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Methods of Approaching an Accurate Three-dimensional Potential Solution for a Wing

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Summary.—There is a great need for more accurate data on the aerodynamic derivatives of swept-back wings in order to solve problems of stability, control and flutter. As one step in the search for these data the estimation of the three-dimensional potential solution is essential, and if it is to be of value the degree of accuracy of any approximation must be known beyond question.

This report gives attention to some fundamental aspects of the vortex-sheet theory for determining the distribution of lift on a finite wing. The accuracy and limitations of some existing approximate forms of the theory are discussed. With special reference to the labour of computation an iterative approach to an accurate solution is suggested, and the general mathematical expression for the distribution of lift required to give an exact solution for a Vee wing is considered.

It is proposed—

(*a*) That, with the specific purpose of checking the Falkner (R. & M. 1910[7], 1943) vortex-lattice theory, the iterative procedure should be applied to a wing of constant chord with acute hyperbolic leading and trailing edges (*see* Fig. 3).

(*b*) That by choosing suitable functions calculations should be undertaken to determine a reliable potential solution for a Vee wing (*see* Fig. 2) in an inclined stream.

(*c*) That further study is needed before calculations can usefully be undertaken to improve the accuracy of existing methods of estimating the characteristics of deflected controls.

1. *Introduction.*—The problem will be expressed in rectangular co-ordinates (x, y, z), referred to a convenient point on the chord of the wing in the plane of symmetry. The undisturbed fluid velocity is supposed to be uniform of magnitude V in the direction of the axis of x. The axis of z points vertically downwards. The components of fluid velocity are denoted by $(V + u, v, w)$. Within a region of irrotational flow a velocity potential Φ exists. For an inviscid fluid there will be bound vorticity on the wing surface, shed vorticity in the wake and irrotational flow elsewhere. Outside the vortical region it is permissible to write

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial x} &= V + u \\ \frac{\partial \Phi}{\partial y} &= v \\ \frac{\partial \Phi}{\partial z} &= w \end{aligned}\right\} \qquad \ldots \qquad (1)$$

where $\Phi(x, y, z)$ is a continuous function. Furthermore, for an incompressible fluid Φ will satisfy Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \,. \qquad \ldots \qquad (2)$$

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In the first place vortex-sheet theory assumes that the wing is infinitely thin. In the second place the theory is a linear one; and it is supposed that the induced components of velocity (u, v, w) are so small compared with V that the second-order terms are negligible. It is apparent that the theory breaks down in the neighbourhood of a stagnation point.

Let the under suffices a, b denote values immediately above and below the wing surface. From considerations of energy the fluid pressures p on opposite surfaces are related by

$$\begin{aligned} p_b - p_a &= \tfrac{1}{2}\rho\,[\{(V + u_a)^2 + v_a^2 + w_a^2\} - \{(V + u_b)^2 + v_b^2 + w_b^2\}] \\ &= \rho V(u_a - u_b) \\ &= \rho V \frac{\partial}{\partial x}(\Phi_a - \Phi_b) \text{ from equation (1)}, \qquad \ldots \qquad (3) \end{aligned}$$

where ρ is the fluid density.

The direction of flow on the solid boundary must follow the contour of the wing surface defined by $z(x, y)$. Thus

$$w = \frac{\partial \Phi}{\partial z} = V \frac{\partial z}{\partial x} \qquad \ldots \qquad (4)$$

at all points of the plan form, and z/c, where c is the wing chord, is of the same order as w/V. Now from equation (2),

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial z^2} &= -\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} \\ &= -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = O\left(\frac{w}{c}\right) \end{aligned}$$

at all points inside the perimeter of the plan form. Then the variation in w with z is

$$\frac{\partial \Phi}{\partial z} - \left(\frac{\partial \Phi}{\partial z}\right)_{z=0} = \frac{\partial^2 \Phi}{\partial z^2} \cdot z = O\left(\frac{wz}{c}\right),$$

which is negligible. For the purpose of determining w, therefore, the vorticity may be confined to the plane $z = 0$.

Thus the assumption of a linear theory leads to two distinct simplifications:—

(a) The complete distribution of lift, given by $(p_b - p_a)$ in equation (3), is determined directly from the values of the velocity potential on the wing surface.

(b) The field of flow near the wing is represented by the velocity potential $\Phi(x, y, z)$, defined in equation (1) and continuous except across a cut C in the plane $z = 0$ bounded by the leading edge of the wing and extending to infinity in the wake.

A unique velocity potential satisfying equation (2) is (Lamb, 1932)[1],

$$\Phi(x_1, y_1, z_1) = Vx_1 + \frac{1}{4\pi}\int_C\!\!\int (\Phi_a - \Phi_b)\frac{\partial}{\partial z_1}\left(\frac{1}{r}\right) dx\, dy\,, \qquad \ldots \qquad (5)$$

where $r^2 = (x - x_1)^2 + (y - y_1)^2 + z_1^2$ and $(\Phi_a - \Phi_b)$ is the required discontinuity in Φ across the cut at $(x, y, 0)$. Since $p_b = p_a$ in the wake, to the first order it follows that

$$\frac{\partial}{\partial x}(\Phi_a - \Phi_b) = 0\,,$$

and in the wake $(\Phi_a - \Phi_b)$ is a function of y only determined by its value at the trailing edge.

Equations (4) and (5) determine a unique potential solution for the flow in the neighbourhood of an infinitely thin wing of any plan form, bent or cambered to the first order. In the course of this introduction it has also been assumed that the flow is steady, incompressible and inviscid. Non-uniform flow has been considered by W. P. Jones (R. & M. 2117[2], 1945). The solution for compressible flow may be related to an equivalent incompressible potential solution by the linear perturbation theory described by Goldstein and Young (R. & M. 1909[3], 1943). The effect of thickness/chord ratio will be determined to the first order by replacing each wing section by a cambered plate whose two-dimensional pressure distribution[4] (Glauert, 1937) is equivalent to that of the wing section at its particular incidence. On a similar basis it is possible to approximate to the solution with a viscous fluid by using empirical two-dimensional characteristics of the wing section (R. & M. 2730[5]). However this will not account for the pressure gradient across the finite region of stagnant fluid in the wake and its influence on the free shed vorticity. With certain limitations, therefore, the steady, compressible viscous flow past wings of finite thickness may be determined from a potential solution, given by equations (4) and (5), for an equivalent thin wing.

2. *General Potential Solution.*—The potential solution for a thin wing gives a distribution of lift per unit area

$$p_b - p_a = \rho V \frac{\partial}{\partial x} (\Phi_a - \Phi_b) , \qquad \text{(eqn.3)}$$

where $(\Phi_a - \Phi_b)$ is determined from the equations

$$\lim_{z_1 \to 0} \frac{\partial \Phi}{\partial z_1} = V \frac{\partial z}{\partial x} \qquad \text{(eqn.4)}$$

$$\Phi(x_1, y_1, z_1) = Vx_1 + \frac{1}{4\pi} \int_C \int (\Phi_a - \Phi_b) \frac{\partial}{\partial z_1} \left(\frac{1}{r}\right) dx\, dy , \qquad \text{(eqn.5)}$$

where $z(x, y)$ is the contour of the wing surface relative to the undisturbed stream, and C is bounded by the leading edge of the wing and extends to infinity in the wake, where $(\Phi_a - \Phi_b)$ is a function of y only determined by its value at the trailing edge. Equations (4) and (5) determine $(\Phi_a - \Phi_b)$ uniquely over the plan form for any prescribed $z(x, y)$. A solution is obtained in three operations, which will be considered in turn in sections 2.1, 2.2 and 2.3.

(1) assuming a general form for $(\Phi_a - \Phi_b)$ with arbitrary coefficients,

(2) substituting the expression for $(\Phi_a - \Phi_b)$ in equation (5),

(3) determining the arbitrary coefficients from equation (4).

In each operation there are practical and mathematical difficulties to consider. To elucidate the latter is to complicate the former; and this report will examine what additional labour is necessary in order to assess and improve the accuracy of existing approximate methods.

2.1. The choice of the form for $(\Phi_a - \Phi_b)$ is dependent on any discontinuities in the problem such as occur—

(*a*) near the perimeter of the plan form,

(*b*) at abrupt changes in $\partial z/\partial x$.

It is clearly desirable that the general form for $(\Phi_a - \Phi_b)$ should be capable of producing tangential flow along the complete vortex sheet. A particular solution in the chosen form may then be considered as an approximation to the unique potential solution of the problem.

If it is assumed that both the leading and trailing edges of the wing form smooth curves, the pressure distribution over a cambered or symmetrical wing without deflected flaps, for which $\partial z/\partial x$ is a continuous function of x and y, may be represented by the double series

$$\frac{p_b - p_a}{\rho V^2} = \sum_{n=0}^{N} \sum_{m=1}^{M} C_{nm} A_m(y)\, \Gamma_n(\theta)\,, \qquad (6)$$

where θ is the usual chordwise angular co-ordinate related to (x, y) by

$$x = R(y) - \frac{c(y)}{2} \cos \theta\,,$$

where $x = R(y)$ represents a curve through the mid-chord points,

$$\Gamma_0(\theta) = 2 \cot \tfrac{1}{2}\theta\,,$$

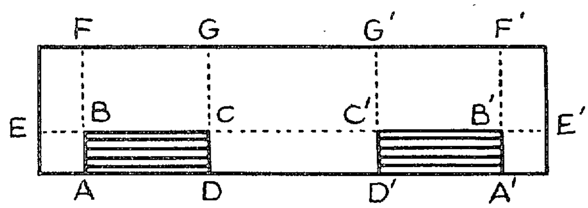
$$\Gamma_n(\theta) = -\,2 \sin n\theta \text{ for } n \geqslant 1\,,$$

$$A_m(y) = \frac{s}{c}\left(\frac{y}{s}\right)^{m-1}\left[1 - \left(\frac{y}{s}\right)^2\right]^{1/2},$$

where s is the wing semi-span, and C_{nm} is an arbitrary constant coefficient.

The form (6) is used by Jones (R. & M. 2145[6], 1943) and Falkner (R. & M. 1910[7], 1943) but will not suffice when there are important irregularities in the plan form. This complication (*a*) is discussed in section 4 of the report.

Complication (*b*) arises when the vortex-sheet theory is applied to the problem of wing loading due to deflected control surfaces. The basic pressure distribution should then be modified to give the necessary discontinuities in $\partial z/\partial x$ round the perimeters of the control surfaces. Such a function can be constituted, but in the opinion of the author with $(\Phi_a - \Phi_b)$ so expressed the evaluation of w becomes unmanageable. There remains the option of introducing discontinuities in w, where they should not exist or smoothing out discontinuities in w, where they should exist. In the accompanying diagram the control surfaces are represented by ABCD, A′B′C′D′.



In the approximate theory of Ref. 8 (1946), the author decided to admit the chordwise discontinuity by using equation (6) with an infinite series in $n (N = \infty)$, but to obviate the spanwise discontinuity by taking the given polynomial form for A_m and a finite number of values of m (six). This only produced discontinuities in w across the line EE′. Across most of BC, C′B′, the discontinuities were approximately correct and across most of EB, CC′, B′E′, the discontinuities were quite small. This type of solution, however, is only practicable when the chord ratio, E, of the control is constant along its span. With a variable E it is only possible to take a finite value of N admitting no discontinuities in w whatsoever. The apparent alternative is to supplement the functions $A_m(y)$ in equation (6) with the appropriate spanwise logarithmic functions of Multhopp[9] (1938) on the lines of the P functions recommended by Falkner (R. & M. 2591[14], 1947). This would produce discontinuities in w in a spanwise direction across AF, DG, D′G′, A′F′. The author would oppose this alternative for three reasons:—

(*a*) The unwelcome introduction of the discontinuities across BF, CG, C′G′, B′F′.

(*b*) The complication of the evaluation of w, over sizeable strips of the plan form containing AF, DG, D′G′, A′F′.

(*c*) The fact that the only advantage from this alternative, a more accurate determination of the spanwise distribution of lift, can always be assessed from the wavy polynomial solution by comparing corresponding solutions by the lifting-line theory[9].

The remainder of the report is concerned with problems for which $\partial z/\partial x$ is a continuous function of x and y. The form of $(\Phi_a - \Phi_b)$ determined by equations (3) and (6) will suffice, unless there are important irregularities in the plan form (*see* section 4). It is usual to take between four and eight spanwise variables according to the detail required. In the symmetrical problem, for example, $7 \leqslant M \leqslant 15$ with zero coefficients for the even values of m. The choice of N depends on the application of the problem. If only the spanwise distribution of lift is required, there is a strong argument for taking $N = 0$ or 1, and when the aerodynamic centre is also required, $N = 1$ or 2. But of considerable importance is the estimation of control hinge moments and the values of $b_1(\partial C_H/\partial \alpha)$ will presumably be covered by this solution provided N is large enough, 3 or 4 perhaps. Particularly near the trailing edge little reliance can be put on the two-dimensional form of the pressure distribution, as the conditions of flow close to the free vorticity are very critical. Since rigorous justification of the approximations involved in taking small values of N is quite lacking, the author is undertaking a numerical calculation* of the wing loading on a delta wing of aspect ratio $A = 3$ in which only the form of $(\Phi_a - \Phi_b)$ near the leading edge is derived from two-dimensional theory.

2.2. The main difficulty in the vortex-sheet theory lies in the evaluation of w. Having selected the form of $(\Phi_a - \Phi_b)$, the existing methods of obtaining approximate solutions of the general problem may be divided into the two types:—

(*a*) a method in which the flow near the wing due to the arbitrary pressure distribution is represented by that due to a finite number of vortices of finite strength,

(*b*) a method in which the flow at the wing surface is determined by the direct evaluation of the limiting normal component

$$w = \frac{\partial \Phi}{\partial z_1} \text{ as } z_1 \to 0 \,,$$

where Φ is given by equation (5).

The following approximate solutions will be considered:

In type (*a*)—

(i) Lifting-line theory[9] (Multhopp, 1938).
(ii) L-Method of Weissinger[10] (1942).
(iii) Vortex-lattice theory (R. & M. 1910[7], 1943).

In type (*b*)—

(iv) Method of Jones (R. & M. 2145[6], 1943).
(v) Numerical evaluation for a particular delta wing ($A = 3$).*

The methods of (i) and (ii) are only of use in determining the spanwise distribution of lift. They both assume that $N = 0$ and yield a solution in at most one computer-day. Solution (i) is shorter, easier to handle and applicable to problems with deflected controls, but takes no account of sweep and even for straight wings is subject to errors of 6 to 15 per cent. Solution (ii), though inapplicable to controls, has the merit of an established accuracy of the order $\frac{1}{2}$ per cent for a rectangular wing ($A = 5$). For the limited purpose of determining the spanwise load the problem can present a simplified picture, when it is assumed that w is linear along each chord. As emphasised by Weissinger it is then rigorously correct to take $N = 0$ and to use the value of w at three-quarter chord. Provided $\partial z/\partial x$ is a linear function of x, the approximation of Ref. 8 leads to a solution for which $N = 1$, and will illustrate this concept. If equation (6) is written in the form

$$\frac{p_b - p_a}{\rho V^2} = \sum_{m=1}^{M} A_m(y)[(C_{0m} - \tfrac{1}{2}C_{1m})\, 2 \cot \tfrac{1}{2}\theta + C_{1m}(-2 \sin \theta + \cot \tfrac{1}{2}\theta)] \,, \quad .. \quad (7)$$

* This will be published in R. & M. 2819.

then the second term in the square bracket of equation (7) produces no circulation and, in accordance with two-dimensional principles as it is assumed, gives zero w at three-quarter chord. The lift per unit span,

$$\rho V K(y) = \pi \rho V^2 c(y) \sum_{m=1}^{M} A_m(y)(C_{0m} - \tfrac{1}{2}C_{1m}) ,$$

then determines the boundary conditions along the line through the three-quarter chord points

$$x = R(y) + \tfrac{1}{4}c(y) .$$

The problem is thus reduced to that of a single variable $K(y)$. Moreover Weissinger has shown,

(A) that the assumption of linearity in w leads to an error of only $\frac{1}{2}$ per cent in the C_L of a rectangular wing ($A = 5$),

(B) that a single vortex of strength $K(y)$ placed along the quarter-chord line

$$x = R(y) - \tfrac{1}{4}c(y)$$

with an associated trailing vortex wake of strength $-dK/dy$ per unit span will determine w at

$$x = R(y) + \tfrac{1}{4}c(y)$$

within the accuracy of (A) for straight and swept-back wings ($A = 5$) of constant chord.

Though the accuracy of (A) and (B) may deteriorate with decreasing aspect ratio and increasing wing taper, it would appear that with these limitations Weissinger's L-method is in satisfactory agreement with his more accurate F-method (Ref. 10, Fig. 6).

In solution (iii), Falkner greatly diminishes the practical difficulty of evaluating w by substituting a rectangular network of vortices for the vortex sheet. He uses equation (6) with $N = 2$, and the solution is of wider application than (ii) and requires about four computer-days. Van Dorn and De Young[11] (1947) have compared the results of (ii) and (iii) for wings of medium taper with angles of sweepback -45 deg $\leqslant \Lambda \leqslant 46$ deg. The lift-curve slopes $\partial C_L/\partial \alpha$ agree within 2 per cent for $|\Lambda| \leqslant 30$ deg, but Falkner's value lies 7 per cent below Weissinger's for $\Lambda = -45$ deg ($A = 2\cdot 99$) and 8 per cent above Weissinger's for $\Lambda = +46$ deg ($A = 3\cdot 45$). It follows that there is a factor associated with sweep, which is incorrectly accounted for by one or both of these methods. Weissinger's method is apparently inaccurate when it includes the boundary condition at a kink in the three-quarter chord locus. The discrepancies in the values of $\partial C_L/\partial \alpha$ for large sweep may be attributed to Weissinger's use of a solving point at the median section. But serious consideration must also be given to the objection, raised by Schlichting and Thomas[12] (1947), that the rectangular vortex network used by Falkner is inadequate for swept wings on account of the error in the downwash due to a ' staircase vortex '.

From the mathematical standpoint two features of the vortex-lattice theory can be criticised. For any prescribed pressure distribution the configuration of the vortex network is quite arbitrary. Moreover the strengths of the individual horseshoe vortices are determined from certain selected two-dimensional principles and are effectively dependent on the positions at which the downwash is required. The uniqueness of the solution and the basis of mathematical reasoning are lost. Jones (R. & M. 2225[13], 1946) has shown that, for a rectangular wing ($A = 6$), Falkner's approximate downwash distributions agree well with the results of exact vortex-sheet theory for points along the mid-chord axis of the wing over the inner part of the span, but that Falkner's values are about 5 per cent low at $0\cdot 8$ span and are likely to be in greater error towards the tip. However, it should be emphasized, in the first place, that in the presence of free vorticity, the use of two-dimensional principles may be inadequate for determining control hinge moments even in this favourable case and for wings of low aspect ratio may introduce far-reaching errors not localized near the trailing edge, and, in the second place, that at large sweeps the rectangular configuration of the vortex network is suspect[12]. Exact calculations of downwash would provide isolated checks on the boundary flow conditions of any given approximate solution.

The accurate evaluation of w is extremely laborious. The choice of method lies between (iv) and (v). In (iv) Jones has expressed the double integral (5) as the limiting value of a single spanwise integral of elliptic functions, and the subsequent evaluation requires careful supervision.

The method of (v) will be published in R. & M. 2819. First of all the potential difference, given by equations (3) and (6),

$$(\Phi_a - \Phi_b) = Vc(y) \sum_{n=0}^{N} \sum_{m=1}^{M} C_{nm} A_m(y) K_n(\theta) , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where

$$A_m(y) = \frac{s}{c(y)} \left(\frac{y}{s}\right)^{m-1} [1 - (y/s)^2]^{1/2} ,$$

$$K_0(\theta) = \theta + \sin\theta ,$$

$$K_1(\theta) = \tfrac{1}{2}(\tfrac{1}{2}\sin 2\theta - \theta) ,$$

and for $n \geqslant 2$, $K_n(\theta) = \frac{1}{2}\left(\frac{\sin(n+1)\theta}{n+1} - \frac{\sin(n-1)\theta}{n-1}\right)$,

is tabulated at the corners of a uniform rectangular network covering the plan form. Then if (x_1, y_1) is chosen at one of these corners, the numerical evaluation of equation (5) is achieved by splitting the area of integration into the rectangle of four times the linear dimensions of the network symmetrically surrounding (x_1, y_1) and the remainder of the plan form and the wake. By using polynomial representations of $(\Phi_a - \Phi_b)$ the contribution to w from the rectangle is expressed as a linear function of the 25 values of $(\Phi_a - \Phi_b)$ distributed inside and on the rectangle, and the remainder is obtained by direct integration outside it. The accuracy of this process for the delta wing ($A = 3$) is considered to be about $0 \cdot 1$ per cent provided that (x_1, y_1) is not too close to the wing tip or leading edge. The numerical operations are straightforward but lengthy and sensitive to small computational errors.

A satisfactory potential solution of the general problem for a particular plan form would probably need 24 arbitrary constants, *e.g.*, equation (8) with $N = 3$ and 6 values of m. Accurate calculations of w at any position in terms of the 24 arbitrary coefficients would require 10 to 20 computer-days, the choice of method depending primarily on the plan form of the wing. After w had been obtained at 24 points of the plan form, it would remain to solve the 24 resulting simultaneous equations for each specified set of values of $\partial z/\partial x$ in accordance with equation (4). Thus it could not take less than 1 computer-year to establish one satisfactory solution. How much of this labour could be avoided, if it were merely required to determine the spanwise distribution of lift on a wing of large sweep, is not clear. Provided that there is no kink at the median section, the problem is equivalent to Weissinger's F-method[10] (1942), which would employ either (iv) or (v) to evaluate w at the three-quarter chord, putting $N = 0$ in equation (8). It should be possible by this means to arrive at the spanwise distribution of lift in 25 computer-days, and to confirm the accuracy of Weissinger's L-method, *i.e.*, (ii) for tapered wings of small aspect ratio. Jones has suggested to the author that some light would be thrown on the problem if an attempt were made to assess the errors in any approximate solution and then to improve the solution by iteration. A numerical process is briefly outlined in section 3 to make clear the labour involved.

2.3. From the practical point of view it is plain that the boundary condition (4) cannot be satisfied at more than a small number of representative positions. It is usual to solve as many linear simultaneous equations as there are constants C_{nm} in equation (8), each equation expressing the condition (4) at one point of the plan form. Very roughly the labour involved is $0 \cdot 002L^3$ computer-days, where L is the number of equations. Ill-conditioned equations may arise through an unfortunate choice of the L positions or through an unsatisfactory form of $(\Phi_a - \Phi_b)$ discussed in section 2.1. In the former case it is best to compute the boundary conditions at additional positions and to examine the solution obtained by normalization, which is clearly described and illustrated in R. & M. 2591[14], section 7. It is more practicable to save the extra labour in evaluations of w by reducing the number of constants C_{nm} to L', solving the curtailed L equations by normalization and comparing that solution with the ill-conditioned one.

There always remains uncertainty as to how the flow behaves between the selected positions and near to the perimeter of the plan form. In the approximate theories of Ref. 6 and Ref. 8, this difficulty is partly overcome by assuming that the normal velocity $w_{nm}(\theta, y)$ induced by each term $A_m(y)\Gamma_n(\theta)$ in equation (6) along the chord of each wing section is proportional to the corresponding function in two-dimensional theory. Thus if equation (6) is written in the form

$$\frac{p_b - p_a}{\rho V^2} = \sum_{m=1}^{M} A_m(y) \Big[(C_{0m} - \tfrac{1}{2}C_{1m})\, 2 \cot \frac{\theta}{2} + C_{1m} \Big(- 2 \sin \theta + \cot \frac{\theta}{2} \Big) + \sum_{n=2}^{N} C_{nm}(- 2 \sin n\theta) \Big] , \quad \ldots \quad (9)$$

the normal velocity is immediately expressible in the form

$$w = V \sum_{m=1}^{M} [(C_{0m} - \tfrac{1}{2}C_{1m}) W_{0m}(y) + C_{1m} W_{1m}(y)(\tfrac{1}{2} + \cos \theta) + \sum_{n=2}^{N} C_{nm} W_{nm}(y) \cos n\theta] . \quad \ldots \quad (10)$$

It follows from equations (4) and (10) that

$$\left.\begin{aligned} \sum_{m=1}^{M} (C_{0m} - \tfrac{1}{2}C_{1m}) W_{0m}(y) &= \frac{1}{\pi}\int_0^{\pi} (1 - \cos \theta)\, \frac{\partial z\,(x,y)}{\partial x}\, d\theta \\ \text{and for } n \geqslant 1, \qquad & \\ \sum_{m=1}^{M} C_{nm} W_{nm}(v) &= \frac{2}{\pi}\int_0^{\pi} \cos n\theta\, \frac{\partial z\,(x,y)}{\partial x}\, d\theta \end{aligned}\right\} \quad \ldots \quad (11)$$

Subsequently to Ref. 8, the author has shown that an additional linear variation in w due to the trailing vorticity must also be included, and, as remarked earlier in this report, it is not certain whether quadratic chordwise variation in $w_{0m}(\theta, y)$ should be considered for swept wings. But these considerations only complicate the process by which w is expressed in the form (10) and would lead to additional terms in $(C_{0m} - \frac{1}{2}C_{1m})$ in the first two or three of equations (11). It remains to satisfy (11) at various sections along the span. This reduces both the risk of ill-condition and the number of variables in each set of linear simultaneous equations.

3. *Iterative Potential Solution.*—Jones has recommended a solution of vortex-sheet theory by iteration. The numerical process described here will make clear the labour involved.

If $w = V \dfrac{\partial z}{\partial x}$ represents the known downwash distribution and $K = (\Phi_a - \Phi_b)$ denotes the required circulation or discontinuity in velocity potential, then from equation (5) at all points of the wing

$$4\pi w = \lim_{z_1 \to 0} \int_C\!\!\int K \frac{\partial^2}{\partial z_1{}^2}\Big(\frac{1}{r}\Big)\, dx\, dy\, . \quad \ldots \quad (12)$$

Now let K_A denote an approximation to K obtained by any convenient method. Next evaluate the exact downwash distribution w_A corresponding to K_A as given directly by equation (12). In general since K_A is only approximately equal to K, w_A will differ from w. This difference will give an indication of the accuracy of K_A. If the accuracy is insufficient the solution can be improved by finding K_B the approximate solution of (12), when w is replaced by $(w - w_A)$. Then $K = K_A + K_B$ is an improved solution. By repeating this process several times it should be possible to derive a solution to any required accuracy.

One could use any approximate method, however crude, with such a scheme. To reduce the computational labour, however, it would be best to start with a fairly accurate K_A, estimated by Falkner's method (R. & M. 1910[7]) for instance. In more detail Falkner assumes equation (8) with $N = 2$,

$$\begin{aligned} K &= Vc(y) \sum_{m=1}^{M} A_m(y)[C_{0m}(\theta + \sin\theta) + \tfrac{1}{2}C_{1m}(\tfrac{1}{2}\sin 2\theta - \theta) \\ &\quad + \tfrac{1}{2}C_{2m}(\tfrac{1}{3}\sin 3\theta - \sin\theta)] \\ &= \sum_{i=1}^{3M} C_i K_i \,, \end{aligned} \qquad (13)$$

where K_i are simple doublet distributions of a particular type and C_i are constants to be determined. The substitution of K in equation (12) yields the exact relation

$$w = \sum_{i=1}^{3M} C_i W_i \,, \qquad (14)$$

where W_i represents the exact downwash distribution corresponding to K_i as given by (12). To calculate W_i exactly for a large number of simple distributions would be very laborious. However by the use of Falkner's vortex-lattice theory the values of W_i can be calculated approximately without much difficulty. Let $W_i(F)$ represent these approximate values and replace equation (14) by the relation

$$w = \sum_{i=1}^{3M} C_i W_i(F) \qquad (15)$$

which can be satisfied in general at $3M$ points on the wing. In practice odd and even values of m are treated separately as the corresponding symmetrical and antisymmetrical contributions to the problem are independent. This leads to a system of L simultaneous equations, where respectively

$$L = \tfrac{3}{2}(M + 1) \text{ and } \tfrac{3}{2}M \,.$$

These equations can be expressed in matrix notation as

$$\left\{ w_j \right\} = \left[W_{ij}(F) \right] \left\{ C_i \right\} , \qquad (16)$$

where $[W_{ij}(F)]$ represents a square matrix of order L and $\{w_j\}$ and $\{C_i\}$ denote columns. Hence

$$\left\{ C_i \right\} = \left[W_{ij}(F) \right]^{-1} \left\{ w_j \right\} ; \qquad (17)$$

and on substituting C_i in equation (13) the first approximation K_A is given by

$$K_A = \left[K_1, K_2, \ldots K_L \right] \left[W_{ij}(F) \right]^{-1} \left\{ w_j \right\} . \qquad (18)$$

Similarly it follows that the approximate solution for K_B is given by substituting $\{(w - w_A)_j\}$ for $\{w_j\}$ in equation (18), w_A being obtained by substituting K_A for K in equation (12). For a particular lattice the inverse matrix $[w_{ij}(F)]^{-1}$ is invariant and the simple distributions K_1, K_2. . . . K_L would be used throughout. The accuracy of solutions obtained in this way would be automatically assessed at each stage and the rate of convergence of the iterative process would indicate whether the chosen distributions K_i were suitable.

The labour in calculating w_A, w_B, etc., depends on the form of K_i. If equation (13) is used, Jones (R. & M. 2145[6]) has reduced the corresponding integral for w to the form

$$4\pi w = V \lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \Big[- 4 \int_{-s}^{s} \frac{z_1}{c} \Big\{ \{C_0(y) - \tfrac{1}{2}C_1(y)\}(P_0' + P_1) + C_1(y)(\tfrac{1}{4}P_2 + \tfrac{1}{2}P_1)$$

$$+ C_2(y)(\tfrac{1}{6}P_3 - \tfrac{1}{2}P_1) \Big\} \, dy - z_1 \iint_{\text{wake}} \frac{\pi c}{r^3} \{C_0(y) - \tfrac{1}{2}C_1(y)\} \, dx \, dy \Big] , \quad \ldots \quad \ldots \quad (19)$$

where $C_n(y) = \sum_{m=1}^{M} C_{nm} A_m(y) \; (n = 0, 1, 2)$,

and P_0', P_1, P_2, P_3 are expressible in terms of elliptic integrals and for any given plan form depend only on y and the point (x_1, y_1) at which w is required. As it is remarked in section 4, the integral (19) will diverge if the slope of the leading or trailing edge is discontinuous at the section $y = y_1$. In the case of swept plan forms, for example, there would normally be singularities in w along the whole section $y = 0$ owing to the unsuitable choice of K in equation (13), and in this neighbourhood the value of the solution would be seriously impaired. The evaluation of (19) could only be undertaken by an experienced computer, who with careful supervision would take at least $7L$ days to complete the calculations for L points (x_1, y_1). But the additional labour for each subsequent iteration would only amount to L computer-days. If it is supposed that I iterations will produce a satisfactory solution the exact calculations of downwash would involve $L(6 + I)$ computer-days.

If it is necessary to consider a more general form of K than (13), K_i now being a more general function of the two co-ordinates x and y, the limit of the surface integral (12) should be evaluated by an entirely numerical process such as (v) in section 2.2. In this case there is no computational advantage in working with the separate terms K_i. Instead K_A would be evaluated at corners of a uniform rectangular network over the plan form in about 8 computer-days. Then provided that the L points are chosen not too close to the leading edge or the wing tip, the technique in use for the calculations of w for the delta wing ($A = 3$) would serve for any plan form. The amount of preparatory work would be small as the constants of integration near the singularity and the values of $\dfrac{\partial^2}{\partial z_1^2}\left(\dfrac{1}{r}\right)$ would be unchanged. It is estimated that the labour would amount to $(8 + 4L)$ computer-days for each iteration.

Assuming that the solution (18) has been obtained in the first place, the labour in determining K_B lies almost entirely in evaluating the L quantities $(w - w_A)_j$. For instance, consider $L = 24$. Table 1 shows that, if satisfactory, evaluation by means of equation (19) would be more economical provided that more than one iteration is issued.

TABLE 1

I	1	2	3	4
$L(6 + I)$	168	192	216	240
$I(8 + 4L)$	104	208	312	416

The solution could of course be obtained directly without approximation from the linear equations (14), if the W_i's were known exactly. But the calculations corresponding to a large number of distributions K_i would be very laborious and if $L = 24$, for instance, they would probably require $15 \times 24 = 360$ computer-days. So it is thought that the present iterative method would be relatively easier, since only the exact downwash calculations corresponding to K_A, K_B and possibly K_C would have to be carried out in order to obtain sufficient accuracy. This scheme would certainly check the accuracy of a solution by vortex-lattice theory and could be used to improve it, if necessary. The main disadvantage of the scheme is that it is based on the special type of distributions K_i defined in equation (13) which may not always be the most suitable.

4. *Form of the Pressure Distribution.*—On the basis of vortex-sheet theory the distribution of pressure difference is directly related to the discontinuity in velocity potential by equation (3). The choice of form for $(\Phi_a - \Phi_b)$ is discussed in section 2.1. Special care is needed:—

(*a*) near the perimeter of the plan form,

(*b*) at abrupt changes in $\partial z/\partial x$;

otherwise it is impossible to fulfil the boundary conditions determined by equations (4) and (5). Provided that the problem is not complicated by (*b*) it is usual to work with equation (6) or the equivalent form (8), which can reasonably be expected to serve near the perimeter of the plan form unless there are important irregularities there.

4.1. From equation (4),

$$w = \lim_{z_1 \to 0} \frac{\partial \Phi}{\partial z_1} = \frac{1}{4\pi} \lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \left[\int_C \int (\Phi_a - \Phi_b) \frac{\partial}{\partial z_1} \left(\frac{1}{r} \right) dx\, dy \right] . \quad \ldots \quad (20)$$

A necessary and sufficient condition for the convergence of w is that $(\Phi_a - \Phi_b)$ should be differentiable at (x_1, y_1). Put

$$(\Phi_a - \Phi_b) = (\Phi_a - \Phi_b)_1 + (x - x_1) \left\{ \frac{\partial(\Phi_a - \Phi_b)}{\partial x} \right\}_1$$

$$+ (y - y_1) \left\{ \frac{\partial(\Phi_a - \Phi_b)}{\partial y} \right\}_1 + R . \quad \ldots \quad (21)$$

Then w will be convergent or divergent according as

$$- \frac{\partial}{\partial z_1} \left[\int_S \int (\Phi_a - \Phi_b) \frac{z_1}{r^3}\, dx\, dy \right]$$

is finite or infinite, where S is any finite area enclosing (x_1, y_1). Let S denote the rectangle

$$\left. \begin{array}{l} |x - x_1| \leqslant \xi \\ |y - y_1| \leqslant \eta \end{array} \right\} . \quad \ldots \quad (22)$$

The first term of equation (21) contributes

$$- (\Phi_a - \Phi_b)_1 \lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \left(4 \tan^{-1} \frac{\xi\eta}{r z_1} \right) ,$$

where $r^2 = \xi^2 + \eta^2 + z^2_1$; and this tends to a finite limit

$$+ \frac{4(\xi^2 + \eta^2)^{1/2}}{\xi\eta} (\Phi_a - \Phi_b)_1 .$$

The contributions of the second and third terms of (21) vanish identically. It follows that

$$\lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \left[\int_S \int - R \, . \, \frac{z_1}{r^3}\, dx\, dy \right]$$

indicates the convergence of w. Since $(\Phi_a - \Phi_b)$ is differentiable at (x_1, y_1), it follows that

$$R = O(r) .$$

It is clearly sufficient for convergence that

$$R = O(r^{1+\lambda}) \text{ or } o(r^{1+\lambda}) ,$$

where $\lambda > 0$, for then there exists a number X such that throughout S,

$$|R| < X r^{1+\lambda}.$$

Now

$$\left| \lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \left[\int_S \int - \frac{R z_1}{r^3}\, dx\, dy \right] \right| \leqslant \lim_{z_1 \to 0} \left[\int_S \int |R| \left| \frac{\partial}{\partial z_1} \left(- \frac{z_1}{r^3} \right) \right| dx\, dy \right]$$

$$\leqslant \lim_{z_1 \to 0} \left[\int_S \int X \frac{r\, d\theta\, dr}{r^{2-\lambda}} \right] \text{ in polar co-ordinates}$$

$$\leqslant \frac{X}{\lambda} 2\pi (\xi^2 + \eta^2)^{\lambda/2} .$$

Thus w will converge.

But it is important to note that continuity in $(\Phi_a - \Phi_b)$ is not a sufficient condition. For if it is supposed that

$$(\Phi_a - \Phi_b) = (\Phi_a - \Phi_b)_1 + X[(x - x_1)^2 + (y - y_1)^2]^{1/2} \qquad \ldots \quad (23)$$

throughout S, then from equations (20), (22) and (23)

$$w = (\Phi_a - \Phi_b)_1 \frac{(\xi^2 + \eta^2)^{1/2}}{\pi \xi \eta} - \frac{1}{4\pi} \lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \left[\int_S \int \frac{z_1 X r}{(r^2 + z_1^2)^{3/2}}\, dx\, dy \right]$$

$$+ \{\text{finite contribution from the area } (C - S)\} , \qquad \ldots \quad (24)$$

where r now denotes $[(x - x_1)^2 + (y - y_1)^2]^{1/2}$. The second term of the right-hand side is transformed in polar co-ordinates to

$$- \frac{X}{4\pi} \lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \left[\int_S \int \frac{z_1 r^2\, d\theta\, dr}{(r^2 + z_1^2)^{3/2}} \right] .$$

It is convenient to replace S by a circular area S',

$$r \leqslant \rho ,$$

since clearly the contribution from $(S - S')$ is finite. Thus the term becomes

$$- \tfrac{1}{2} X \lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \left[\int_0^\rho \frac{z_1 r^2\, dr}{(r^2 + z_1^2)^{3/2}} \right]$$

$$= + \tfrac{1}{2} X \lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \left[\frac{\rho z_1}{(\rho^2 + z_1^2)^{1/2}} - z_1 \log \frac{(\rho^2 + z_1^2)^{1/2} + \rho}{z_1} \right] ,$$

which will produce a logarithmic infinity.

4.2. Now consider the behaviour of w, as given by equation (20), when $(\Phi_a - \Phi_b)$ is given by equation (8) and there is a discontinuity in the direction of the leading or trailing edge of the wing at $y = y_1$. It follows from the definition of θ in equation (6), that

$$\tfrac{1}{2} c(y) \sin \theta \frac{\partial \theta}{\partial y} = \tfrac{1}{2} \cos \theta \frac{dc(y)}{dy} - \frac{dR(y)}{dy} . \qquad \ldots \quad (25)$$

There will be discontinuities c_1', R_1' in the values of $\dfrac{dc}{dy}$, $\dfrac{dR}{dy}$ at $y = y_1$, one of which may disappear. Unless

$$c_1' \cos \theta = 2R_1' , \qquad \ldots \quad (26)$$

there will be a corresponding discontinuity in $\partial\theta/\partial y$. It follows from equation (8) that

$$\frac{\partial}{\partial y} (\Phi_a - \Phi_a) = Vs \sum_{n=0}^{N} \sum_{m=1}^{M} C_{nm} \left[\frac{d}{dy} \left\{ \left(\frac{y}{s} \right)^{m-1} \left[1 - \left(\frac{y}{s} \right)^2 \right]^{1/2} \right\} K_n(\theta) \right.$$

$$\left. + \left(\frac{y}{s} \right)^{m-1} \left[1 - \left(\frac{y}{s} \right)^2 \right]^{1/2} \frac{dK_n}{d\theta} \frac{\partial \theta}{\partial y} \right] . \qquad \ldots \quad (27)$$

Therefore with the possible exception of a position defined by equation (26), there will be finite spanwise discontinuities $K'(x)$, say, in $\frac{\partial}{\partial y}(\Phi_a - \Phi_b)$ at all points of the section $y = y_1$. Then $(\Phi_a - \Phi_b)$ may be expressed in the form

$$(\Phi_a - \Phi_b) = f(x, y) + \tfrac{1}{2}K'(x)\,|y - y_1|, \qquad (28)$$

where $f(x, y)$ is a differentiable function of x and y. From equation (20) the corresponding downwash velocity at (x_1, y_1) is

$$w = \text{finite contribution from } \left\{ f(x, y) + \frac{K'(x) - K'(x_1)}{2}\,|y - y_1| \right\}$$

$$- \frac{K'(x_1)}{8\pi} \lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \left[\int_C \int |y - y_1| \cdot \frac{z_1}{r^3}\, dx\, dy \right]. \qquad (29)$$

Similarly to the instance in equation (24), it can readily be shown that the contribution to the integral in equation (29) from the area S, defined in equation (22), is

$$\frac{K'(x_1)}{2\pi} \lim_{z_1 \to 0} \frac{\partial}{\partial z_1} \left[z_1 \log \frac{z_1\{(\xi^2 + \eta^2 + z_1^{\,2})^{1/2} + \xi\}}{(\eta^2 + z_1^{\,2})^{1/2}\{(\xi^2 + z_1^{\,2})^{1/2} + \xi\}} \right],$$

which will diverge as the limit is taken.

It follows that the use of equation (8) will necessarily produce a logarithmically infinite downwash velocity along a section at which the direction of the leading or trailing edge is discontinuous. It seems unsatisfactory that conditions at the perimeter of the plan form may compel violations of the boundary conditions in the centre of the plan form. For small discontinuities of sweep and sudden changes of taper this consideration is probably unimportant. In the opinion of the author such irregularities are trivial in that they can scarcely improve the flow and that any beneficial aerodynamic properties associated with such plan forms would not suffer if the changes of sweep or taper were made smooth; and it would seem preferable to modify the wing rather than the method of the theory if a special theoretical comparison were required. But the large angularity that frequently occurs near the root of swept-back wings will have an influence on the chordwise pressure distribution, the importance of which can only be determined by abandoning the form of the velocity potential in equation (8).

4.3. When considering the problem of a uniform incidence applied to any plan form, the objections of the preceding sections may be overcome by assuming a solution of the general form

$$\frac{\Phi_a - \Phi_b}{lV} = \sum_n F_n \left\{ \frac{y}{l} \right\} G_n \left\{ f\left(\frac{x}{l}, \frac{y}{l} \right) \right\}, \qquad (30)$$

where l is a length,

F_n, G_n are arbitrary functions of a single variable,

$f\left(\frac{x}{l}, \frac{y}{l}\right)$ is a function differentiable at all points inside the plan form and monotonic in x/l such that

$f\left(\frac{x}{l}, \frac{y}{l}\right) = 0$ is the equation of the leading edge

$f\left(\frac{x}{l}, \frac{y}{l}\right) = 1$ is the equation of the trailing edge.

There are necessary conditions that

$$\left.\begin{aligned} F_n\left(\frac{y}{l}\right) &= 0, \text{ when } y = s \\ G_n(f) &= 0, \text{ when } f = 0 \\ \frac{dG_n}{df} = G_n' &= 0, \text{ when } f = 1. \end{aligned}\right\}$$

The general velocity potential may be expressed similarly to equation (8),

$$\frac{\Phi_a - \Phi_b}{Vs} = \sum_{n=0}^{N} \sum_{m=1}^{M} C_{nm} \left(\frac{y}{s}\right)^{m-1} \left[1 - \left(\frac{y}{s}\right)^2\right]^{1/2} K_n(\Theta) , \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

where

$$\cos \Theta = 1 - 2f\left(\frac{x}{l}, \frac{y}{l}\right)$$

replaces the definition of θ in equation (6). It is clear that, if $\Theta = \theta$,

$$\begin{aligned} f\left(\frac{x}{l}, \frac{y}{l}\right) &= \tfrac{1}{2}(1 - \cos \theta) \\ &= \tfrac{1}{2} - \frac{R(y) - x}{c(y)} \end{aligned}$$

will not satisfy the condition of differentiability, when equation (8) is unsuitable.

For complicated plan forms a function $f\left(\frac{x}{l}, \frac{y}{l}\right)$ would be difficult to construct. To illustrate the types of wings that could be represented conveniently, consider the functions

$$\left.\begin{aligned} f_1(\psi) &= \frac{x^2 - y^2 \tan^2 \psi}{l^2} \\ f_2(\psi) &= \frac{x\{(x + d \tan \psi)^2 - y^2 \tan^2 \psi\}}{l(l + d \tan \psi)^2} \\ f_3(\psi) &= 1 - \frac{[(l - x)^2 + y^2]^{1/2}\{(l - x) + \sin \psi[(l - x)^2 + y^2]^{1/2}\}}{l^2(1 + \sin \psi)} . \end{aligned}\right\} \quad \ldots \quad (32)$$

For each of the three functions the corresponding leading and trailing edges together with other intermediate curves of the family

$$f = \text{constant}$$

have been plotted in Fig. 1 in the case $\psi = 45$ deg.

Fig. 1a shows a family of rectangular hyperbolae, $f_1 =$ constant, with the leading edge as asymptotes.

Fig. 1b indicates the possibility of representing a Pterodactyl wing with functions of the type f_2.

Fig. 1c shows that $f_3 =$ constant, will represent a family of curves asymptotic to a V-shaped trailing edge.

By the use of the functions $f_1(\psi)$ and $f_3(\psi)$ any wing of constant sweep and taper can be represented in the form

$$f = \frac{f_1(\Lambda_L)}{1 + f_1(\Lambda_L) - f_3(\Lambda_T)} ,$$

where Λ_L and Λ_T are the angles of sweepback of the leading and trailing edges respectively.

Thus from equation (32)

$$f\left(\frac{x}{l}, \frac{y}{l}\right) = \frac{1}{1+P}, \quad \ldots \quad (33)$$

where
$$P = \frac{[(l-x)^2 + y^2]^{1/2}\{(l-x) + \sin \Lambda_T\, [(l-x)^2 + y^2]^{1/2}\}}{(x^2 - y^2 \tan^2 \Lambda_L)(1 + \sin \Lambda_T)} .$$

Instead of equation (33) there are slight advantages in taking

$$f\left(\frac{x}{l}, \frac{y}{l}\right) = \frac{1}{1+\sqrt{P}} . \quad \ldots \quad (34)$$

It is then convenient that

$$f\left(\frac{x}{l}, 0\right) = \frac{x}{l} ;$$

and when $(\Phi_a - \Phi_b)$ is given by equation (30), it is necessary that $G_n(f)$ should be a polynomial function such that

$$\left.\begin{aligned} G_n(f) &= 0, \text{ when } f = 0 \\ G_n'(f) &= 0 \text{ and } G_n''(f) = 0, \text{ when } f = 1 \end{aligned}\right\} .$$

Provided that $y \neq 0$, $G_n'(0) \neq 0$ and $G_n'''(1) \neq 0$, it may then be verified that near the leading edge, $x = x_L = y \tan \Lambda_L$,

$$\frac{\Phi_a - \Phi_b}{lV} = O\left(\frac{x - x_L}{l}\right)^{1/2} ,$$

and that near the trailing edge, $x = x_T = l + y \tan \Lambda_T$,

$$\frac{\partial}{\partial x}\left[\frac{\Phi_a - \Phi_b}{V}\right] = O\left(\frac{x_T - x}{l}\right)^{1/2} .$$

The evaluations of downwash due to a velocity potential given by equation (30) cannot be carried out by the method of Jones (R. & M. 2145[6]), unless equation (31) with $\Theta = \theta$ is valid. A purely numerical process such as the method (v) of section 2.2 would have to be adopted. In the opinion of the author such a calculation should be undertaken, say, for a wing derived from equation (34), as shown in Fig. 2. The labour of computation would depend on the number of positions of the plan form at which the exact boundary conditions are required. It is estimated that if L variables are admitted into equation (30) and the direction of flow is established at L positions, the calculations would involve approximately $8 + 4L^{3/2} + 0{\cdot}002L^3$ computer-days which is expressed in tabular form.

TABLE 2

L	6	12	18	24
$8 + 4L^{3/2} + 0{\cdot}002L^3$	67	178	325	506

It is considered that the fundamental knowledge gained would justify the heavy labour of computation in the case $L = 18$, say.

5. *Concluding Remarks.*—There is a great need for more accurate data on the aerodynamic derivatives of swept-back wings in order to solve problems of stability, control and flutter. As one step in the search for these data the estimation of the three-dimensional potential solution is essential, and if it is to be of value the degree of accuracy of any approximation must be known beyond question.

The fundamental derivative $\partial C_L/\partial \alpha$ may be readily determined by the methods of Weissinger[10] and Falkner (R. & M. 1910[7]), but for an unknown reason the two estimates differ by about $7\frac{1}{2}$ per cent for wings of 45 deg sweep (Van Dorn and De Young[11]). It is concluded that there are only two possible explanations; Weissinger's use of a solving point at the median section may be unsound for wings of large sweep and Falkner's rectangular vortex network may be inadequate on account of the error in the downwash due to a 'staircase vortex' (Schlichting and Thomas[12]).

In the first place it is suggested that, with the specific purpose of checking the Falkner[7] vortex-lattice theory, the iterative procedure, described in section 3, would lead to greater accuracy in the detailed pressure distribution, especially over the parts of the wing which would form control surfaces. It seems very doubtful whether iteration would help in improving the pressures at the root of a Vee wing; and it is therefore proposed that a wing of constant chord with acute hyperbolic leading and trailing edges (*see* Fig. 3) should be selected for the necessary calculations, which would probably involve labour amounting to at most 8 computer-months.

It has been demonstrated in section 4 that, if the pressure distribution is taken in the usual form (equation (6)) which is assumed by Weissinger[10], Falkner[7] and Jones (R. & M. 2145[6]), there is necessarily a logarithmically infinite downwash at virtually all points of a wing section at which the direction of the leading or trailing edge is discontinuous. It seems unsatisfactory that the shape of the perimeter may compel violations of the boundary conditions in the centre of the plan form. The large angularity of Vee wings may have an influence on the chordwise pressure distribution, the importance of which can only be determined by abandoning equation (6).

In the second place, therefore, it is suggested that by choosing suitable functions for the distribution of the velocity potential over the plan form it is possible to solve the problem of the Vee wing exactly. It is proposed that calculations should be undertaken to discover the distribution of pressure on a wing, such as that shown in Fig. 2, in an inclined stream. A reliable solution would require at least 13 computer-months ; and it is considered that the fundamental knowledge gained would amply justify the heavy labour of computation.

The characteristics of deflected controls are complicated by the introduction of partial spanwise and chordwise discontinuities, which, in the present state of knowledge, cannot practicably be incorporated in an exact theory. It is therefore expedient to create discontinuities in downwash where they should not exist or to smooth them out where they should exist. The approximate method, described in Ref. 8, is a compromise which, subject to the modification mentioned in section 2.3, is satisfactory for straight wings, though possibly less accurate for swept ones. A typical calculation by this method involves about $1\frac{1}{2}$ computer-months. A more accurate approach to the problem of deflected controls requires further study.

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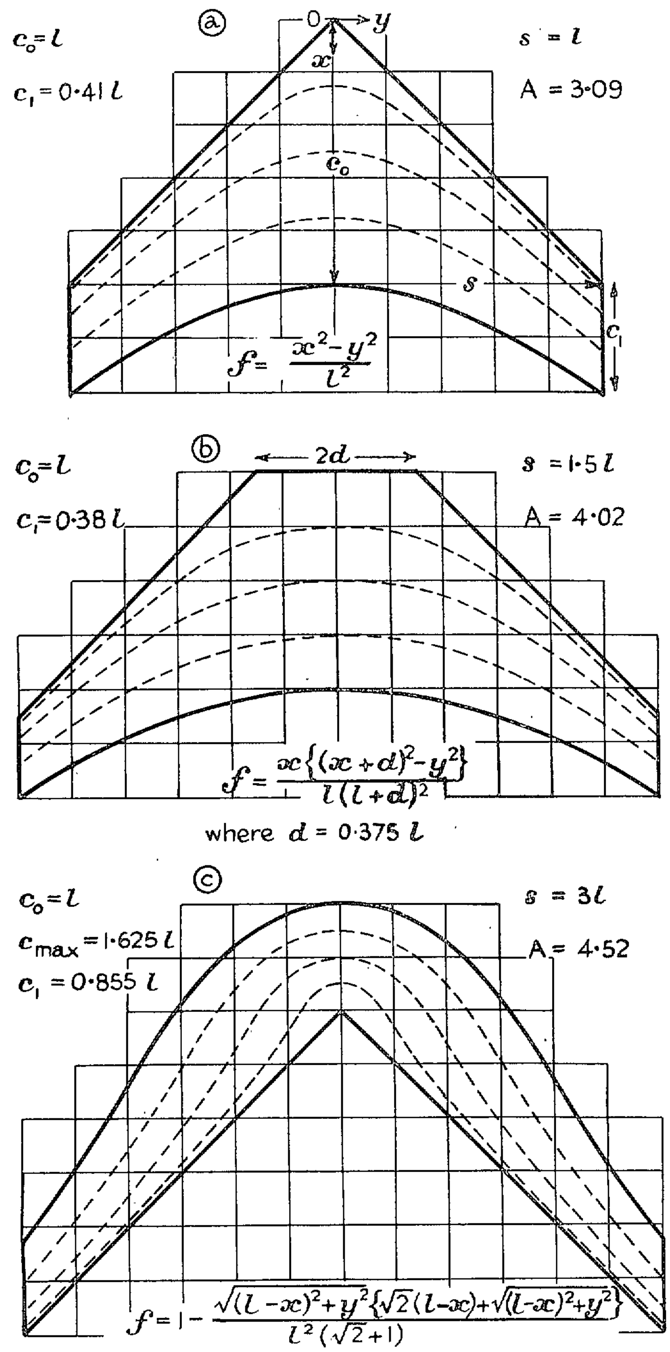

FIG. 1. Representative plan forms. Leading and trailing edges are defined by $f = 0$ and 1 respectively.

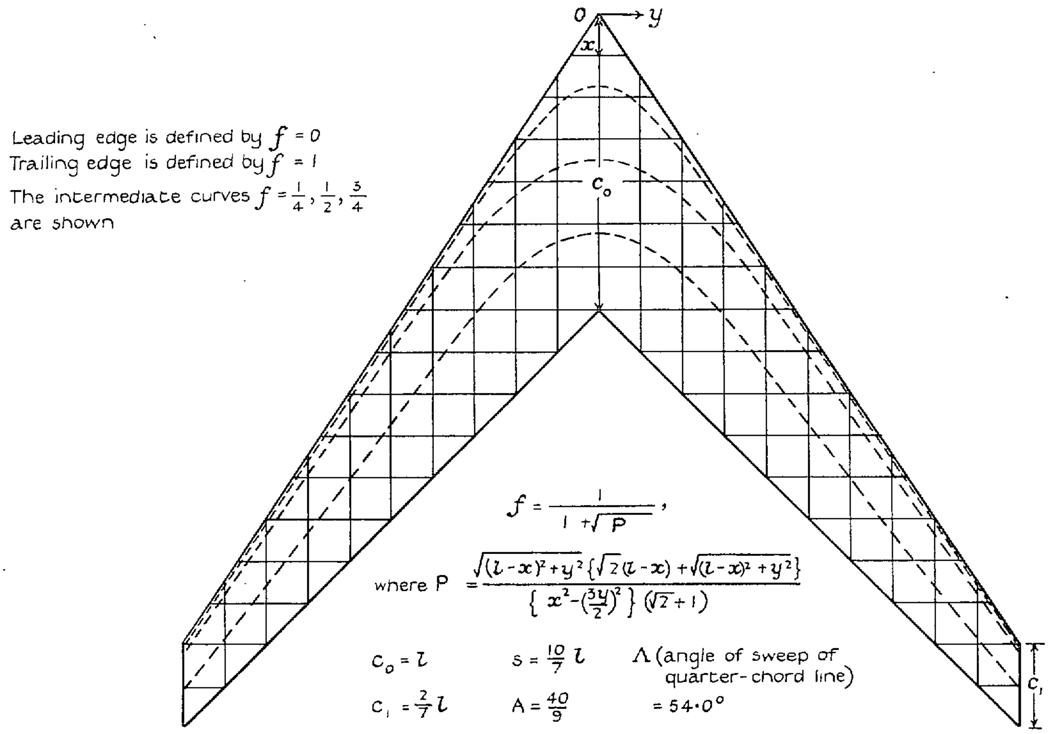

FIG. 2. Selected plan form.

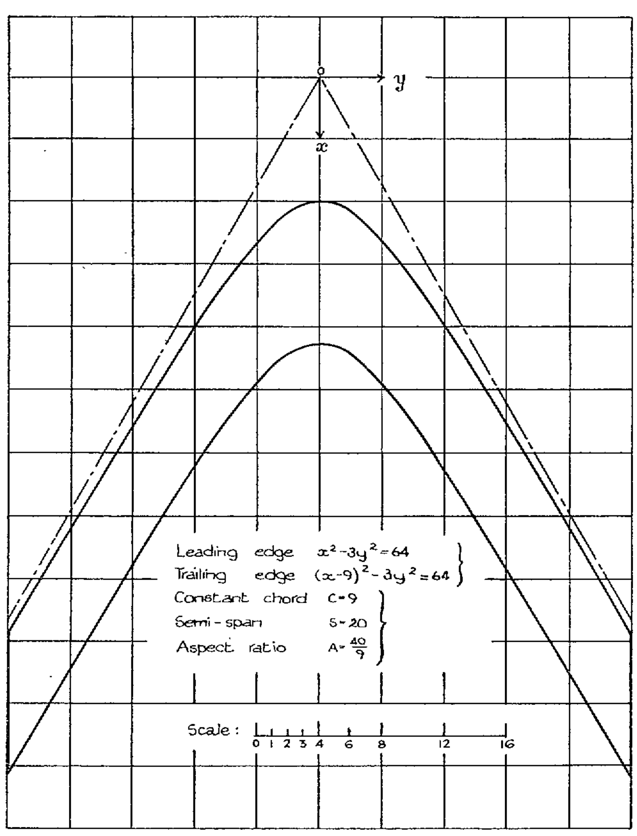

FIG. 3. Suitable plan form for the proposed iterative calculation.

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