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The Calculation of Whirling Speeds of a System of Rotors Keyed to Co-axial Shafts

By

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The Calculation of Whirling Speeds of a System of Rotors Keyed to Co-axial Shafts

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T. S. WILSON, B.A., B.Sc., of D. Napier & Son, Ltd.

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1. *Introduction.*—The whirling of shafts carrying rotors is a subject which has attracted the attention of many engineers and mathematicians notably Dunkerley[4], Chree[1], Stodola[8], Jeffcott[5] and Morris[6,7] during the past fifty years. The last mentioned writer has given some valuable historical surveys and criticisms in addition to his own elucidation of several aspects of the general problem.

The main purpose of this paper is to bring the calculation of whirling speeds of an important class of systems within the scope of the iterative technique of Duncan and Collar[2,3], and to demonstrate by theory and example that problems involving large numbers of degrees of freedom may thereby be efficiently dealt with. It would appear that the power of this iterative method is not so widely appreciated as it might be. One erroneous belief is that the utility of the method ceases whenever slow convergence of the iteration ensues. An additional refinement of procedure, which the writer has exploited, allows two or more modes to be extracted more or less simultaneously from an iteration which is converging slowly.

The idealised system which will be under review is an example of what has been termed a 'semi-rigid' system[3]. It is to be thought of as derived from an actual system of shafts and rotors by collecting the mass into rigid body units at several well chosen positions, and then determining, as realistically as is possible from the original continuous system, the elastic couplings associated with the several freedoms of the rigid bodies.

The formulation of this elastic problem presents the real difficulty in these, and indeed in most other, vibration problems. It is usually assumed for simplicity that lateral flexibility arises solely from the bending of shafts, taking such to be covered by the general theory of thin rods. That is the course adopted by the writer, with the additional arbitrary principle that when any debatable points arise, such as the type of fixing provided by a bearing, the more flexible alternative is selected. Such procedure, however, cannot always give a satisfactory solution of the real elastic problem. Particularly is this so in aircraft installations where the engine structure supporting the bearings may have, of itself, a flexibility comparable with that arising from the bending of shafts. In this connection, the importance of static tests of flexibility upon existing systems of shafts, and their supports, should not be overlooked.

The two main features of the modern theory of whirling, in contradistinction to some of the earlier notions[1,4] are that any point of the flexural axis of a shaft is allowed to have a small but general vibratory movement, and that the steady rotation, imparted by the drive, at any section is invariably about the tangent to the flexural axis at that section at any instant[6,7]. In the sequel, the terms 'shaft rotation' or 'steady rotation' will refer to this rotation due to the drive, and is not to be confused with rotary motions in plane closed curves which points of the flexural axis may have and which will be referred to as 'whirling'.

* D. Napier & Son, Report AQ. IV. y. 18, received 1st April, 1949.

With emphasis on numerical aspects, the paper deals therefore with idealised systems of rotors, including sometimes mere points of mass, rigidly attached to a nest of light shafts having a common flexural axis, and having axial symmetry as regards lateral flexibility. Nominally the flexural axis contains the centroids of all the rotors and is coincident with a principal inertia axis of each rotor. Small departures from this ideal state of assembly concerning each centroid and each inertia axis are known as static and dynamic unbalance respectively, and will be taken into consideration as some of the influences responsible for the excitation of the system when in a state of steady rotation about the flexural axis. In the steady state, the shafts are not restricted to have a common rotational speed, but it is assumed that the steady state has only one degree of rotary freedom, in which case it is specified by the rotational speed of any one shaft. Only rotors having axial symmetry as regards mass are dealt with.

The writer feels that any value of this paper would have been enhanced if he had been able to cite a personal experience of actual installations which have exhibited whirling. Such experience is essential in order to give guidance on the importance of forms of excitation other than unbalance, brought about for example by blades of a propeller or turbine having to pass close to spaced obstacles in the flow of air or of gases[7]. The reader who wishes to see an account of experimental work verificative of the now accepted theory of whirling is referred to pages 429 to 470 and pages 1113 to 1143 of *Steam and Gas Turbines*, the authorized translation by Loewenstein of Stodola's standard work[8].

2. *The General Problem Briefly Described.*—In the general theory of any dynamical system vibrating about a state of stable equilibrium, the term 'normal mode' is used to denote any one of the critical configurations or shapes in which the system may pulsate freely without external agency. Associated with each normal mode is a definite frequency of vibration, referred to as a 'natural frequency'.

In the case of whirling, which concerns flexural vibration about a state of steady rotation, the words normal mode and natural frequency have a direct physical meaning for those systems which are axially symmetrical as regards both mass and elastic properties. For such systems, a normal mode denotes any plane shape into which the flexural axis or axes may be bent such that rotation of this plane about the axis of symmetry is *freely* possible. The definite angular velocity ω with which this plane naturally rotates determines the corresponding natural frequency $\omega/2\pi$, and is not to be confused with, nor arbitrarily assumed to be equal to and in the same sense as, the steady angular velocity Ω of one of the shafts. For any given steady state of rotation, represented by the angular velocity Ω, there are the same number m of positive (*i.e.*, in the same sense as Ω) natural angular velocities of whirl, $\omega_1, \omega_2, \ldots\ldots \omega_m$, as there are negative angular velocities $-\omega_{-1}, -\omega_{-2}, \ldots\ldots -\omega_{-m}$, each of these $2m$ natural angular velocities being associated with a modal shape. It is convenient to think of these modes as numbered 1, 2, ... m in such a way that the members of each of the two sets of moduli $(\omega_1, \omega_2, \ldots\ldots \omega_m)$ and $(\omega_{-1}, \omega_{-2}, \ldots\ldots \omega_{-m})$ are in order of ascending magnitude, and to use the terms 'forward' and 'reverse' to describe the sense of the whirl.

Only when axial inertia is not insignificant, and gyroscopic effects are therefore present, do these natural angular velocities of whirl vary with the imposed rotation of the shafts. As the imposed rotation is increased, the angular velocities of the forward whirls increase, and those of the reverse whirls decrease, the corresponding modal shapes suffering some change.

Any natural whirling state of the axially symmetrical system cannot be sustained at any speed of the drive unless suitable forcing excitation is present. If Ω denote, as before, the steady speed of one shaft, the case of a general excitation, periodic in one revolution of the given shaft, will be envisaged. There will thus be considered the possibility of excitations of frequencies $\Omega/2\pi$, $2\Omega/2\pi$, ... $n\Omega/2\pi$, the integer n being referred to as the 'order number'.

The term 'whirling speed' is used in this paper to denote the value of Ω at which any excitation causes resonance of any one of the natural whirling states. Two parameters, *viz.*, modal number and order number are therefore involved in the array of whirling speeds for any system of the type considered here.

In the case of systems which are axially unsymmetrical, there is the possibility of continuous regions of steady speed within which the free vibrations are theoretically unstable, but to what extent this is of practical importance the writer has no knowledge. It is not intended in this paper to deal with such problems, though they do in fact lend themselves to treatment by the iterative methods dealt with in Section 8.

3. *Description of the Whirling Properties of some Simple Systems.*—Although the acceptable theory of whirling was finally established about thirty years ago, there still exists much confusion of thought concerning such basic problems as that of a rotating thin shaft, or of a single heavy symmetrical rotor on a light shaft.

One of the most widespread beliefs is that a perfectly balanced thin uniform shaft, if disturbed laterally whilst running at one of the critical speeds associated with out-of-balance excitation, will then assume the whirling state to an ever increasing degree. In actual fact, and as pointed out in section 2, sustained whirling of the single uniform shaft is a forced circular motion of each point of its axis, the usual source of excitation being lack of balance of the shaft about its own axis. If the perfectly balanced shaft, thin enough to make gyroscopic effects negligible, be plucked or struck whilst rotating at any speed, the resulting *free* vibration is substantially in the plane of the disturbance and is indistinguishable in other noticeable matters from the vibration resulting from plucking the shaft whilst at rest.

It is also fallacious to regard the balance of a naturally straight shaft as modified by any bending produced by invariable forces, such as gravity for example. The notion that the whirling of the horizontal shaft is directly influenced by gravity, is undoubtedly stimulated by those elementary treatments which tacitly employ Rayleigh's principle and make use of the gravity shape as the approximation to the fundamental whirling shape. In consideration of bent shafts as influencing whirling, contrast should be apparent between one shaft whose central line is naturally curved, and another whose central line is naturally straight but is being elastically bent by gravity. The former shaft is unbalanced as its flexural axis is not its central line.

Let attention now be given to the case of a light shaft arranged horizontally on two supports and carrying an over-hung balanced disc. When the shaft is not rotating, flexural vibration of the shaft in a vertical plane will involve a linear vertical oscillation of the centroid of the disc together with a tilting oscillation of the disc about its horizontal diameter. There will thus be, under these conditions, two normal modes of vibration having, in general, differing frequencies $\omega_1/2\pi$ and $\omega_2/2\pi$. In the first mode, the vertical sinusoidal motion of any point of the shaft, including the centroid of the disc, may be regarded as the superposition of two circular motions of equal radius but having the opposite angular velocities ω_1 and $-\omega_1$. Thinking similarly of the other mode leads to the idea that four natural circular whirls of angular velocities ω_1 and ω_2 in the forward sense, and ω_1 and ω_2 in the reverse sense, are associated with the system in its non-rotating state. Now suppose the shaft and disc to be rotating with any steady angular velocity Ω. There are now, as before, four natural circular whirls, two forward and two reverse, but, owing to the gyroscopic effect of the spinning disc, these are no longer equal and opposite in pairs. The change in the natural angular velocities was described in section 2. Moreover, there are now four distinct modal shapes of the shaft. If the shaft of such a system be disturbed laterally whilst in steady rotation, all four of the natural whirls will in general appear, so that the resulting free motion will not be in one plane, neither will it appear circular, except perhaps after sufficient time has elapsed to allow the three most heavily damped ones to disappear.

As an example of whirling speeds, suppose that the overhung disc system just described was excited by giving one of the bearings a sinusoidal vertical movement of prescribed amplitude and of the same frequency $\Omega/2\pi$ as that of the shaft revolution. There are thus circular excitations of angular velocities Ω and $-\Omega$ acting on the system. As Ω is slowly increased, the first resonance to occur would be the first reverse whirl, characterised by any point of the axis of the shaft describing a circle with angular velocity Ω in the reverse sense. At higher speeds resonances would be obtained with the first forward whirl and with the second reverse whirl, but, for a relatively thin disc, the frequency of the excitation would never actually overtake the frequency of the second forward whirl, however high Ω became.

The experience such as this with a system having appreciable axial inertia, in contrast to the thin shaft for example, discounts the impression given by some authorities that there is no distinction between lateral vibration of shafts when rotating and when not rotating.

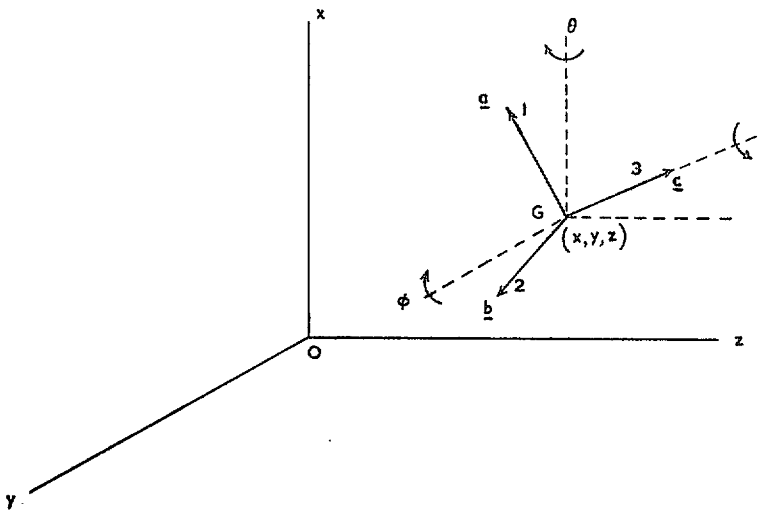

FIG. 1. Pictorial representation of the disturbed rotor.

4. *The Dynamic Loads Produced by a Disturbed Balanced Rotor.*—Ox, Oy, Oz (Fig. 1) form a right-handed system of orthogonal axes fixed in space. The unit vectors **a**, **b** and **c**, radiating from G, the centroid of the rotor, are along the principal axes G1, G2, G3 of the rotor, the corresponding inertias being A, B and C (lb in. sec^2). In the steady state, G is coincident with O, and the unit vector **c** lies along Oz, the flexural axis of the system to which the rotor belongs. The mass of the rotor is M (lb in.$^{-1}$ sec^2).

At the instant t the co-ordinates of G relative to the fixed frame are x, y, z, assumed to be small, and the vector **c** is slightly inclined to Oz.

If α, β, θ and ϕ are small, and ψ is an angle of any magnitude, then $(\cos\psi, \sin\psi, -\beta)$, $(-\sin\psi, \cos\psi, -\alpha)$ and $(\phi, \theta, 1)$ are each unit vectors, the first two being orthogonal to the first degree of smallness. Each of the first two will be orthogonal to the third if

$$\left.\begin{aligned} \alpha &= -\phi\sin\psi + \theta\cos\psi, \\ \text{and } \beta &= \phi\cos\psi + \theta\sin\psi. \end{aligned}\right] \qquad \ldots \quad (4.1)$$

The unit vectors **a**, **b** and **c** may thus be given to this degree of accuracy by

$$\left.\begin{aligned} \mathbf{a} &= (\cos\psi,\ \sin\psi,\ -\beta), \\ \mathbf{b} &= (-\sin\psi,\ \cos\psi,\ -\alpha), \\ \mathbf{c} &= (\phi,\ \ \theta,\ \ 1\ ), \end{aligned}\right] \qquad \ldots \quad (4.2)$$

together with the relationships (4.1). It will be noticed that the vector **c** fulfils the condition of being only slightly inclined to Oz.

The angular velocity vector at G will be expressible as $\tilde{\omega}_1\mathbf{a} + \tilde{\omega}_2\mathbf{b} + \tilde{\omega}_3\mathbf{c}$ where the components $\tilde{\omega}_{1,2,3}$ are given by

$$\left.\begin{aligned} \tilde{\omega}_1 &= \mathbf{c}\,.\,\dot{\mathbf{b}} = -\,\dot{\mathbf{c}}\,.\,\mathbf{b} = \dot{\phi}\sin\psi - \dot{\theta}\cos\psi, \\ \tilde{\omega}_2 &= \mathbf{a}\,.\,\dot{\mathbf{c}} = -\,\dot{\mathbf{a}}\,.\,\mathbf{c} = \dot{\phi}\cos\psi + \dot{\theta}\sin\psi, \\ \tilde{\omega}_3 &= \mathbf{b}\,.\,\dot{\mathbf{a}} = -\,\dot{\mathbf{b}}\,.\,\mathbf{a} = \dot{\psi}. \end{aligned}\right] \qquad \ldots \quad (4.3)$$

Owing to the fact that the angle ψ, unlike the angles θ and ϕ, is not restricted to be small, some care is needed when describing these three angles geometrically. θ is an anti-clockwise rotation about an axis through G parallel to the fixed axis Ox, whilst ϕ is a clockwise rotation about an axis through G parallel to Oy. ψ is a clockwise rotation about the principal axis G3 in its actual position. This definition of the angles may be shown to be unique by demonstrating analytically that it is immaterial in what sequence these rotations are performed, on the understanding that when the rotation ψ is made, it must be about the line in which G3 then lies (*see* Fig. 1).

It will be supposed that the angular velocity $\dot{\psi}$ consists of a constant part Ω upon which is superimposed a small variable $\dot{\varepsilon}$. Accordingly the angle ψ, as defined above, may be written

$$\psi = \Omega t + \gamma + \varepsilon, \qquad (4.4)$$

where γ is a constant phase angle, and its corresponding derivatives $\dot{\psi}$ and $\ddot{\psi}$ are

$$\dot{\psi} = \Omega + \dot{\varepsilon} \qquad (4.5)$$

and $$\ddot{\psi} = \ddot{\varepsilon}. \qquad (4.6)$$

It may be remarked that it is immaterial whether the small rotation ε is considered to be about G3 or about a line through G parallel to Oz.

Resolving the linear momentum of the rotor into components (p_x, p_y, p_z) parallel to the fixed axes, and taking their rates of change, it is clear that

$$\left.\begin{aligned} \dot{p}_x &= M\ddot{x}, \\ \dot{p}_y &= M\ddot{y}, \\ \dot{p}_z &= M\ddot{z}. \end{aligned}\right] \qquad (4.7)$$

The angular momenta about G1, G2, G3 may be written down and then resolved into components $(p_\theta, p_\phi, p_\varepsilon)$ about the fixed axes in the sense of positive θ, ϕ and ε. Provided x, y and z are small as premised, the momenta $(p_\theta, p_\phi, p_\varepsilon)$ will, to the first degree, be also the angular momenta of the rotor about the fixed axes Ox, Oy, Oz. Taking rates of change, it may be shown that, to the first degree of smallness,

$$\left.\begin{aligned} \dot{p}_\theta = {} & \{A' + B' \cos 2(\Omega t + \gamma)\}\ddot{\theta} - \{B' \sin 2(\Omega t + \gamma)\}\ddot{\phi} \\ & - 2\{B' \sin 2(\Omega t + \gamma)\}\Omega\dot{\theta} - \{C' + 2B' \cos 2(\Omega t + \gamma)\}\Omega\dot{\phi}, \\ \dot{p}_\phi = {} & -\{B' \sin 2(\Omega t + \gamma)\}\ddot{\theta} + \{A' - B' \cos 2(\Omega t + \gamma)\}\ddot{\phi} \\ & + \{C' - 2B' \cos 2(\Omega t + \gamma)\}\Omega\dot{\theta} + 2\{B' \sin 2(\Omega t + \gamma)\}\Omega\dot{\phi}, \\ \dot{p}_\varepsilon = {} & C'\ddot{\varepsilon}, \end{aligned}\right] \qquad (4.8)$$

$$\left.\begin{aligned} \text{where } A' &= \tfrac{1}{2}(A + B), \\ B' &= \tfrac{1}{2}(A - B), \\ \text{and } C' &= C, \end{aligned}\right] \qquad (4.9)$$

and substitution has been made for ψ and its derivatives.

A dynamic load is defined to be the reversed rate of change of any linear or angular momentum, and corresponds with the positive sense of some linear or angular variable. Thus the dynamic loads corresponding to $x, y, z, \theta, \phi, \varepsilon$ of Fig. 1 are $-\dot{p}_x$, $-\dot{p}_y$, $-\dot{p}_z$, $-\dot{p}_\theta$, $-\dot{p}_\phi$, $-\dot{p}_\varepsilon$, respectively.

5. *Dynamic Loads due to Static and Dynamic Unbalance, and to Other Effects.*—The unbalance may be thought of as due to the attachment to the ideal rotor of the masses shown in Fig. 2.

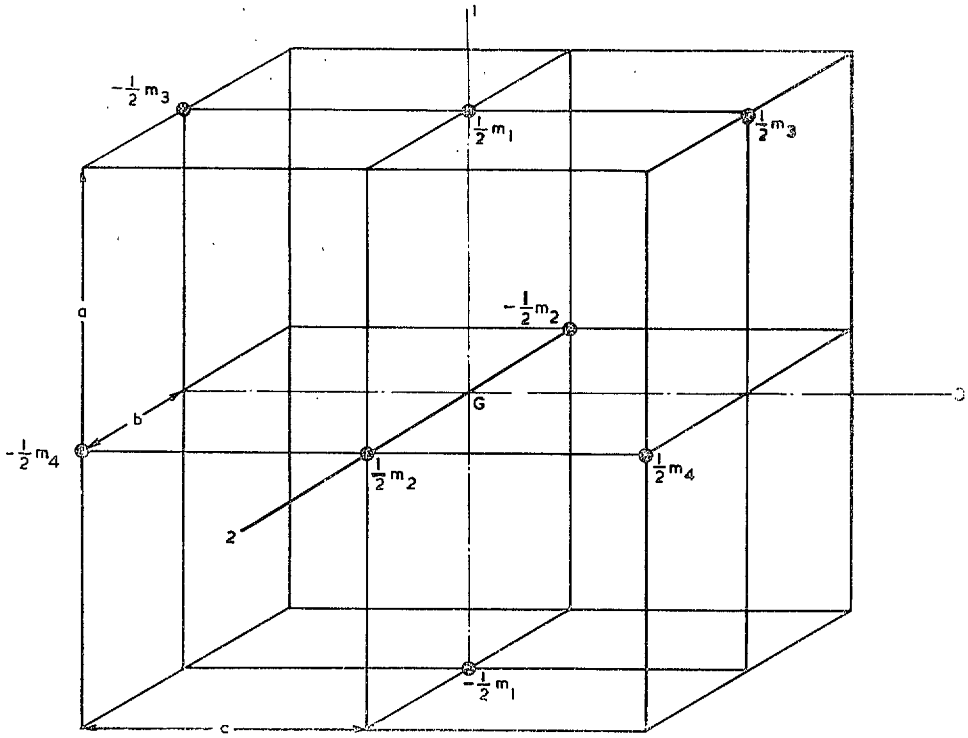

FIG. 2. Representation of unbalance by means of particles.

The masses of the particles and their positions relative to the axes G1, G2, G3 of the rotor are as follows: $\frac{1}{2}m_1$ at $(a, 0, 0)$, $-\frac{1}{2}m_1$ at $(-a, 0, 0)$, $\frac{1}{2}m_2$ at $(0, b, 0)$, $-\frac{1}{2}m_2$ at $(0, -b, 0)$, $\frac{1}{2}m_3$ at $(a, 0, c)$, $-\frac{1}{2}m_3$ at $(a, 0, -c)$, $\frac{1}{2}m_4$ at $(0, b, c)$, and $-\frac{1}{2}m_4$ at $(0, b, -c)$.

It may be verified that these particles make no addition to the mass of the rotor, nor do they affect its inertias A, B and C. They do, however, shift the centroid of the whole assembly to the point $(m_1a/M,\ m_2b/M,\ 0)$ and introduce products of inertia $F = m_1bc$ and $G = m_3ac$ corresponding to the pairs of axes G2, G3 and G3, G1 respectively.

The dynamic loads due to this lattice of particles and corresponding to x, y, z, θ, ϕ, ε are found to be

$$\left.\begin{array}{c} m_1a\Omega^2\cos(\Omega t+\gamma) - m_2b\Omega^2\sin(\Omega t+\gamma), \\ m_1a\Omega^2\sin(\Omega t+\gamma) + m_2b\Omega^2\cos(\Omega t+\gamma), \\ 0, \\ m_3ca\Omega^2\sin(\Omega t+\gamma) + m_4bc\Omega^2\cos(\Omega t+\gamma), \\ m_3ca\Omega^2\cos(\Omega t+\gamma) - m_4bc\Omega^2\sin(\Omega t+\gamma), \\ 0, \end{array}\right] \qquad \ldots \quad (5.1)$$

respectively, when the effects of the small disturbances are ignored. The omitted terms have some significance when the unbalance is large, but it is not proposed to deal with this question in the present paper.

Any loading which is independent of the displacements of the rotor will be referred to as external loading.

Notice that the external loads given by (5.1) are of the form

$$
\begin{array}{lll}
& P \cos \Omega t - Q \sin \Omega t, & \text{corresponding to } x, \\
& P \sin \Omega t + Q \cos \Omega t, & \quad ,, \qquad ,, \; y, \\
& K \cos \Omega t - L \sin \Omega t, & \quad ,, \qquad ,, \; \phi, \\
\text{and} & K \sin \Omega t + L \cos \Omega t, & \quad ,, \qquad ,, \; \theta.
\end{array}
$$

If two similar lattices of particles could exist around G, one rotating with angular velocity $n\Omega$ and the other with $-n\Omega$, the external loads would be of the form

$$
\left.\begin{array}{l}
P_n \cos n\Omega t - Q_n \sin n\Omega t + P_{-n} \cos n\Omega t + Q_{-n} \sin n\Omega t, \\
P_n \sin n\Omega t + Q_n \cos n\Omega t - P_{-n} \sin n\Omega t + Q_{-n} \cos n\Omega t, \\
K_n \cos n\Omega t - L_n \sin n\Omega t + K_{-n} \cos n\Omega t + L_{-n} \sin n\Omega t, \\
K_n \sin n\Omega t + L_n \cos n\Omega t - K_{-n} \sin n\Omega t + L_{-n} \cos n\Omega t,
\end{array}\right] \quad \cdots \quad (5.2)
$$

corresponding to x, y, ϕ, θ respectively.

The most general kind of pure nth order external loading corresponding to x and y respectively would be $A \cos n\Omega t + B \sin n\Omega t$ and $C \cos n\Omega t + D \sin n\Omega t$. The first two forms of (5.2) show clearly that this generality has been achieved with $P_n = \frac{1}{2}(A + D)$, $Q_n = \frac{1}{2}(C - B)$, $P_{-n} = \frac{1}{2}(A - D)$ and $Q_{-n} = \frac{1}{2}(B + C)$. Similarly the last two forms of (5.2) give the most general kind of pure nth order loading corresponding to ϕ and θ.

The superposition of loadings, formed from (5.2) by giving n all integer values from 1 upwards, will be a representation by Fourier series of external loading which is generally periodic in one revolution of the rotor, and which might arise, for instance, from aerodynamic effects due to the presence of blades attached to the rotor.

It follows, therefore, that the possibility of such generally periodic external loading of the rotor will be catered for by taking loads

$$
\left.\begin{array}{ll}
\sum_{n=-\infty}^{n=+\infty} (P_n \cos n\Omega t - Q_n \sin n\Omega t) & \text{along } Ox \\
\sum_{n=-\infty}^{n=+\infty} (P_n \sin n\Omega t + Q_n \cos n\Omega t) & \quad ,, \quad Oy \\
\sum_{n=-\infty}^{n=+\infty} (K_n \cos n\Omega t - L_n \sin n\Omega t) & \text{about } Oy \\
\sum_{n=-\infty}^{n=+\infty} (K_n \sin n\Omega t + L_n \cos n\Omega t) & \quad ,, \quad Ox
\end{array}\right] \quad \cdots \quad (5.3)
$$

The order number n will be referred to by its modulus, together with the descriptive term forward or reverse depending upon its sign.

6. *The Assumed System and its Elastic Properties.*—The general type of system to be considered consists of m rotors, similar to the one dealt with above, and carried by a system of co-axial shafts having a common flexural axis. The centroids of the rotors nominally lie upon this axis at points 1, 2, 3, ... r, ... s, ... etc. Any one of these points, say r, is considered to be rigidly attached to the rth rotor which is itself rigidly attached to one of the shafts over a certain length usually considered small in comparison with the length of the shaft itself.

In relation to the fixed axes of Fig. 1, the flexural axis of the shafts will lie initially along Oz, regarded for convenience as horizontal, and Ox will be thought of as vertical.

Associated with each of the points r will be small displacements of type x, y, z, θ, ϕ and ε, as defined in section 4.

The transverse elastic properties of the system, involving displacements of type x, y, ϕ and θ, are supposed to be such that if $(uv)_{rs}$ denote the deflection of type v at point s due to unit loading of type u at point r, then, for all r and s,

$$\left.\begin{array}{l}
(xx)_{rs} = (yy)_{rs}, \\
(x\phi)_{rs} = (y\theta)_{rs}, \\
(\phi x)_{rs} = (\theta y)_{rs}, \\
(\phi\phi)_{rs} = (\theta\theta)_{rs}, \\
(xy)_{rs} \;..\; (yx)_{rs} = 0, \\
(x\theta)_{rs} = (\theta x)_{rs} = (y\phi)_{rs} = (\phi y)_{rs} = 0, \\
(\theta\phi)_{rs} = (\phi\theta)_{rs} = 0,
\end{array}\right] \qquad \ldots \qquad (6.1)$$

together with others which follow as a necessary consequence of Rayleigh's reciprocal relationships. The equations (6.1) define what was referred to in section 1 as 'axial symmetry regarding lateral flexibility'.

Assuming that the non-zero flexibilities $(yy)_{rs}$, $(y\theta)_{rs}$, $(\theta y)_{rs}$ and $(\theta\theta)_{rs}$ arise solely from the bending of shafts, a general treatment of their numerical evaluation will be outlined in section 9.

Meanwhile it should be noted that flexibility due to the structure itself, and determined experimentally, is not excluded from the flexibilities provided that it obeys (6.1).

7. *Formation of the General Equations of Motion, and Those for the Particular Case of Axial Symmetry of Mass.*—Letting a suffix r be used to distinguish the properties of the rth rotor from those of others, the displacements x_s, y_s, ϕ_s and θ_s of the sth rotor at time t will be given by

$$x_s + \sum_{r=1}^{m} [(\dot{p}_x)_r\,(yy)_{rs}] + \sum_{r=1}^{m} [(\dot{p}_\phi)_r\,(\theta y)_{rs}] =$$

$$\sum_{n=-\infty}^{\infty} \sum_{r=1}^{m} \left[\begin{array}{l} \{(P_n)_r \cos(n\Omega_r t) - (Q_n)_r \sin(n\Omega_r t)\}(yy)_{rs} \\ +\{(K_n)_r \cos(n\Omega_r t) - (L_n)_r \sin(n\Omega_r t)\}(\theta y)_{rs} \end{array}\right], \qquad \ldots \qquad (7.1)$$

$$y_s + \sum_{r=1}^{m} [(\dot{p}_y)_r\,(yy)_{rs}] + \sum_{r=1}^{m} [(\dot{p}_\theta)_r\,(\theta y)_{rs}] =$$

$$\sum_{n=-\infty}^{\infty} \sum_{r=1}^{m} \left[\begin{array}{l} \{(P_n)_r \sin(n\Omega_r t) + (Q_n)_r \cos(n\Omega_r t)\}(yy)_{rs} \\ +\{(K_n)_r \sin(n\Omega_r t) + (L_n)_r \cos(n\Omega_r t)\}(\theta y)_{rs} \end{array}\right], \qquad \ldots \qquad (7.2)$$

$$\phi_s + \sum_{r=1}^{m} [(\dot{p}_x)_r\,(y\theta)_{rs}] + \sum_{r=1}^{m} [(\dot{p}_\phi)_r\,(\theta\theta)_{rs}] =$$

$$\sum_{n=-\infty}^{\infty} \sum_{r=1}^{m} \left[\begin{array}{l} \{(P_n)_r \cos(n\Omega_r t) - (Q_n)_r \sin(n\Omega_r t)\}(y\theta)_{rs} \\ + \{(K_n)_r \cos(n\Omega_r t) - (L_n)_r \sin(n\Omega_r t)\}(\theta\theta)_{rs} \end{array}\right], \qquad \ldots \qquad (7.3)$$

$$\theta_s + \sum_{r=1}^{m} [(\dot{p}_y)_r\, (y\theta)_{rs}] + \sum_{r=1}^{m} [(\dot{p}_\theta)_r\, (\theta\theta)_{rs}] =$$

$$\sum_{n=-\infty}^{\infty} \sum_{r=1}^{m} \left[\begin{array}{l} \{(P_n)_r \sin (n\Omega_r t) + (Q_n)_r \cos (n\Omega_r t)\}(y\theta)_{rs} \\ + \{(K_n)_r \sin (n\Omega_r t) + (L_n)_r \cos (n\Omega_r t)\}(\theta\theta)_{rs} \end{array} \right], \quad \ldots \quad (7.3)$$

in which the relationships (6.1) appear, and $(\dot{p}_x)_r$, $(\dot{p}_y)_r$, $(\dot{p}_\phi)_r$ and $(\dot{p}_\theta)_r$ are taken from (4.7) and (4.8).

Suppose now that one of the shafts be chosen as the reference shaft, that its steady rotation is Ω, and that only the external loading of frequency $n\Omega/2\pi$ is acting upon one rotor of that shaft.

When there is axial symmetry of mass, and therefore $A_r = B_r$ for all values of r, it follows from (4.7) and (4.8) that

$$\left. \begin{aligned} (\dot{p}_x)_r &= M_r\ddot{x}_r, \\ (\dot{p}_y)_r &= M_r\ddot{y}_r, \\ (\dot{p}_\phi)_r &= A_r\ddot{\phi}_r + C_r\Omega_r\dot{\theta}_r, \\ (\dot{p}_\theta)_r &= A_r\ddot{\theta}_r - C_r\Omega_r\dot{\phi}_r. \end{aligned} \right] \quad \ldots \quad (7.5)$$

The trial substitutions

$$\left. \begin{aligned} x_r &= X_r \cos n\Omega t, \\ y_r &= X_r \sin n\Omega t, \\ \phi_r &= \Phi_r \cos n\Omega t, \\ \theta_r &= \Phi_r \sin n\Omega t, \end{aligned} \right] \quad \ldots \quad (7.6)$$

give

$$\left. \begin{aligned} -(\dot{p}_x)_r &= n^2\Omega^2 M_r X_r \cos n\Omega t, \\ -(\dot{p}_y)_r &= n^2\Omega^2 M_r X_r \sin n\Omega t, \\ -(\dot{p}_\phi)_r &= n^2\Omega^2\left(A_r - \frac{C_r}{n_r}\right) \Phi_r \cos n\Omega t, \\ -(\dot{p}_\theta)_r &= n^2\Omega^2\left(A_r - \frac{C_r}{n_r}\right) \Phi_r \sin n\Omega t, \end{aligned} \right] \quad \ldots \quad (7.7)$$

where

$n_r = (\Omega/\Omega_r)n$ = number of complete cycles of the excitation per revolution of the *rth* shaft. .. .. (7.8)

Hence if only excitation of the P and K type is present, the equations (7.1), (7.2), (7.3) and (7.4) may be satisfied by choosing the X_r and Φ_r so that

$$\left. \begin{aligned} &\sum_{r=1}^{m} \{M_r (yy)_{rs} X_r\} + \sum_{r=1}^{m} \left[\left(A_r - \frac{C_r}{n_r}\right)(\theta y)_{rs}\Phi_r\right] - (1/n^2\Omega^2)X_s \\ &\qquad = -\{P_n(yy)_{rs} + K_n(\theta y)_{rs}\}/n^2\Omega^2, \end{aligned} \right.$$

and

$$\left. \begin{aligned} &\sum_{r=1}^{m} \{M_r (y\theta)_{rs} X_r\} + \sum_{r=1}^{m} \left[\left(A_r - \frac{C_r}{n_r}\right)(\theta\theta)_{rs}\Phi_r\right] - (1/n^2\Omega^2)\Phi_s \\ &\qquad = -\{P_n(y\theta)_{rs} + K_n(\theta\theta)_{rs}\}/n^2\Omega^2. \end{aligned} \right] \quad \ldots \quad (7.9)$$

Each of the $2m$ expressions on the left-hand sides of the set of equations, given by (7.9) when s takes all its m values, can vanish when $1/n^2\Omega^2$ is a latent root of the matrix product

$$\begin{bmatrix} [yy], & [\theta y] \\ [y\theta], & [\theta\theta] \end{bmatrix} \begin{bmatrix} [M], & [0] \\ [0], & [J] \end{bmatrix}, \qquad (7.10)$$

in which $[yy]$, $[\theta y]$, $[y\theta]$, and $[\theta\theta]$ each denotes an $m \times m$ square matrix formed from $(yy)_{rs}$, $(\theta y)_{rs}$, $(y\theta)_{rs}$ and $(\theta\theta)_{rs}$, $[M]$ denotes a diagonal matrix of the masses, and $[J]$ is also a diagonal matrix whose diagonal elements are $A_1 - C_1/n_1$, $A_2 - C_2/n_2$, etc. These diagonal elements of $[J]$ will sometimes be referred to as ' equivalent inertias '.

When Ω is such that $(1/n^2\Omega^2)$ is a latent root of (7.10), a whirling speed has been reached, and the balance of the equations (7.9) is undertaken by the usual device of assuming suitable small damping loads which are linear expressions in the velocities, and which limit the amplitudes X_r, Φ_r to finite values, in addition to giving a continuous change of π in the phase lag of the response of the particular normal mode behind the particular excitation as the whirling speed is passed through. As interest here lies only in the whirling speeds, however, a general discussion of damping will be avoided.

The same matrix (7.10) is obtained as a result of the trial substitutions

$$x_r = -Y_r \sin n\Omega t,$$

$$y_r = Y_r \cos n\Omega t,$$

$$\phi_r = -\Theta_r \sin n\Omega t,$$

and

$$\theta_r = \Theta_r \cos n\Omega t,$$

in (7.5) with the object of balancing the excitation of the Q and L type of frequency $n\Omega/2\pi$.

For convenience in numerical work, the matrix (7.10) is compiled after expressing the flexibilities in micro-inches (or micro-radians) per lb (or per lb in.), working with weights (lb) in place of masses, and expressing all inertias in the units lb in.^{2} The whirling speeds will then be given whenever $\lambda = 10^6 g/n^2\Omega^2$ is a latent root of the matrix so modified.

When all the elements of $[J]$ are positive, all the latent roots must be necessarily positive, and the corresponding whirling speeds N_{1n}, N_{2n}, —— etc., written in ascending order of magnitude, are then calculable from the equation

$$N_{rn} = \frac{187{,}711}{|n|\sqrt{(\lambda_{rn})}} \text{ r.p.m.}, \qquad (7.11)$$

where λ_{1n}, λ_{2n}, *etc.*, are the latent roots written in descending order of magnitude, and the numerator is based upon taking $g = 386 \cdot 4$ in./sec^2.

When, however, $[J]$ contains a certain number of negative elements, there will be this same number of negative latent roots, which are not of interest in connection with whirling speeds. It may for instance be found that $\lambda_{11} = -200$, $\lambda_{21} = 100$ and $\lambda_{31} = 25$ in some problem concerning one large thin disc and another of negligible inertia. In this case there are only two whirling speeds under the postulated first order forward excitation, *viz.*, at 18,771 r.p.m. and 37,542 r.p.m.

8. *The Iterative Technique.*—In order to simplify the description of the method, the notation of section 7 will be changed in such a way that there is no symbolic distinction between deflection and slope, force and couple, and between weight and inertia. The symbol y_s will denote either a

linear displacement or a slope, and the corresponding flexibilities will be written f_{rs}. The symbol w_s will denote the weight if y_s is a deflection, or the equivalent inertia if y_s is a slope. The dynamic matrix (7.10), modified by using weights in place of masses, would be, for a system having four variables y_1, y_2, y_3 and y_4,

$$\begin{bmatrix} w_1f_{11} & w_2f_{12} & w_3f_{13} & w_4f_{14} \\ w_1f_{21} & w_2f_{22} & w_3f_{23} & w_4f_{24} \\ w_1f_{31} & w_2f_{32} & w_3f_{33} & w_4f_{34} \\ w_1f_{41} & w_2f_{42} & w_3f_{43} & w_4f_{44} \end{bmatrix} \qquad \ldots \qquad (8.1)$$

in which $f_{rs} = f_{sr}$.

The extraction of the latent root of highest modulus, and the corresponding mode, of (8.1) would be according to the following prescription:—

Underneath the dynamic matrix write down a row of 4 numbers. These can be chosen arbitrarily, but if the calculator has a rough idea of the mode he should write down this guess. The largest of these numbers should be 1. Let this row be (a_1, a_2, a_3, a_4). Concentrate on the first row of (8.1) and add together the products of corresponding terms from this row and from the row (a_1, a_2, a_3, a_4). Enter the result, y_1, underneath a_1. Now take the second row of the matrix and use it in a similar way, entering the resulting y_2 under a_2. Carry on this process until all four rows of the matrix have been employed, thus completing the row (y_1, y_2, y_3, y_4) underneath (a_1, a_2, a_3, a_4). Now make a note at the side of the y-row, of the value of the member of this row, say y_m, which is of greatest modulus, and then form a new a-row underneath the y-row by dividing throughout the latter row by y_m. Repeat the sequence of operations upon the new a-row, thus giving underneath it a new y-row and a new y_m. From this y-row and y_m a new a-row is obtained, and so on. The work of passing from one a-row to the next is described as one round of iteration. The rounds are normally continued until two successive a-rows are identical to the order of accuracy employed. This last a-row is the mode corresponding to the latent root λ_1, whose value is the last recorded value of y_m.

If during the extraction of λ_1, the iterative scheme just described does not appear to be converging after five rounds, the calculator should modify the scheme as follows:—

Starting with the sixth a-row, determine the sixth y-row, but omit the step of dividing through this row by its y_m, and continue with the y-row instead. The resulting row is, in turn, operated upon in like manner, and so on. At least two rounds of this modified iteration must be completed, thus giving to hand a succession of rows which will be denoted by **6**, **7**, **8**, etc. First see if numbers p_1 and p_2 can be found so that

$$\mathbf{8} - p_1\mathbf{7} + p_2\mathbf{6} = 0; \qquad \ldots \qquad (8.2)$$

i.e., this equation must hold for the first, second, third, etc., entries in rows **6**, **7** and **8**. Obviously, if all the equations (four in this case) represented by (8.2) were written down they would be more than is necessary to determine unique values for p_1 and p_2. Rather than take a least-square solution, it is advisable to solve for p_1 and p_2 from any two of the equations and then to test the remainder with the values obtained. If the test is satisfactory, it means that λ_1 and λ_2 may be calculated as the two roots of the quadratic equation

$$\lambda^2 - p_1\lambda + p_2 = 0. \qquad \ldots \qquad (8.3)$$

Further, the modes are given by

$$\left.\begin{array}{ll} & \mathbf{7} - \lambda_2\mathbf{6}, \text{ corresponding to } \lambda_1, \\ \text{and} & \mathbf{7} - \lambda_1\mathbf{6}, \qquad ,, \qquad ,, \ \lambda_2. \end{array}\right] \qquad \ldots \qquad (8.4)$$

If the test (8.2) fails, complete a further round of this modified iteration and see if p_1, p_2, and p_3 can be found so that

$$\mathbf{9} - p_1\mathbf{8} + p_2\mathbf{7} - p_3\mathbf{6} = 0, \qquad (8.5)$$

and, if so, λ_1, λ_2 and λ_3 may be calculated as roots of the cubic

$$\lambda^3 - p_1\lambda^2 + p_2\lambda - p_3 = 0. \qquad (8.6)$$

In this case the modes are

$$\left.\begin{array}{lll} \mathbf{8} - (\lambda_2 + \lambda_3)\mathbf{7} + \lambda_2\lambda_3\mathbf{6}, & \text{corresponding to} & \lambda_1, \\ \mathbf{8} - (\lambda_3 + \lambda_1)\mathbf{7} + \lambda_3\lambda_1\mathbf{6}, & \quad ,, \qquad ,, & \lambda_2, \\ \mathbf{8} - (\lambda_1 + \lambda_2)\mathbf{7} + \lambda_1\lambda_2\mathbf{6}, & \quad ,, \qquad ,, & \lambda_3. \end{array}\right] \qquad (8.7)$$

The calculator should not rely upon such separation of even the first mode without testing it by a further round of iteration according to the original scheme. If this is not satisfactory he will have to continue the iteration.

The theory underlying this numerical technique will be dealt with in a separate paper.

If the calculator has been obliged to separate the modes according to the methods summarised in equations (8.2) to (8.7), and has achieved a satisfactory result for λ_1 and its mode, he should not be content to assume that the second or higher mode has been equally well separated. This assumes of course that he is interested in the modes higher than the first. If, on the other hand, the second mode is required, but no other, it may be sufficient to test the suspected second mode by a round of iteration according to the original scheme. But if this fails, the calculator must remove, by a method given later in this section, the effects of the first mode from the original matrix. In the meantime it will be convenient to deal with a simplifying procedure based on Rayleigh's principle.

It frequently happens in whirling problems that only the first latent root of the matrix is required and no special interest attaches to the corresponding modal shape. In such a case, Rayleigh's principle, appealed to after each round of ordinary iteration, will often enable the latent root to be deduced to a given accuracy at a stage of the iteration previous by many rounds to that at which y_m would have the same accuracy. The Rayleigh approximation to λ is deduced from an a-row and the y-row which comes from it after operating once with the matrix, and is given by

$$\lambda \simeq \frac{w_1y_1^2 + w_2y_2^2 + w_3y_3^2 + \ldots.}{w_1y_1a_1 + w_2y_2a_2 + w_3y_3a_3 + \ldots.}. \qquad (8.8)$$

A convenient way of using this to evaluate the row $(w_1y_1, w_2y_2, w_3y_3 \ldots.)$ on a separate sheet of paper, offering this up to the a-row and y-row in turn, and using (8.8) as $\lambda \simeq \{\Sigma(wy)y\}/\{\Sigma(wy)a\}$.

It remains to describe the method of reduction whereby, from a knowledge of the first latent root λ_1 and its mode $(\alpha_1, \alpha_2, \alpha_3 \ldots.)$, there can be deduced a new matrix whose latent roots are, in order of descending moduli, λ_2, λ_3, λ_4, 0, and whose modes are the second, third, fourth and first modes respectively of (8.1). Iteration using this new matrix would proceed as it did for the original matrix, but λ_2 is now the privileged latent root. The reduction is performed as follows:—

Underneath the row

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \qquad (8.9)$$

write down the row

$$(w_1\alpha_1,\ w_2\alpha_2,\ w_3\alpha_3,\ w_4\alpha_4). \qquad \ldots \qquad (8.10)$$

The summation of products of corresponding terms in (8.9) and (8.10) gives $\Sigma w\alpha^2$, and this is immediately divided into λ_1 to give

$$\mu_1 = \lambda_1/(\Sigma w\alpha^2). \qquad \ldots \qquad (8.11)$$

An additional row

$$(\mu_1\alpha_1,\ \mu_1\alpha_2,\ \mu_1\alpha_3,\ \mu_1\alpha_4) \qquad \ldots \qquad (8.12)$$

is then written down, and then a 4×4 matrix is compiled by taking the row (8.10) and multiplying throughout by the first, second, third, etc., members of the row (8.12); this matrix is therefore

$$\begin{bmatrix} \mu_1 w_1\alpha_1^2, & \mu_1 w_2\alpha_2\alpha_1, & \mu_1 w_3\alpha_3\alpha_1, & \mu_1 w_4\alpha_4\alpha_1 \\ \mu_1 w_1\alpha_1\alpha_2, & \mu_1 w_2\alpha_2^2, & \mu_1 w_3\alpha_3\alpha_2, & \mu_1 w_4\alpha_4\alpha_2 \\ \mu_1 w_1\alpha_1\alpha_3, & \mu_1 w_2\alpha_2\alpha_3, & \mu_1 w_3\alpha_3^2, & \mu_1 w_4\alpha_4\alpha_3 \\ \mu_1 w_1\alpha_1\alpha_4, & \mu_1 w_2\alpha_2\alpha_4, & \mu_1 w_3\alpha_3\alpha_4, & \mu_1 w_4\alpha_4^2 \end{bmatrix} \qquad \ldots \qquad (8.13)$$

A new matrix is then compiled by subtracting corresponding members of (8.13) from those of (8.1) thus giving

$$\begin{bmatrix} w_1(f_{11} - \mu_1\alpha_1^2), & w_2(f_{12} - \mu_1\alpha_1\alpha_2), & w_3(f_{13} - \mu_1\alpha_1\alpha_3), & w_4(f_{14} - \mu_1\alpha_1\alpha_4) \\ w_1(f_{12} - \mu_1\alpha_1\alpha_2), & w_2(f_{22} - \mu_1\alpha_2^2), & w_3(f_{23} - \mu_1\alpha_2\alpha_3), & w_4(f_{24} - \mu_1\alpha_2\alpha_4) \\ w_1(f_{13} - \mu_1\alpha_1\alpha_3), & w_2(f_{23} - \mu_1\alpha_2\alpha_3), & w_3(f_{33} - \mu_1\alpha_3^2), & w_4(f_{34} - \mu_1\alpha_3\alpha_4) \\ w_1(f_{14} - \mu_1\alpha_1\alpha_4), & w_2(f_{24} - \mu_1\alpha_2\alpha_4), & w_3(f_{34} - \mu_1\alpha_3\alpha_4), & w_4(f_{44} - \mu_1\alpha_4^2) \end{bmatrix} \qquad (8.14)$$

The matrix (8.14) is the required reduced matrix. It may be worked with in the same way as was (8.1), and may be checked by operating with it upon (8.9) when a row of numbers should result which are acceptable as zeros.

In describing the iterative methods, the writer has tacitly assumed that an ordinary Brunsviga type of mechanical or electrical calculating machine is available. Some examples will be found in section 10.

9. *Numerical Work in Connection with the Elastic Problem for Co-axial shafts.*—The basic problem concerns a single shaft of length L, constant flexural rigidity EI, and encastré at one end O (Fig. 3). The flexibilities at any other point A are

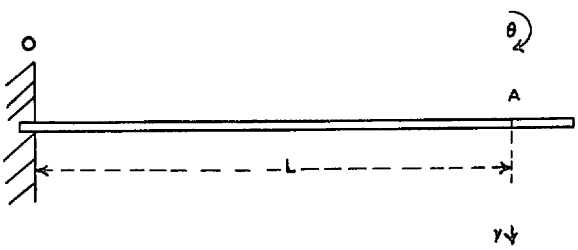

FIG. 3. Single built-in shaft.

$$\left.\begin{aligned} (yy)_{AA} &= L^3/3EI, \\ (y\theta)_{AA} &= (\theta y)_{AA} = L^2/2EI, \\ (\theta\theta)_{AA} &= L/EI, \end{aligned}\right] \qquad (9.1)$$

the loads and deflections being positive when measured downwards, and the couples and rotations being positive when clockwise.

Consider now the case of several shafts OA, AB, BC, ..., (Fig. 4), having flexural rigidities $(EI)_1$, $(EI)_2$, $(EI)_3$, ..., lengths L_1, L_2, L_3, ... joined rigidly at A, B, C, ... forming a single straight shaft encastré at the end O. It is required to find the flexibilities corresponding

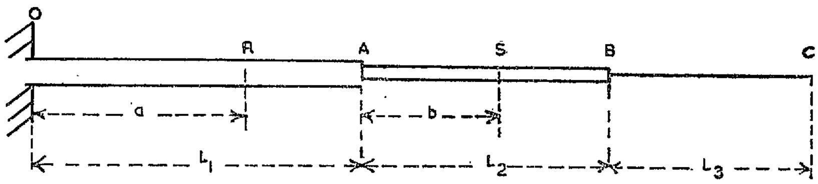

Fig. 4. Composite built-in shaft.

to two points R and S, where OR $= a$ and AS $= b$. It is clear at once that, with unit load at R, the displacement and rotation at R will be $(yy)_{RR} = a^3/3(EI)_1$ and $(y\theta)_{RR} = a^2/2(EI)_1$. The corresponding movements at S, *viz.*, $(yy)_{RS}$ and $(y\theta)_{RS}$, may be deduced by observing that, since the load is at R, the whole shaft beyond R will be straight with a slope $(y\theta)_{RR}$. Hence $(yy)_{RS} = (yy)_{RR} + (L_1 - a + b)\,(y\theta)_{RR}$, and $(y\theta)_{RS} = (y\theta)_{RR}$. Similarly, with unit couple at R, it would be found that $(\theta y)_{RR} = a^2/2(EI)_1$, $(\theta\theta)_{RR} = a/(EI)_1$, $(\theta y)_{RS} = (\theta y)_{RR} + (L_1 - a + b)\,(\theta\theta)_{RR}$, and $(\theta\theta)_{RS} = (\theta\theta)_{RR}$. The flexibilities at A are $(yy)_{AA} = L_1^3/3(EI)_1$, $(y\theta)_{AA} = (\theta y)_{AA} = L_1^2/2(EI)_1$, and $(\theta\theta)_{AA} = L_1/(EI)_1$.

With unit load at S, the whole shaft OA is first kept undeflected by applying at A a load -1 and a couple $-b$, under which conditions the deflection and rotation at S will be $b^3/3(EI)_2$ and $b^2/2(EI)_2$ respectively. A is then released by first applying at A a load $+1$ giving a further deflection $\{(yy)_{AA} + b(y\theta)_{AA}\}$ and rotation $(y\theta)_{AA}$ at S, and then applying a couple $+b$ at A, giving the additional deflection $b\{(\theta y)_{AA} + b(\theta\theta)_{AA}\}$ and rotation $b(\theta\theta)_{AA}$ at S. Similarly, with unit couple at S, OA is first kept undeflected by applying a couple -1 at A, after which A is released by reversing this couple. In this manner it would be found that

$$\left.\begin{aligned}
(yy)_{SS} &= b^3/3(EI)_2 + (yy)_{AA} + 2b(y\theta)_{AA} + b^2(\theta\theta)_{AA},\\
(y\theta)_{SS} &= b^2/2(EI)_2 + (y\theta)_{AA} + b(\theta\theta)_{AA},\\
(\theta y)_{SS} &= (y\theta)_{SS},\\
(\theta\theta)_{SS} &= b/(EI)_2 + (\theta\theta)_{AA}.
\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (9.2)$$

If the point S were in the next shaft BC, the preliminary calculation of the flexibilities at the joint B would be required. These are given from (9.2) by putting $b = L_2$.

Simple extension of this method of fixing and releasing joints in turn, taking account all the while of movements at any chosen points of a composite shaft, enable flexibilities relating to these points to be calculated.

The next problem for consideration is a composite shaft A B C .... D E simply supported at A and E, interest lying in flexibilities

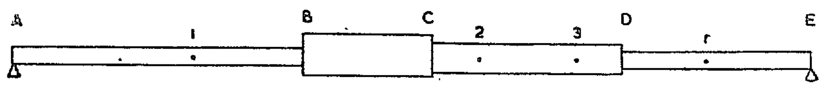

Fig. 5. Composite shaft simply supported at two points.

at any points 1, 2, 3, ... Consider first the application of a unit load or couple at 1. Since there are only two simple supports, the loads P at A and Q at E, acting on the shaft, are known from statical considerations. With the bearings A and E released but the point C encastré, the shafts are allowed to bend under the action of the unit load at 1 and the loads P and Q at A and E, record being made of the deflections of interest at the points 1, 2, 3 ...... and also the deflections y_A and y_E at A and E. Finally the whole shaft is given a rigid-body movement represented by a displacement δ of C and a rotation θ about C, in such a way that A and E are brought back to their original undeflected position. That is, δ and θ are chosen so that

$$\delta + (\mathrm{CE})\theta + y_E = 0,$$

and

$$\delta - (\mathrm{CA})\theta + y_A = 0.$$

Of frequent occurrence is the uniform shaft simply supported at two points, and the results for this are quoted with reference to Fig. 5(a). A and B are the two supports, and R and S the two given points lying within the supports, with AR $= a$, RS $= b$, SB $= c$ and AB $= L$. The results

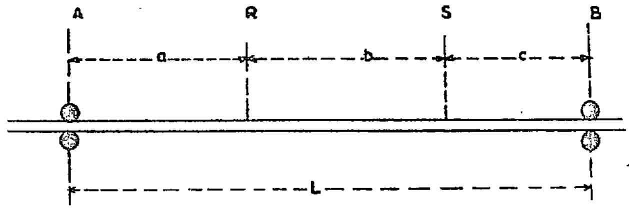

Fig. 5a. Uniform shaft simply supported at two points.

are

$$\left.\begin{aligned} EI(yy)_{RR} &= \{a^2(b+c)^2\}/3L, \\ EI(y\theta)_{RR} &= EI(\theta y)_{RR} = \{a(b+c)(b+c-a)\}/3L, \\ EI(\theta\theta)_{RR} &= \{a^2 - a(b+c) + (b+c)^2\}/3L, \end{aligned}\right] \quad \ldots \quad (9.2)$$

$$\left.\begin{aligned} EI(yy)_{RS} &= \{ac(L^2 - a^2 - c^2)\}/6L, \\ EI(y\theta)_{RS} &= \{a(a^2 + 3c^2 - L^2)\}/6L, \\ EI(\theta y)_{RS} &= \{c(L^2 - c^2 - 3a^2)\{/6L, \\ EI(\theta\theta)_{RS} &= \{3a^2 + 3c^2 - L^2\}/6L, \end{aligned}\right] \quad \ldots \quad (9.3)$$

from which other useful formulae such as

$$EI(\theta\theta)_{BB} = L/3 \quad \ldots \quad (9.4)$$

may be obtained as special cases.

The elastic systems dealt with thus far have been 'just stiff', *i.e.*, encastré at only one point, or simply supported at two points. It is however a relatively simple matter to introduce additional constraints into the system illustrated in Fig. 5. Suppose for instance that points 2 and 3 are required to be simple supports in addition to the ones at A and E. Knowing the flexibilities at 2 and 3 when the system is only just stiff, application of unit load at point r for instance would need loads Y_2 and Y_3 given simultaneously by

$$Y_2(yy)_{22} + Y_3(yy)_{32} + (yy)_{r2} = 0,$$

and

$$Y_2(yy)_{23} + Y_3(yy)_{33} + (yy)_{r3} = 0,$$

in order to produce the conditions of simple support at 2 and 3. Under the new four-point support, unit load at r will produce at s the deflection

$$Y_2(yy)_{2s} + Y_3(yy)_{3s} + (yy)_{rs},$$

and the rotation

$$Y_2(y\theta)_{2s} + Y_3(y\theta)_{3s} + (y\theta)_{rs}.$$

Consideration may now be given to just-stiff systems of two co-axial shafts, of which, when only simple supports are used, there are six types illustrated in Figs. 6 to 11. After each type has been introduced, a sufficient outline of the corresponding elastic problem will be given.

Type 1. Two external bearings A and B on outer shaft. Inner shaft bearing on outer at C and D. No external bearings on inner shaft.

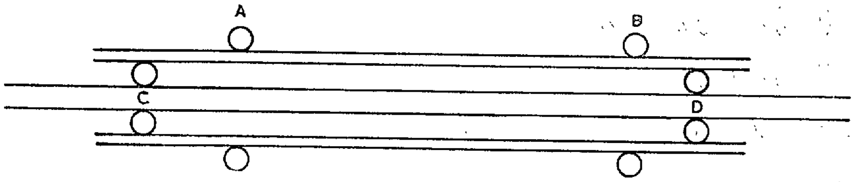

FIG. 6. First type of co-axial shafts.

For any load or couple applied to the outer shaft the deflections of that shaft may be found as for a shaft with two supports. For such loading the inner shaft will deflect as a rigid body with known movements at C and D. For loading applied to the inner shaft and with C and D temporarily held by known loads, deflections of the inner shaft may be determined. Knowing the flexibilities of the outer shaft at C and D, the loads holding the inner shaft at C and D may then be reversed giving an additional known rigid body movement of the inner shaft.

Type 2. Two external bearings A and B on outer shaft. Inner shaft bearing on outer at one point C. One external bearing F on inner shaft.

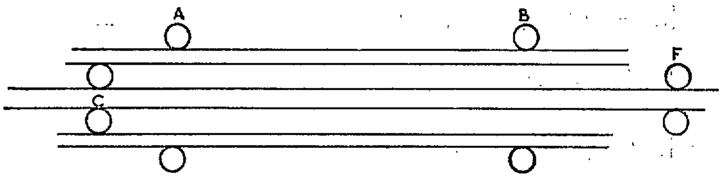

FIG. 7. Second type of co-axial shafts.

Similar remarks to those relating to the first type apply, but this is slightly easier to deal with in so far as any rigid body movements given to the inner shaft are in fact rotations about F.

Type 3. Two external bearings A and B on outer shaft, and two external bearings E and F on inner shaft.

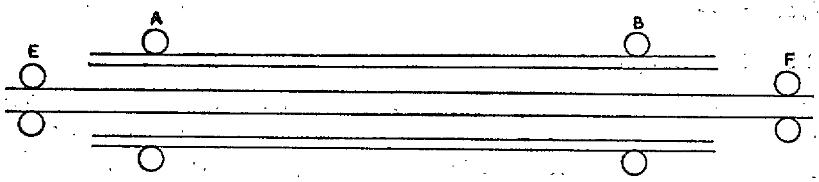

FIG. 8. Third type of co-axial shafts.

The inner and outer elastic systems are clearly independent, each being an example of a shaft with two simple supports.

Type 4. One external bearing A on outer shaft. Inner shaft bearing on outer at C and D. One external bearing F on outer shaft.

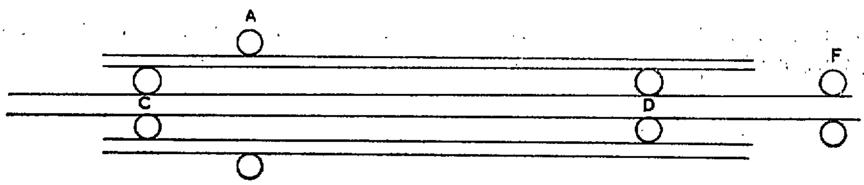

FIG. 9. Fourth type of co-axial shafts.

Since there are only two external bearings at A and F, the loads exerted on the shafts at these two points, when any load or couple is applied to the shafts at some point, may be determined statically. Thence, by considering statically the outer and inner shafts in turn, the equal and opposite loads on the two shafts at each point C and D may be determined. If now C be kept in a fixed position, the bearing D removed, but the loads at D on each shaft retained, the deflections of the two shafts under these conditions may be found. To reproduce the original problem, all that is then necessary is to rotate the inner shaft as a rigid body about F until contact between inner and outer shafts at D is restored.

Type 5. One external bearing B on outer shaft. Inner shaft bearing on outer at one point C. Two external bearings E and F on inner shaft.

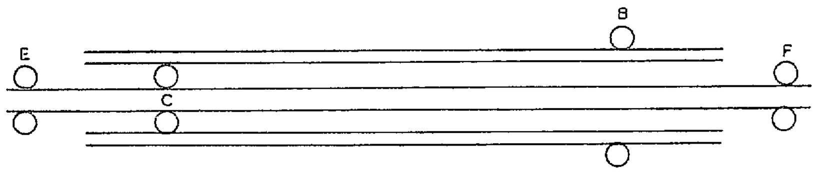

FIG. 10. Fifth type of co-axial shafts.

This is comparable elastically with the second type. Under a load or couple applied to the inner shaft, the deflections of the inner shaft will be found as for a shaft with two supports. The corresponding deflections of the outer shaft will be due to a rigid body rotation about B. Loading on the outer shaft may be dealt with by fixing and then releasing bearing C.

Type 6. Two external bearings E and F on inner shaft. Outer shaft bearing on inner at C and D. No external bearings on outer shaft.

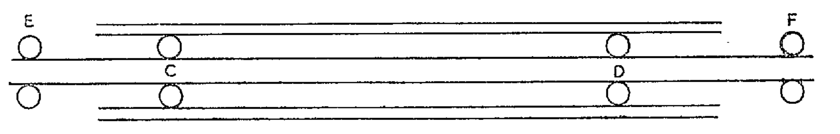

FIG. 11. Sixth type of co-axial shafts.

This is comparable elastically with the first type. The deflections of the inner shaft due to a load or couple applied to that shaft may be found as for a shaft simply supported at two points. The corresponding deflections of the outer shaft are due to a rigid body movement defined by the known deflections at C and D. A load or couple on the outer shaft may be dealt with by first fixing and then releasing the bearings C and D.

Additional fixing, such as extra bearings, may be introduced into all these types by including the proposed points in the scheme of flexibilities for the just-stiff system. Then, for any applied loading, the reactions needed to satisfy the required conditions may be determined.

Systems of more than two co-axial shafts may be dealt with in the same general manner.

10. *Numerical Examples.*

Example. An artificial example illustrative of the iterative technique and method of reduction.

Suppose that the flexibility matrix appropriate to four variables y_1, y_2, y_3 and y_4 was

$$\begin{bmatrix} 261, & 255, & -30, & -45 \\ 255, & 258, & -30, & -45 \\ -30, & -30, & 12, & 10 \\ -45, & -45, & 10, & 19 \end{bmatrix} \quad \ldots\ldots \quad (10.1)$$

the units being micro-inches (or micro-radians) per lb (or per lb in.) and that the 'weights' associated with the variables were 1, 2, 3 and 4 lb (or lb in.[2]) respectively. The dynamic matrix is accordingly

$$\begin{bmatrix} 261, & 510, & -90, & -180 \\ 255, & 516, & -90, & -180 \\ -30, & -60, & 36, & 40 \\ -45, & -90, & 30, & 76 \end{bmatrix} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10.2)$$

compiled by multiplying the successive columns of (10.1) by 1, 2, 3 and 4.

Starting with the arbitrary mode (1, 1, − 1, − 1) the iteration, using (10.2), would proceed:—

				y_m
1	1	−1	−1	
1041	1041	−166	−241	1041
1	1	− 0·159462	− 0·231508	
827·02302	827·02302	−105·000952	−157·738468	827·02302
1	1	− 0·126962	− 0·190295	
816·679680	816·679680	−102·182432	−153·271280	816·679680
1	1	− 0·125119	− 0·187676	
816·042390	816·042390	−102·011324	−153·016946	816·042390
1	1	− 0·125007	− 0·187511	
816·002610	816·002610	−102·000692	−153·001046	816·002610
1	1	− 0·125000	− 0·187501	
816·000180	816·000180	−102·000040	−153·000076	816·000180
1	1	− 0·125000	− 0·187500	
816·000000	816·000000	−102·000000	−153·000000	816
1	1	− 0·125000	− 0·187500	

Thus $\lambda_1 = 816$, and the mode corresponding to it is (1, 1, − 1/8, − 3/16), or, more conveniently, (16, 16, − 2, − 3).

Following the method given in Section 8, the calculations would be:—

(8.9):— (16, 16, − 2, − 3).

(8.10):— (16, 32, − 6, − 12).

(8.11):— $\mu_1 = 816/\{256 + 512 + 12 + 36\} = 816/816 = 1.$

(8.12):— $(16, \quad 16, \quad -2, -3)$.

(8.13):— $$\begin{bmatrix} 256, & 512, & -96, & -192 \\ 256, & 512, & -96, & -192 \\ -32, & -64, & 12, & 24 \\ -48, & -96, & 18, & 36 \end{bmatrix}$$

(8.14):— $$\begin{bmatrix} 5, & -2, & 6, & 12 \\ -1, & 4, & 6, & 12 \\ 2, & 4, & 24, & 16 \\ 3, & 6, & 12, & 40 \end{bmatrix} \quad \cdots\cdots\cdots \quad (10.3)$$

To illustrate the correctness of the principles used in the method of separation of modes, the modified iteration using the matrix (10.3) will be commenced with the mode (3, 0, 4, 2) which is void of the first normal mode. The iteration will be found to proceed:—

(1):—	(3,	0,	4,	2)
(2):—	(63,	45,	134,	137)
(3):—	(2673,	2565,	5714,	7547)
(4):—	(133083,	132435,	273494,	393857).

It may be verified that

$$(4) - 73(3) + 1218(2) - 4896(1) = (0, 0, 0, 0),$$

and thus λ_2, λ_3 and λ_4 are the roots of the cubic

$$\lambda^3 - 73\lambda^2 + 1218\lambda - 4896 = 0,$$

giving $$\lambda_2 = 51, \quad \lambda_3 = 16, \quad \lambda_4 = 6.$$

The corresponding modes may be found according to (8.7) which shows

$$(3) - 22(2) + 96(1) = (1575, \quad 1575, \quad 3150, \quad 4725),$$
$$(3) - 57(2) + 306(1) = (0, \quad 0, \quad -700, \quad 350),$$
$$(3) - 67(2) + 816(1) = (900, \quad -450, \quad 0, \quad 0),$$

or, more simply,

$(1, \quad 1, \quad 2, \quad 3)$ corresponding to λ_2,

$(0, \quad 0, \quad -2, \quad 1)$,, ,, λ_3,

and $(2, \quad -1, \quad 0, \quad 0)$,, ,, λ_4.

Example 2. Elastic problem of type 2.

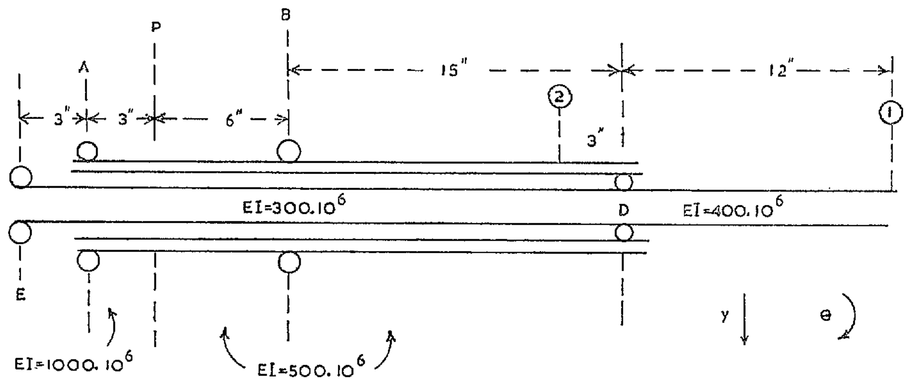

FIG. 12. Example of second type of co-axial shafts.

The outer shaft has two external bearings A and B where AB = 9 in. At point P where PB = 6 in. there is a change of section of the outer shaft, the two portions BP and PA having flexural rigidities $500 \cdot 10^6$ and $1000 \cdot 10^6$ lb in.^{2} respectively. The remaining portion of the outer shaft has flexural rigidity $500 \cdot 10^6$ lb in.^{2}. The point of interest on the outer shaft is 2 where B2 = 12 in. The inner shaft has an external bearing at E where EA = 3 in., and bears on the outer shaft at D where BD = 15 in. Between these two bearings the flexural rigidity is $300 \cdot 10^6$ lb in.^{2} The overhung portion of this shaft has flexural rigidity $400 \cdot 10^6$ lb in.^{2} and the point of interest is 1 lying at a distance 12 in. from D.

Considering the outer shaft first, a clockwise couple of 1 lb in. at B will involve loads of + 1/9 lb and − 1/9 lb on the shaft at A and B respectively. With A and B free to move but with the above loads acting, and with P encastré,

$$y_A = \tfrac{1}{9}\left(\frac{3^3}{3 \times 1000}\right) = 0 \cdot 001, \ \mu\text{-in.},$$

$$y_B = 1\left(\frac{6^2}{2 \times 500}\right) - \tfrac{1}{9}\left(\frac{6^3}{3 \times 500}\right) = 0 \cdot 020, \ \mu\text{-in.},$$

and

$$\theta_B = 1\left(\frac{6}{500}\right) - \tfrac{1}{9}\left(\frac{6^2}{2 \times 500}\right) = 0 \cdot 008, \ \mu\text{-radn},$$

where μ is used as an abbreviation for 10^{-6}.

A rigid body movement represented by a displacement of $- 0 \cdot 001$, μ-in. at A, together with a rotation about A of $- (0 \cdot 019)/9 = - 0 \cdot 00\dot{2}\dot{1}$, μ-radn, would bring A and B back to their correct positions, from which it follows that $(\theta\theta)_{BB} = 0 \cdot 008 - 0 \cdot 00\dot{2}\dot{1} = 0 \cdot 005\dot{8}$, μ radn/lb in.

Hence

$$(yy)_{22} = 1\left(\frac{12^3}{3 \times 500}\right) + \{12(\theta\theta)_{BB}\}12 \qquad = 2 \cdot 0, \ \mu\text{-in./lb},$$

$$(y\theta)_{22} = 1\left(\frac{12^2}{2 \times 500}\right) + 12(\theta\theta)_{BB} \qquad = 0 \cdot 214{,}666, \ \mu\text{-radn/lb},$$

$$(yy)_{2D} = (yy)_{22} + 3(y\theta)_{22} \qquad = 2 \cdot 644, \ \mu\text{-in./lb},$$

$$(y\theta)_{21} = (yy)_{2D}/27 \qquad = 0 \cdot 097{,}925, \ \mu\text{-radn/lb},$$

$$(yy)_{21} = 39(y\theta)_{21} \qquad = 3\cdot 819{,}111,\ \mu\text{-in./lb},$$

$$(\theta y)_{22} = (y\theta)_{22} \qquad = 0\cdot 214{,}666,\ \mu\text{-in./lb in.},$$

$$(\theta\theta)_{22} = 1\left(\frac{12}{500}\right) + (\theta\theta)_{BB} \qquad = 0\cdot 029{,}888,\ \mu\text{ radn/lb in.},$$

$$(\theta y)_{2D} = (\theta y)_{22} + 3(\theta\theta)_{22} \qquad = 0\cdot 304{,}333,\ \mu\text{-in./lb in.},$$

$$(\theta\theta)_{21} = (\theta y)_{2D}/27 \qquad = 0\cdot 011{,}271,\ \mu\text{-radn/lb in.},$$

$$(\theta y)_{21} = 39(\theta\theta)_{21} \qquad = 0\cdot 439{,}592,\ \mu\text{-in./lb in.},$$

$$(yy)_{DD} = 1\left(\frac{15^3}{3 \times 500}\right) + \{15(\theta\theta)_{BB}\}15 \qquad = 3\cdot 575,\ \mu\text{-in./lb},$$

$$(y\theta)_{D1} = (yy)_{DD}/27 \qquad = 0\cdot 132{,}407,\ \mu\text{-radn/lb},$$

$$(yy)_{D1} = 39(y\theta)_{D1} \qquad = 5\cdot 163{,}888,\ \mu\text{-radn/lb}.$$

A load of 1 lb at 1 causes a load of + 13/9 lb on the outer shaft at D; similarly a couple of 1 lb in. at 1 gives at D on outer shaft a load of + 1/27 lb. Hence

$$(yy)_{11} = 1\left(\frac{12^3}{3 \times 400}\right) + 12\left(\frac{27}{3 \times 300}\right)12 + \tfrac{13}{9}(yy)_{D1} = 13\cdot 218{,}950,\ \mu\text{-in./lb},$$

$$(y\theta)_{11} = 1\left(\frac{12^2}{2 \times 400}\right) + 12\left(\frac{27}{3 \times 300}\right) + \tfrac{13}{9}(y\theta)_{D1} \qquad = 0\cdot 731{,}255,\ \mu\text{-radn/lb},$$

$$(\theta y)_{11} = 1\left(\frac{12^2}{2 \times 400}\right) + \left(\frac{27}{3 \times 300}\right)12 + \tfrac{1}{27}(yy)_{D1} \qquad = 0\cdot 731{,}255,\ \mu\text{-in./lb in.},$$

$$(\theta\theta)_{11} = 1\left(\frac{12}{400}\right) + \left(\frac{27}{3 \times 300}\right) + \tfrac{1}{27}(y\theta)_{D1} \qquad = 0\cdot 064{,}903,\ \mu\text{-radn/lb in.}$$

Other flexibilities such as $(\theta y)_{12}$, $(yy)_{12}$, $(\theta\theta)_{12}$, etc., may be be determined to serve as checks upon $(y\theta)_{21}$, $(yy)_{21}$, $(\theta\theta)_{21}$, etc.

Example 3. Whirling of contra-rotating propeller system.

A hypothetical example will be considered by imagining a rigid propeller of weight 486 lb and of polar moment of inertia 364,500 lb/in.^{2} at each of the stations 1 and 2 of the co-axial shaft system of Example 2. The inner shaft will be imagined to be rotating clockwise looking along it from left to right, and the outer shaft is rotating anticlockwise with the same angular velocity at any instant. Each propeller, for the purpose of this example will be assumed to be so thin axially that its diametral inertia is half its polar moment of inertia. It must also be postulated that the number of blades in each propeller is three or more in order that the condition of axial symmetry of mass shall apply. The distribution of weight along the shafts is ignored in this example.

Taking the variables in the sequence y_1, y_2, θ_1, θ_2, the flexibility matrix is, from Example 2,

$$\begin{bmatrix} 13\cdot 218{,}950, & 3\cdot 819{,}111, & 0\cdot 731{,}255, & 0\cdot 439{,}592, \\ 3\cdot 819{,}111, & 2\cdot 000{,}000, & 0\cdot 097{,}925, & 0\cdot 214{,}666, \\ 0\cdot 731{,}255, & 0\cdot 097{,}925, & 0\cdot 064{,}903, & 0\cdot 011{,}271, \\ 0\cdot 439{,}592, & 0\cdot 214{,}666, & 0\cdot 011{,}271, & 0\cdot 029{,}888 \end{bmatrix} \quad \ldots \quad \ldots \quad (10.4)$$

in micro-units of deflection per unit load.

The two sources of excitation which must be considered are (i) unbalance of the 'outer' propeller, and (ii) unbalance of the 'inner' propeller. The 'inertias' to be taken are therefore

486, 486, 546750, $-$ 182250 for excitation (i),

and 486, 486, $-$ 182250, 546750 " " (ii),

since the equivalent diametral inertia of each propeller is $\{182250 - 364500/n\}$, where n is the order number which is $+1$ for the exciting propeller and -1 for the other; see relationship (7.8).

Accordingly the two dynamic matrices are

$$\begin{bmatrix} 6424{\cdot}410 & 1856{\cdot}088 & 399813{\cdot}750 & -80115{\cdot}750 \\ 1856{\cdot}088 & 972{\cdot}000 & 53541{\cdot}000 & -39123{\cdot}000 \\ 355{\cdot}390 & 47{\cdot}592 & 35486{\cdot}250 & -2054{\cdot}250 \\ 213{\cdot}642 & 104{\cdot}328 & 6162{\cdot}750 & -5447{\cdot}250 \end{bmatrix} \quad \ldots \quad \ldots \quad (10.5)$$

for excitation (i), and

$$\begin{bmatrix} 6424{\cdot}410 & 1856{\cdot}088 & -133271{\cdot}250 & 240347{\cdot}250 \\ 1856{\cdot}088 & 972{\cdot}000 & -17847{\cdot}000 & 117369{\cdot}000 \\ 355{\cdot}390 & 47{\cdot}592 & -11828{\cdot}750 & 6162{\cdot}750 \\ 213{\cdot}642 & 104{\cdot}328 & -2054{\cdot}250 & 16341{\cdot}750 \end{bmatrix} \quad \ldots \quad \ldots \quad (10.6)$$

for excitation (ii). It should be noted that in compiling these matrices, the flexibilities were taken in rational instead of the decimal form of (10.4), and that the matrices (10.5) and (10.6) are 'exact'.

Using the matrix (10.5), and commencing with the mode 1, 0·09, 0·09, 0, the iteration proceeded:—

				y_m
1	0·09	0·09	0	42574·695
1·000000	0·158833	0·081350	0·018266	37780·672
1·000000	0·149585	0·085023	0·016730	39355·081
1·000000	0·149896	0·085003	0·016824	39340·131
1·000000	0·149840	0·085012	0·016815	39344·346
1·000000	0·149844	0·085012	0·016815	39344·354

in which the intermediate y-rows have been omitted as indeed they may in the actual process if y_m is evaluated first. The mode appears in the last row together with $\lambda_1 = 39344{\cdot}354$ giving $N_1 = 946$ r.p.m.

The iteration with the matrix (10.6) was commenced upon the mode 1, 0·36, 0·03, 0·06. After five complete rounds, as shown by the next Table, it was clear that at least another mode was active, and the separation method was resorted to.

The iteration proceeded:—

				y_m
1	0·36	0·03	0·06	17515·299
1·000000	0·497436	0·022119	0·066789	20452·421
1·000000	0·478369	0·025866	0·064115	19274·975
1·000000	0·486877	0·024245	0·065261	19782·237
1·000000	0·483072	0·024970	0·064748	19555·255
1·000000	0·484750	0·024650	0·064974	

Completing two further rounds of the modified iteration led to the following equations for p_1 and p_2:—

$$385{,}458{,}594\cdot74905 - 19{,}655\cdot33457p_1 + p_2 = 0, \quad \ldots \quad \ldots \quad \ldots \quad (a)$$

$$186{,}691{,}010\cdot43033 - 9{,}513\cdot26986p_1 + 0\cdot484750p_2 = 0, \quad \ldots \quad \ldots \quad \ldots \quad (b)$$

$$9{,}532{,}126\cdot92672 - 487\cdot30005p_1 + 0\cdot024650p_2 = 0, \quad \ldots \quad \ldots \quad \ldots \quad (c)$$

$$25{,}023{,}320\cdot14868 - 1{,}275\cdot11460p_1 + 0\cdot064974p_2 = 0. \quad \ldots \quad \ldots \quad \ldots \quad (d)$$

Solving for p_1 and p_2 from (a) and (b) gave

$$p_1 = 10921\cdot80092$$

and $$p_2 = -170{,}786{,}943\cdot55952$$

with equation errors of only $0\cdot00000$, $+ 0\cdot03069$, $+ 34\cdot63357$ and $+ 61\cdot46646$ in (a), (b), (c) and (d) respectively. The test (8.2) was therefore satisfactory, and the deduced roots were $\lambda_1 = 19624\cdot52914$ and $\lambda_2 = -8702\cdot72822$. The deduced first mode was

$$1\cdot000000, \quad 0\cdot484233, \quad 0\cdot024749, \quad 0\cdot064904,$$

and a further round of iteration reproduced this except for some errors of 1 in the last place. It followed that the whirling speed (corresponding to $\lambda_1 = 19624\cdot5$) was 1340 r.p.m.

It may be noticed from either matrix (10.5) or (10.6) that if inertia effects are neglected, the two corresponding latent roots are given by

$$(6424\cdot410 - \lambda)(972\cdot000 - \lambda) = (1856\cdot088)^2;$$

λ_1 is $6996\cdot274$ on this basis, corresponding to a 1st order whirling speed of 2244 r.p.m.

11. *Concluding Remarks.*—An interesting topic which has not been included in the paper is the coupling which exists between flexural, longitudinal and torsional vibration when the system of the type considered has appreciable unbalance. Such effects were not apparent in the theory because all dynamic loads due to vibratory displacements of the unbalance particles were ignored.

An attempt has been made to indicate the importance of gyroscopic effects in some cases, and a numerical technique for dealing with practical problems was developed.

Attention is drawn to the important example of a pair of contra-rotating propellers arranged on co-axial shafts. The particular danger associated with this arrangement appears to be the whirling speed associated with unbalance of the propeller driven by the outer shaft.

The examples were chosen in such a way as to illustrate various points of numerical technique, and it was decided on that account to avoid anything spectacular in the way of large numbers of variables. To give some idea of the time involved in more complicated examples, reference may be made to an interesting problem recently worked out by the writer. It was of a similar kind to Example 3, but one for which there had to be several point-masses to give a reasonable allowance for the inertia effects of the shafts, whose weight was not negligible compared with the three rotors affixed to them. Altogether there were 13 degrees of freedom, and the particular solution sought was complicated by the fact that the first three latent roots of the 13×13 dynamic matrix were approximately -193, -182 and $+85$. The third root was the one required, and its mode was successfully separated from the iteration in just under two days.

12. *Acknowledgements*.—The author's belief in the value of the iterative method has been further strengthened by the experience of his colleagues in the Napier Calculations Office, particularly V. C. Allen and D. H. L. Inns, who have applied the technique to a variety of vibration problems. The author has been helped towards a full appreciation of the whirling problem by the many valuable talks he has had with Capt. J. Morris of the Royal Aircraft Establishment. To Mr. H. Sammons, the Managing Director of the Napier Company, thanks are due for permission to publish the paper.

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