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Summary.—Exact solutions are given for the inviscid flow past two cylindrical profiles in the centre of a stream of limited depth. The first of these relates to a nearly circular cylinder and the second to a thin section giving a constant pressure drop over the greater part of its surface. The stream has either parallel walls, constant pressure walls, or the boundaries may be partly parallel and partly of the constant pressure type. For the thin profiles the changes of thickness ratio required to give the same pressure distribution as in an unlimited flow are found.

Introduction.—The present paper describes a contribution to the theory of two-dimensional wind-tunnel interference on bodies set at zero incidence in the centre of the flow. The work is limited to incompressible flow but it is hoped to extend section 2, which describes the flow past slender profiles, to subsonic compressible flow in a later paper. The irrotational flow has been found between parallel walls and also in a free jet past two cylindrical sections. The first of these is a circular cylinder and the second is a slender profile with a uniform pressure drop over the greater part of the boundary. No account is taken of the wake which exists behind a body in a real fluid so that the tunnel interference represented in the present paper is that usually referred to as solid blockage and the effects of wake blockage are not considered.

The free-streamline method is employed for the calculation of the flow of a jet past both bodies so that the outward displacement of the streamline is correctly represented and the approximations associated with the application of the method of images to these problems are avoided. The results are not limited therefore to examples in which the profile dimensions are small compared with the width of the jet. The free-streamline method has an additional advantage as the solutions can readily be adapted to deal with problems in which the outer boundaries of the flow consist partly of parallel walls and partly of free streamlines.

The main application of the present work is to the theory of the use of adjustable or shaped walls in two-dimensional wind tunnels. The adjustable walls may be set under experimental conditions to represent a straight-walled channel or may be adjusted so that the pressure along the walls is constant. Interest centres mainly in the correct positioning of the walls between these limits to remove the effects of tunnel constraint. This subject has been considered in detail, for compressible flow, by Lock and Beavan (R. & M. 2005[1]) (1944) who employed a doublet to represent the aerofoil, for the calculation of solid blockage, and then used the method of images to calculate the interference effects at the walls. The present work, though limited to incompressible flow, gives additional information for bodies whose dimensions are large compared with the depth of the stream and which cannot therefore be dealt with satisfactorily by a linearized method.

In section 1 the flow past a nearly circular cylinder in a stream of limited width is considered. The flow between parallel walls has been given by Lamb[2] (1932) and a direct transformation is now developed which converts this flow into that of a free jet past the cylinder. In each case the distortion of the cylindrical boundary from the circular shape is negligible for diameters less than half the initial width of the stream. The maximum velocities over the cylindrical boundaries are compared for the two cases, and in addition the displacement of the wall of the free jet is compared with that of the corresponding streamline of an unconstrained flow.

Section 2 deals with the flow past a profile which has a constant pressure drop over the greater part of its surface. The unconstrained flow past a series of sections of this type has been given in an earlier paper, Whitehead (R. & M. 2161[3]) (1942), and the simplest of the shapes described there is selected for the present development. The flow between parallel walls is obtained by minor modifications of the method for the unlimited flow, but the flow of the free jet past the section is a somewhat more difficult problem. In this section the thickness ratios of profiles giving identical pressure reductions are compared for the three types of flow. The relative displacements of the wall of the free jet and of the corresponding streamlines of the unconstrained flow are also compared for two examples.

Notation

a, e, d	Constants
b	Distance of the stagnation points from the origin in the t-plane
c	Distance of source and sink from the origin in the t-plane for flow with partly parallel walls
C	Constant
f, g	Semi-diameters of nearly circular cylinders
h	Width of channel, initial width of jet
k	Modulus of the elliptic integrals
k'	Complementary modulus $\sqrt{(1-k^2)}$
K, E	Elliptic integrals of the first and second kinds:
M, M_1	Constants
p	Constant defining the distances of the source and sink from the origin in the t-plane
q	$e^{-\pi K'/K}$
q_0	Resultant velocity on the curved wall of the thin sections
q_1	Maximum velocity on the channel wall
t	Complex variable
U	Initial velocity of flow in channel or jet
u	Argument of Jacobi's elliptic functions
$\bar{u}$, $\bar{v}$	Velocity components divided by U
$W = \phi + i\psi$	Complex potential function
$2x_0$, $2(y_0 + y_1)$	Chord and thickness of profiles of section 2
z	Complex variable in plane of channel and jet flow
α	Particular value of u defined by the relation $\mathrm{sn}\,\alpha = 1/pk$
β, γ	Constants
μ_0	Strength of doublet in the t-plane
μ, ν	Constants
$\Theta(u)$	Jacobi's theta functions
$\zeta = \xi + i\eta$	Complex variable defined by the relation $\zeta = \log \dfrac{1}{U}\dfrac{dW}{dz}$

1. *The Flow Past a Circular Cylinder.*—1.1. *Flow in a Channel with Parallel Walls.*—The flow between parallel walls past a nearly circular cylinder has been found by Lamb*, who combined a uniform flow with a row of doublets spaced equally along a line perpendicular to the flow. The same flow is derived in the present paper by an alternative method using a logarithmic transformation which facilitates comparison with the corresponding flow of a free jet past a doublet. The flow in the upper half of the straight-walled channel of width h, in a z_1-plane, is transformed into the upper half of a t-plane, as shown in Fig. 1. In the latter plane the doublet is also at the origin and the points corresponding to the ends of the channel are located on the real axis unit distance on either side of the doublet. The flow in the t-plane, corresponding to a flow from left to right in the z_1-plane, is therefore from a source at the point $(-1, 0)$ past a doublet at the origin to a sink, of strength equal to that of the source, at the point $(1, 0)$. Thus the potential function in the t-plane corresponding to a flow with velocity U in the z_1-plane is given by

$$W = \phi + i\psi = \frac{Uh}{2\pi} \log \frac{1+t}{1-t} + \frac{\mu_0}{2\pi t}, \qquad \text{(1)}$$

and the streamline $\psi = 0$ forms an oval boundary enclosing the doublet. The strength of the doublet μ_0 is conveniently expressed in terms of b, the distance of the stagnation points from the origin in the t-plane, so that $\dfrac{\mu_0}{2U} = \dfrac{hb^2}{1 - b^2}$.

The transformation to the z_1-plane is carried out using the relation

$$\frac{2\pi z_1}{h} = \log \frac{1+t}{1-t}, \qquad \text{(2)}$$

which may be expressed alternatively in the form $t = \tanh \dfrac{\pi z_1}{h}$. Hence the potential function in the z_1-plane becomes

$$W = U\left(z_1 + \frac{h}{\pi} \frac{b^2}{1-b^2} \coth \frac{\pi z_1}{h}\right), \qquad \text{(3)}$$

and this gives the flow past a boundary of nearly circular cross-section surrounding the doublet. The horizontal diameter $2f_1$ is represented by the length $2b$ in the t-plane and is therefore given by the relation

$$\frac{\pi f_1}{h} = \tanh^{-1} b .$$

This may expand in a series which converges rapidly for small value of b, thus

$$\frac{\pi f_1}{h} = b + \frac{b^3}{3} + \frac{b^5}{5} + \frac{b^7}{7} + \ldots\ldots \qquad \text{(4)}$$

The vertical diameter $2g_1$ is found from equation (3) by substituting $z_1 = ig_1$ and equating ψ to zero. This gives the relation

$$\frac{\pi g_1}{h} \tan \frac{\pi g_1}{h} = \frac{b^2}{1-b^2},$$

which may be expressed in the form of a series similar to (4)

$$\frac{\pi g_1}{h} = b + \frac{b^3}{3} + \frac{7}{45} b^5 + \frac{69}{945} b^7 + \ldots\ldots \qquad \text{(5)}$$

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* And also by N. A. V. Piercy (*Aerodynamics.* 1947), using a conformal transformation different from that given below.

The boundary differs only slightly from the circular form, even if the diameter is as large as half the width of the channel.

The complex velocity ratio $\bar{u} - i\bar{v}$, given generally by $\frac{1}{U}\frac{dW}{dz_1}$ is determined directly from (3), and takes the form

$$\frac{1}{U}\frac{dW}{dz_1} = \frac{1 - b^2 \coth^2 \frac{\pi z_1}{h}}{1 - b^2} \quad \ldots \quad (6)$$

On the upper wall where $z_1 = x_1 + ih/2$ the expression reduces to

$$\bar{u} = \frac{1 - b^2 \tanh^2 \frac{\pi x_1}{h}}{1 - b^2}, \quad \ldots \quad (7)$$

and its maximum value directly above the doublet is $1/(1 - b^2)$. On the cylindrical boundary the maximum value of the velocity occurs when $z_1 = ig_1$ and, after substituting from equation (5), this can be expressed in the form

$$\bar{u} = 2 + \tfrac{2}{3}b^2 + \tfrac{34}{45}b^4 + \ldots . \quad \ldots \quad (8)$$

Since the first term represents the velocity on the cylinder in an unlimited flow the further terms show the increases arising from the presence of the straight boundaries.

1.2. *Flow of a Two-dimensional Jet past a Circular Cylinder.*—The flow of a jet with constant pressure along its boundaries past a doublet is found by transforming the same flow in the t-plane shown in Fig. 1a, into a ζ-plane which is defined by,

$$\zeta = \xi + i\eta = \log \frac{1}{U}\frac{dW}{dz} . \quad \ldots \quad (9)$$

The appropriate ζ-plane boundary is shown in Fig. 2a and consists entirely of straight lines. The corresponding flow past the doublet in the z-plane is indicated in Fig. 2b. The straight lines AB and AH in the ζ-plane parallel to the ξ axis correspond to the parts of the axis of symmetry in the z-plane between the stagnation points on the oval boundary. They are separated by distance π from the lines BC and HG which represent the remainder of the axis of symmetry because the flow along them in the z-plane is in the opposite direction. The short barrier CDEFG lying along the η axis corresponds to the free streamline wall of the jet. The ζ-plane boundary for the flow between parallel walls, already described, differs only in that the section CDEFG is replaced by a short extension of BC and HG along the ξ axis to the right of the origin up to a point E_1 whose co-ordinates are $\left(\log \frac{1}{1 - b^2}, 0\right)$.

The transformation from the t to the ζ-planes is found by the method of Schwarz and Christoffel which gives, for this example, the relationship,

$$\frac{d\zeta}{dt} = m \frac{t^2 - d^2}{t(t^2 - b^2)\sqrt{(t^2 - 1)}} .$$

The points on the free-streamline wall which have the maximum vertical component of velocity are defined by the new constant d whose value is greater than unity. The expression is integrated by use of the substitution $s = \sqrt{(t^2 - 1)}$ and the relation between ζ and s takes the form

$$\zeta = im \left[\frac{d^2}{2b^2} \log \frac{s - i}{s + i} + \frac{b^2 - d^2}{2b^2\sqrt{(1 - b^2)}} \log \frac{s - i\sqrt{(1 - b^2)}}{s + i\sqrt{(1 - b^2)}}\right] + C . \quad (10)$$

The constants m and d are found from the conditions that the walls AB and AH are distant 2π apart from that BC and HG coincide instead of merely being parallel to one another. The first condition gives

$$\frac{imd^2}{2b^2} = 1 \text{ and the second } \frac{-\,im\,(b^2 - d^2)}{2b^2\sqrt{(1 - b^2)}} = 1\,.$$

The constant C is zero since ζ is zero when s is zero. The transformation formula, expressed in terms of t, becomes therefore

$$\zeta = \log \frac{\sqrt{(t^2 - 1)} + i}{\sqrt{(t^2 - 1)} - i} \cdot \frac{\sqrt{(t^2 - 1)} - i\sqrt{(1 - b^2)}}{\sqrt{(t^2 - 1)} + i\sqrt{(1 - b^2)}}\,, \qquad \qquad (11)$$

and, after substituting from (9), dW/dz may be found in terms of t. dW/dt is determined from equation (1) and, if μ_0 is expressed in terms of b, this gives the relation

$$\frac{1}{U}\frac{dW}{dt} = -\frac{h(t^2 - b^2)}{\pi(1 - b^2)\,t^2(t^2 - 1)}\,. \qquad \qquad (12)$$

Combining equations (9), (11) and (12) to eliminate W, dz/dt is found in terms of t so relating the z-plane, in which the flow of a jet of width h past a doublet is required, to the t-plane in which the potential function is known. The transformation takes the form

$$\frac{dz}{dt} = -\frac{h\{\sqrt{(t^2 - 1)} + i\sqrt{(1 - b^2)}\}^2}{\pi(1 - b^2)(t^2 - 1)\{\sqrt{(t^2 - 1)} + i\}^2}\,, \qquad \qquad (13)$$

and is integrated after making the substitution $t = \tanh \frac{z_1\pi}{h}$. Comparison with equation (2) shows that this changes (13) into a direct transformation from the flow in the z_1-plane past the doublet between parallel walls, shown in Fig. 1b, into the flow of a jet past a doublet as indicated in Fig. 2b. dz/dz_1 is then given by

$$\frac{dz}{dz_1} = \left(\frac{1/\sqrt{(1 - b^2)} + \cosh \frac{z_1\pi}{h}}{1 + \cosh \frac{z_1\pi}{h}}\right)^2\,, \qquad \qquad (14)$$

and becomes after integration

$$z = z_1 + \frac{2h}{\pi}\left(\frac{1}{\sqrt{(1 - b^2)}} - 1\right)\tanh\frac{z_1\pi}{2h} + \frac{h}{2\pi}\left(\frac{1}{\sqrt{(1 - b^2)}} - 1\right)^2\left(\tanh\frac{z_1\pi}{2h} - \frac{1}{3}\tanh^3\frac{z_1\pi}{2h}\right). \quad (15)$$

The horizontal diameter $2f$ and the vertical diameter $2g$ of the resulting cylindrical boundary may be found directly from (15) by substituting for z_1 the values f_1 and ig_1, appropriate to the flow between parallel walls. The resulting expressions are

$$\frac{\pi f}{h} = b + \frac{5}{6}b^3 + \frac{61}{80}b^5 + \frac{481}{672}b^7 + \ldots\,, \qquad \qquad (16)$$

and

$$\frac{\pi g}{h} = b + \frac{5}{6}b^3 + \frac{577}{720}b^5 + \frac{7967}{10{,}080}b^7 + \ldots\,. \qquad \qquad (17)$$

In this case the difference between the diameters is even smaller than for the flow between parallel walls. This is due partly to a reduction in the difference between the coefficients of b^5 in (16) and (17) as compared with (4) and (5) and partly to the fact that a given value of the parameter b corresponds to a relatively larger cylinder for the flow in a free jet.

The velocities in the flow are found by dividing equation (6) by equation (14). The maximum value of $\bar{u}$ on the boundary at the point $z = ig$ takes the form

$$\bar{u} = 2 - \tfrac{1}{3} b^2 - \tfrac{73}{360} b^4 \ldots\ldots \qquad (18)$$

Comparison with equation (8) shows that the coefficient of b^2 is opposite in sign and has one half of its previous value. This is in agreement with the results obtained by Lock (R. & M. 1275[4]) (1929) and Glauert (R. & M. 1566[5]) (1933) for small bodies, using the method of images. The coefficient of b^4 in (18) is smaller than in (8) so that for larger bodies the effects of the two types of constraint will no longer bear a ratio of two to one. To illustrate the changes which occur with the size of the cylinder, a comparison of the velocity ratio is plotted against the ratio of the diameter of the cylinder to the initial breadth of the channel, in Fig. 3. Curve (*a*) shows results for the parallel walled channel and curve (*b*) the corresponding values for the free jet. When the cylinder diameter is $h/4$ it is seen that the ratio of the tunnel constraint effects increases to approximately 2·5.

The co-ordinates of the free-streamline boundary are determined directly from equation (15) by the substitution $z_1 = x_1 + ih/2$. The maximum displacement will clearly occur at $x_1 = 0$ and its value is given by

$$\frac{\delta\pi}{h} = 2\left(\frac{1}{\sqrt{(1 - b^2)}} - 1\right) + \frac{2}{3}\left(\frac{1}{\sqrt{(1 - b^2)}} - 1\right)^2 . \qquad (19)$$

It is plotted, in Fig. 4, against g/h and is compared with the displacement of the corresponding streamline of the unconstrained flow past a circular cylinder. For a streamline initially $h/2$ from the axis the displacement for an unlimited flow is given in terms of the radius a of the cylinder as

$$\frac{\delta\pi}{h} = \frac{\pi}{4}\left[\sqrt{(1 + (4a/h)^2)} - 1\right] . \qquad (20)$$

The ratio of the values of the displacement is $\pi/2$ for small cylinders but differs less from unity as the size increases, as is shown in Fig. 4.

1.3. *Flow with Partly Rigid Walls and Partly Free-Streamline Walls.*—A modified arrangement in which the boundaries of the flow are free-streamlines near the doublet but consist of parallel walls upstream and downstream of the central section can also be found using the transformation formulae (9) and (11). The change required to produce the new configuration in the z-plane, shown in Fig. 5a, consists merely of shifting the source and sink in the t-plane to points P and Q distant c from the origin, c being less than unity. The source and sink then lie to the left of the origin in the ζ-plane, as indicated in Fig. 5b, so that the velocity of the flow at infinity is changed from U, which now becomes the velocity on the free-streamline part of the wall, to the lower value

$$U_2 = U \frac{1 + \sqrt{(1 - c^2)}}{1 - \sqrt{(1 - c^2)}} \cdot \frac{\sqrt{(1 - b^2)} - \sqrt{(1 - c^2)}}{\sqrt{(1 - b^2)} + \sqrt{(1 - c^2)}} .$$

Equation (1) is then replaced by the relation

$$W = \frac{Uh}{2\pi}\left[\log \frac{c + t}{c - t} + \frac{2b^2 c}{(c^2 - b^2)t}\right] , \qquad (21)$$

and equation (12) by

$$\frac{dW}{dt} = \frac{Uhc^3}{\pi} \cdot \frac{(t^2 - b^2)}{t^2(c^2 - t^2)(c^2 - b^2)} . \qquad (22)$$

The width of the stream away from the body increases to hU/U_2 as the strengths of the source and sink have not been changed. The transformation from the t to the z-planes therefore becomes

$$\frac{dz}{dt} = \frac{c^3h}{\pi(c^2 - b^2)} \cdot \frac{\{\sqrt{(t^2 - 1)} + i\sqrt{(1 - b^2)}\}}{(c^2 - t^2)\{\sqrt{(t^2 - 1)} + i\}^2} . \qquad (23)$$

This equation may be integrated by splitting it into partial fractions but certain features of the flow can be found without this step. The length of the free-streamline wall above the doublet is given by the relation $(\phi_C - \phi_G)/U$ and this may be evaluated directly from equation (21). The maximum velocity on the cylindrical boundary is found from equations (21) and (11). The co-ordinates of the point in the t-plane where the streamline $\psi = 0$ crosses the imaginary axis are first found, and the corresponding real value of ζ follows. The velocity is then found from (9) and may be compared with the value $2U_2$ for an unconstrained flow. For this type of flow, particular examples will exist in which the pressures on the cylinder differ only slightly from those in an unlimited flow.*

2. *The Flow Past a Slender Section.*—2.1. *Flow in an Unlimited Stream.*—This section describes the methods of calculating the flow past slender profiles with a constant pressure reduction over the greater part of the surface. The flow past aerofoils of this type in an unlimited stream has been considered in an earlier paper (R. & M. 2161[3]) and the flow past similar profiles midway between parallel walls and in a free jet are given below. In the latter example a similar modification to that used for the circular cylinder yields boundaries which are partly parallel and partly free streamline. The thickness ratio of sections giving identical pressure distributions are compared for the various cases instead of calculating the change of velocity due to tunnel constraint for a single profile.

The simplest of the sections described in R. & M. 2161[3] is employed, and is shown in Fig. 6. The boundary is doubly symmetrical and has straight sections perpendicular to the stream at the front and rear which are joined by curved walls along which the velocity is constant at a value higher than that of the undisturbed stream. The relative length of the straight section of the boundary decreases rapidly when the thickness ratio is reduced and the thin profiles of this type differ only slightly from an elliptical cylinder.

The flow past the cylinder profile in the z-plane in an unlimited flow is defined by the relation

$$\frac{dW}{dz} = q_0 \frac{\sqrt{(W^2 - 1)} - W\sqrt{(1 - k^2)}}{\sqrt{(k^2W^2 - 1)}} , \qquad (24)$$

which is obtained by simplifying the general equation (7) of R. & M. 2161[3]. The stream function ψ is zero on the cylindrical boundary, as well as on the x-axis, and ϕ is zero on the vertical axis of symmetry. The limits of the curved wall, along which the velocity has a constant value q_0, are defined by $\phi = \pm 1$ and the stagnation points correspond to $\phi = \pm 1/k$. The constant k is found from the condition that the velocity of the undisturbed flow, corresponding to large values of ϕ, is equal to U, so that

$$\frac{U}{q_0} = \frac{1 - \sqrt{(1 - k^2)}}{k} \text{ or } k = \frac{2q_0}{U} \Big/ 1 + \left(\frac{q_0}{U}\right)^2 . \qquad (25)$$

The horizontal projection $2x_0$ and the vertical projection y_0 of the curved wall, and the length of the straight wall y_1 adjacent to the stagnation point are obtained by integrating (24) along the streamline $\psi = 0$. The equations expressed in terms of the non-dimensional velocity components take the general forms

$$x = \frac{1}{U}\int \frac{\bar{u}}{\bar{u}^2 + \bar{v}^2}\, d\phi \text{ and } y = \frac{1}{U}\int \frac{\bar{v}}{\bar{u}^2 + \bar{v}^2}\, d\phi .$$

* Further computation has shown that this occurs for a small cylinder when the length of the free-streamline l is about $1 \cdot 51h$, the value of l/h increases only slowly with the size of the cylinder.

The lengths x_0, y_0, y_1 are therefore given by

$$x_0 = \frac{1}{q_0}\int_0^1 \frac{\sqrt{(1-\phi^2)}}{\sqrt{(1-k^2\phi^2)}}\,d\phi\,,\; y_0 = \frac{1}{q_0}\int_0^1 \frac{\phi\sqrt{(1-k^2)}\,d\phi}{\sqrt{(1-k^2\phi^2)}}\,, \qquad \ldots \quad \ldots \quad \ldots \quad (26)$$

and

$$y_1 = \frac{1}{q_0}\int_1^{1/k} \frac{\sqrt{(1-k^2\phi^2)}\,d\phi}{\sqrt{(\phi^2-1)} - \phi\,.\,\sqrt{(1-k^2)}}\,.$$

These relations can be expressed in terms of the elliptic integrals of modulus k and of the complementary modulus k', defined by $\sqrt{(1-k^2)}$. Employing the usual notation for the complete elliptic integrals, they take the final forms

$$x_0 = \frac{1}{q_0k^2}\left[E - k'^2K\right],\quad y_0 = \frac{k'(1-k')}{q_0k^2}\,,$$

and

$$y_1 = \frac{1}{q_0k^2}\left[E' - k^2K' + k'^2\right]. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

The thickness ratio of the resulting profile is therefore

$$\frac{E' - k^2K' + k'}{E - k'^2K}\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

2.2. *Flow Between Parallel Walls.*—The flow past a similar profile in the centre of a parallel-walled channel of width h is found readily by using transformations similar to those employed for the isolated aerofoil. The configuration in the ζ-plane defined in equation (9) is shown in Fig. 7a together with the corresponding z-plane diagram in Fig. 7b. The straight sections of the boundary are AB and CD and BC is the constant velocity portion. PEQ is the straight wall of the channel. The ζ-plane is transformed into the upper half of a t-plane in which the flow is from a source at P to a sink at Q and in which E lies at infinity. The relation between the ζ-plane and the t-plane is identical with that connecting ζ and W for the profile in an unlimited stream. Hence for the present case, expressing ζ in terms of dW/dz

$$\frac{dW}{dz} = q_0\,\frac{\sqrt{(t^2-1)} - k't}{\sqrt{(k^2t^2-1)}}\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

The points A and D correspond to $t = \pm 1/k$ and B and C to $t = \pm 1$. On the wall PQ the velocity is greater then U and has its maximum value above the centre of the profile at E. This point corresponds to the end of the barrier in the ζ-plane, whose co-ordinates are defined as $\left(\log\frac{q_1}{U}\,,0\right)$. Since the corresponding value of t is infinite, k may be expressed in terms of q_0 and q_1 from the relation (25), q_1 replacing U. The points P and Q at the origin in the ζ-plane correspond to $t = \pm p$ and, as at these points the velocity in the z-plane is U, p is found in terms of k from the relation

$$1 = \frac{q_0}{U}\,.\,\frac{\sqrt{(p^2-1)} - k'p}{\sqrt{(k^2p^2-1)}}\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

If k'_0 is defined as the value of k' appropriate to the section giving the same velocity q_0 for flow unconstrained by the presence of the walls, this equation may be expressed in the form

$$p = \frac{k'_0}{\sqrt{\{(k'_0 + k')(k'_0 - k')\}}}\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

The potential function in the t-plane for the source at the point $t = -p$ and the sink at the point $t = p$ with a boundary lying along the real axis is given, for this case, by

$$W = \frac{Uh}{2\pi} \log \frac{p+t}{p-t}, \quad \ldots \quad (32)$$

the strength of the source being chosen to give a channel of width h. The flow in the z-plane is found from the t-plane by eliminating W between equations (29) and (32). From (32)

$$\frac{dW}{dt} = -\frac{Uh}{\pi} \cdot \frac{p}{t^2 - p^2},$$

and hence

$$\frac{dz}{dt} = -\frac{\pi q_0}{Uhp} \cdot \frac{\{\sqrt{(t^2-1)} - k't\}(t^2 - p^2)}{\sqrt{(k^2t^2-1)}}. \quad \ldots \quad (33)$$

The dimensions of the cylindrical boundary are found by integrating this relation for real values of t. The resulting expressions for x_0 and y_0, which define the dimensions of the curved part of the profile, are given by

$$x_0 = \frac{Uhp}{\pi q_0} \int_0^1 \frac{(1-t^2)dt}{(t^2-p^2)\sqrt{(1-k^2t^2)}} \text{ and } y_0 = \frac{Uhp}{\pi q_0} \int_0^1 \frac{k't\,dt}{(t^2-p^2)\sqrt{(1-k^2t^2)}}.$$

The first of these may be expressed in terms of elliptic integrals and the second in circular functions by the relations

$$x_0 = \frac{Uh}{\pi q_0}\left[\frac{K}{p} - \frac{\sqrt{(p^2-1)}}{\sqrt{(k^2p^2-1)}}\left(K \,.\, E(\alpha) - \alpha \,.\, E\right)\right], \quad \ldots \quad (34)$$

where

$$\operatorname{sn} \alpha = 1/pk,$$

and

$$y_0 = \frac{Uhpk'}{\pi q_0 \sqrt{(k^2p^2-1)}}\left[\tan^{-1}\frac{1}{\sqrt{(k^2p^2-1)}} - \tan^{-1}\frac{k'}{\sqrt{(k^2p^2-1)}}\right]. \quad \ldots \quad (35)$$

The height of the straight wall AB is given by the relation

$$y_1 = \frac{Uhp}{\pi q_0}\int_1^{1/k} \frac{\sqrt{(1-k^2t^2)}\,dt}{\{\sqrt{(t^2-1)} - k't\}\,(t^2-p^2)},$$

which may be rearranged in the form

$$y_1 = \frac{Uh}{q_0}\left[\frac{pk'}{\pi\sqrt{(p^2k^2-1)}} \cdot \tan^{-1}\frac{k'}{\sqrt{(k^2p^2-1)}} + \frac{\sqrt{(p^2-1)}}{\sqrt{(k^2p^2-1)}} \cdot \frac{\alpha}{2K}\right] - x_0\frac{K'}{K}. \quad (36)$$

Equations (34), (35) and (36) enable the thickness ratios of the profiles to be determined. It is convenient to select initially the velocity on the profile q_0 and q_1, the maximum velocity on the channel wall. For a given value of q_0 the chord of the section increases as q_1 is increased and the thickness ratio falls.

2.3. *Flow in a Free Jet.*—The flow within a jet with constant pressure walls past the profiles is next considered. The problem is more complicated than those dealt with earlier as the transformation from the ζ to the t-planes involves the introduction of elliptic functions and the boundary dimensions must be determined finally by numerical integration. The ζ-plane and

z-plane boundaries for this flow are shown in Figs. 8a and 8b. The transformation of the upper half of the t-plane into the ζ-plane is found by the Schwarz-Christoffel method and takes the form

$$\frac{d\zeta}{dt} = iM \frac{t^2 - e^2}{(t^2 - a^2)\sqrt{\{(t^2-1)(k^2t^2-1)\}}} . \qquad (37)$$

The points B and C at the end of the curved wall of the profile are again situated at $t = \pm 1$. In this problem, however, k is defined differently and the points $t = \pm 1/k$ correspond to P and Q, the positions at which the source and sink are located. The stagnation points now correspond to $t = \pm a$. Equation (37) is integrated after making the substitution $t = \operatorname{sn} u$, this transforms the upper half of the t-plane into a rectangle of sides $2K$ and K' in the u-plane. The points on the rectangle corresponding to the profile and the wall of the jet are indicated in Fig. 8c. The constant a defining the stagnation points, is expressed for convenience in terms of a new constant μ by the relation

$$a = \operatorname{sn}(K + iK' - i\mu) = \frac{1}{k \operatorname{sn}(K - i\mu)},$$

and e is similarly replaced by ν

$$e = \operatorname{sn}(\nu + iK') = \frac{1}{k \operatorname{sn} \nu} .$$

With these substitutions (37) becomes

$$\frac{d\zeta}{du} = iM_1 \left[\frac{1 - k^2 \operatorname{sn}^2 \nu \, . \operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2(K - i\mu) \operatorname{sn}^2 u} \right],$$

and may be integrated in terms of theta functions giving the relation

$$\zeta = iM_1 \left[\gamma u + i\beta \log \frac{\Theta_1(u + i\mu)}{\Theta_1(u - i\mu)} \right] + C . \qquad (38)$$

The real constants γ and β are defined in terms of μ and ν as follows

$$i\beta = \frac{\operatorname{sn}^2(K - i\mu) - \operatorname{sn}^2 \nu}{2 \operatorname{sn}(K - i\mu) \operatorname{cn}(K - i\mu) \operatorname{dn}(K - i\mu)},$$

and

$$\gamma = 1 + 2i\beta \frac{\Theta'(K - i\mu)}{\Theta(K - i\mu)} .$$

The value of ζ at the point N is $\log \frac{q_0}{U}$ and is equal to the constant C in equation (38), since the other terms vanish for $u = 0$. At the point $u = K$ the corresponding value of ζ is $\log \frac{q_0}{U} + i\pi/2$, and hence it may be shown that

$$M_1\gamma = \pi/2K . \qquad (39)$$

Both the points $u = K + iK'$ and $u = iK'$ correspond to $\zeta = 0$ and these conditions lead to the two relations between the constants

$$\gamma K = - \beta\pi , \text{ and } \log \frac{q_0}{U} = \frac{\pi}{2} \frac{K' - \mu}{K} . \qquad (40)$$

Expressing ζ in terms of dW/dz from equation (9) and substituting for γ and β from (39) and (40), the final relation between dW/dz and u takes the form

$$\frac{dW}{dz} = q_0 \left(\frac{\Theta_1(u + i\mu)}{\Theta_1(u - i\mu)} \right)^{1/2} e^{i\pi u/2K} . \qquad (41)$$

The parameter k is the main variable defining the ratio of the chord of the section to the initial width of the jet h. Small values of k correspond to aerofoils of small chord and as k approaches unity the chord increases rapidly, as also does the ratio K/K'. The constant μ is found directly from equation (40) but ν is required only for determining the point on the free streamline wall of the jet which has the maximum vertical velocity component.

The flow in the t-plane, and hence in the u-plane from a source at P to an equal sink at Q is given by

$$W = \frac{Uh}{2\pi} \log \frac{1 + kt}{1 - kt} . \qquad (42)$$

As in earlier examples the strength of the source is taken as Uh so that a jet of initial width h and velocity U is obtained. Differentiating equation (42) and eliminating W from (41) yields the relation

$$\frac{dz}{dt} = \frac{Uhk}{\pi q_0} \cdot \frac{e^{-i\pi u/2K}}{\mathrm{dn}^2\, u} \left(\frac{\Theta_1(u - i\mu)}{\Theta_1(u + i\mu)} \right)^{1/2} . \qquad (43)$$

For the curved section BC, $1 > t > -1$ and u is real, lying in the range $K > u > -K$. Values of u corresponding to given values of t are therefore found directly from tables of elliptic integrals. When u is real $\left| \dfrac{\Theta_1(u - i\mu)}{\Theta_1(u + i\mu)} \right|$ is equal to unity and argument 2ω is found by expanding the theta functions in terms of $q = e^{-\pi K'/K}$. Tan ω then takes the form

$$\tan \omega = \frac{2 \sum_1^\infty q^{m^2} \sin \dfrac{m\pi u}{K} \cdot \sinh \dfrac{m\pi\mu}{K}}{1 + 2 \sum_1^\infty q^{m^2} \cos \dfrac{m\pi u}{K} \cosh \dfrac{m\pi\mu}{K}} \qquad (44)$$

q is small for sections whose chord is less than h, the width of the jet, and the series converge rapidly. Using values of ω found from (44), equation (43) gives $2x_0$ and y_0 which are the horizontal and vertical projections of the curved wall BC. They are expressed in a form suitable for numerical integration by

$$x_0 = \frac{Uhk}{\pi q_0} \int_0^1 \frac{\cos\left(\dfrac{\pi u}{2K} - \omega\right)}{1 - k^2 t^2}\, dt \text{ and } y_0 = \frac{Uhk}{\pi q_0} \int_0^1 \frac{\sin\left(\dfrac{\pi u}{2K} - \omega\right)}{1 - k^2 t^2}\, dt . \qquad (45)$$

The length of the straight section of the boundary AB in the vicinity of the stagnation point is found similarly. In this case the limits for u are K and $K + i(K' - \mu)$, and the argument of the theta functions is imaginary. Integration along AB in the u-plane gives, for y_1, the value

$$y_1 = \frac{Uhk}{\pi q_0} \int_0^{K' - \mu} e^{\pi(\lambda - K')/2K} \frac{\mathrm{sn}\,(\lambda, k')}{\mathrm{cn}\,(\lambda, k')}$$

$$\left[\frac{\sinh\left(\dfrac{K' + \mu - \lambda}{2K}\right)\pi - q^2 \sinh 3\left(\dfrac{K' + \mu - \lambda}{2K}\right)\pi \ldots .}{\sinh\left(\dfrac{K' - \mu - \lambda}{2K}\right)\pi - q^2 \sinh 3\left(\dfrac{K' - \mu - \lambda}{2K}\right)\pi \ldots .} \right]^{1/2} d\lambda . \qquad (46)$$

which may be evaluated numerically after making a substitution, of the form

$$\sinh\left(\frac{K' - \mu - \lambda}{2K}\right)\pi = (1 - \sqrt{s})^2 ,$$

to eliminate the infinite values of the integrand in the vicinity of $\lambda = K' - \mu$.

To illustrate the effects of tunnel constraint on the flow past these sections, the dimensions have been calculated for a series of profiles all having a velocity of $1 \cdot 15U$ over the curved surface. In an unlimited flow the section producing this velocity has a thickness ratio of 16 per cent. Smaller thicknesses are required for flow between parallel walls, as is shown by the curve (a) of Fig. 9. The thickness ratio is reduced to slightly more than 10 per cent when the chord is equal to h, the distance between the walls. For flow with constant pressure boundaries, as in a free jet, a larger thickness ratio is needed to produce the chosen velocity distribution though the changes are numerically smaller than for sections of equal chord between parallel walls. The corresponding curve is (b) in Fig. 9.

Calculations of the maximum displacement of the wall of the jet due to the presence of the aerofoil, show that the displacement bears an almost constant ratio to that of the corresponding streamline of an unconstrained flow. For sections of small chord the ratio is $\pi/2$ as in the case of the circular cylinder, but falls slightly to $1 \cdot 53$ for a chord of $0 \cdot 55h$ and a thickness ratio of $17 \cdot 4$ per cent, and to $1 \cdot 50$ for a chord of $0 \cdot 82h$ and a thickness ratio of $18 \cdot 9$ per cent. These values have been calculated by integrating along the line NE perpendicular to the flow.

Conclusions.—Methods of calculating the irrotational flow past a circular cylinder and an oval section in channels of finite width are presented. The channel walls may either be parallel or may be shaped so as to give constant pressure and to correspond therefore to flow in a free jet. The free streamline method is employed for the latter calculations so that the displacement of the walls is correctly represented, and the results are not limited to bodies which are small compared with the width of the stream.

For a circular cylinder between parallel walls, the maximum velocity on the boundary rises sharply as the diameter is increased and a much smaller change of opposite sign occurs in the free jet. For the constant pressure profiles the variation of the thickness ratio with the chord, which is required to give a particular pressure distribution, is determined.

The ratio of the maximum displacement of the free-streamline wall of the jet to that of the corresponding streamline of an unconstrained stream is found in each case. The ratio decreases as the size of the body increases for both examples but the effect is considerable only in the case of the circular cylinder.

It is shown that the method of calculating the flow within a free jet may be modified to give flow bounded by partly parallel walls and partly constant-pressure walls.

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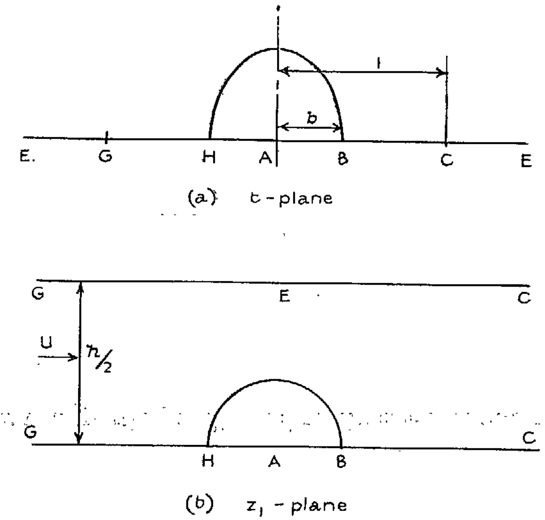

FIG. 1. Flow past circular cylinder between parallel walls.

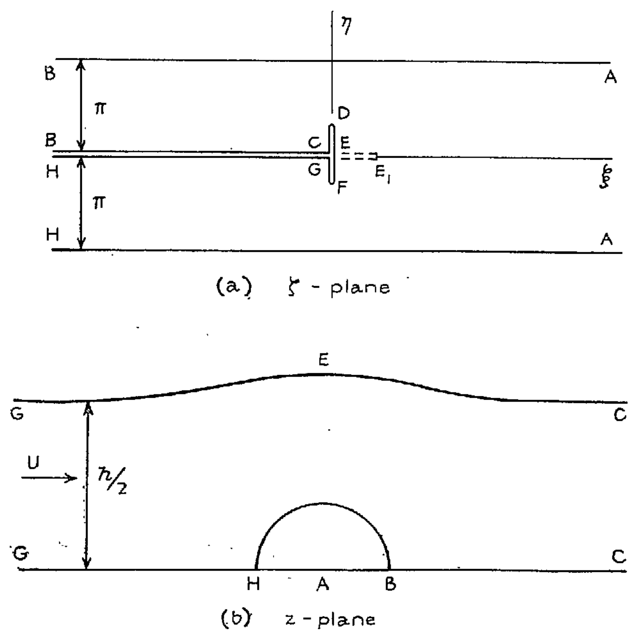

FIG. 2. Flow past a cylinder in a free jet.

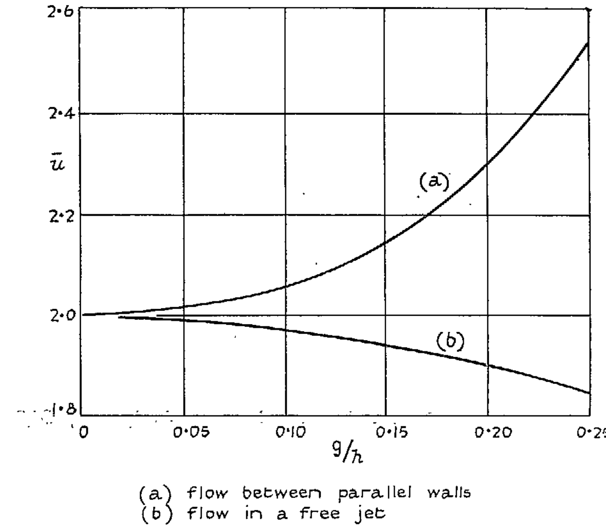

FIG. 3. Maximum velocity of flow past a circular cylinder.

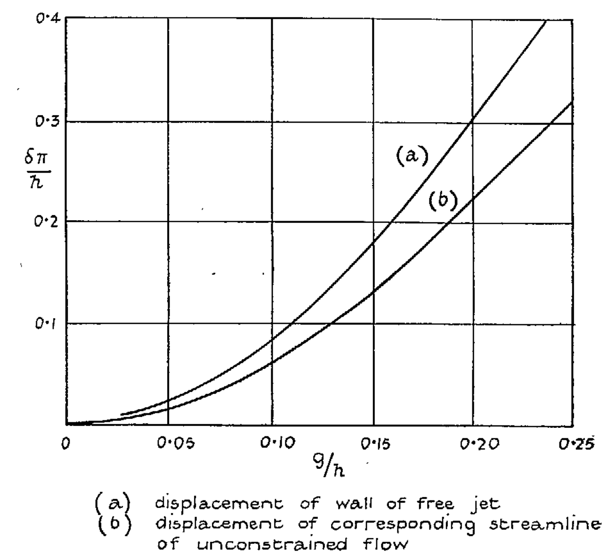

FIG. 4. Comparison of maximum displacements of streamlines.

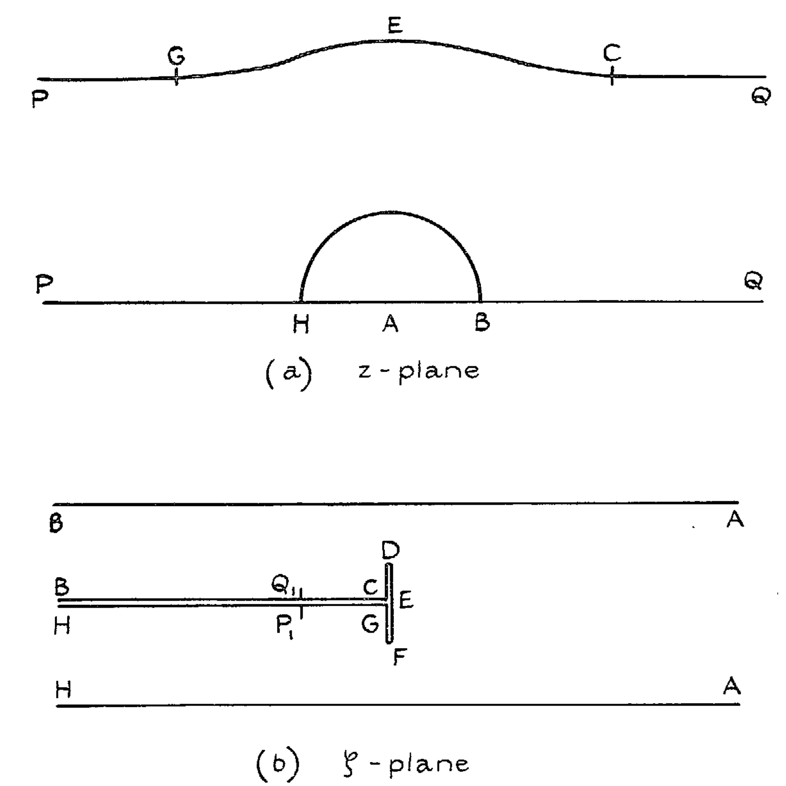

FIG. 5. Flow with partly parallel and partly constant-pressure walls.

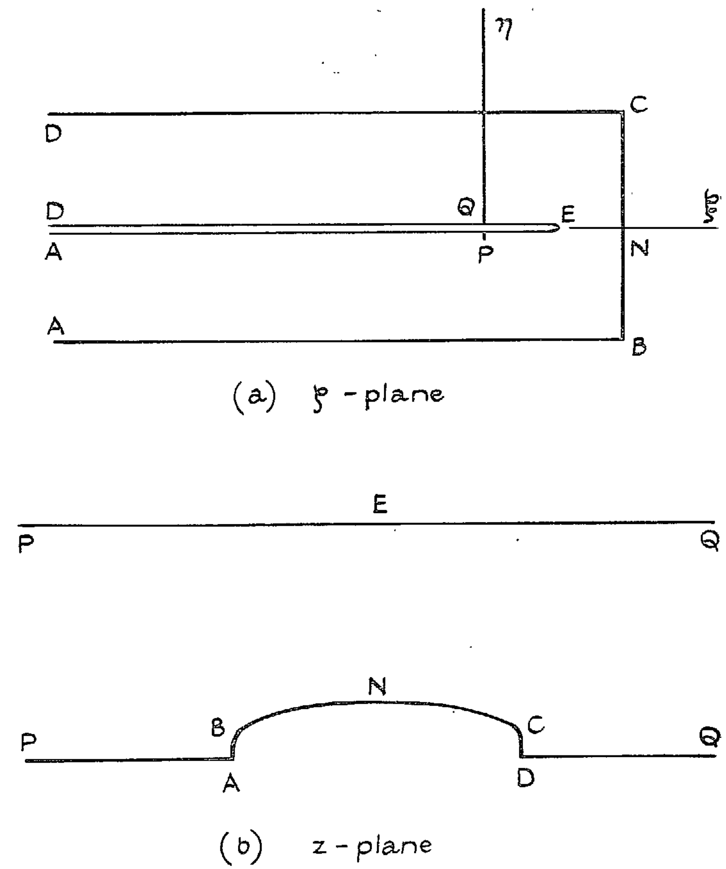

FIG. 7. Flow between parallel walls.

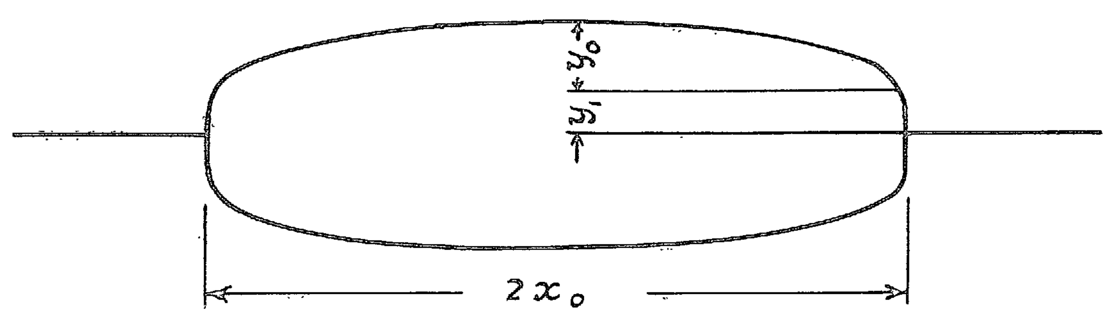

FIG. 6.

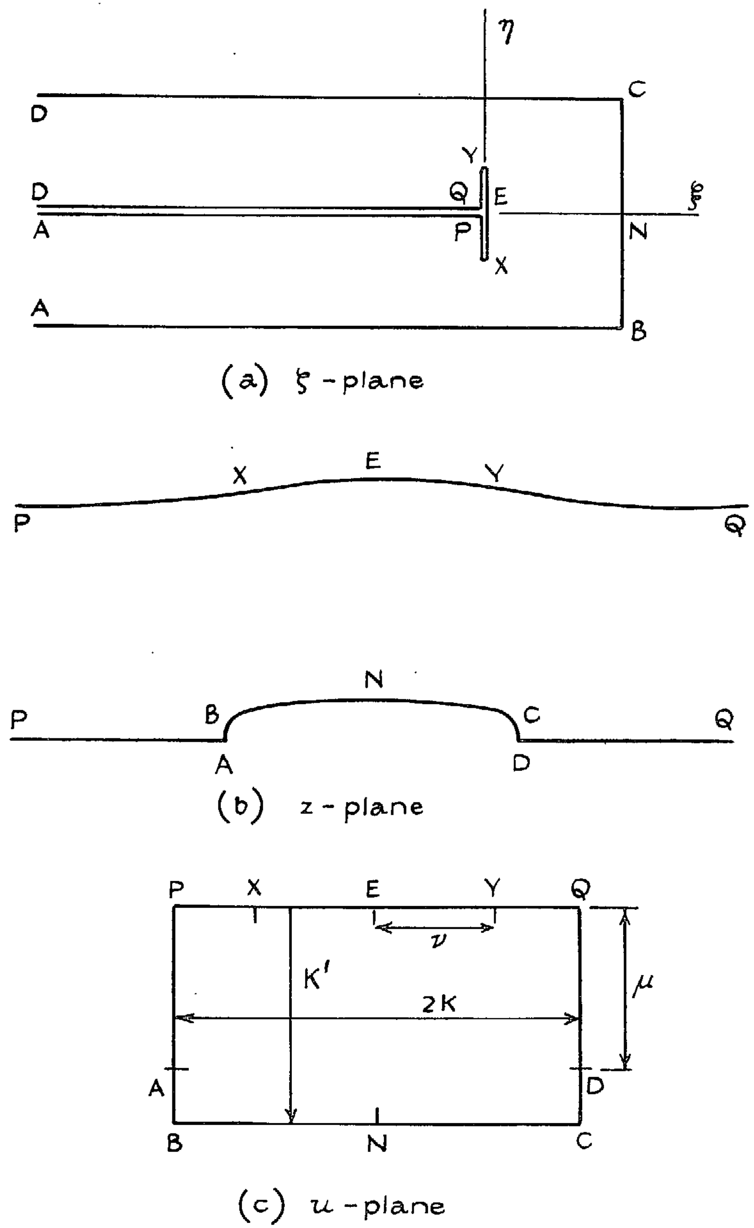

FIG. 8. Flow in a free jet.

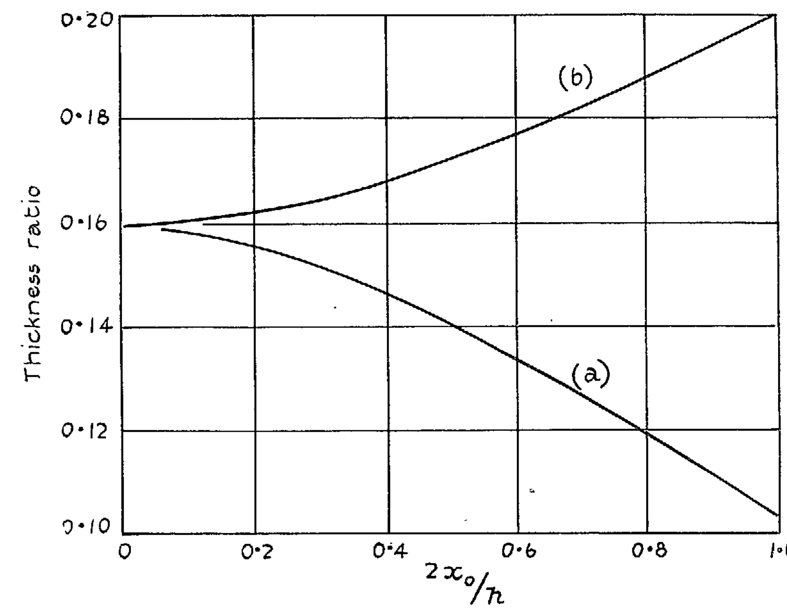

FIG. 9. Variation of thickness ratio for sections giving $q_0 = 1 \cdot 15U$.

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