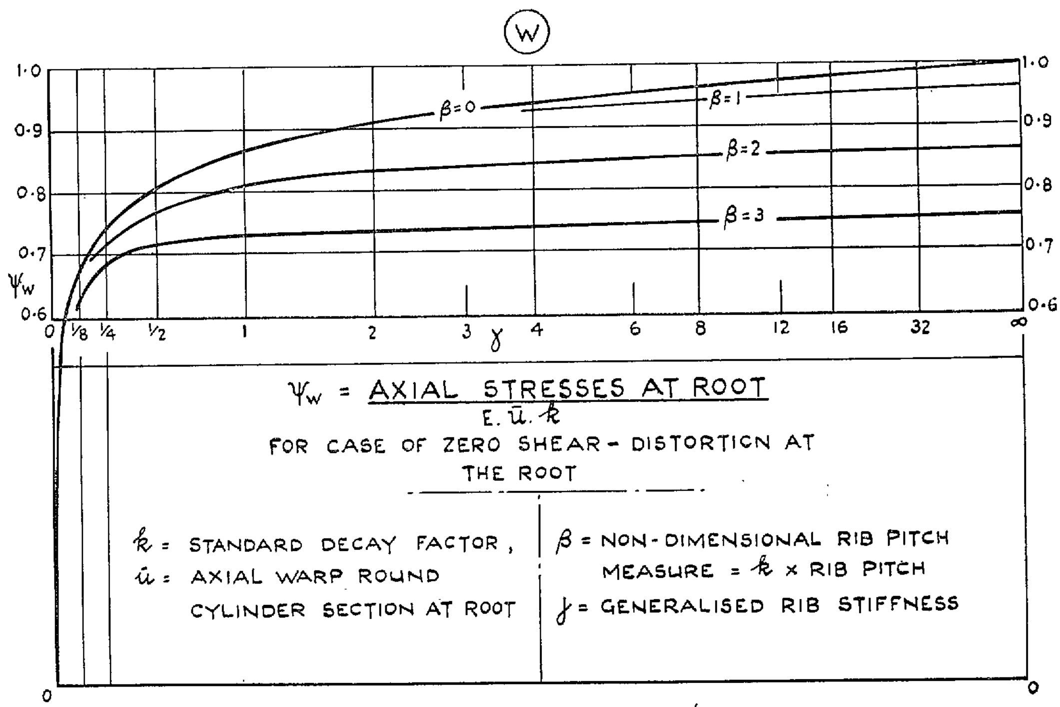


FIG. 2. Stresses at root due to axial warping at root.

C

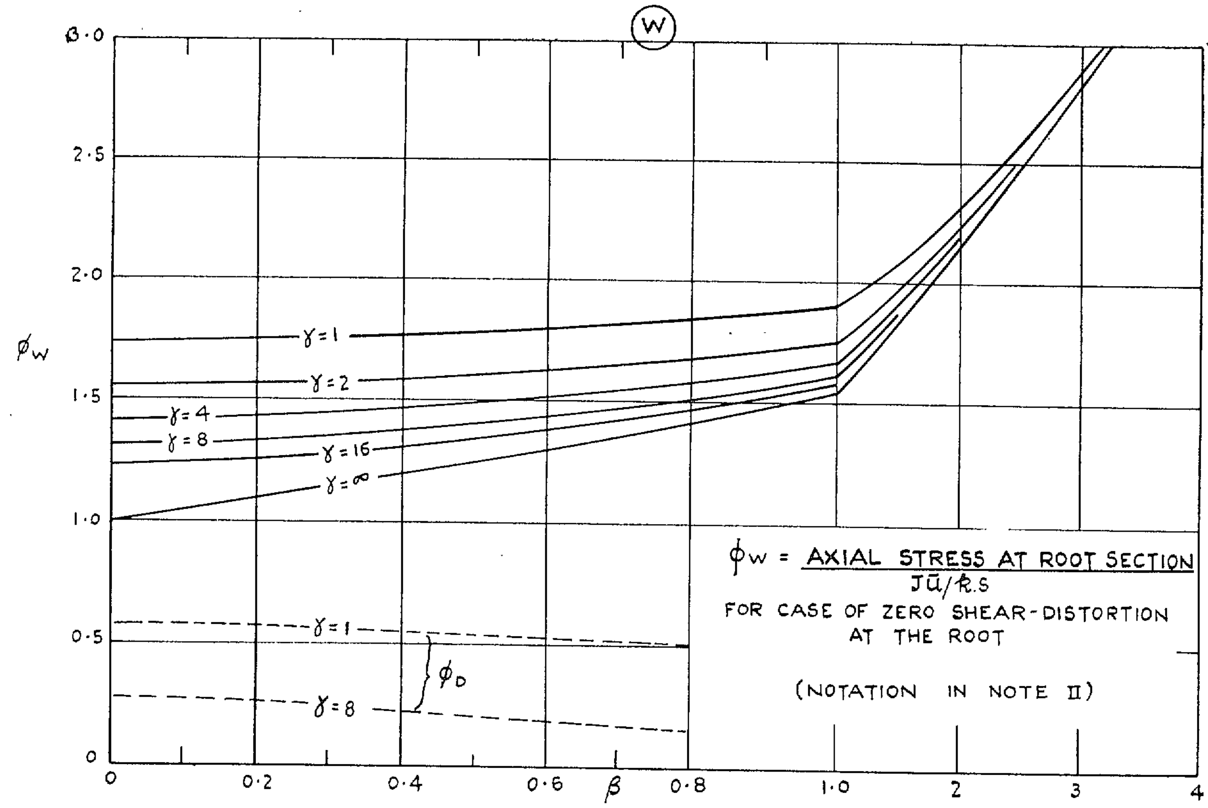


FIG. 3. Ratio of root direct stress to root shear stress due to axial warping at the root.

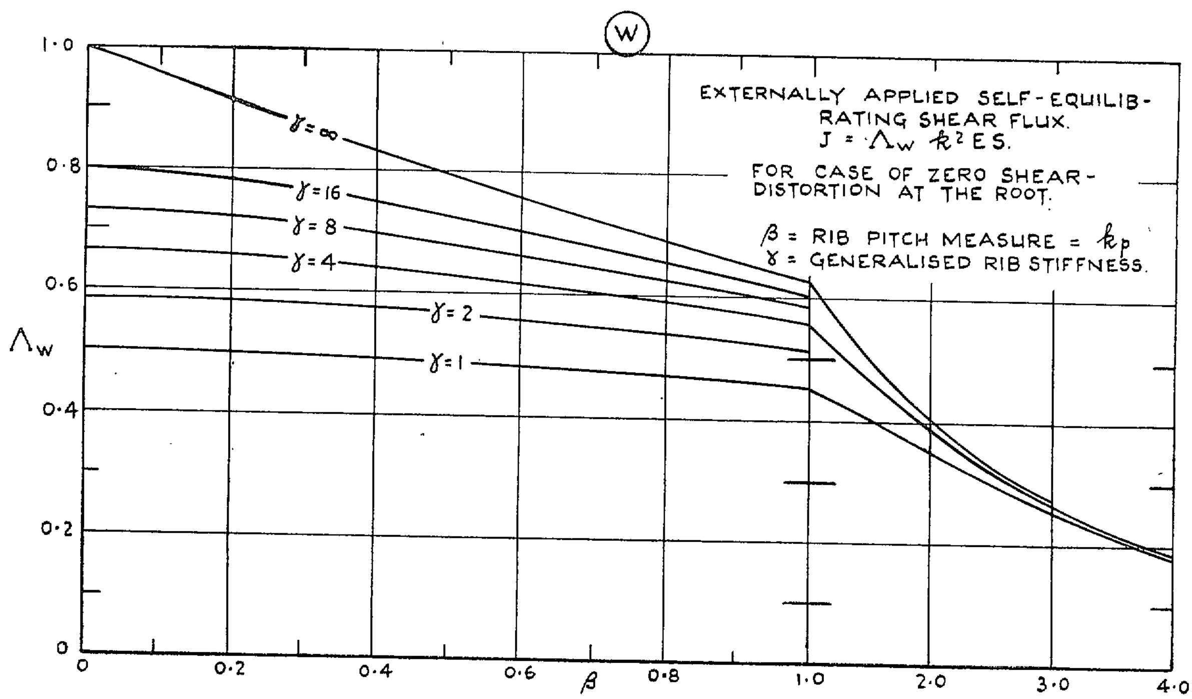


FIG. 4. Root shears corresponding to axial warping at the root.

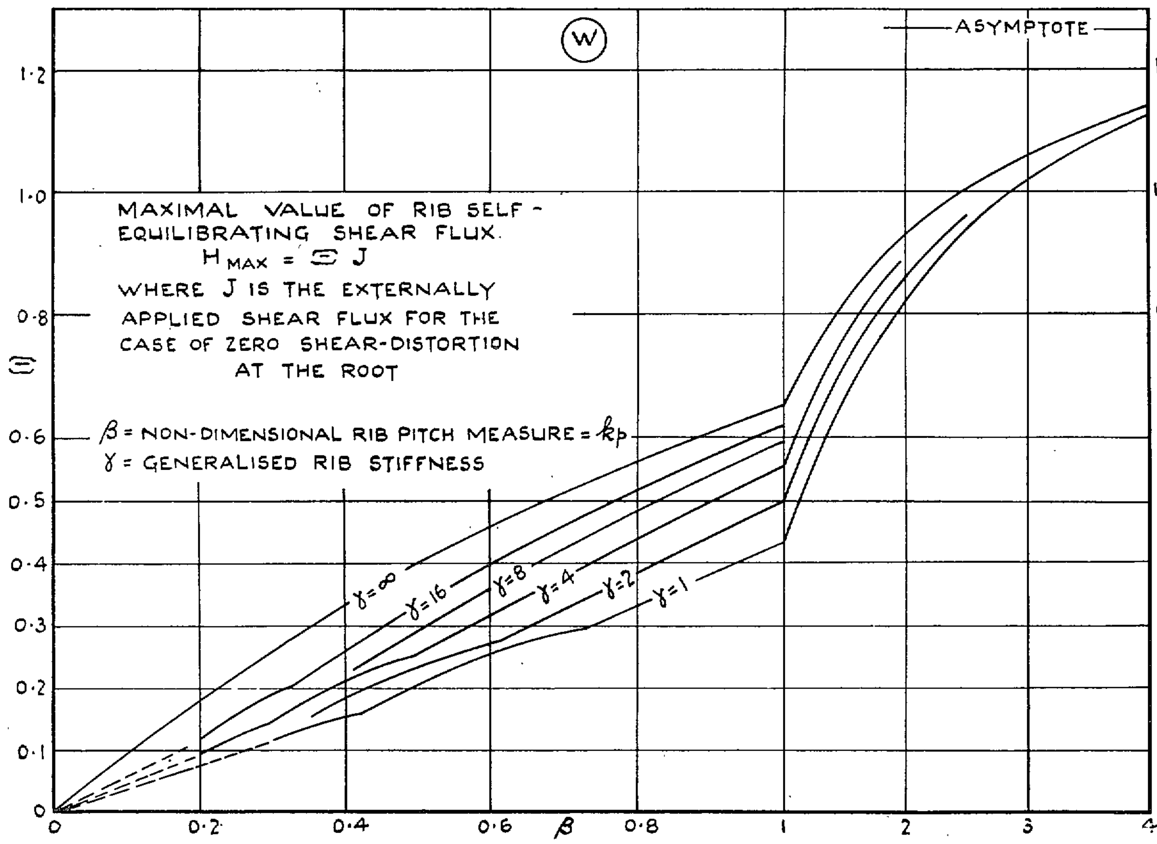


FIG. 5. Peak rib shears due to axial warping at root.

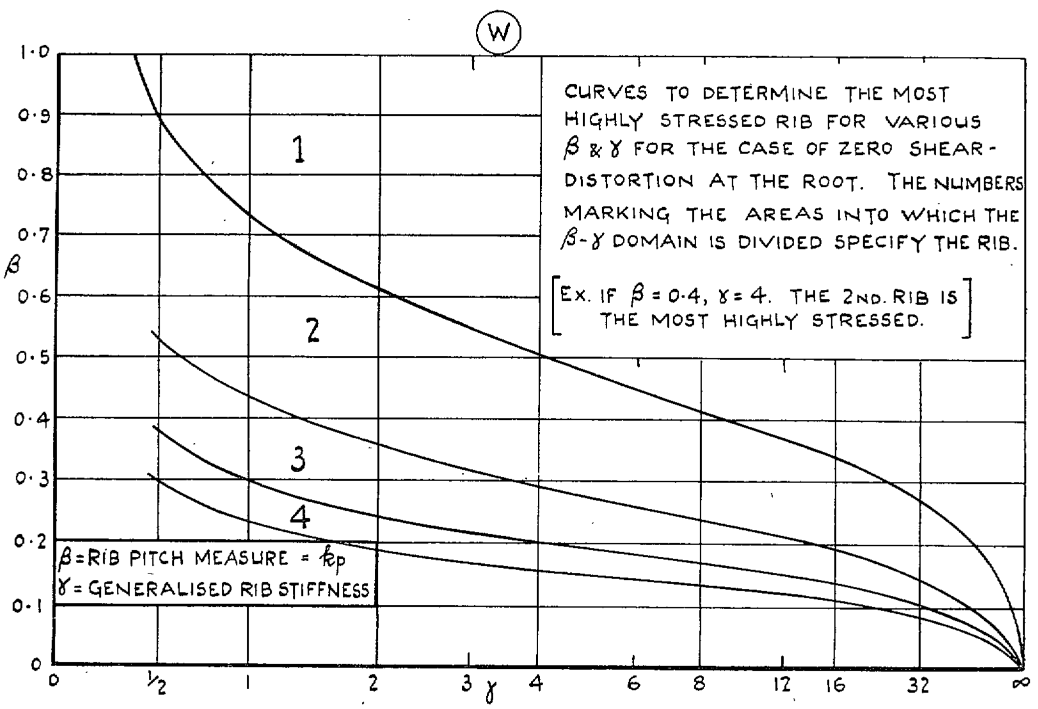


FIG. 6. Most highly stressed rib due to axial warping at root.

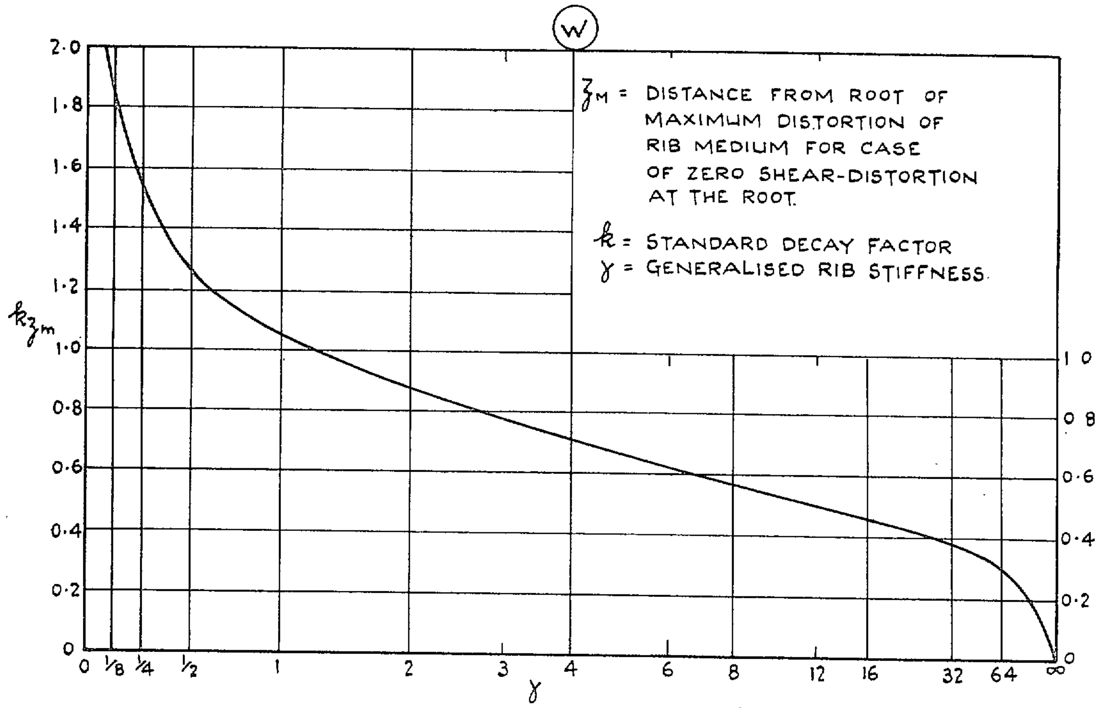


FIG. 7. Position of maximum shear-distortion of rib medium due to axial warping at the root.

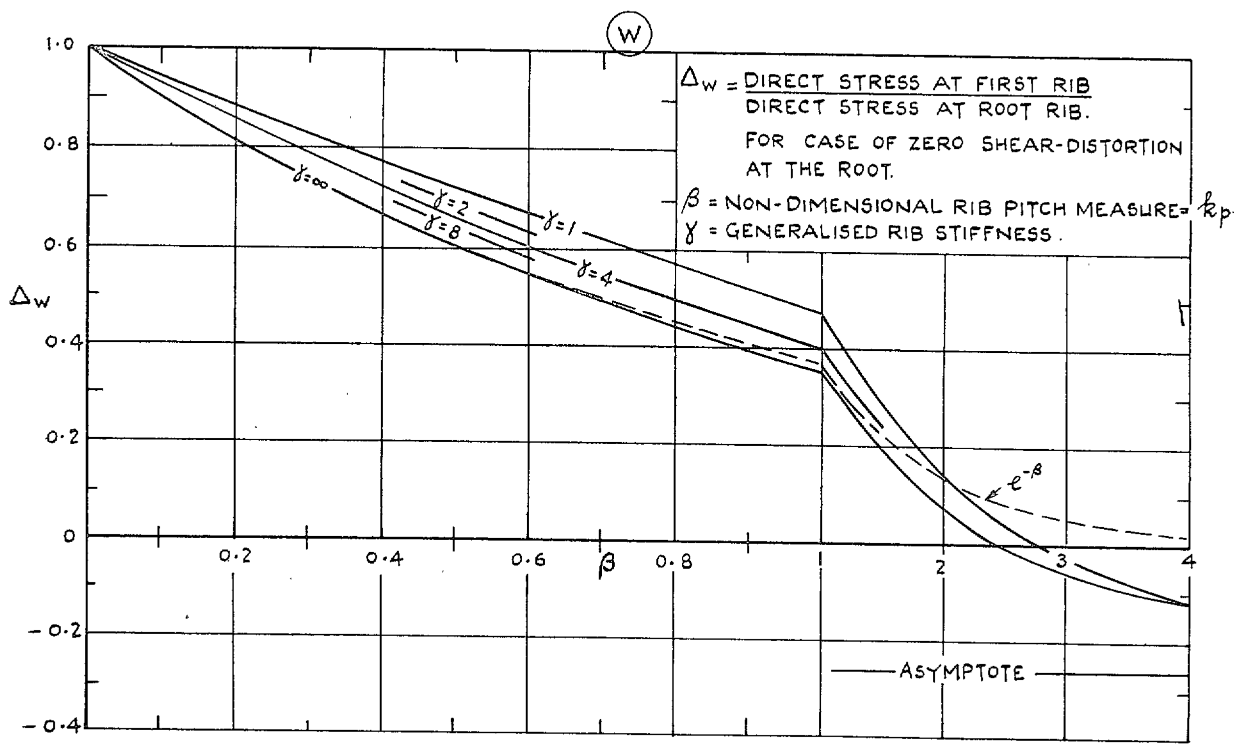


FIG. 8. Curves for determining the axial stresses in the cylinder at the 1st rib section due to axial warping at the root.

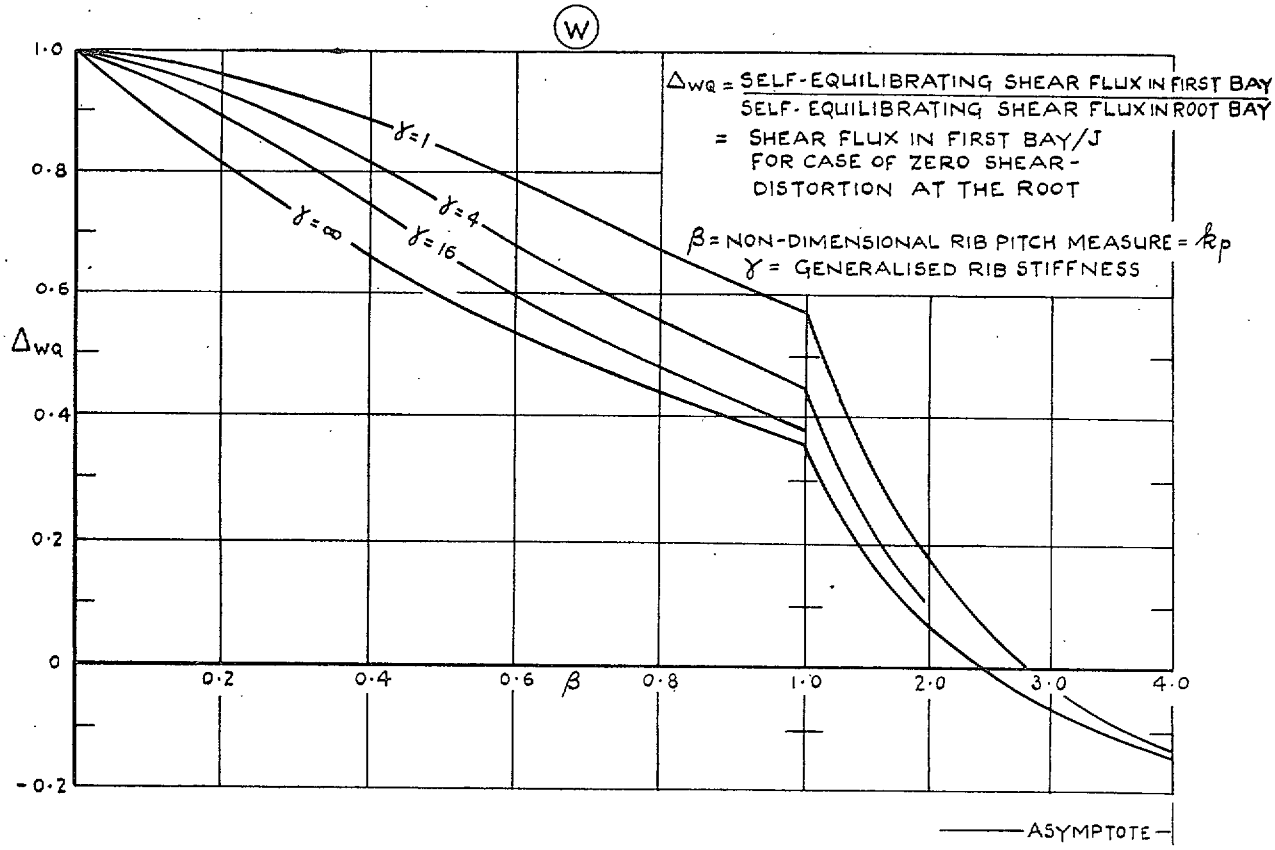


FIG. 9. Δ_{WQ}–β–γ family of curves.

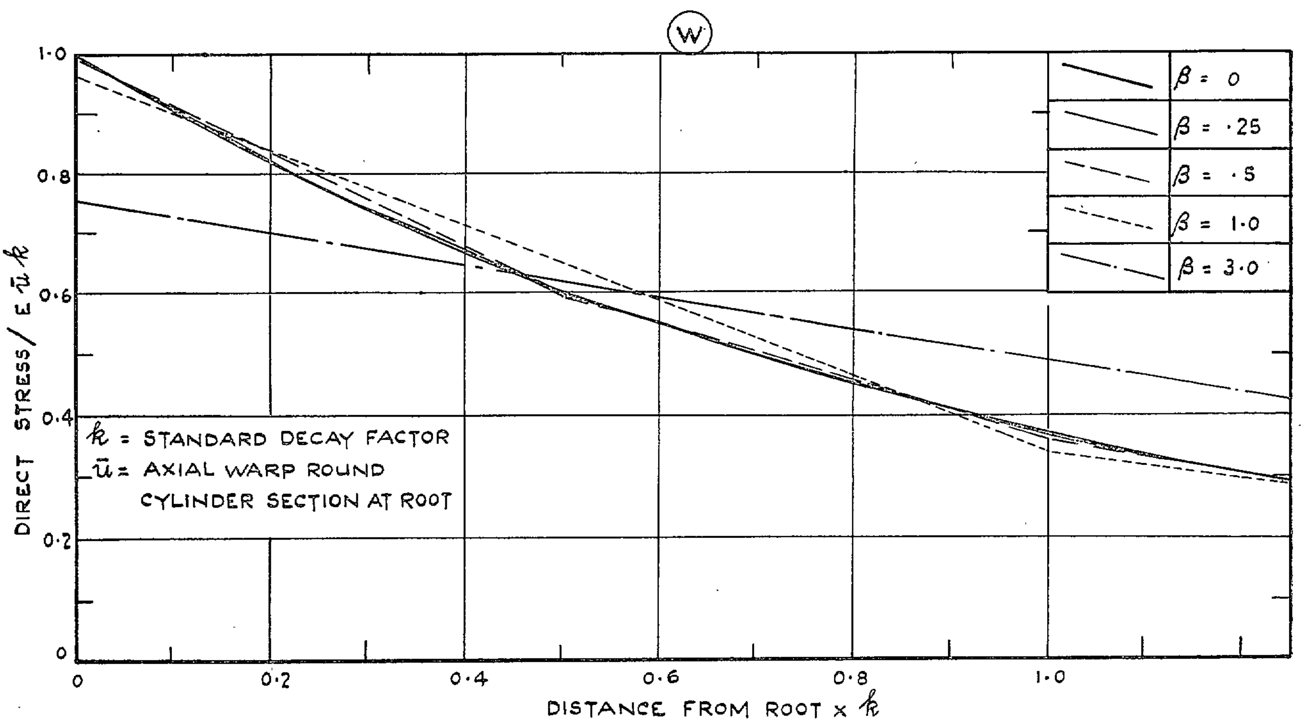


FIG. 10. Effect of varying rib-pitch. Each rib infinitely stiff. (Axial warp but no shear-distortion at root.)

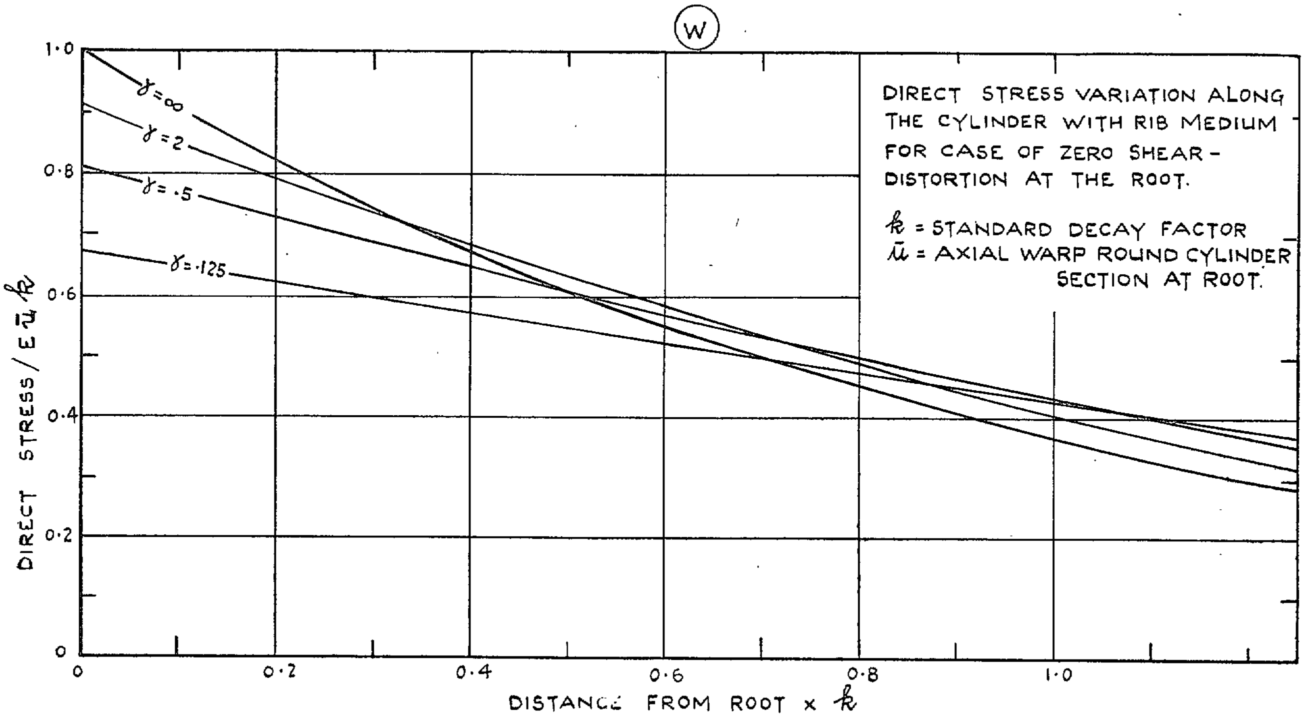


FIG. 11. Effect of varying rib-medium stiffness. (Axial warp at the root).

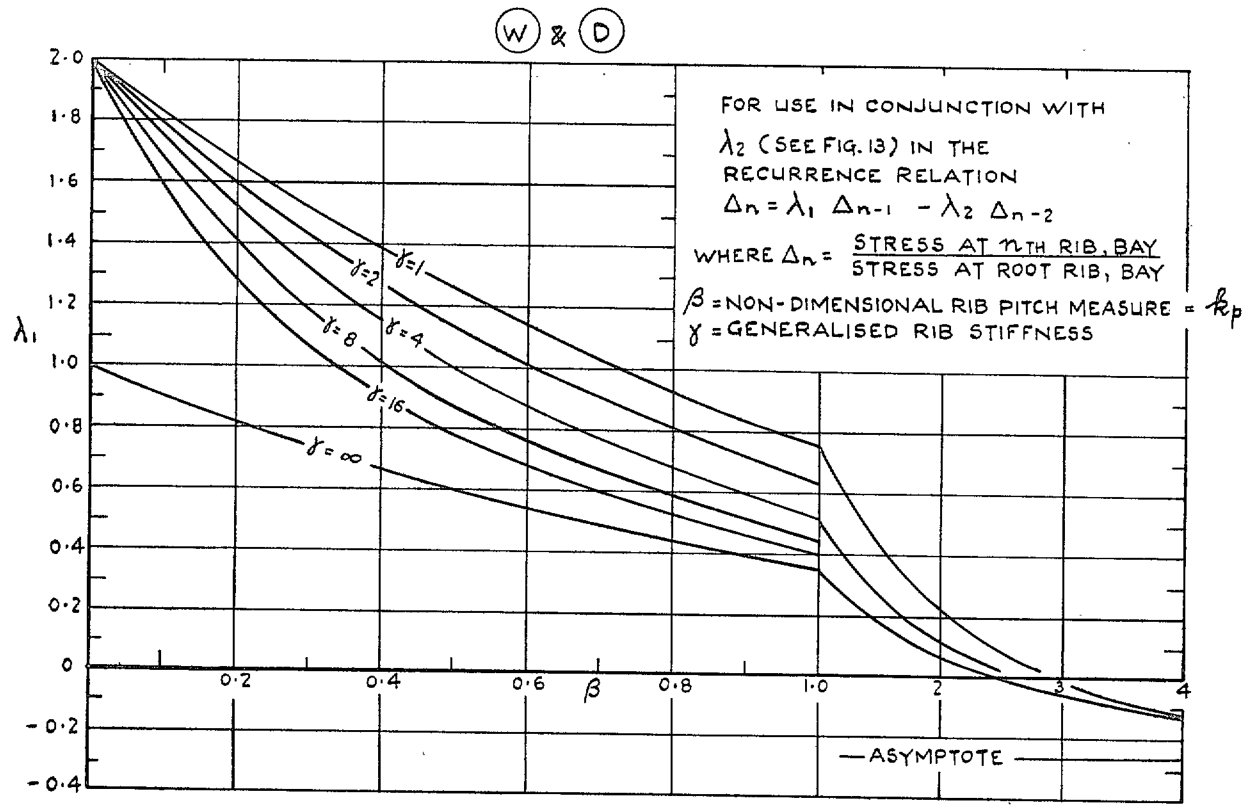


FIG. 12. λ_1–β–γ family of curves.

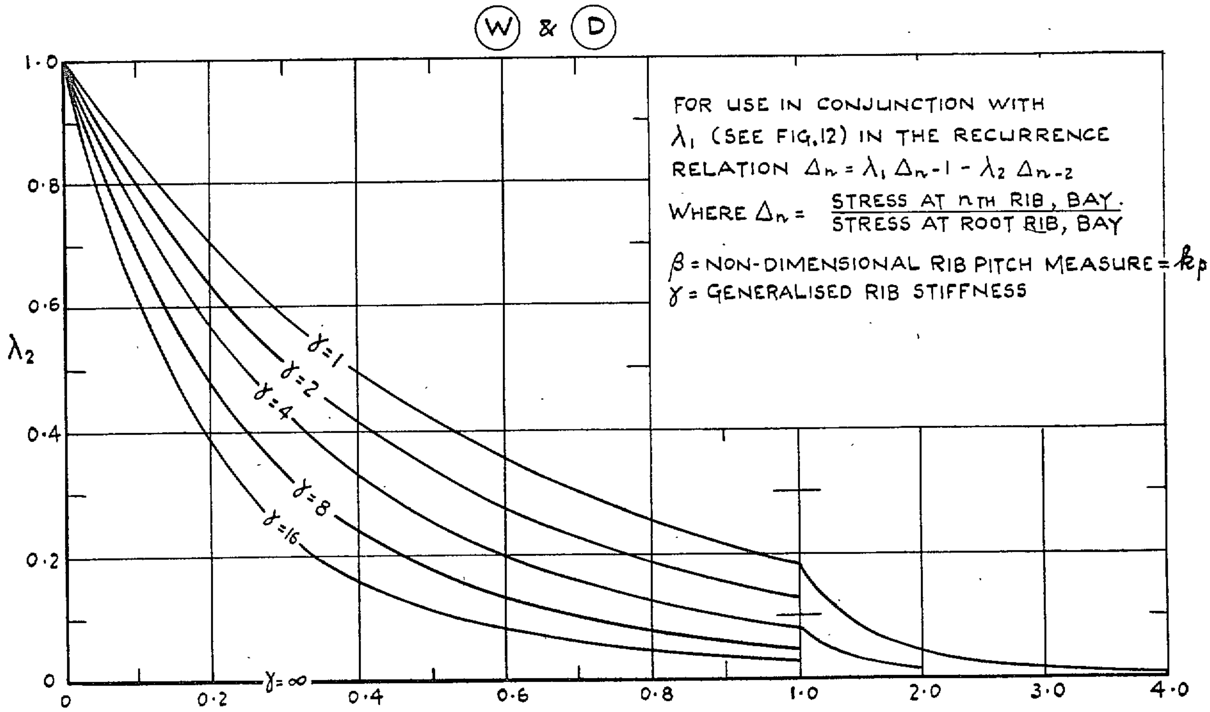


FIG. 13. λ_2–β–γ family of curves.

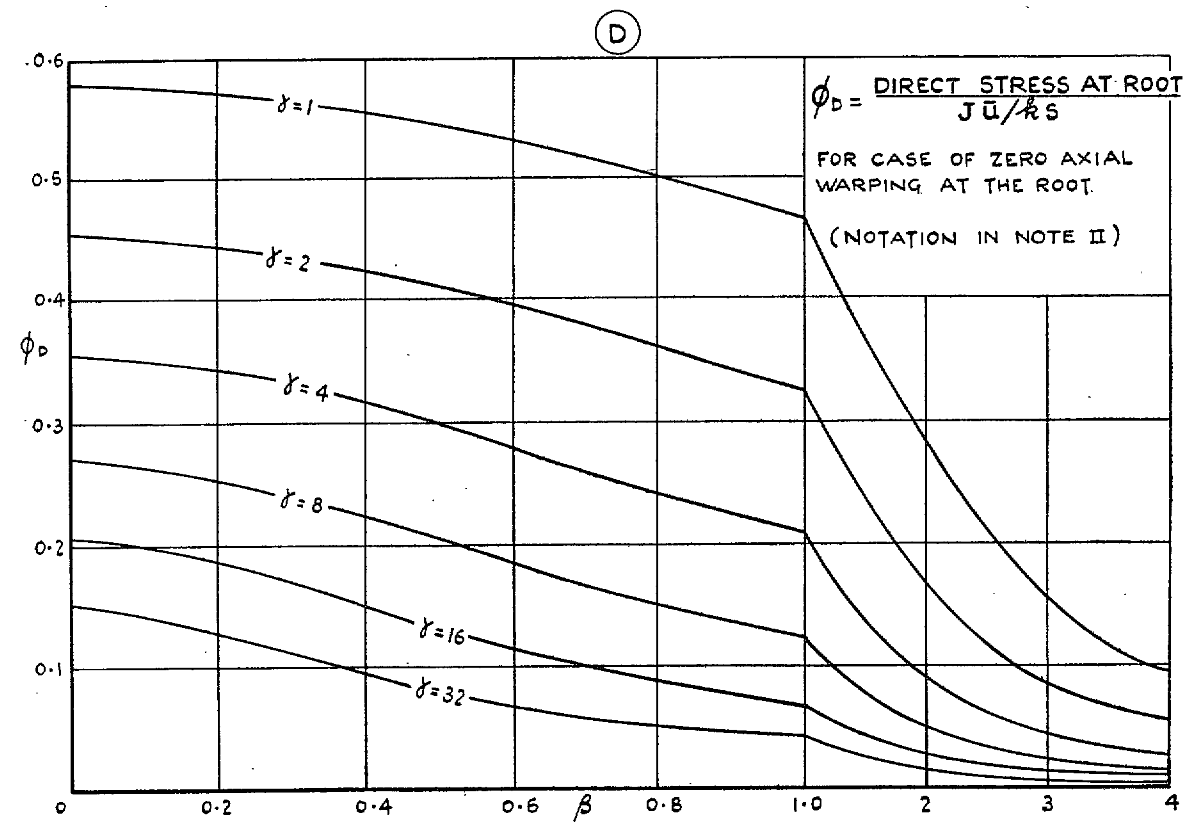


FIG. 14. Direct stress at root due to shear-distortion at root.

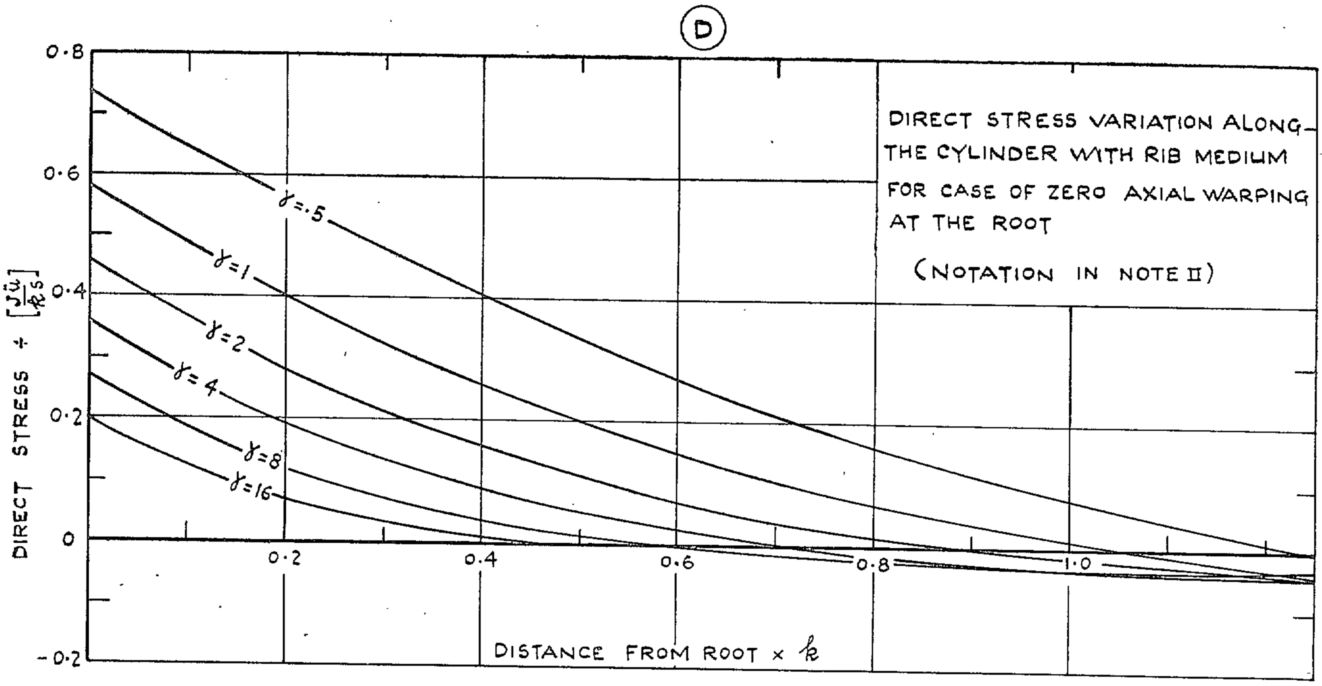


FIG. 15. Effect of varying rib-medium stiffness (due to shear-distortion at the root).

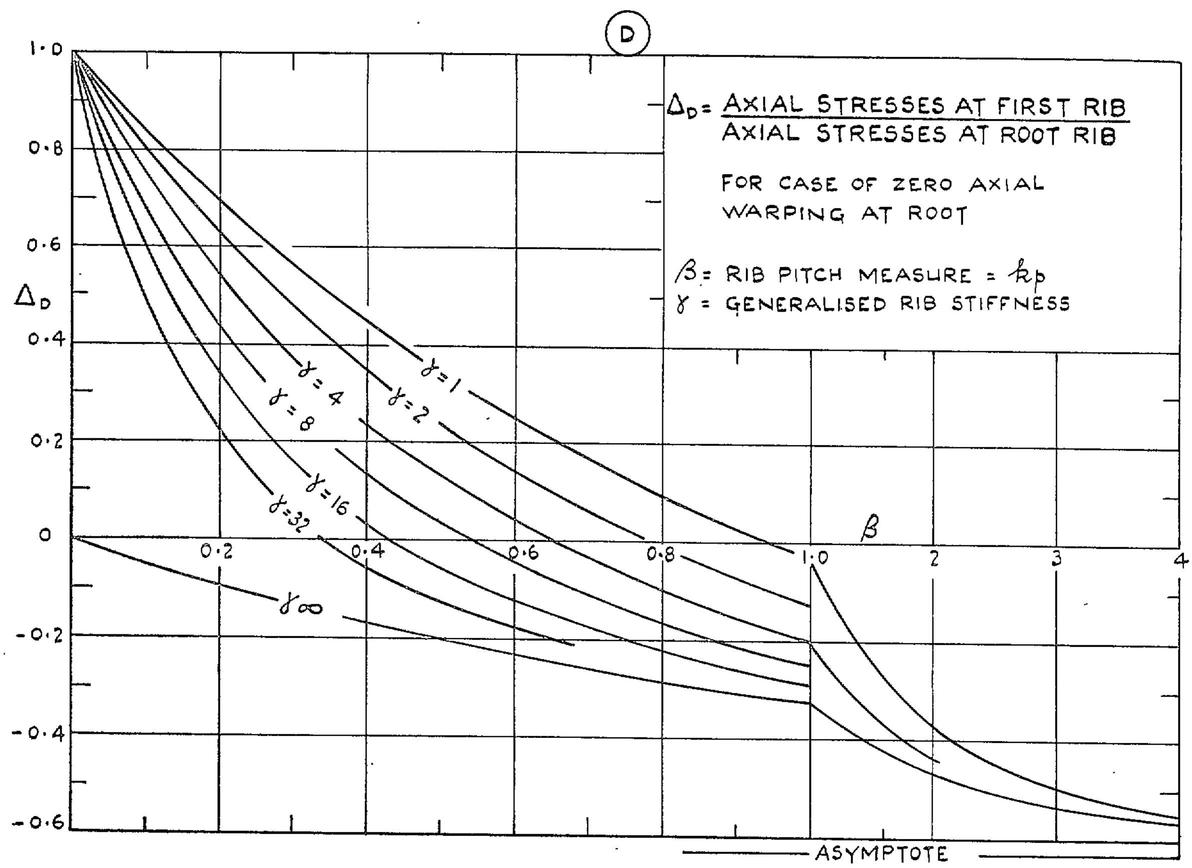


FIG. 16. Δ_D–β–γ family of curves.

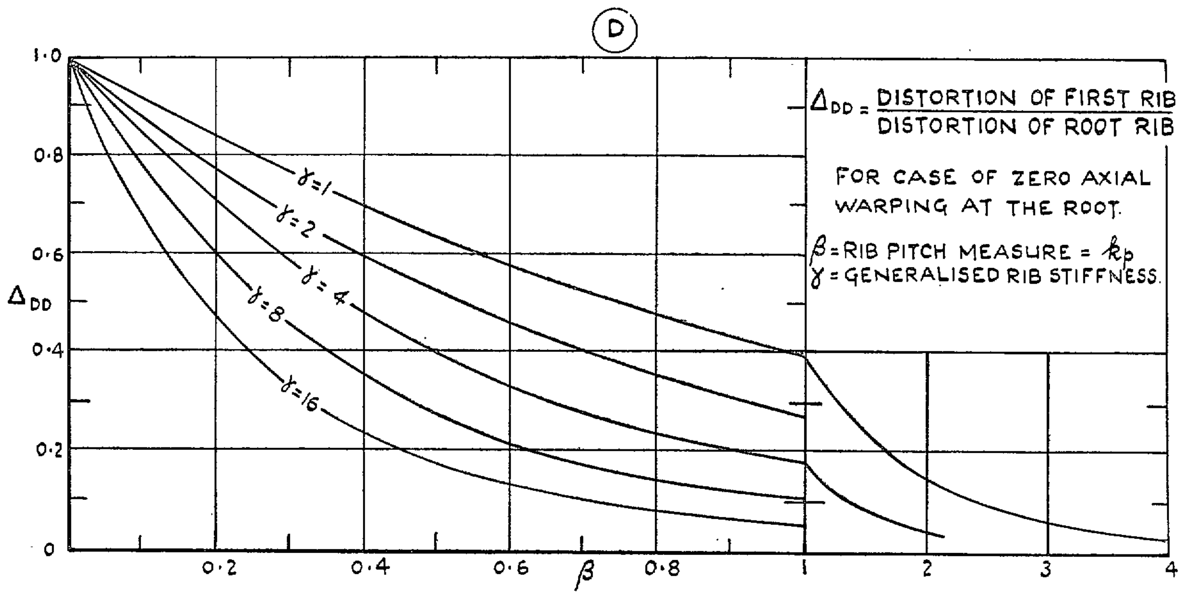


FIG. 17. Δ_{DD}–β–γ family of curves.

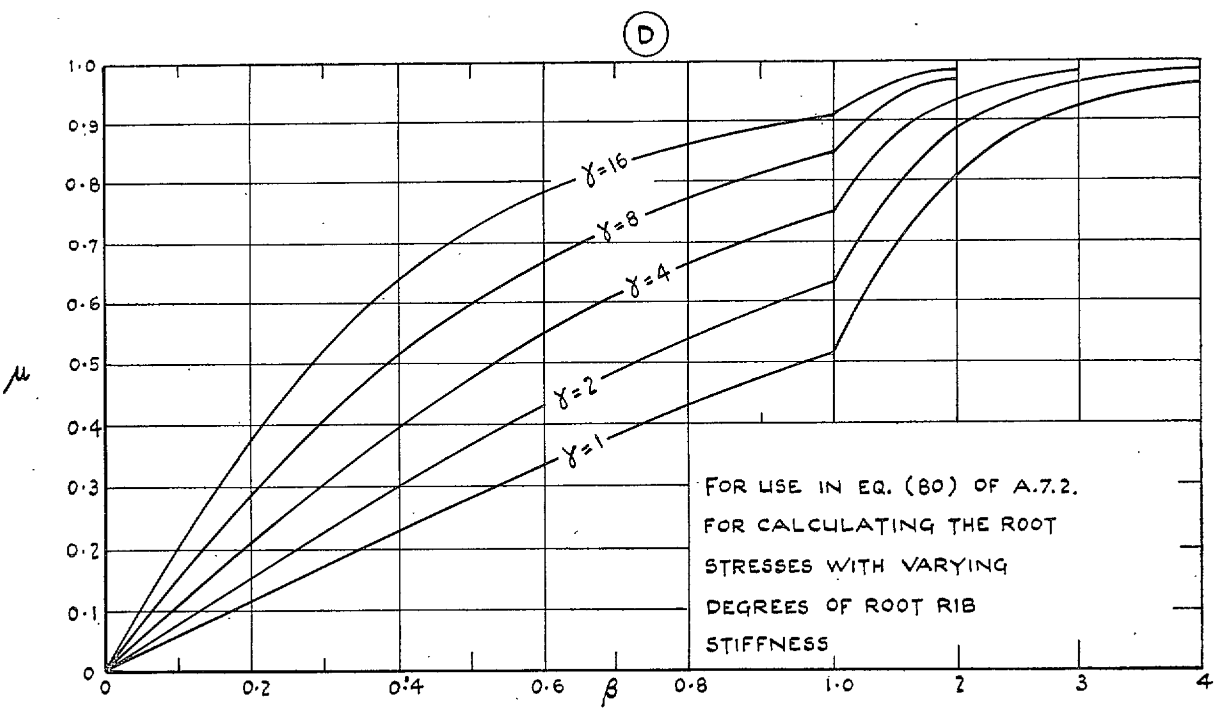


FIG. 18. μ–β–γ family of curves.

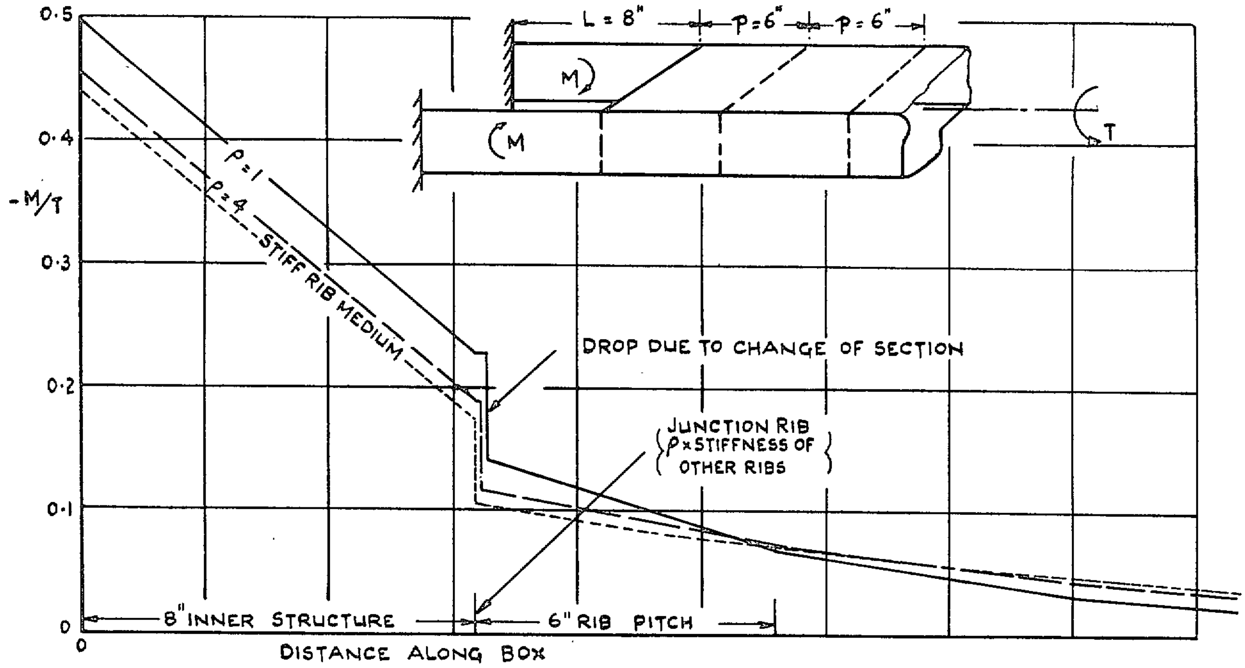


FIG. 19. Influence of junction rib stiffness on spar bending moments. (Example with 8-in. cut-out at root end.)

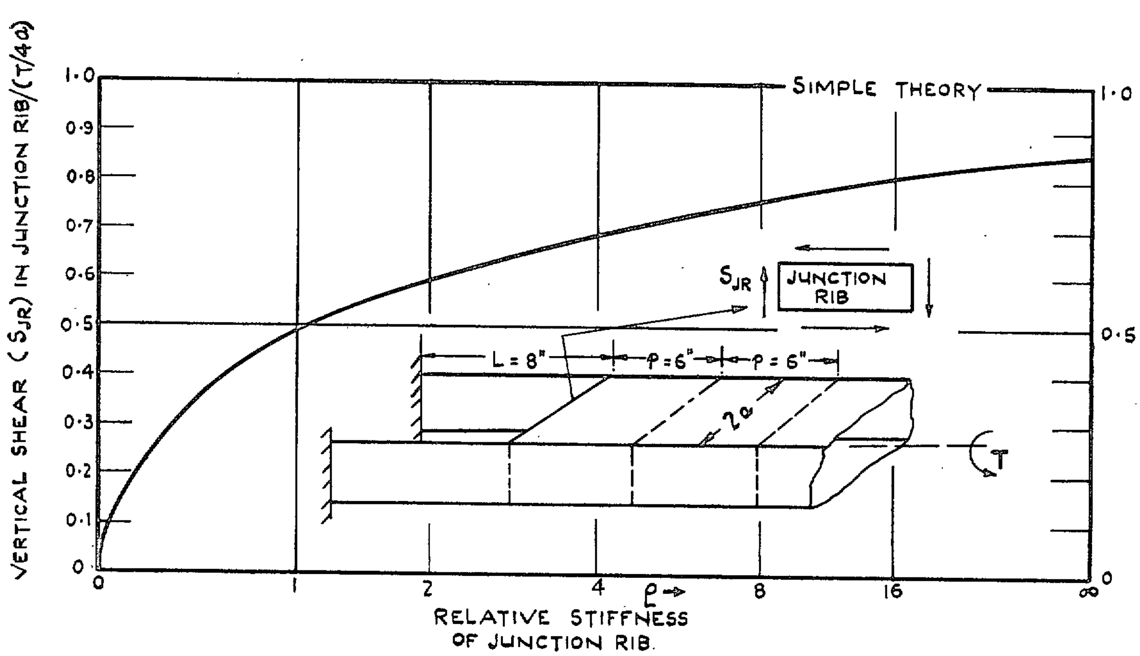


FIG. 20. Variation of junction rib load with stiffness. ($L = 8$ in.)

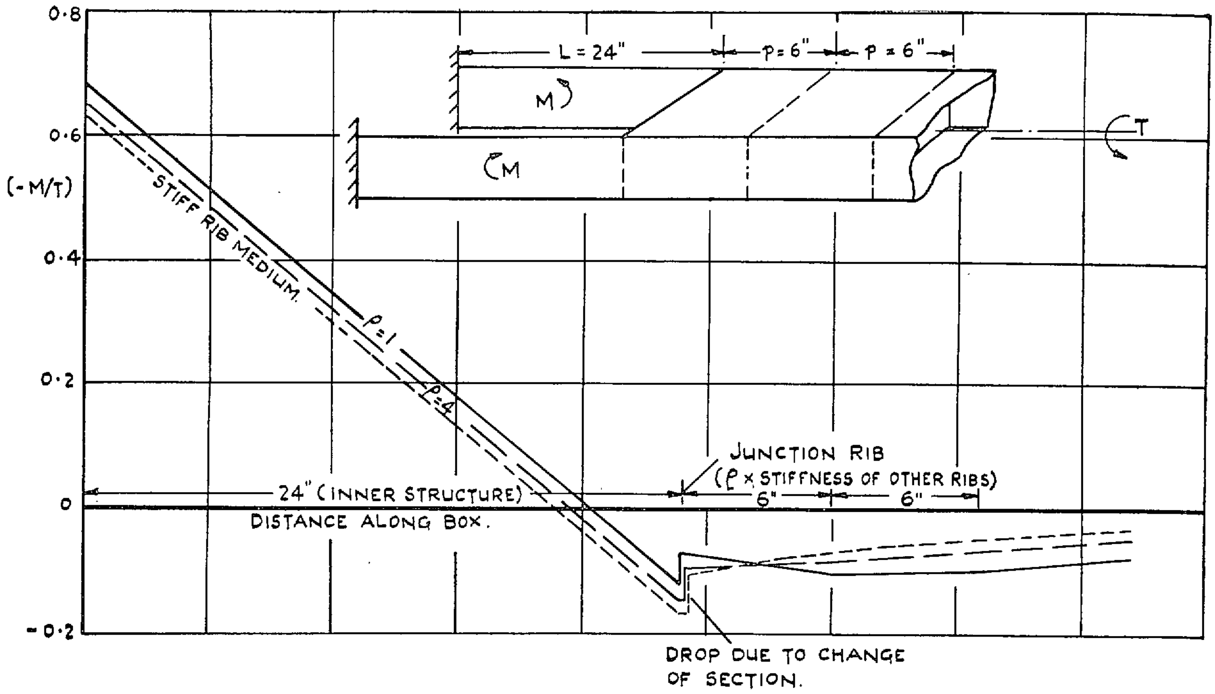


FIG. 21. Influence of junction rib stiffness on spar bending moments. (Example with 24-in. cut-out at root end.)

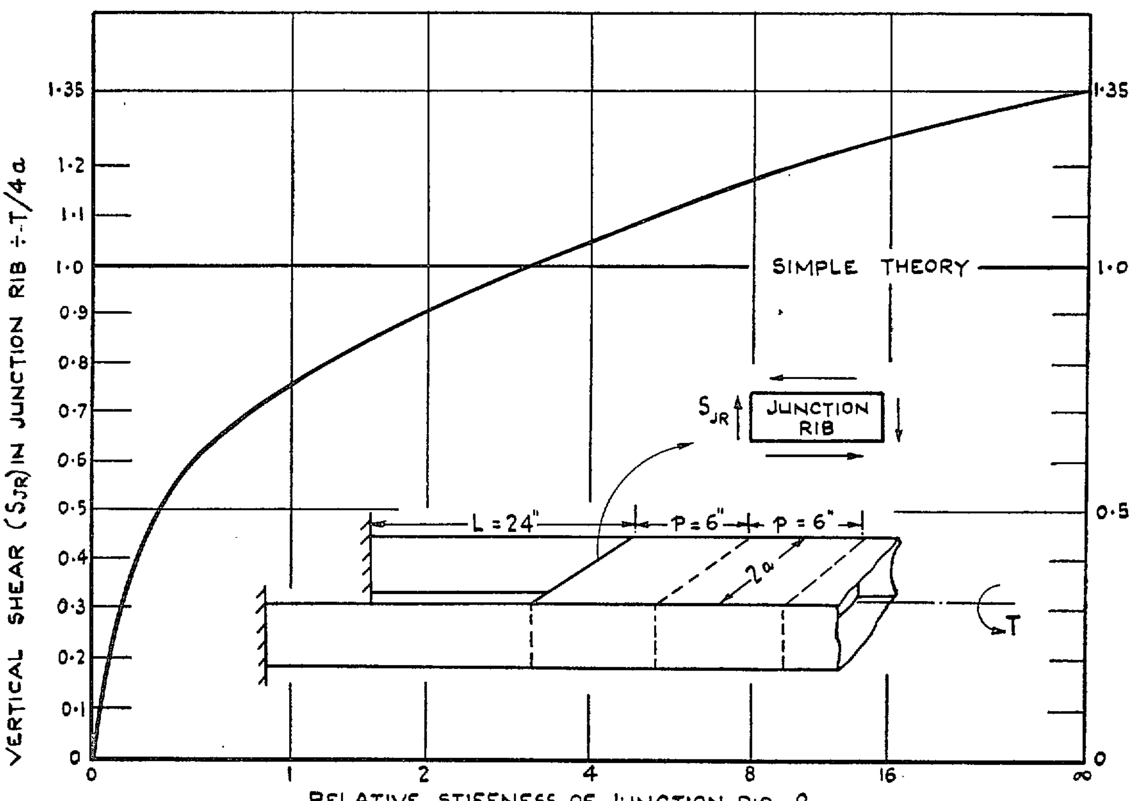


FIG. 22. Variation of junction rib load with stiffness. ($L = 24$ in.)

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