R. & M. No. 2243
(9829)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

On Certain Types of Boundary-layer Flow with Continuous Surface Suction

By

B. THWAITES, B.A.,
of the Aerodynamics Division, N.P.L.

LONDON: HIS MAJESTY'S STATIONERY OFFICE
Price 1*s.* 3*d.* net

On Certain Types of Boundary-layer Flow with Continuous Surface Suction

By

B. THWAITES, B.A.,
of the Aerodynamics Division, N.P.L.

Reports and Memoranda, No. 2243
July, 1946

Summary.—In this report, two matters are dealt with which were left in an unsatisfactory state in the Appendices of Refencence 1. The first concerns the conditions obtaining near the front of a flat plate in a uniform stream with constant continuous suction through the plate. We now satisfactorily prove that the boundary-layer velocity profile tends to the well-known Blasius profile as the front end of the plate is approached. The second matter concerns the solution of the boundary-layer equations of motion when " similar " velocity profiles are assumed—it is shown that only two types of outside stream velocity distributions lead to " similar " profiles, under ordinary conditions.

1. *Flat Plate, Uniform Stream, Constant Suction.*—1.1. In the usual notation, the equations of steady motion of flow within the boundary layer are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = UU' + \nu\frac{\partial^2 u}{\partial y^2}. \qquad (1)$$

The equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2)$$

allows the use of the stream function ψ for which

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x}. \qquad (3)$$

We are here concerned with the flow past a flat boundary in a uniform stream, when there is a constant velocity at and normal to the boundary. The boundary is the positive part of the x-axis, *i.e.* $y = 0$, $x \geqslant 0$. This flow has been exhaustively studied (Reference 2 gives a complete review of the work on the problem) and no exact solution has been found in finite terms. It has always been assumed that as $x \to +0$ the velocity profiles of the boundary layer tend to Blasius' profile. No indication has yet been found of the German reasons for this assumption, and no valid proof has yet been given. The proof in Appendix I of Reference 1 assumes the answer and we now remedy this unsatisfactory state of affairs as follows.

New independent variables (ξ, η) are introduced for which

$$\left.\begin{aligned} \xi &= -\frac{v_0}{2}\left(\frac{x}{U\nu}\right)^{1/2} \\ \eta &= \tfrac{1}{2}\left(\frac{U}{\nu x}\right)^{1/2} y \end{aligned}\right\}. \qquad (4)$$

The coefficients of $x^{1/2}$ and $yx^{-1/2}$ in these co-ordinates were chosen not only to make ξ, η non-dimensional, but also to obtain an equation whose boundary conditions are independent of the ratio v_0/U.

v_0 is the velocity at and normal to the boundary. For suction with which we are really concerned, v_0 is negative.

Suppose $\psi = (\nu U x)^{1/2} f(\xi, \eta)$ .. .. (5)

in which f is a function of ξ and η and is to be determined. Using (3) we obtain

$$\left.\begin{aligned} u &= \tfrac{1}{2} U \frac{\partial f}{\partial \eta}, \quad v = \tfrac{1}{2}\left(\frac{U\nu}{x}\right)^{1/2}\left(\eta \frac{\partial f}{\partial \eta} - f\right) + \frac{v_0}{4}\frac{\partial f}{\partial \xi}, \\ \frac{\partial u}{\partial y} &= \frac{U}{4}\left(\frac{U}{\nu x}\right)^{1/2} \frac{\partial^2 f}{\partial \eta^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{U}{8}\left(\frac{U}{\nu x}\right)\frac{\partial^3 f}{\partial \eta^3}, \\ \frac{\partial u}{\partial x} &= -\frac{1}{4}\frac{U}{x}\eta\frac{\partial^2 f}{\partial \eta^2} - \frac{1}{8}\frac{U v_0}{(U\nu x)^{1/2}}\frac{\partial^2 f}{\partial \eta \partial \xi}. \end{aligned}\right\} \quad \ldots \quad (6)$$

Using the expressions derived in (6), the equation of motion (1) becomes

$$-\frac{1}{16}\frac{U^2 v_0}{(U\nu x)^{1/2}}\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial \xi} - \frac{1}{8}\frac{U^2}{x} f \frac{\partial^2 f}{\partial \eta^2} + \frac{U v_0}{16}\left(\frac{U}{\nu x}\right)^{1/2}\frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial \xi} = \frac{\nu U^2}{8\nu x}\frac{\partial^3 f}{\partial \eta^3}, \quad \ldots \quad (7)$$

which simplifies to

$$\frac{\partial^3 f}{\partial \eta^3} + f\frac{\partial^2 f}{\partial \eta^2} = \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial \xi} - \frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial \xi}\right). \quad \ldots \quad (8)$$

The boundary conditions which must be satisfied are

$$\left.\begin{aligned} u &= 0 \\ v &= v_0 \end{aligned}\right\} y = 0, \quad u = U, \quad y = \infty. \quad \ldots \quad (9)$$

Two of these conditions give

$$\left.\begin{aligned} \frac{\partial f}{\partial \eta} &= 0, \ \eta = 0, \\ \frac{\partial f}{\partial \eta} &= 2, \ \eta = \infty. \end{aligned}\right\} \quad \ldots \quad (10)$$

For the third, we have from (6) that at $y = 0$

$$v = \left[-\tfrac{1}{2}\left(\frac{U\nu}{x}\right)^{1/2} f + \frac{v_0}{4}\frac{\partial f}{\partial \xi}\right]_{y=0} = \frac{v_0}{4}\left(\frac{f}{\xi} + \frac{\partial f}{\partial \xi}\right)_{y=0}. \quad \ldots \quad (11)$$

We require $v = v_0$ at $y = 0$.

If therefore $f = 2\xi$ for $\eta = 0$, (11) gives

$$v = \frac{v_0}{4}\left(\frac{2\xi}{\xi} + 2\right) = v_0.$$

The third boundary condition of (9) therefore requires that

$$f = 2\xi, \ \eta = 0. \quad \ldots \quad (12)$$

The equation of motion (1) has thus been transformed into

$$\frac{\partial^3 f}{\partial \eta^3} + f\frac{\partial^2 f}{\partial \eta^2} = \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial \xi} - \frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial \xi}\right), \qquad (13)$$

with the boundary conditions

$$\left.\begin{aligned} f &= 2\xi \\ \frac{\partial f}{\partial \eta} &= 0 \end{aligned}\right\} \eta = 0, \frac{\partial f}{\partial \eta} = 2, \eta = \infty. \qquad (14)$$

The solution is required for positive values of ξ and η.

1.2 The solution of (13) appears to be regular in the neighbourhood of the origin, and hence is expansible in powers of ξ for small values of ξ. Let us put therefore

$$f = f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \ldots. \qquad (15)$$

By substitution of this expression for f in (13) and equation of coefficients of powers of ξ to zero, we get successively

$$\left.\begin{aligned} &f_0''' + f_0 f_0'' = 0, \\ &f_1''' + f_1 f_1'' - f_0' f_1' + 2f_0'' f_1 = 0, \\ &f_2''' + f_0 f_2'' - 2f_0' f_2' + 3f_0'' f_2 = (f_1')^2 - 2f_1 f_1'', \end{aligned}\right\} \qquad (16)$$

and so on. Dashes denote differentiation with respect to η.

The boundary conditions become

$$\left.\begin{aligned} &f_0(0) = 0, f_0'(0) = 0, f_0'(\infty) = 2, \\ &f_1(0) = 2, f_1'(0) = 0, f_1'(\infty) = 0, \\ &f_n(0) = 0, f_n'(0) = 0, f_n'(\infty) = 0, n \geqslant 2. \end{aligned}\right\} \qquad (17)$$

The nth equation in (16), $n > 0$, is linear in $f_n(\eta)$. The first equation is non-linear and is Blasius' well-known equation. The form of (15) shows at once that as $\xi \rightarrow +0$, the velocity profiles tend to the Blasius profile. This question therefore appears to have been cleared up satisfactorily.

2. *On Similar Profiles*.—In Appendix II of Reference 1, it was shown that the usual method of "similar" profiles could not be applied to the constant suction problem considered in §1. This is, of course, hardly surprising since we now know the profiles are different at the two ends of the boundary. However on extension of the analysis in Appendix II, Reference 1 gives the result that there are only three general cases of boundary-layer motion in which "similar" profiles exist. We now go on to find these three cases.

For simplicity, let us use the independent variables x/ν, y/ν, so that the equation of motion (1) becomes, using x and y for the new variables

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = UU' + \frac{\partial^2 u}{\partial y^2}, \qquad (18)$$

the dash denoting differentiation with respect to x/ν.

Suppose that we have a flow in which the velocity distributions through the boundary-layer are similar for each x.

We can write, without loss of generality,

$$\psi = p(x)\ F(yf(x)) + g(x) \qquad (19)$$

and we suppose that neither $p(x)$ or $f(x)$ is identically zero.

This gives

$$\left.\begin{aligned} &u = pfF', \quad v = -p'F - ypf'F' - g'(x), \\ &\frac{\partial u}{\partial y} = pf^2F'', \quad \frac{\partial^2 u}{\partial y^2} = pf^3F''', \\ &\frac{\partial u}{\partial x} = (pf' + p'f)F' + ypff'F''. \end{aligned}\right\} \quad \ldots \quad (20)$$

Herein dashes on F denote differentiation with respect to $\eta \equiv yf(x)$.

We have the two boundary conditions

$$u = 0, y = 0 \text{ and } u = U, y = \infty. \quad \ldots \quad (21)$$

We may suppose that $F'(\infty) = 1$, without loss of generality.

Then from (20),

$$\left.\begin{aligned} U &= pf, \\ U' &= pf' + p'f. \end{aligned}\right\} \quad \ldots \quad (22)$$

Using the expressions of (20) and (22), the equation of motion (18) becomes

$$fpF' \left[(pf' + p'f)F' + ypff'F''\right] - \left[p'F + ypf'F' + g'\right] pf^2F'' = pf(pf' + p'f) + pf^3F''', \ldots \quad (23)$$

or

$$[(F')^2 - 1]\,(pf' + p'f) - p'fFF'' - f^2F''' - fg'F'' = 0. \ldots \quad (24)$$

This equation is soluble for $F(\eta)$ if and only if

$$\frac{fg'}{d} = \frac{p'f + pf'}{c} = \frac{p'f}{b} = \frac{f^2}{a}, \quad \ldots \quad (25)$$

or

$$\frac{g'}{d} = \frac{p' + p\dfrac{f'}{f}}{c} = \frac{p'}{b} = \frac{f}{a}, \quad \ldots \quad (26)$$

a, b, c, d being numerical constants, for the coefficients of the F's in (24) cannot be expressed as functions of η since they do not contain y.

(24) then becomes

$$aF''' + bF''F + c\,[1 - (F')^2] + dF'' = 0. \quad \ldots \quad (27)$$

From (26) since $f \neq 0$, $a \neq 0$, for otherwise there would be no motion at all. We may further suppose that $d \neq 0$, since this would mean $g' = 0$, or the addition of a constant to ψ which is not significant. We also have

$$f = \frac{a}{b}p' \quad \ldots \quad (28)$$

and

$$p' = \frac{b}{d}g', \quad \ldots \quad (29)$$

and thence

$$\frac{(p')^2 + pp''}{c} = \frac{p'}{b},$$

or

$$\frac{p''}{p'} = \left(\frac{c}{b} - 1\right)\frac{p'}{p}. \quad \ldots \quad (30)$$

We must now distinguish between the following cases :—

(i) $\underline{c = 2b,\ c \neq 0,}$ then $\dfrac{p''}{p'} = \dfrac{p'}{p}$, ... (31)

whence
$$\left.\begin{aligned} p &= Be^{Ax}, \\ f &= AB\frac{a}{b}e^{Ax}, \end{aligned}\right\} \qquad (32)$$

and $U = AB^2\dfrac{a}{b}e^{2Ax}$. .. .. (33)

In this case
$$\frac{a}{b}F''' + FF''\left(1 + \frac{d}{b}\right) + 2\,[1 - (F')^2] = 0 \qquad (34)$$

Here the stream velocity is an exponential function of x, as is also the boundary-layer thickness. This solution was independently found earlier by E. J. Watson.

(ii) $\underline{c = b = 0,}$ then $p' = 0$.

and
$$\left.\begin{aligned} p' &= 0. \\ \frac{f'p}{f} &= 0 \end{aligned}\right\} \qquad (35)$$

Thus since $p \neq 0, f' = 0$ .. .. (36)

and $f = C, p = D$.

Also, $g' = \dfrac{d}{a}f$, from equation (26).

Therefore $g' = \dfrac{d}{a}C$,

whence $g = \dfrac{d}{a}Cx$, ... (37)

the arbitrary constant being insignificant in the value of the stream function.

Equation (27) becomes
$$aF''' + dF'' = 0 \qquad (38)$$

whence $F' = 1 - e^{-(d/a)\eta}$, ... (39)

the constants having been disposed of to ensure that the two boundary conditions $F'(0) = 0$, $F'(\infty) = 1$ are satisfied.

In this case, the velocity at and normal to the boundary is equal to $-g'$, or $-(d/a)C$. This is a constant, equal, say, to v_0. Further the stream velocity at infinity, U is equal to pf and is constant.

Thus the equation (39) can be rewritten as
$$\frac{u}{U} = 1 - e^{v_0 y}, \qquad (40)$$

which is the Griffith and Meredith asymptotic suction profile.

(iii) $\underline{c \neq 2b,}$ then $\dfrac{p''}{p'} = \left(\dfrac{c}{b} - 1\right)\dfrac{p'}{p}$ .. .. (41)

$$\log p' = \left(\frac{c}{b} - 1\right)\log p + \text{constant},$$

$$p' = Ap^{(c/b)-1} \quad \ldots \quad (42)$$

$$p^{1-(c/b)}\,dp = A\,dx\,,$$

$$\frac{1}{2-(c/b)}\,p^{2-(c/b)} = Ax + B,$$

$$p = \left[(Ax+B)\left(2-\frac{c}{b}\right)\right]^{b/(2b-c)} \quad \ldots \quad (43)$$

At the front end of the boundary, $x = 0$, we assume the boundary layer to be of either zero or infinite thickness, and hence $B = 0$.

(43) then gives

$$f = \frac{a}{b}A\left[Ax\left(2-\frac{c}{b}\right)\right]^{(c-b)/(2b-c)}, \quad \ldots \quad (44)$$

whence

$$U = pf = \frac{a}{b}A\left[Ax\left(2-\frac{c}{b}\right)\right]^{c/(2b-c)} \quad \ldots \quad (45)$$

Here U is proportional to some power of x.

This is the general motion considered by Falkner and Skan in which $U = cx^m$.

The boundary conditions, $u = 0$ at $y = 0$ and $u = U$ at $y = \infty$ become, as usual,

$$F'(0) = 0,\ F'(\infty) = 1.$$

The normal velocity at the boundary is, from (20)

$$v_0 = -\,(p'F(0) + g')$$

$$-\,p'\left(F(0) + \frac{d}{b}\right).$$

If therefore $F(0) = -\,d/b$, we have the solution of flow along an impermeable wall, and if $F(0) \neq -\,d/b$ there is a distribution of suction along the wall proportional to p'. If $U \propto x^m$, then the corresponding suction distribution to give " similar " profiles is $v_0 \propto x^{(m-1)/2}$.

Several authors have pointed out the possibility of solving the boundary-layer equations of motion under this particular type of stream and suction velocity distributions. In particular Holstein in Germany has carried out the computation in a number of cases.

In (i), (ii) and (iii), above, we have therefore shown that solutions of the boundary-layer equations to give " similar " velocity profiles exist in only three general cases. With each case, there is an associated distribution of boundary suction velocity, for which " similar " profiles also exist.

Conclusion.—The assertion that the boundary-layer velocity profiles tend to Blasius' profile as the front end of a plate in a uniform stream under constant continuous surface suction is approached has, we believe, now been satisfactorily proved.

It has also been shown that " similar " profiles can be obtained in boundary-layer flow in only three general cases of stream velocity distributions, in each of which there is associated a particular distribution of surface suction giving also similar profiles.

REFERENCES

No.	*Author*	*Title, etc.*
1	B. Thwaites	On the Flow past a Flat Plate with Uniform Suction. A.R.C. 9391. February, 1946. (To be published.)
2	B. Thwaites	Notes on German Theoretical Work on Porous Suction. A.R.C. 9672. June, 1946. (Unpublished.)
3	H. Holstein	Similar Laminar Boundary Layers on Permeable Walls. U.M.3050. 1943.

(81516) Wt. 11 5/48 Hw.