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An Approximate Solution of the Compressible Laminar Boundary Layer on a Flat Plate

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Summary.—Following a major assumption that enthalpy and velocity are dependent only on local conditions, an enthalpy-velocity relation

$$\frac{i}{i_1} = \frac{i_p}{i_1} - \sigma^{1/3}\left(\frac{i_p}{i_1} - \frac{i_e}{i_1}\right)\frac{u}{u_1} - \sigma\,\frac{\gamma - 1}{2}\,M_1{}^2\left(\frac{u}{u_1}\right)^2$$

is obtained for the laminar boundary layer on a flat plate where subscripts p refer to the plate, 1 to the free stream and e to the equilibrium temperature condition at the plate. When compared with general results, this relation (exact for Prandtl number $\sigma = 1$) gives a close approximation to Crocco's numerical results[2] for $\sigma = 0{\cdot}725$ and $1{\cdot}25$, up to $u/u_1 = 0{\cdot}8$.

Using the above relation in conjunction with the approximate viscosity-temperature relation

$$\frac{\mu}{\mu_1} = C\,\frac{T}{T_1}$$

suggested by Chapman and Rubesin[4], and with Young's[3] suggested first approximation for shearing stress

$$\frac{\tau}{\tau_0} = \left\{1 - \left(\frac{u}{u_1}\right)^2\right\}^{1/2}$$

it is shown that close approximations to displacement thickness and velocity distribution are given by

$$\frac{1}{2}\,\frac{\delta^*}{x}\,(Re_x)^{1/2} = \frac{C}{F_0}\left\{\left(A - \frac{D}{2}\right)\frac{\pi}{2} - \left(B + 1\right)\right\}$$

and

$$\frac{1}{2}\,\frac{y}{x}\,(Re_x)^{1/2} = \frac{C}{F_0}\left\{\left(A - \frac{D}{2}\right)\sin^{-1} z + \left(\frac{Dz}{2} + B\right)\left(1 - z^2\right)^{1/2} B\right\}$$

where
$$z = u/u_1$$
$$A = T_p/T_1$$
$$B = \sigma^{1/3}\left(\frac{T_p}{T_1} - \frac{T_e}{T_1}\right)$$
$$D = \sigma\,\frac{\gamma - 1}{2}\,M_1{}^2$$
$$\frac{Te}{T_1} = 1 + \sigma^{1/2}\,\frac{\gamma - 1}{2}\,M_1{}^2$$

and
$$F_0 = c_f\,(Re_x)^{1/2} = 0{\cdot}664\,\sqrt{C}$$

which serves to define C.

These have the advantage of being algebraic in form whereas previous results have involved complex numerical integrations for individual cases.

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1. *Introduction.*—The solution of the differential equations for the laminar boundary layer in a compressible fluid is made extremely difficult by the fact that the density (ρ), viscosity (μ) and thermal conductivity (k) all vary with temperature, so that the equations of motion and energy become inter-dependent. Even to obtain numerical solutions it has generally been necessary to make restrictive assumptions and lengthy calculations.

The simplest results to date have been given by Howarth[1] who assumed both that the Prandtl number $\left(\sigma = \frac{c_p \mu}{k}\right)$ was equal to unity and that there was a linear variation of viscosity with temperature. Even after these simplifications, the complete evaluation of the layer still involves graphical or numerical integrations, and is only possible with any accuracy for the simple case of the flat plate in the absence of pressure gradients.

The flat plate problem has received the attention of many workers, but as yet all solutions have been purely numerical for particular cases. The object of the present note is to present an approximate, analytical solution which is more general, has the merits of simplicity and shows clearly the effects of changes in working conditions.

Having presented the fundamental equations of the boundary layer in section 2, approximate formulae are derived for the enthalpy-velocity relation in section 3 and for the variation of shearing stress across the layer in section 4. The absolute values of the latter depend on a constant which is derived in section 5 in conjunction with the skin friction coefficient. These formulae are sufficient to determine the remaining characteristics of the layer, as is shown in sections 6 to 8.

2. *Fundamental Equations for the Laminar Boundary Layer on a Flat Plate in Compressible Flow.*—In this case, and if it is assumed that the pressure does not vary along the plate, then the boundary layer differential equations of momentum and energy can be expressed in the form,

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \quad \ldots \quad (1)$$

$$\rho u \frac{\partial i_H}{\partial x} + \rho v \frac{\partial i_H}{\partial y} = \frac{\partial}{\partial y}\left\{\frac{k}{c_p}\frac{\partial}{\partial y}(i + \tfrac{1}{2}\sigma u^2)\right\} \quad \ldots \quad (2)$$

where x is measured along the plate

y is measured normal to the plate

u and v are the components of velocity in the directions of x and y

ρ is the density

μ is the dynamic viscosity

k is the thermal conductivity

c_p is the specific heat at constant pressure

σ is the Prandtl number $\left(= \frac{c_p \mu}{k}\right)$

i is the enthalpy ($= Jc_pT$ where T is the 'static' temperature) and $i_H = i + \frac{1}{2}u^2$ ($= Jc_pT_H$ where T_H is the total temperature)

μ, k and c_p all vary with temperature, but it is assumed in the above that the Prandtl number

$$\sigma = \frac{c_p \mu}{k}$$

is constant.

ρ is linked with the pressure and temperature by the equation of state

$$p = \rho RT$$

and since p is everywhere constant then

$$\rho \propto \frac{1}{T} \qquad \qquad (3)$$

Thus, even in this simple case, equations 1 and 2 are inter-dependent and this fact makes their solution much more difficult than in the isothermal case of classical hydrodynamics (when ρ, μ, k and c_p are constant).

Numerical results for particular cases have only been obtained after lengthy calculations and by making restrictive assumptions. An approximate analytical solution is investigated in the following sections.

3. *Relation Between Enthalpy and Velocity.*—If $\sigma = 1$, then inspection shows that equations 1 and 2 for the variables u and i_H are identical in form and the well known result

$$i_H = au + b$$

follows, where a and b are constants.

In the case $\sigma \neq 1$, assume as an approximation that enthalpy and velocity are dependent only on local conditions* in which case we may assume that v, $\partial u/\partial x$ and $\partial i/\partial x$ can be neglected in comparison with the absolute values of u and i. If so, then approximately

$$v = 0$$
$$u = u(y)$$
$$i = i(y)$$

and equations 1 and 2 reduce to

$$\frac{d}{dy}\left(\mu \frac{du}{dy}\right) = 0$$

$$\frac{d}{dy}\left\{\frac{k}{c_p}\frac{d}{dy}\left(i + \tfrac{1}{2}\sigma u^2\right)\right\} = 0$$

hence

$$\mu \frac{du}{dy} = \text{constant} = \tau_0 \text{ (local skin friction)}$$

and

$$\frac{k}{c_p}\frac{d}{dy}\left(i + \tfrac{1}{2}\sigma u^2\right) = \text{constant} = -Jq_0$$

(where q_0 is the local heat transfer rate).

Then

$$-\frac{Jq_0}{\tau_0} = \frac{1}{\sigma}\frac{\dfrac{d}{dy}\left(i + \tfrac{1}{2}\sigma u^2\right)}{\dfrac{du}{dy}}$$

$$= \frac{1}{\sigma}\frac{d}{du}\left(i + \tfrac{1}{2}\sigma u^2\right)$$

* This is equivalent to an assumption of parallel flow made with success in the treatment of the incompressible turbulent boundary layer.

and by integration

$$-\frac{Jq_0}{\tau_0}\,\sigma u = i + \tfrac{1}{2}\sigma u^2 + A. \qquad (4)$$

The inner portion of the boundary layer is of more importance than the outer in any calculations, so choose A to fit the wall condition,

$$u = 0, \quad i = i_p.$$

Then equation 4 becomes

$$i = i_p - \sigma\frac{Jq_0}{\tau_0}\,u - \tfrac{1}{2}\sigma u^2 \qquad (5)$$

Now, Crocco[2] has shown that in high-speed flow

$$\frac{Jq_0}{\tau_0} \simeq \frac{i_p - i_e}{u_1}\,\sigma^{-2/3} \qquad (6)$$

where i_e is an equilibrium value of i at the wall, given approximately by

$$i_e = i_1 + \sigma^{1/2}\,\tfrac{1}{2}u_1^2 \qquad (7)$$

and subscript '1' refers to conditions at the outer edge of the boundary layer (free-stream conditions in the present case).

Substitution of equation 6 in equation 5 gives

$$i = i_p - \sigma^{1/3}(i_p - i_e)\frac{u}{u_1} - \tfrac{1}{2}\sigma u_1^2\left(\frac{u}{u_1}\right)^2$$

or

$$\frac{i}{i_1} = \frac{i_p}{i_1} - \sigma^{1/3}\left(\frac{i_p}{i_1} - \frac{i_e}{i_1}\right)\frac{u}{u_1} - \sigma\,\frac{\gamma-1}{2}\,M_1^2\left(\frac{u}{u_1}\right)^2. \qquad (8)$$

Equation 8 has been derived on the assumption that enthalpy and velocity are dependent only on local conditions. Its worth as an approximation* in the usual case when $v \neq 0$ will now be estimated by comparison with Crocco's numerical results[2]. The latter were obtained for $\mu \propto T$, but in general Ref. 2 shows that the variation of enthalpy with velocity is almost independent of the viscosity law chosen. Crocco's enthalpy-velocity relation can be written as

$$\frac{i}{i_1} = \frac{i_p}{i_1} - \left(\frac{i_p - i_e}{i_1}\right) f_1\left(\sigma, \frac{u}{u_1}\right) - \sigma\,\frac{\gamma-1}{2}\,M_1^2 f_2\left(\sigma, \frac{u}{u_1}\right) \qquad (9)$$

where, in the notation of Ref. 2,

$$f_1\left(\sigma, \frac{u}{u_1}\right) = \Theta_\sigma^I\left(\frac{u}{u_1}\right)$$

$$f_2\left(\sigma, \frac{u}{u_1}\right) = 2J_\sigma\left(\frac{u}{u_1}\right)$$

and values of the latter functions are given in Tables 3 and 4 of that report[2].

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* It may be noted that equation 8 is exact when $\sigma = 1$.

Comparison of f_1 and f_2 with the corresponding quantities $\left\{ \sigma^{1/3} \left(\frac{u}{u_1}\right) \text{ and } \left(\frac{u}{u_1}\right)^2 \right\}$ of equation 8 is made in Table 1 and Fig. 1 for $\sigma = 0 \cdot 725$ and $1 \cdot 25$. These show that equation 8 forms a good approximation to the enthalpy-velocity distribution at least up to $u/u_1 = 0 \cdot 8$, where f_1 is within 3 per cent and f_2 is within 4 per cent of the corresponding exact values.

For air at temperatures less than 400 deg K it is sufficiently accurate to take c_p = constant in which case equation 8 becomes the temperature-velocity relation

$$\frac{T}{T_1} = \frac{T_p}{T_1} - \sigma^{1/3} \left(\frac{T_p - T_e}{T_1}\right) \frac{u}{u_1} - \sigma \frac{\gamma - 1}{2} M_1^2 \left(\frac{u}{u_1}\right)^2 \qquad \text{(10)}$$

where from equation 6

$$\frac{T_e}{T_1} = 1 + \sigma^{1/2} \frac{\gamma - 1}{2} M_1^2 .$$

This will be assumed to be the case in the remainder of this note.

4. *Variation of Shearing Stress across the Boundary Layer.*—The local shearing stress (τ) in the boundary layer is given by

$$\tau = \mu \frac{\partial u}{\partial y}. \qquad \text{(11)}$$

For the purposes of section 3 it was assumed that this was constant across the layer and equal to the wall value τ_0. In general this is not the case and to find the variation of τ it is necessary to solve equation 1 (the equation of motion).

Now Crocco[2] has shown that in the case of the flat plate, when $\partial p / \partial x = 0$, it is possible to reduce equation 1 to an ordinary differential equation in terms of the variable u, provided $\partial i / \partial x = 0$, *i.e.*, provided i is a function of u alone. The non-dimensional form of this equation is

$$F F'' + 2 \frac{\rho}{\rho_1} \cdot \frac{\mu}{\mu_1} \cdot \frac{u}{u_1} = 0 \qquad \text{(12)}$$

where F is a function of u/u_1 and a prime denotes differentiation with respect to u/u_1. The function F is given by

$$F = C_\tau \, (Re_x)^{1/2} \qquad \text{(13)}$$

where $\quad C_\tau = \frac{\tau}{\frac{1}{2}\rho_1 u_1^2}$ and $Re_x = \frac{\rho_1 u_1 x}{\mu_1}$.

(Thus, at the wall

$$F = F_0 = c_f \, (Re_x)^{1/2} \qquad \text{(13a)}$$

where $\quad c_f = \frac{\tau_0}{\frac{1}{2}\rho_1 u_1^2}$ is the local skin friction coefficient.)

The boundary conditions to be satisfied by solutions of equation 12 are derived in Ref. 2 as

$$\left.\begin{aligned} \frac{u}{u_1} &= 0, \quad F' = 0 \\ \frac{u}{u_1} &= 1, \quad F = 0 \end{aligned}\right\} \qquad \text{(14)}$$

Finally, when F has been obtained as a function of u/u_1, the velocity field can be obtained by integration of equation 11, which in the present case can be transformed to read

$$\frac{\partial\left\{\frac{y}{x}\,(Re_x)^{1/2}\right\}}{\partial\left(\frac{u}{\mu_1}\right)} = \frac{2}{F}\cdot\frac{\mu}{\mu_1} \quad \ldots \quad (15)$$

Equation 12 shows that F as a function of u/u_1 is influenced by the choice of viscosity law. If $\mu \propto T$, then equation 12 becomes (by virtue of equation 3)

$$FF'' + 2\frac{u}{u_1} = 0 \quad \ldots \quad (12a)$$

and a numerical solution for this case has been obtained by Crocco[2].

Before going further, it is best to consider how μ varies with T (and hence with u/u_1).

4.1. *Variation of Viscosity with Temperature*.—The best representation to date of the variation of viscosity with temperature is given by Sutherland's formula

$$\mu = \mu_0\left(\frac{T}{273}\right)^{1/2}\frac{1 + T_c/273}{1 + T_c/T}$$

where μ_0 is the viscosity at zero centigrade and T_c is a characteristic temperature for the gas. For air, T_c can be taken as 116 deg K.

From this we obtain

$$\frac{\mu}{\mu_1} = \left(\frac{T}{T_1}\right)^{1/2}\frac{1 + T_c/T_1}{1 + T_c/T} \quad \ldots \quad (16)$$

and this formula is usually approximated to by a power law variation

$$\frac{\mu}{\mu_1} = \left(\frac{T}{T_1}\right)^n \quad \ldots \quad (17)$$

where the index n depends on the free-stream temperature and is chosen so that the power law variation (equation 17) is tangential to Sutherland's variation (equation 16) at that temperature. This approximation is only valid for temperatures in the neighbourhood of the free-stream temperature, as has been illustrated by Crocco[2].

Neither equation 16 nor equation 17 is particularly amenable for use in analytical evaluation of the boundary layer.

Recently, Chapman and Rubesin[4] have proposed the form

$$\frac{\mu}{\mu_1} = C\frac{T}{T_1} \quad \ldots \quad (18)$$

where C is a constant chosen to suit the range of temperatures under consideration. In their work[4] they take C to be given by

$$C = \left(\frac{\mu_p}{\mu_1}\right)\left(\frac{T_1}{T_p}\right)$$

where μ_p/μ_1 is given by Sutherland's formula (equation 16).

In the present note we shall use this form (equation 18) but shall make use of Crocco's results[2] to define C (section 5, below).

4.2. *Variation of Shearing Stress.*—Taking

$$\frac{\mu}{\mu_1} = C\frac{T}{T_1} \qquad (18)$$

and since

$$\frac{\rho}{\rho_1} = \frac{T_1}{T} \qquad (3)$$

we obtain, from equation 12,

$$FF'' + 2C\frac{u}{u_1} = 0$$

or, since C is a constant

$$\frac{F}{\sqrt{C}}\left(\frac{F}{\sqrt{C}}\right)'' + 2\frac{u}{u_1} = 0 \qquad (12b)$$

subject to the boundary conditions

$$\left.\begin{aligned} \frac{u}{u_1} &= 0, \left(\frac{F}{\sqrt{C}}\right)' = 0 \\ \frac{u}{u_1} &= 1, \left(\frac{F}{\sqrt{C}}\right) = 0 \end{aligned}\right\} \qquad (14a)$$

By comparison of equations 12a and 12b, and the boundary conditions of equations 14 and 14a we see that Crocco's solution[2] for F of equation 12a, when $C = 1$, should also be valid in the general case if

$\dfrac{F}{\sqrt{C}}$ is substituted for F.

In particular, this means that

$$F_0 = c_f\,(Re_x)^{1/2} = 0{\cdot}664\sqrt{C} \qquad (19)$$

and also that $\dfrac{F}{F_0}\left(= \dfrac{\tau}{\tau_0}\right)$, as a function of u/u_1 should not be affected by changes in C.

Now Young[3] has pointed out that a suitable first approximation to Crocco's values[2] of $\dfrac{F}{F_0}$ is given by

$$\frac{F}{F_0} = \left\{1 - \left(\frac{u}{u_1}\right)^2\right\}^{1/2} \qquad (20)$$

(It can be shown that this is equivalent to Lamb's approximation[6] for velocity distribution in incompressible flow, *see* section 6.1.1 below.)

A comparison between equation 20 and Crocco's solution is given in the following table and in Fig. 2.

$\frac{u}{u_1}$	0·1	0·3	0·5	0·7	0·8	0·9	0·95	1·0
$\frac{F}{F_0}$ (Crocco)	0·9992	0·9795	0·9036	0·7252	0·5751	0·3596	0·2123	0
$\left\{1-\left(\frac{u}{u_1}\right)^2\right\}^{1/2}$	0·9950	0·9539	0·8660	0·7141	0·6000	0·4359	0·3123	0

These show that the approximation given by equation 29 is within 5 per cent up to $u/u_1 = 0 \cdot 8$.

We now have approximate expressions for the enthalpy-velocity and the shearing stress-velocity relations, given by equations 10 and 20 respectively, and in conjunction with equation 15 these are sufficient for the complete solution of the laminar boundary layer provided the constant C in the viscosity-temperature relation (equation 18) is known. The constant C is determined in the next section.

Meanwhile a check on the errors introduced by using the approximate expression for the stress distribution is given by calculating c_f from the momentum integral equation

$$c_f = \frac{\tau_0}{\frac{1}{2}\rho_1 u_1^{\,2}} = 2\,\frac{d\theta}{dx} \qquad \ldots \qquad (21)$$

(when $\frac{\partial p}{\partial x} = 0$).

where

$$\theta = \int_0^\delta \frac{\rho}{\rho_1}\,\frac{u}{u_1}\left(1 - \frac{u}{u_1}\right) dy$$

$$= \frac{2x}{(Re_x)^{1/2}} \int_0^1 \frac{\rho u}{\rho_1 u_1}\left(1 - \frac{u}{u_1}\right)\frac{1}{F} \times \frac{\mu}{\mu_1}\, d\!\left(\frac{u}{u_1}\right)$$

when the variable of integration is changed from y to u/u_1, using equation 15.

Also $$\frac{\rho}{\rho_1} = \frac{T_1}{T} \qquad \ldots \qquad (3)$$

and $$\frac{\mu}{\mu_1} = C\,\frac{T}{T_1} \qquad \ldots \qquad (18)$$

so that we obtain

$$\theta = \frac{2Cx}{(Re_x)^{1/2}} \int_0^1 \frac{\frac{u}{u_1}\left(1 - \frac{u}{u_1}\right)}{F}\, d\!\left(\frac{u}{u_1}\right)$$

and $$F = F_0\left\{1 - \left(\frac{u}{u_1}\right)^2\right\}^{1/2} \qquad \ldots \qquad (20)$$

so that finally

$$\theta = \frac{2Cx}{F_0\,(Re_x)^{1/2}}\,\frac{4-\pi}{4} \qquad \ldots \qquad (22)$$

and substitution of equation 22 in equation 21 gives

$$F_0 = c_f(Re_x)^{1/2}$$

$$= 0\cdot 655\sqrt{C}$$

which is within $1\frac{1}{2}$ per cent of the more exact value

$$F_0 = 0\cdot 664\sqrt{C} \qquad \ldots \qquad (19)$$

obtained as a generalisation of Crocco's result[2] in equation 19.

The latter will be used in all the subsequent calculations.

5. *Skin Friction. Determination of the Constant C.*—By taking

$$\frac{\mu}{\mu_1} = C\,\frac{T}{T_1} \qquad \ldots \qquad (18)$$

section 4 has shown that

$$F_0 = c_f(Re_x)^{1/2}$$

$$= 0\cdot 664\sqrt{C}\,. \qquad \ldots \qquad (19)$$

Comparison of Crocco's final results[2] (based on Sutherland's formula) with equation 19 shows that

$$C = C(T_1, \frac{T_p}{T_1}, M_1).$$

Now, if equation 18 is to be a suitable approximation to Sutherland's formula, then it is reasonable to suppose that C is chosen so that the two formulae are in agreement for some value of T between T_1 and T_p, *i.e.*,

$$C = \left(\frac{T'}{T_1}\right)^{1/2} \frac{1 + T_c/T_1}{T'/T_1 + T_c/T_1} \qquad \ldots \qquad (23)$$

and the problem is then to determine T'/T_1 as a function of T_1, T_p/T_1 and M_1.

5.1. *Value of C when* $\sigma = 0\cdot 725$.—By analysis of Crocco's results (which are for $\sigma = 0\cdot 725$), Johnson and Rubesin of the University of California have obtained the approximate formula for skin friction

$$c'_f(Re'_x)^{1/2} = 0\cdot 664 \qquad \ldots \qquad (24)$$

where primes denote that density and viscosity are to be evaluated at an 'intermediate' temperature

$$T' = 0 \cdot 42T_1(1 + 0 \cdot 076M_1^2) + 0 \cdot 58T_p \quad \ldots \quad (25)$$

and viscosity is to be evaluated by Sutherland's formula.

Now

$$c'_f = \frac{\rho_1}{\rho'} c_f$$

$$= \frac{T'}{T_1} c_f \text{ by equation 3,}$$

and

$$Re_x' = \frac{\rho'}{\rho_1} \cdot \frac{\mu_1}{\mu'} Re_x$$

$$= \frac{1}{C} \cdot \left(\frac{T_1}{T'}\right)^2 \text{ by equations 3 and 18,}$$

so that equation 24 can be altered to read

$$c_f\sqrt{Re_x} = 0 \cdot 664\sqrt{C}$$

which is of the form of equation 19, and C is to be evaluated from equation 23 at a temperature T' given by equation 25.

Thus equations 23 and 25 could be used to determine the constant C, when $\sigma = 0 \cdot 725$.

On the other hand, from a semi-empirical analysis and generalisation of the same results[2], Young[3] has advanced the approximate formula

$$c_f\sqrt{Re_x} = 0 \cdot 664 \left[0 \cdot 45 + 0 \cdot 55 \frac{T_p}{T_1} + 0 \cdot 09 (\gamma - 1)M_1^2\sigma^1/^2\right]^{(n-1)/2} \quad \ldots \quad (26)$$

where it is assumed that

$$\frac{\mu}{\mu_1} = \left(\frac{T}{T_1}\right)^n \quad \ldots \quad (17)$$

Now if equations 18 and 17 are to agree at a temperature T', then

$$C = \left(\frac{T'}{T_1}\right)^{n-1} \quad \ldots \quad (27)$$

and substitution of equation 27 in equation 19 gives

$$c_f\sqrt{Re_x} = 0 \cdot 664 \left(\frac{T'}{T_1}\right)^{(n-1)/2} \quad \ldots \quad (19a)$$

The similarity of equations 26 and 19a is evident and indicates that

$$\frac{T'}{T_1} = 0 \cdot 45 + 0 \cdot 55 \frac{T_p}{T_1} + 0 \cdot 09(\gamma - 1)M_1^2\sigma^{1/2} \quad \ldots \quad (28)$$

by Young's analysis. As already mentioned, Crocco's results[2] are for $\sigma = 0 \cdot 725$, and with this value equation 28 becomes

$$\frac{T'}{T_1} = 0 \cdot 45 \left\{1 + 0 \cdot 068 M_1^2\right\} + 0 \cdot 55 \frac{T_p}{T_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28a)$$

which should be comparable with the estimate of Johnson and Rubesin given by equation 25. Comparison is made in Fig. 3, and shows reasonable agreement.

It should be noted that Young's formula (equation 26) gives

$$F_0 = c_f \sqrt{Re_x} = \text{constant}$$

when $n = 1$, *i.e.*, when

$$\frac{\mu}{\mu_1} = \frac{T}{T_1}$$

which is assumed to be the case when T_1 is of the order of 116 deg K, whereas Crocco's results show that $c_f \sqrt{Re_x}$ is variable and such a variation appears if Sutherland's formula is used for evaluating C, as in Johnson and Rubesin's approach. What Young's formula does is to give close agreement with Crocco's results if the ratio

$$\frac{F_0}{F(0)_{n=1}}$$

is considered.

To obtain the absolute value of F_0, the variation of $\sqrt{C}$ with T/T_1 according to Sutherland's formula should be used. This is shown in Fig. 4 for two values of T_1, and the error in using a power law approximation is evident. The appropriate value of T'/T_1 should therefore be obtained from equation 25 and should be used in conjunction with equation 23.

5.2. *Extension to Other Values of* σ.—Equation 10 gives

$$\frac{T}{T_1} = \frac{T_p}{T_1} - \sigma^{1/3} \left(\frac{T_p}{T_1} - \frac{T_e}{T_1}\right) \frac{u}{u_1} - \sigma \frac{\gamma - 1}{2} M_1^2 \left(\frac{u}{u_1}\right)^2 \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

where $$\frac{T_e}{T_1} = 1 + \sigma^{1/2} \frac{\gamma - 1}{2} M_1^2.$$

Equations 25 and 28a give values of T'/T_1 at $\sigma = 0 \cdot 725$ and it is easily shown that they can be derived from equation 10 by putting $\sigma = 0 \cdot 725$ and

$$\frac{u}{u_1} = 0 \cdot 468, \left(\frac{u}{u_1}\right)^2 = 0 \cdot 273$$

in the case of equation 25, and

$$\frac{u}{u_1} = 0 \cdot 502, \left(\frac{u}{u_1}\right)^2 = 0 \cdot 317$$

in the case of equation 28a, *i.e.*, an approximation to T'/T_1 would be the value of T/T_1 at some u/u_1 either between $0 \cdot 468$ and $0 \cdot 522$ $(= \sqrt{0 \cdot 273})$ in the case of equation 25 or between $0 \cdot 502$ and $0 \cdot 563$ $(= \sqrt{0 \cdot 317})$ in the case of equation 28a. Either set of values is within the fully valid range of equation 10.

This suggests that equation 10 with appropriate values for u/u_1 and $(u/u_1)^2$ might give a better estimate than Young's generalisation[3] (equation 28) for the variation of T'/T_1 with σ. Johnson and Rubensin's analysis was based on Sutherland's formula so for that reason we shall take

$$\frac{u}{u_1} = 0\cdot 468,\ \left(\frac{u}{u_1}\right) = 0\cdot 273.$$

This gives the general formula

$$\frac{T'}{T_1} = \frac{T_p}{T_1} - 0\cdot 468\sigma^{1/3}\left(\frac{T_p}{T_1} - \frac{T_e}{T_1}\right) - 0\cdot 273\,\sigma\,\frac{\gamma - 1}{2}\,M_1^2 \qquad \ldots \qquad \ldots \qquad (29)$$

and C is to be obtained from equation 23, or the Sutherland curves in Fig. 4.

If air is the working fluid, with $\sigma = 0\cdot 72$, it will be sufficiently accurate to obtain T'/T_1 from the broken curves of Fig. 3.

We are now in a position to evaluate the velocity distribution and displacement thickness of the boundary layer.

6. *Velocity Distribution.*—From equation 15

$$\frac{\partial\left(\frac{y}{x}\sqrt{Re_x}\right)}{\partial\left(\frac{u}{u_1}\right)} = \frac{2}{F}\cdot\frac{\mu}{\mu_1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (15)$$

we obtain

$$\frac{1}{2}\frac{y}{x}\sqrt{Re_x} = \int_0^{u/u_1}\frac{\mu/\mu_1}{F}\,d\left(\frac{u}{u_1}\right). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (30)$$

Put $\qquad \eta = \dfrac{1}{2}\dfrac{y}{x}\sqrt{Re_x},\ z = \dfrac{u}{u_1}$.

Equation 10 gives

$$\frac{T}{T_1} = A - Bz - Dz^2 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (10)$$

where

$$\left.\begin{aligned} A &= \frac{T_p}{T_1} \\ B &= \sigma^{1/3}\,\frac{T_p - T_e}{T_1} \\ D &= \sigma\,\frac{\gamma - 1}{2}\,M_1^2 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (31)$$

and $\dfrac{T_e}{T_1} = 1 + \sigma^{1/2}\,\dfrac{\gamma - 1}{2}\,M_1^2$ from equation 6 with c_p constant. Then, by using equation 18 for μ/μ_1 and equation 20 for F/F_0, we obtain from equation 30

$$\begin{aligned} \eta &= \frac{C}{F_0}\int_0^z \frac{A - Bz - Dz^2}{(1 - z^2)^{1/2}}\,dz \\ &= \frac{C}{F_0}\left\{\left(A - \frac{D}{2}\right)\sin^{-1} z + \left(\frac{Dz}{2} + B\right)(1 - z^2)^{1/2} - B\right\} \qquad \ldots \quad \ldots \quad \ldots \qquad (32) \end{aligned}$$

as an approximation to the velocity distribution across the boundary layer.

6.1. *Comparison of Approximate Velocity Distribution with Particular Exact Distributions.*—
6.1.1. $M_1 = 0$. $T_p = T_1$.—In this case, equation 31 gives

$$A = 1, B = 0, D = 0.$$

Furthermore, $T' = T_1$, hence $C = 1$ and $F_0 = 0 \cdot 664$ from equation 19. Then equation 32 gives

$$z = \sin (0 \cdot 664\, \eta) . \qquad (32a)$$

This is compared with the Blasius (numerical) distribution for incompressible flow in Table 2 and Fig. 5, and shows good agreement over the whole range, the maximum discrepancy being of the order of 2 per cent.

Note that if the velocity distribution is of the form of equation 32, *i.e.*,

$$z = \sin F_0 \eta$$

then substituting for z and η we obtain

$$F_0 \frac{1}{2} \cdot \frac{y}{x} \sqrt{Re_x} = \sin^{-1} \frac{u}{u_1} .$$

Taking $u/u_1 = 1$ at $y = \delta$ this gives

$$F_0 \frac{1}{2} \cdot \frac{\delta}{x} \cdot \sqrt{Re_x} = \frac{\pi}{2}$$

and hence $$\frac{u}{u_1} = \sin \frac{\pi}{2} \frac{y}{\delta}$$

which is Lamb's approximation[6] for the velocity distribution in incompressible flow.

Thus there is correspondence between Young's approximation

$$\frac{F}{F_0} = \left\{ 1 - \left(\frac{u}{u_1} \right)^2 \right\}^{1/2} \qquad (20)$$

for the distribution of shearing stress in compressible flow and Lamb's approximation[6] for velocity distribution in incompressible flow.

6.1.2. $M_1 = 2 \cdot 5$, $T_p = T_e$, $C = 1$, $\sigma = 1$.—The case $C = 1$, $\sigma = 1$ has been analysed by Howarth[1] under zero heat-transfer conditions and a velocity distribution can be obtained from his results by numerical integration.

For $C = 1$, $\sigma = 1$ and zero heat transfer, we have

$$A = \frac{T_e}{T_1} = 1 + \frac{\gamma - 1}{2} M_1^2, B = 0, C = 1, D = \frac{\gamma - 1}{2} M_1^2$$

and equation 32 gives

$$\eta = 1 \cdot 506 \left\{ \left(1 + \frac{\gamma - 1}{4} M_1^2 \right) \sin^{-1} z + \frac{\gamma - 1}{4} M_1^2 z (1 - z^2)^{1/2} \right\} . \qquad (32b)$$

Values computed from equation 32b are compared in Table 2 and Fig. 5 with values computed from Howarth's analysis. The case $M_1 = 2 \cdot 5$ has been chosen, and the agreement is sufficient for all practical purposes, the maximum discrepancy in u/u_1 being of the order of 2 per cent.

6.1.3. $M_1 = 5 \cdot 0$, $T_p = \frac{1}{4} T_1$, $C = 1$, $\sigma = 0 \cdot 7$.—This is a case of compressible flow with heat transfer for which Hantzsche and Wendt[5] give a velocity distribution in graphical form.

Using the approximate formula (equation 32) we have under the above conditions

$$A = \tfrac{1}{4}$$

$$\frac{T_e}{T_1} = 1 + \sigma^{1/2} \frac{\gamma - 1}{2} M_1^2 = 5 \cdot 18$$

$$B = -4 \cdot 38$$

$$D = 3 \cdot 5$$

and since $\quad C = 1,\ F_0 = 0 \cdot 664.$

Hence

$$\eta = 1 \cdot 506\{4 \cdot 38 - 1 \cdot 5 \sin^{-1} z - (4 \cdot 38 - 1 \cdot 75\, z)(1 - z^2)^{1/2}\} \quad \ldots \quad \ldots \quad (32c)$$

from which

$z =$	0·1	0·3	0·5	0·7	0·8	0·9	0·95	1·0
$\eta =$	0·0675	0·366	0·843	1·451	1·813	2·227	2·485	3·033

These values are plotted in Fig. 6, where they are compared with the curve taken from Hantzsche and Wendt's report[5]. The agreement is within 1 per cent.

6.2. *General Remarks.*—Equation 32 has been shown to give a sufficiently good approximation to the velocity distribution across the boundary layer in three representative cases. It also possesses the advantage of being algebraic in form and is relatively simple to evaluate, by comparison with earlier estimates which have involved graphical or numerical integration.

7. *Displacement Thickness.*—The displacement thickness δ^* is given by

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy,$$

i.e.,

$$\frac{1}{2} \frac{\partial^*}{x} \sqrt{Re_x} = \frac{C}{F_0} \int_0^1 \frac{A - (B + 1)\, z - Dz^2}{(1 - z^2)^{1/2}}\, dz$$

where $z = u/u_1$ and substitutions have been made from equations 3, 10, 15, 18, 20 and 31. By integration

$$\frac{1}{2} \frac{\delta^*}{x} \sqrt{Re_x} = \frac{C}{F_0} \left\{\left(A - \frac{D}{2}\right) \frac{\pi}{2} - (B + 1)\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

where, as before
$$\left.\begin{aligned} A &= \frac{T_p}{T_1} \\ B &= \sigma^{1/3}\,\frac{T_p - T_e}{T_1} \\ D &= \sigma\,\frac{\gamma - 1}{2}\,M_1^2 \end{aligned}\right\} \qquad \ldots \quad (31)$$

and $\dfrac{T_e}{T_1} = 1 + \sigma^{1/2}\,\dfrac{\gamma - 1}{2}\,M_1^2$ from equation 6.

7.1. *Comparison with Particular Exact Values.*—7.1.1. $M_1 = 0$, $T_p = T_1$.—In this case $A = 1$, $B = 0$, $D = 0$ and $F_0 = 0\cdot664$, $C = 1$.

Hence
$$\frac{1}{2}\,\frac{\delta^*}{x}\,(Re_x)^{1/2} = 1\cdot506\left(\frac{\pi}{2} - 1\right),$$

i.e.,
$$\frac{\delta^*}{x}\,(Re_x)^{1/2} = 1\cdot7193$$

as compared with the Blasius value
$$\frac{\delta^*}{x}\,(Re_x)^{1/2} = 1\cdot7208$$

which shows agreement within $0\cdot1$ per cent.

(Note that if $F_0 = 0\cdot655$, as determined in section 4.2, then $\dfrac{\delta^*}{x}\,(Re_x)^{1/2} = 1\cdot7429$).

7.1.2. $T_p = T_e$, $\sigma = 1$, $C = 1$.—From Howarth's analysis[1], using the integral formula for δ^* and by numerical integration we can obtain the formula
$$\frac{\delta^*}{x}\,(Re_x)^{1/2} = 1\cdot7208\left(1 + 1\cdot385\,\frac{\gamma - 1}{2}\,M_1^2\right) \qquad \ldots \quad (34)$$

under the above conditions.

Under the same conditions, from equation 31
$$A = \frac{T_e}{T_1} = 1 + \frac{\gamma - 1}{2}\,M_1^2$$
$$B = 0$$
$$D = \frac{\gamma - 1}{2}\,M_1^2$$

and $\qquad F_0 = 0\cdot664$ from equation 19.

Then equation 33 becomes
$$\frac{\delta^*}{x}\sqrt{Re_x} = 3\cdot012\left[\left(1 + \frac{\gamma - 1}{4}\,M_1^2\right)\frac{\pi}{2} - 1\right]$$
$$= 1\cdot7208\left[0\cdot9992 + 1\cdot375\,\frac{\gamma - 1}{2}\,M_1^2\right]. \qquad \ldots \quad (33a)$$

At $M_1 = 5\cdot0$, equation 34 gives

$$\frac{\delta^*}{x}\sqrt{Re_x} = 13\cdot64$$

while equation 33a gives

$$\frac{\delta^*}{x}\sqrt{Re_x} = 13\cdot56,$$

i.e., the agreement is within $0\cdot75$ per cent. This is sufficiently close, because equation 34, derived by numerical integration must also be regarded as an approximation.

Thus equation 33 may be expected to give a sufficiently accurate estimate of displacement thickness under all conditions.

8. *Ratio of Displacement to Momentum Thickness.*—Equation 33 gives

$$\frac{1}{2}\frac{\delta^*}{x}\sqrt{Re_x} = \frac{C}{F_0}\left\{\left(A - \frac{D}{2}\right)\frac{\pi}{2} - (B + 1)\right\}. \quad \ldots \quad (33)$$

Section 4.2, equation 22, gives

$$\frac{1}{2}\frac{\theta}{x}\sqrt{Re_x} = \frac{C}{F_0}\frac{4-\pi}{4} \quad .$$

Hence

$$H = \frac{\delta^*}{\theta} = \frac{\left(A - \frac{D}{2}\right)\frac{\pi}{2} - (B + 1)}{1 - \frac{\pi}{4}} \quad \ldots \quad (35)$$

should give a sufficiently accurate estimate of the ratio of displacement to momentum thickness and we may note that it is independent of the choice of viscosity law.

9. *Summary of Approximate Formulae.*—All the formulae are dependent on $\partial p/\partial x = 0$ and $\partial i/\partial x = 0$.

1. Enthalpy-velocity relation is

$$\frac{i}{i_1} = \frac{i_p}{i_1} - \sigma^{1/3}\left(\frac{i_p}{i_1} - \frac{i_e}{i_1}\right)\frac{u}{u_1} - \sigma\frac{\gamma - 1}{2}M_1^2\left(\frac{u}{u_1}\right)^2$$

which, if c_p is constant becomes the

2. Temperature-velocity relation

$$\frac{T}{T_1} = A - B\frac{u}{u_1} - D\left(\frac{u}{u_1}\right)^2$$

where

$$A = \frac{T_p}{T_1}$$

$$B = \sigma^{1/3}\left(\frac{T_p}{T_1} - \frac{T_e}{T_1}\right)$$

$$D = \sigma\frac{\gamma - 1}{2}M_1^2$$

and $$\frac{T_e}{T_1} = 1 + \sigma^{1/2}\frac{\gamma - 1}{2} M_1^2$$

3. Variation of shearing stress across the boundary layer is

$$\frac{F}{F_0} = \left\{1 - \left(\frac{u}{u_1}\right)^2\right\}^{1/2}$$

where $$F = C_\tau \sqrt{Re_x}$$

$$C_\tau = \frac{\tau}{\frac{1}{2}\rho_1 u_1^2}, \quad Re_x = \frac{\rho_1 u_1 x}{\mu_1}$$

and $$F_0 = c_f \sqrt{Re_x}.$$

4. Local skin friction coefficient

$$F_0 = c_f \sqrt{Re_x} = 0\cdot664 \sqrt{C}$$

where $$C = \left(\frac{T'}{T_1}\right)^{1/2} \frac{1 + T_c/T_1}{T'/T_1 + T_c/T_1}$$

and $$\frac{T'}{T_1} = A - 0\cdot468\, B - 0\cdot273\, D$$

5. Momentum thickness is given by

$$\frac{1}{2}\frac{\theta}{x}\sqrt{Re_x} = \frac{C}{F_0} \cdot \frac{4 - \pi}{4}$$

6. Displacement thickness is

$$\frac{1}{2}\frac{\delta^*}{x}\sqrt{Re_x} = \frac{C}{F_0}\left\{\left((A - \frac{D}{2}\right)\frac{\pi}{2} - (B + 1)\right\}$$

7. Velocity distribution is

$$\frac{1}{2}\frac{y}{x}\sqrt{Re_x} = \frac{C}{F_0}\left\{\left(A - \frac{D}{2}\right)\sin^{-1}z + \left(\frac{Dz}{2} + B\right)(1 - z^2)^{1/2} - B\right\}$$

where $$z = \frac{u}{u_1}.$$

10. *Conclusions*.—1. By assuming that enthalpy and velocity are dependent only on local conditions and by accepting certain relations obtained by Crocco, an approximate enthalpy-velocity relation is obtained for the laminar boundary layer on a flat plate with $\partial p/\partial x = 0$ and $\partial i/\partial x = 0$. This relation gives a close approximation to Crocco's numerical results for $\sigma = 0\cdot725$ and $1\cdot25$ at least up to $u/u_1 = 0\cdot8$.

2. By taking a viscosity-temperature relation of the form

$$\frac{\mu}{\mu_1} = C\,\frac{T}{T_1}$$

as proposed by Chapman and Rubesin, where C is a constant, the variation of shearing stress across the layer when $\partial p/\partial x = 0$ and $\partial i/\partial x = 0$ is shown to be independent of C and an approximation suggested by Young is adopted.

3. The local skin friction coefficient (c_f) can serve to determine C. Approximate formulae for c_f are already available for $\sigma = 0 \cdot 725$ and a new generalisation for other values of σ is suggested.

4. Approximate formulae for displacement thickness and velocity distribution are then derived, which are in very close agreement, at least up to $M_1 = 5 \cdot 0$, with some representative cases obtained by numerical integration of more exact formulae.

LIST OF SYMBOLS

x		Distance measured along surface of plate
y		Distance measured normal to surface of plate
u, v		Components of velocity in the directions x, y
z	$=$	u/u_1 where u_1 is free-stream value of u
ρ		Density
μ		Dynamic viscosity
k		Thermal conductivity
c_p		Specific heat at constant pressure
σ		Prandtl number $\left(= \frac{c_p \mu}{k}\right)$
T		'Static' temperature
i		Enthalpy ($= Jc_pT$, where J is the mechanical equivalent of heat)
i_H	$=$	$i + \frac{1}{2}u^2 \; \{u^2/i = (\gamma - 1)M^2\}$
i_e		Equilibrium value of enthalpy ($= i_1 + \sigma^{1/2} \frac{1}{2} u_1^2$ where subscript 1 denotes free-stream conditions)
τ		local shearing stress in the boundary layer
τ_0		local skin friction
c_f	$=$	$\frac{\tau_0}{\frac{1}{2}\rho_1 u_1^2}$ $C_\tau = \frac{\tau}{\frac{1}{2}\rho_1 u_1^2}$
Re_x		Reynolds number ($= \rho_1 u_1 x/\mu_1$)

LIST OF SYMBOLS—*continued.*

η	$=$	$\dfrac{1}{2}\dfrac{y}{x}\sqrt{Re_x}$
F	$=$	$C_\tau\sqrt{Re_x}$
F_0	$=$	$c_f\sqrt{Re_x}$
C		Defined by $\dfrac{\mu}{\mu_1} = C\dfrac{T}{T_1}$, Sutherland's formula and equation 25 or 28
A, B and D		Constants in enthalpy-velocity relation. Defined by equations 31

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TABLE 1

Comparison of the approximate enthalpy-velocity relation

$$\frac{i}{i_1} = \frac{i_p}{i_1} - \sigma^{1/3}\left(\frac{i_p - i_e}{i_1}\right)\frac{u}{u_1} - \sigma\,\frac{\gamma - 1}{2}\,M_1{}^2\left(\frac{u}{u_1}\right)^2$$

with the exact numerical relation

$$\frac{i}{i_1} = \frac{i_p}{i_1} - \left(\frac{i_p - i_e}{i_1}\right) f_1\left(\sigma, \frac{u}{u_1}\right) - \sigma\,\frac{\gamma - 1}{2}\,M_1{}^2 f_2\left(\sigma, \frac{u}{u_1}\right)$$

obtained by Crocco in Ref. 2 for $\mu \propto T$.

$\frac{u}{u_1}$	$\sigma = 0{\cdot}725$				$\sigma = 1{\cdot}25$			
	$\sigma^{1/3}\frac{u}{u_1}$	f_1	$\left(\frac{u}{u_1}\right)^2$	f_2	$\sigma^{1/3}\frac{u}{u_1}$	f_1	$\left(\frac{u}{u_1}\right)^2$	f_2
0·1	0·0898	0·0892	0·0100	0·0100	0·1077	0·1081	0·0100	0·0100
0·3	0·2694	0·2680	0·0900	0·0902	0·3232	0·3238	0·0900	0·0898
0·5	0·4490	0·4491	0·2500	0·2520	0·5386	0·5371	0·2500	0·2480
0·7	0·6286	0·6373	0·4900	0·5024	0·7540	0·7431	0·4900	0·4792
0·8	0·7184	0·7378	0·6400	0·6674	0·8618	0·8405	0·6400	0·6172
0·9	0·8082	0·8474	0·8100	0·8686	0·9695	0·9301	0·8100	0·7634
0·96	0·8621	0·9244	0·9216	1·0206	1·0341	0·9768	0·9216	0·8490
1·00	0·8980	1·0000	1·0000	1·1742	1·0772	1·0000	1·0000	0·8946

Exact agreement is obtained for $\sigma = 1{\cdot}0$.

TABLE 2

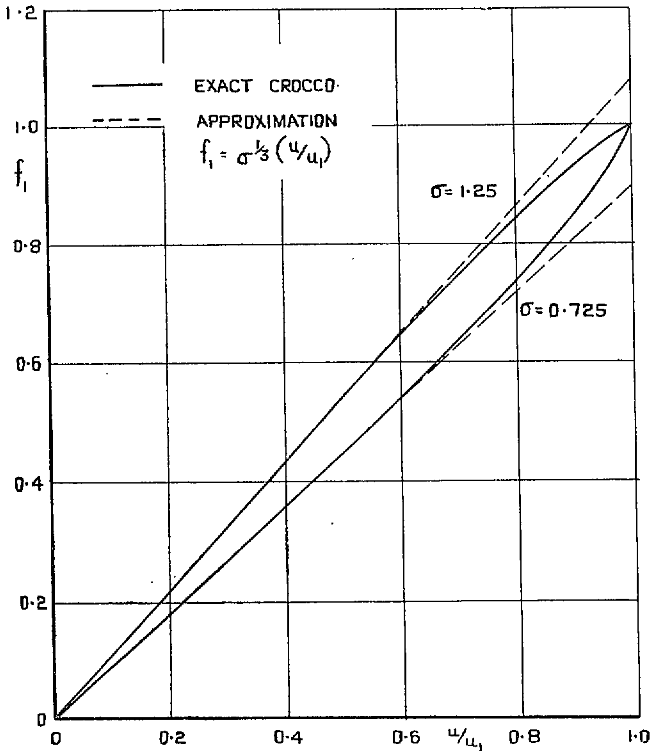
Comparison of approximate (equation 32) and exact velocity distributions

(i) $M_1 = 0 \quad T_p = T_1$

$\eta = \frac{1}{2}\frac{y}{x}\sqrt{Re_x}$	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0	2·365	3·0
$z = \frac{u}{u_1}$ Equation 32	0·1324	0·2625	0·3885	0·507	0·616	0·716	0·801	0·874	0·931	0·972	1·000	—
$z = \frac{u}{u_1}$ Blasius	0·1328	0·2647	0·3938	0·517	0·630	0·729	0·811	0·876	0·923	0·955	0·986	0·999
Equation 32 / Blasius	0·998	0·993	0·988	0·982	0·978	0·982	0·988	0·998	1·007	1·017	1·014	

(ii) $M_1 = 2{\cdot}5,\ \sigma = 1,\ \mu \propto T,\ T_p = T_0$

$\frac{u}{u_1}$	0·1	0·3	0·5	0·7	0·8	0·9	0·99	0·999	1·0
$\frac{1}{2}\frac{y}{x}\sqrt{Re_x}$ Equation 32	0·338	1·015	1·688	2·370	2·715	3·105	3·623	—	3·84
$\frac{1}{2}\frac{y}{x}\sqrt{Re_x}$ from Howarth[1]	0·338	1·004	1·654	2·311	2·678	3·099	3·944	4·49	—



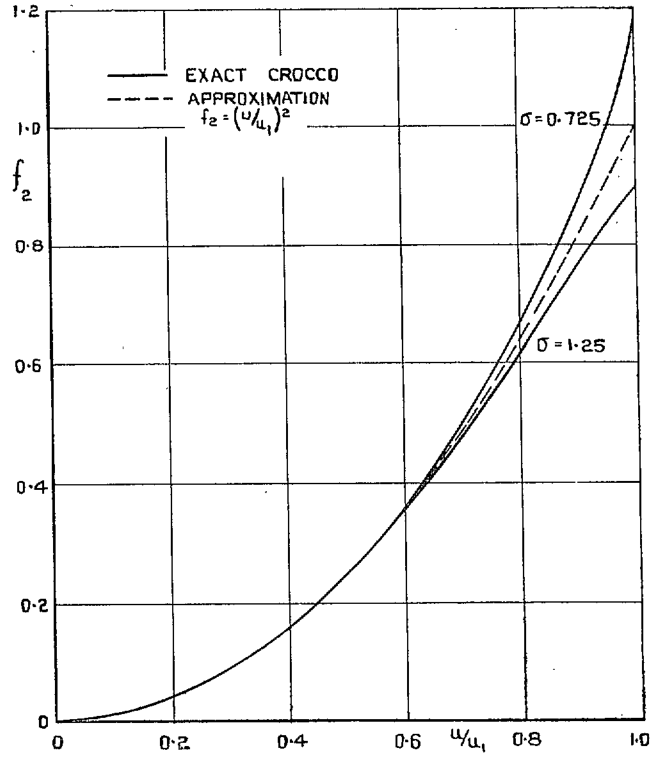


FIG. 1. Variation of terms in enthalpy-velocity distribution.

$$\frac{i}{i_1} = \frac{i_p}{i_1} - \left(\frac{i_p - i_e}{i_1}\right) f_1(\sigma, u/u_1) - \sigma \frac{\gamma - 1}{2} M_1^2 f_2(\sigma, u/u_1)$$

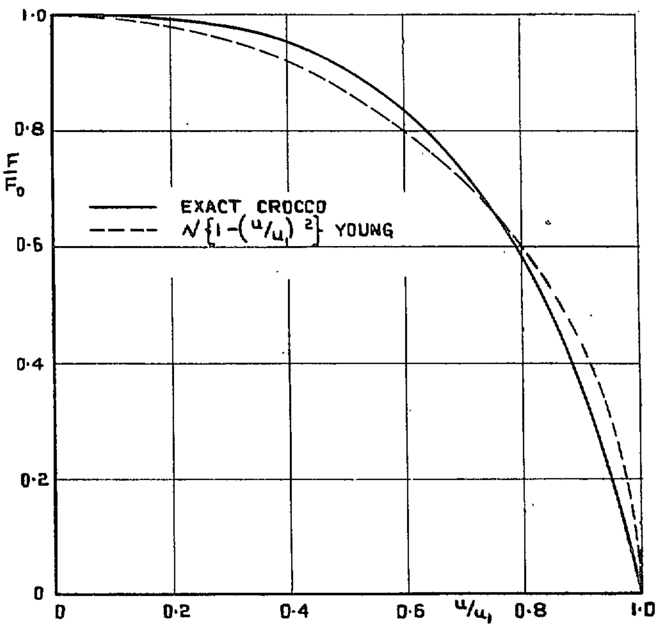


FIG. 2. Variation of shearing stress across the boundary layer.

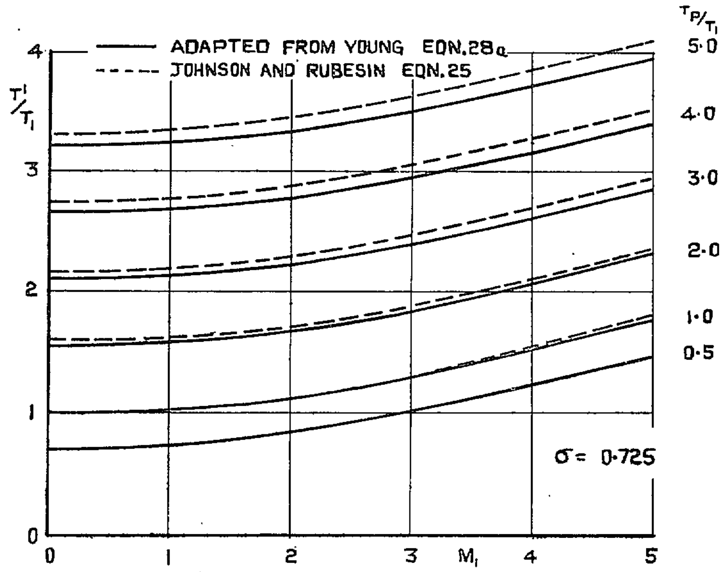


FIG. 3. Temperature ratio for evaluating $\sqrt{C}$ in Fig. 4.

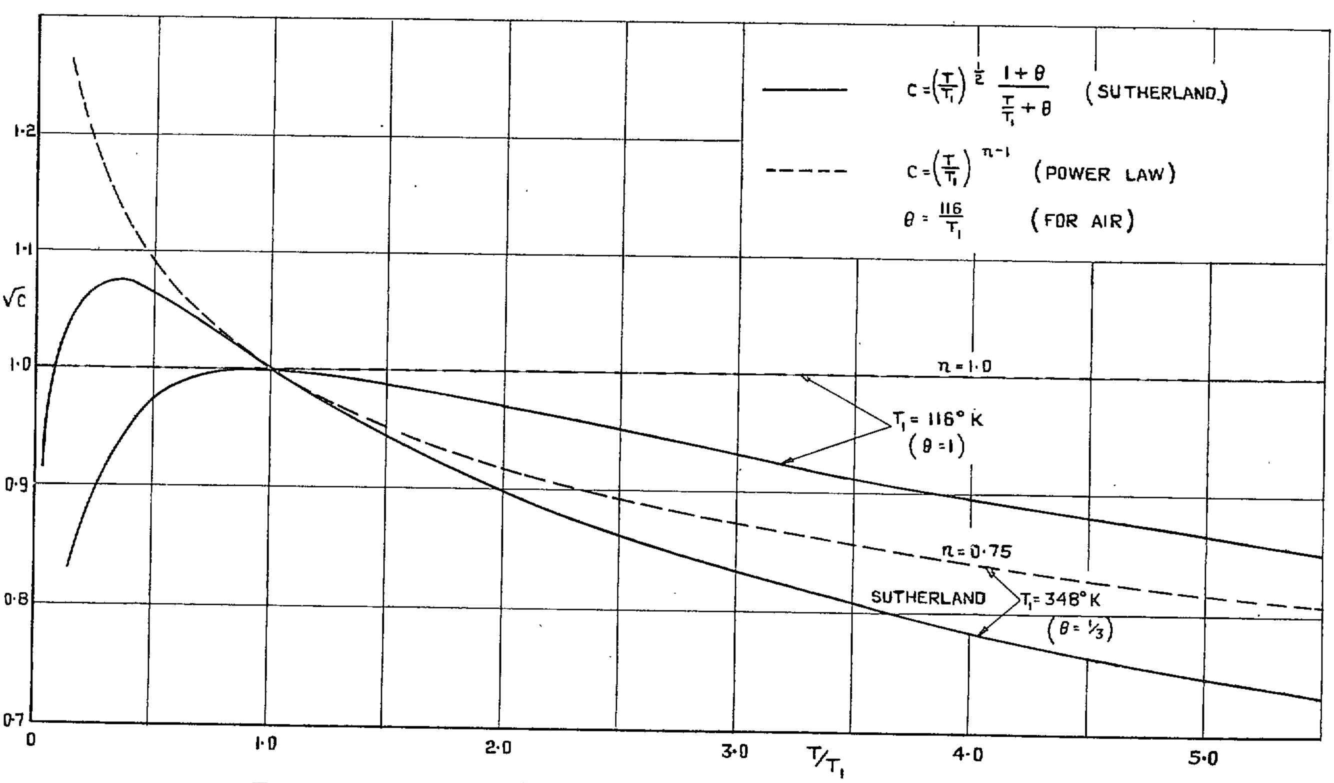


FIG. 4. Variation of $\sqrt{C}$ with T/T_1 according to various viscosity laws.

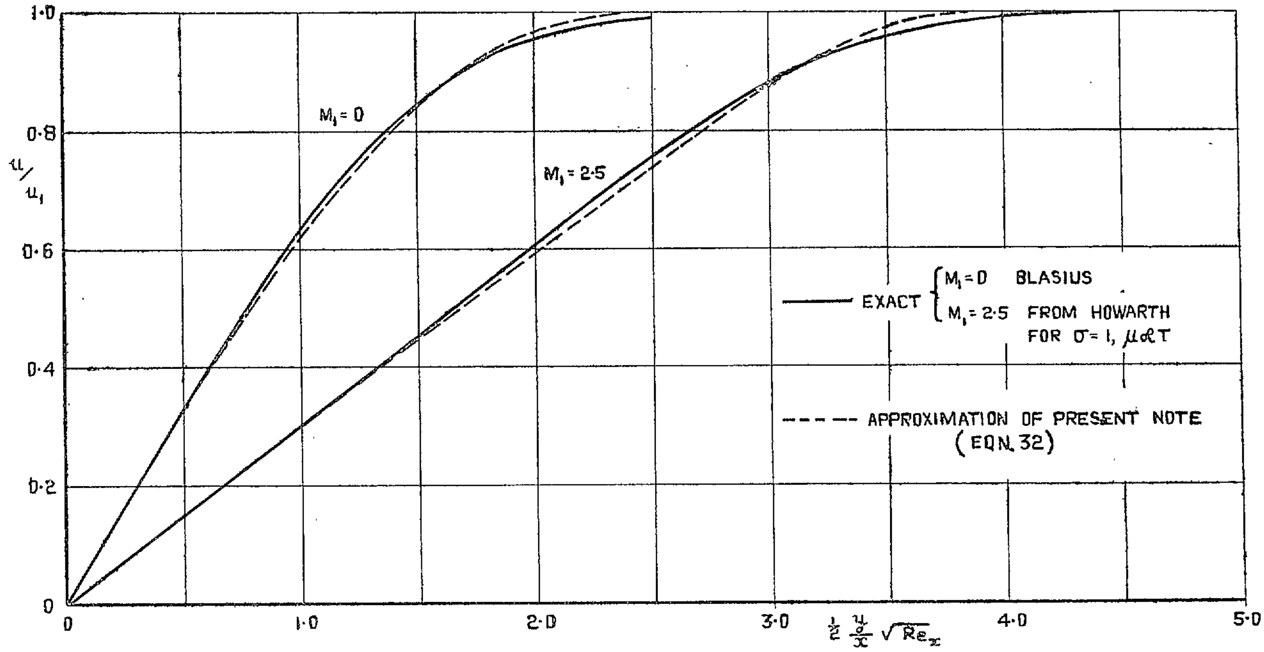


FIG. 5. Comparison of approximate and exact velocity distributions (zero heat transfer).

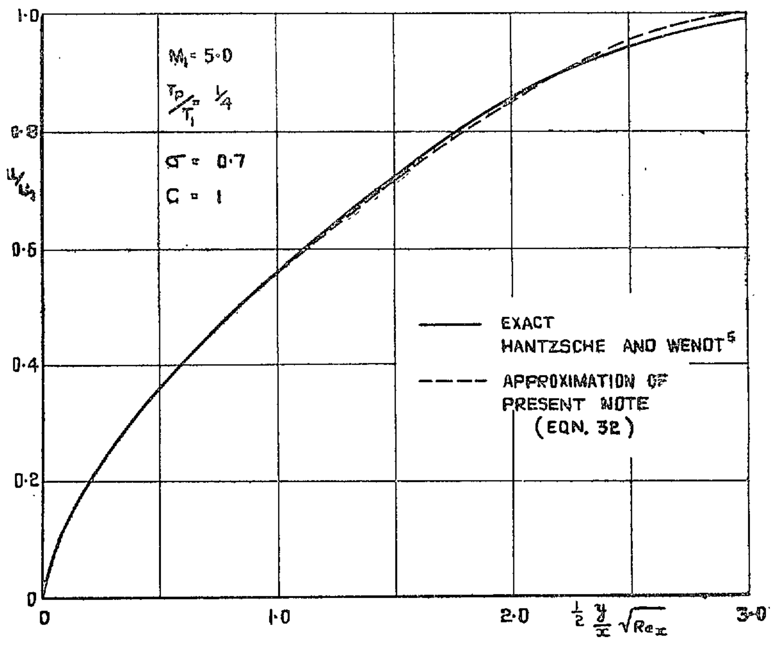


FIG. 6. Comparison of approximate and exact velocity distributions (with heat transfer).

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