C.P. No. 312
(15,101)
A.R.C. Technical Report

C.P. No. 312
(15,101)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
CURRENT PAPERS

The Numerical Integration of the Laminar Compressible Boundary Layer Equations, with Special Reference to the Position of Separation when the Wall is Cooled.

By

G. E. Gadd, B.A., Ph.D,

of the Aerodynamics Division, N.P.L.

LONDON : HER MAJESTY'S STATIONERY OFFICE

1957

THREE SHILLINGS NET

The Numerical Integration of the Laminar Compressible Boundary Layer Equations, with Special Reference to the Position of Separation when the Wall is Cooled.

- By -
G. E. Gadd, B.A., Ph.D.
of the Aerodynamics Division, N.P.L.
Formerly Beit Scientific Research Fellow.

1st August, 1952

Introduction and Summary

A method of integrating the laminar compressible boundary layer equations is described in Section 1. It is assumed that the external velocity at the leading edge of the boundary layer is not zero, but cases with leading edge stagnation points could be dealt with on broadly similar lines. The method enables results of fair accuracy to be obtained even when large intervals of integration are used, and it is thus suitable where the computations are performed by means of a slide rule or mechanical computing machine.

Certain solutions obtained by the method are presented in Section 2. They are all concerned with the position of separation under adverse pressure gradients. The most interesting results, in sub-section 3, are for two cases with the wall cooled to the free stream temperature. The first is for a Mach number of 4, ω (the viscosity-temperature relation index) and σ (the Prandtl number) equal to 1, and with constant adverse external velocity gradient. The second is for a Mach number of 2, ω and σ equal to 1, and with constant adverse pressure gradient. In both cases the boundary layer is found to separate very much less readily than when the wall is thermally insulated.

SECTION I

A Method for the Numerical Solution of Compressible Boundary Layers with Non-zero Leading Edge Velocities.

1. Equations

The governing equations for the viscous stress τ and the enthalpy I in terms of independent variables x, u as used by Crocco (Ref.1) are

$$u \frac{\partial}{\partial x}\left(\frac{\rho\mu}{\tau}\right) + \tau_{uu} - p_x \frac{\partial}{\partial u}\left(\frac{\mu}{\tau}\right) = 0 \qquad \ldots(1)$$

and

$$(1 - \sigma)I_u\tau\tau_u + (I_{uu} + \sigma)\tau^2 - \sigma\rho\mu u I_x + \sigma\mu(I_u + u)p_x = 0 \qquad \ldots(2)$$

with/

with

$$\mu = \mu_d\left(\frac{I}{I_d}\right)^{\omega} \qquad \ldots(3)$$

$$\rho\mu = \rho_d\mu_d\left[1 + \left(1 - \frac{u_1^2}{u_d^2}\right)\frac{\gamma - 1}{2}M_d^2\right]^{\frac{\gamma}{\gamma-1}}\left(\frac{I}{I_d}\right)^{\omega-1} \qquad \ldots(4)$$

$$p_x = -\rho_d u_1 u_{1x}\left[1 + \left(1 - \frac{u_1^2}{u_d^2}\right)\frac{\gamma - 1}{2}M_d^2\right]^{\frac{1}{\gamma-1}} . \qquad \ldots(5)$$

(See list of symbols at end.) The boundary conditions are

$$u = 0, \ \tau\tau_u = p_x\mu_w \ : \ u = u_1, \ \begin{cases}\tau = 0 \\ I = I_1 = I_d + \dfrac{u_d^2}{2} - \dfrac{u_1^2}{2}\end{cases}$$

together with a condition for I at $u = 0$ depending on circumstances.

We transform to independent variables x, η where $\eta \equiv u/u_1$. The equations become

$$\frac{\tau_{\eta\eta}}{u_1^2} + \frac{1}{\tau}\left[\eta u_1 \frac{\partial(\rho\mu)}{\partial x} - \eta^2 u_{1x}\frac{\partial(\rho\mu)}{\partial\eta} - \frac{p_x}{u_1}\mu_\eta\right] - \eta u_1\rho\mu\frac{\tau_x}{\tau^2}$$
$$+ \left[\eta^2\rho\mu \ u_{1x} + \frac{p_x\mu}{u_1}\right]\frac{\tau_\eta}{\tau^2} = 0 \qquad \ldots(6)$$

and

$$(1-\sigma)\frac{I_\eta\tau\tau_\eta}{u_1^2} + \left(\frac{I_{\eta\eta}}{u_1^2} + \sigma\right)\tau^2 - \sigma\rho\mu\eta u_1\left(I_x - \frac{\eta u_{1x}}{u_1}I_\eta\right) + \sigma\mu\left(\frac{I_\eta}{u_1} + \eta u_1\right) p_x = 0 \qquad \ldots.(7)$$

with the boundary conditions

$$\eta = 0, \ \tau\tau_\eta = p_x u_1\mu_w \ : \ \eta = 1, \begin{cases}\tau = 0 \\ I = I_1 = I_d + \dfrac{u_d^2}{2} - \dfrac{u_1^2}{2},\end{cases}$$

2. Change of Parameter

Numerical integration employs finite difference approximations, so it is desirable to deal with functions which are free from singular points where the functions themselves or any of their lower derivatives become infinite. I is free from such singularities with respect to gradients in the x direction, but τ is proportional to $1/\sqrt{x}$ near the leading edge x = 0 if the external velocity is not zero there, and to $\sqrt{x_s - x}$ at the wall near the separation point x_s. (The latter condition was shown probably to apply in incompressible flow by Goldstein (Ref.2), and it can be shown by similar arguments that it also probably applies in the compressible case.) Hence

$$q \equiv \frac{x\tau^2}{\rho_d \mu_d u_d^3}$$

is a more suitable function to deal with than τ. Equation (6) becomes in terms of q

$$2qq_{\eta\eta} - (q_\eta)^2 - \frac{2\eta\rho\mu u_1^3}{\rho_d\mu_d u_d^3}[xq_x - q] + \frac{2u_1^2 xq_\eta}{\rho_d\mu_d u_d^3}\left[\eta^2\rho\mu u_{1x} + \frac{p_x\mu}{u_1}\right]$$

$$+ \frac{4xu_1^2 q}{\rho_d\mu_d u_d^3}\left[\eta u_1\frac{\partial(\rho\mu)}{\partial x} - \eta^2 u_{1x}\frac{\partial(\rho\mu)}{\partial\eta} - \frac{p_x\mu_\eta}{u_1}\right] = 0 \quad \ldots(8)$$

with the boundary conditions

$$\eta = 0,\ q_\eta = \frac{2u_1 x p_x \mu_w}{\rho_d\mu_d u_d^3} \ :\ \eta = 1,\ q = 0.$$

In the numerical solution of this equation and equation (7) for I it is found possible, if great accuracy is not required, to use big steps of integration in the x direction since with smooth pressure distributions q and I usually seem to vary roughly parabolically with x.

3. Leading Edge Values

For the first x - step of the integration we need to know q, q_x, I, and I_x at the leading edge x = 0. At this position the equations for q and q_x are less convenient than those for

$$\sqrt{q} = \frac{\tau}{\mu_d u_d}\sqrt{\frac{\nu_d x}{u_d}} \equiv t$$ and t_x, so we work with the latter functions,

from which q and q_x are easily deduced.

The/

The general equation for t is

$$\frac{t_{\eta\eta}}{u_1^2} - \frac{\eta\rho\mu u_1}{\rho_d\mu_d u_d^3 t^2}\left[x t_x - \frac{t}{2}\right] + \frac{x}{\rho_d\mu_d u_d^3}\left[\eta^2 \rho\mu u_{1x} + \frac{p_x\mu}{u_1}\right]\frac{t_\eta}{t^2}$$

$$+ \frac{x}{\rho_d\mu_d u_d^3 t}\left[\eta u_1 \frac{\partial(\rho\mu)}{\partial x} - \eta^2 u_{1x}\frac{\partial(\rho\mu)}{\partial\eta} - \frac{p_x\mu_\eta}{u_1}\right] = 0.$$

Hence at the leading edge, where $u_1 = u_d$, $I_1 = I_d$, etc.,

$$2tt_{\eta\eta} + \frac{\eta\rho\mu}{\rho_d\mu_d} = 0 \qquad \text{...(9)}$$

and

$$t^2 t_{\eta\eta x} - \frac{3\eta\rho\mu}{2\rho_d\mu_d} t_x + \frac{1}{\rho_d\mu_d u_d}\left\{\left[\eta^2\rho\mu\, u_{1xd} + \frac{p_x\mu}{u_d}\right]t_\eta + \left[\frac{3\eta u_d}{2}\frac{\partial(\rho\mu)}{\partial x} - \eta^2 u_{1xd}\frac{\partial(\rho\mu)}{\partial\eta} + \frac{3\eta}{2}\rho\mu u_{1xd} - \frac{p_x\mu_\eta}{u_d}\right]t\right\} = 0 \qquad \text{...(10)}$$

whilst

$$\mu = \mu_d\left(\frac{I}{I_d}\right)^{\omega}, \quad \rho\mu = \rho_d\mu_d\left(\frac{I}{I_d}\right)^{\omega-1},$$

$$p_x = -\rho_d u_d u_{1xd}, \quad \frac{\partial(\rho\mu)}{\partial x} = \rho_d\mu_d\left(\frac{I}{I_d}\right)^{\omega-1}\left[-\frac{\Upsilon u_{1xd}}{u_d}M_d^2 + \frac{(\omega-1)}{I}I_x\right].$$

Also

$$(I_{\eta\eta} + \sigma u_d^2)\, t + (1-\sigma)\, I_\eta t_\eta = 0 \qquad \text{...(11)}$$

and/

and

$$t^2 I_{\eta\eta x} + (1 - \sigma)\ t t_\eta I_{\eta x} - \sigma\eta \left(\frac{I}{I_d}\right)^{\omega-1} I_x$$

$$= I_\eta (1 - \sigma)\ [t_x t_\eta - t t_{\eta x}] + \frac{\sigma}{u_d} u_{1xd} \left(\frac{I}{I_d}\right)^{\omega-1} \left(\frac{I}{I_d} - \eta^2\right) I_\eta$$

$$+ \sigma\eta u_d u_{1xd} \left(\frac{I}{I_d}\right)^{\omega} - 2\sigma u_d u_{1xd} t^2 \qquad \ldots(12)$$

The boundary conditions are

$$\eta = 0, \begin{cases} t_\eta = 0 \\ t t_{\eta x} = -\dfrac{u_{1xd}}{u_d}\left(\dfrac{I_w}{I_d}\right)^{\omega} \end{cases} ; \quad \eta = 1, \begin{cases} t = t_x = 0 \\ I = I_d,\ I_x = -u_d u_{1xd} . \end{cases}$$

We consider the solution of these equations for various conditions on ω and σ, as follows:

3.1 $\underline{\omega = \sigma = 1}$

Equation (9) becomes

$$2 t t_{\eta\eta} + \eta = 0 \qquad \ldots(13)$$

and equation (11) becomes

$$I_{\eta\eta} + u_d^2 = 0$$

so that

$$I = I_d \left[1 + (1 - \eta^2)\frac{\gamma - 1}{2} M_d^2\right] - (1 - \eta)\ I_{\eta w} .$$

Hence for equation (10) we obtain

$$t^2 t_{\eta\eta x} - \frac{3\eta}{2} t_x = \frac{u_{1xd}}{u_d}\left[(1 - \eta^2) t_\eta - \frac{3\eta}{2} t + \left[(1 - \eta^2)\frac{\gamma - 1}{2} t_\eta + \frac{5\gamma - 2}{2}\eta t\right] M_d^2 - \left[(1 - \eta)\ t_\eta + t\right]\frac{I_{\eta w}}{I_d}\right] \qquad \ldots(14)$$

whilst/

whilst for equation (12) we obtain

$$t^2 I_{\eta\eta x} - \eta I_x = \frac{u_{1xd}}{u_d}\left[u_d^2\eta^3 - 2u_d^2 t^2 + (1-\eta^2)\left(1 + \frac{\gamma - 1}{2}M_d^2\right) I_{\eta w} - (1-\eta)\frac{I_{\eta w}^2}{I_d}\right]. \quad \ldots\ldots(15)$$

From (13) and the boundary conditions for t we see that t is a function of η only: $q = t^2$ is given in Table 1. From (14) and the boundary conditions for t_x it follows that $q_x = 2tt_x$ must be of the form

$$q_x = -\frac{u_{1xd}}{u_d}\left[A + BM_d^2 + C\frac{I_{\eta w}}{I_d}\right],$$

where A, B, C are functions of η only, whilst from (15) and the boundary conditions for I_x it follows that I_x is the form

$$I_x = -\eta^2 u_d u_{1xd} + DI_{xw} + \frac{u_{1xd}}{u_d}\left[E\left(1 + \frac{\gamma - 1}{2}M_d^2\right)I_{\eta w} + F\frac{I_{\eta w}^2}{I_d}\right],$$

where D, E, F are also functions of η only.

Approximate values of the functions A, B, C, D, E, F are given in Table 1.

For certain wall temperature conditions we need to know the relation for $I_{\eta xw}$: it is

$$I_{\eta xw} = -1.640\ I_{xw} - 3.30\frac{u_{1xd}}{u_d}\left(1 + \frac{\gamma - 1}{2}M_d^2\right) I_{\eta w} + 2.66\frac{u_{1xd}}{u_d}\frac{I_{\eta w}^2}{I_d}.$$

3.2 Either ω or $\sigma = 1$

In this case equations (9) to (12) can be solved separately. Thus if $\omega = 1$ the equation for t, (9), is independent of I and that for t_x, (10), is independent of I_x. Hence t can be found from (9), then (11) solved for I, then (10) solved for t_x, and finally (12) solved for I_x. Similarly if $\sigma = 1$ the equation for I, (11), is independent of t and that for I_x, (12), is independent of t_x, so the equations can again be solved successively and independently.

3.3 Either ω or σ near 1

If neither ω nor σ is equal to 1 we have two pairs of simultaneous equations in pairs of unknowns, equations (9) and (11) each involving t and I, and (10) and (12) each involving t_x and I_x. However if ω or σ is near 1 it is probably possible to solve the equations separately by the following iteration method:

If/

If it is σ which is near 1, we put $\sigma = 1$ in equation (11) and solve the resulting equation for a first approximation to I. This approximation is then substituted in equation (9) in place of I in the term $\frac{\rho\mu}{\rho_d\mu_d} = \left(\frac{I}{I_d}\right)^{\omega-1}$, and the equation obtained is solved for a first approximation to t. In turn this approximation is substituted for the t terms of equation (11), which is then solved for a second approximation to I, and so on. The process will probably be convergent, so that eventually a sufficiently close approximation to the true solutions t and I should be obtained. The pair of equations (10) and (12) can then be dealt with similarly, with σ first taken as equal to 1 in equation (12) and the equation solved for a first approximation to I_x, and so on.

If it is ω which is near 1 we put $\omega = 1$ in equation (9) and solve for a first approximation to t as the first step in a similar iteration process.

3.4 Other Cases

In the practical case for air ω and σ are both near to 1 so it should normally be possible to solve equations (9) to (12) by one of the methods described above. However, if the iteration process of sub-section 3.3 fails to converge, or if neither ω nor σ is near 1, the method of solution will probably have to be that of laborious concurrent integration of pairs of simultaneous equations.

4. Away from Leading Edge

Having determined q, q_x, I, and I_x at the leading edge $x = 0$ we can proceed to the first x-step in the integration of equations (8) and (7) for q and I. For a step from x_a to x (x_a of course being 0 for the first step) we make the finite difference approximation

$$q_x(\eta,x) = \frac{2}{x - x_a}\left[q(\eta,x) - q(\eta,x_a)\right] - q_x(\eta,x_a) \qquad \ldots(16)$$

and in general we need also

$$I_x(\eta,x) = \frac{2}{x - x_a}\left[I(\eta,x) - I(\eta,x_a)\right] - I_x(\eta,x_a) \qquad \ldots(17)$$

though for the special case of $\sigma = 1$, no heat transfer to the wall, we know the exact solution for I,

$$I = I_d\left[1 + \left(1 - \eta^2\frac{u_1^2}{u_d^2}\right)\frac{\gamma - 1}{2}M_d^2\right].$$

The/

The approximation for q_x reduces equation (8) to

$$2qq_{\eta\eta} - (q_\eta)^2 + Rq_\eta + Sq + T = 0 \qquad \ldots(18)$$

where

$$\begin{cases} R = \dfrac{2u_1^2 x}{\rho_d \mu_d u_d^3}\left[\eta^2 \rho\mu u_{1x} + \dfrac{p_x \mu}{u_1}\right] \\[2ex] S = \dfrac{2u_1^2}{\rho_d \mu_d u_d^3}\left[2x\left(\eta u_1 \dfrac{\partial(\rho\mu)}{\partial x} - \eta^2 u_{1x} \dfrac{\partial(\rho\mu)}{\partial\eta} - \dfrac{p_x \mu_\eta}{u_1}\right) - \eta\rho\mu u_1 \dfrac{x + x_a}{x - x_a}\right] \\[2ex] T = \dfrac{2\eta\rho\mu u_1^3}{\rho_d \mu_d u_d^3}\left[\dfrac{2x}{x - x_a}\, q(x_a) + x q_x(x_a)\right] \end{cases}$$

and

$$\mu = \mu_d\left(\frac{I}{I_d}\right)^{\omega}$$

$$\rho\mu = \rho_d \mu_d \left[1 + \left(1 - \frac{u_1^2}{u_d^2}\right)\frac{\gamma - 1}{2} M_d^2\right]^{\frac{\gamma}{\gamma-1}} \left(\frac{I}{I_d}\right)^{\omega-1}$$

$$p_x = -\rho_d u_1 u_{1x}\left[1 + \left(1 - \frac{u_1^2}{u_d^2}\right)\frac{\gamma - 1}{2} M_d^2\right]^{\frac{1}{\gamma-1}} .$$

The/

The approximation for I_x reduces equation (7) to

$$(I_{\eta\eta} + \sigma u_1^2)\, q + \tfrac{1}{2}(1-\sigma)\, q_\eta I_\eta - \sigma\eta \frac{u_1^3}{u_d^3} x \left[1 + \left(1 - \frac{u_1^2}{u_d^2}\right)\frac{\gamma - 1}{2} M_d^2\right]^{\frac{1}{\gamma-1}} \left(\frac{I}{I_d}\right)^{\omega}$$

$$\times \left[\frac{2I_d\left[1 + \left(1 - \frac{u_1^2}{u_d^2}\right)\frac{\gamma - 1}{2} M_d^2\right]}{x - x_a} + u_1 u_{1x}\right] - \sigma \frac{u_1^2}{u_d^3} x u_{1x} \left(\frac{I}{I_d}\right)^{\omega-1}$$

$$\times \left[1 + \left(1 - \frac{u_1^2}{u_d^2}\right)\frac{\gamma - 1}{2} M_d^2\right]^{\frac{1}{\gamma-1}} \left[\frac{I}{I_d} - \eta^2 \left[1 + \left(1 - \frac{u_1^2}{u_d^2}\right)\frac{\gamma - 1}{2} M_d^2\right]\right] I_\eta$$

$$+ \sigma \frac{u_1^3}{u_d^3}\eta x \left[1 + \left(1 - \frac{u_1^2}{u_d^2}\right)\frac{\gamma - 1}{2} M_d^2\right]^{\frac{\gamma}{\gamma-1}} \left(\frac{I}{I_d}\right)^{\omega-1} \left[\frac{2I(x_a,\eta)}{x - x_a} + I_x(x_a,\eta)\right] = 0. \quad \ldots(19)$$

The boundary conditions are

$$\eta = 0,\ q_\eta = \frac{2u_1 x p_x \mu_w}{\rho_d \mu_d u_d^3}\ ;\quad \eta = 1 \begin{cases} q = 0 \\ I_x = -u_1 u_{1x}\ . \end{cases}$$

Thus if we have solved up to $x = x_a$, and know q, q_x, I, and I_x there, we have two simultaneous equations in the single independent variable η but each involving two unknown variables, I and q. We consider the solution of these equations in the following cases:

4.1 <u>$\sigma = 1$, No Heat Transfer to Wall</u>

In this case I is known as mentioned above so we have a single straightforward equation in a single unknown q.

4.2 <u>Case Where a Good Guess can be Made for the Temperature Distribution</u>

In this case we have a first approximation for I, and can substitute it for I in equation (18), which can then be solved for a first approximation to q. In turn this approximation can be substituted for q in equation (19), and the resulting equation solved for a second approximation to I, and so on. The iteration process may or may not be convergent.

Circumstances under which it should be possible to guess the temperature distribution fairly closely are:

(a) Conditions close to the $\sigma = 1$, no heat transfer case. The known solution for the latter case can be taken as the first approximation.

(b) In heat transfer cases where the wall temperature is known. The temperature will also be known at $\eta = 1$, and by extrapolating from conditions at x_a and making suitable adjustments to agree with the end point values a fair approximation to the temperature distribution can probably be obtained.

4.3 Other Cases

It will probably be difficult to make a close guess to the temperature distribution if one temperature boundary condition consists of a relation between I and I_η at the wall, as will be the case where radiation and the conductive properties of the wall are governing factors. In such circumstances as these, therefore, or if the iteration process of sub-section 4.2 does not converge, the equations can probably only be solved by laborious concurrent integration.

Having determined q and I (and hence by relations (16) and (17) q_x and I_x) at x_1, the end of the first integration step for which $x_a = 0$, we can take x_1 as the value of x_a for the second step, and similarly we can proceed to successive steps.

5. Summary of the Method and a Note on Singularities at $\eta = 1$

In sub-sections 3 and 4 it has been shown that in favourable circumstances, when we do not have to integrate pairs of equations concurrently, the problem of solving the compressible boundary layer can be reduced to that of solving several second order equations each in the single independent variable η and one unknown (assumed values being taken for any second unknown present) and each having one boundary condition specified at $\eta = 0$ and the other at $\eta = 1$. Such equations can be solved in a number of standard ways. The author used in the original calculations step-by-step methods, but in the subsequent calculations, performed under the supervision of Dr. L. Fox and Mr. C. W. Clenshaw by the Mathematics Division of the N.P.L., relaxation techniques were used. The latter have the advantage of being easily able to deal with the singularities at $\eta = 1$ in q, q_x, I and I_x with respect to gradients in the η direction. With step-by-step methods however the singularities are troublesome and accordingly it was attempted to find analytical solutions for the functions near $\eta = 1$. Such solutions were found only for cases with $\sigma = 1$, and as they involve cumbersome expressions it is not worthwhile to quote them all here. However the solution for $I_{\eta 1}$ (I_η at $\eta = 1$) is perhaps worth stating at the leading edge if $\sigma = 1$

$$I = I_d \left[1 + (1 - \eta^2) \frac{\gamma - 1}{2} M_d^2\right] - (1 - \eta) I_{\eta w}$$

so that

$$I_{\eta 1} = I_{\eta w} - (\gamma - 1) M_d^2 I_d .$$

It/

It is found that elsewhere $I_{\eta 1}$ satisfies the simple first order differential equation

$$- \frac{I_{\eta 1}^2}{I_1} + \left(2 - \frac{u_1^2}{I_1}\right) I_{\eta 1} - \frac{u_1}{u_{1x}} I_{\eta x 1} = 0 \qquad \ldots(20)$$

which can easily be solved numerically.

SECTION II

Calculated Results

A few cases have been calculated by the methods described above. The principal aim was to determine the separation position, where of course the viscous stress at the wall becomes zero. The position of separation was estimated by extrapolating the parabolic relation assumed for $q(0,x)$ in the last x - interval of integration to the position where it indicated q to be zero at the wall.

1. Incompressible Howarth Case

We first considered the case of an incompressible boundary layer with $(u_d - u_1)/x$ constant, for which Howarth (Ref.3) has obtained an accurate solution against which the results obtained by using the present methods with large x-steps of integration could be checked. At separation u_1/u_d is accurately equal to 0.880. The value found by assuming

$$q_x(\eta, x) = \frac{2}{x}\left[q(\eta,x) - q(\eta,0)\right] - q_x(\eta,0)$$ and solving the equation at the

position $u_1/u_d = 0.925$ was 0.895. By performing the calculations in three steps with stations of integration at $u_1/u_d = 0.9625, 0.9250$ and 0.9000 and q assumed to vary parabolically in each interval, the value obtained for u_1/u_d at separation was 0.886. Thus quite good results for separation distance were obtained even with very large x-steps of integration. The computed stress distributions were also fairly accurate as Table 2 shows.

2. $\sigma = 1$, ω near 1, No Heat Transfer Cases

A second problem considered was that of compressible boundary layers with $(u_d - u_1)/x$ constant, no heat transfer to the wall, $M_d = 4$, $\sigma = 1$, and $\omega = 1$, 0.9 and 0.7, the aim being to determine the proportional changes of separation distance due to the changes of ω.

The $\omega = 1$ case can be solved by the Stewartson transformation in conjunction with Thwaites' approximate method for incompressible boundary layers (Refs.4, 5), and in this way it was estimated that separation occurs at $u_1/u_d = 0.940$. By assuming that q varies parabolically with x over the whole boundary layer and integrating at the position $u_1/u_d = 0.960$ we computed by the present methods that separation occurs at $u_1/u_d = 0.951$. Thus it can be assumed that such 1 step calculations give roughly correct results. Moreover if performed with precisely the same intervals of integration, etc., with the 3 different values of ω they should give results subject to approximately the same errors, so that the estimated proportional differences in separation distance should be fairly close/

close to the actual proportional differences. Accordingly 1 step calculations similar to those with $\omega = 1$ were performed with $\omega = 0.9$ and 0.7, though with the slight difference that for the former case $\omega = 1$ the leading edge values of q, q_x, I, and I_x were estimated from Table 1, whereas for the latter cases they had to be computed from the equations. Results (c.f. Table 3) for u_1/u_d at separations were:

ω	1	0.9	0.7
$\dfrac{u_1}{u_d}$	0.951	0.949	0.947

It would appear at first sight that these figures indicate that

$$\frac{d^2}{d\omega^2}\left[1 - \left(\frac{u_1}{u_d}\right)_{sep}\right]$$

is negative at $\omega = 1$, but this is most probably a false impression due partly to inaccuracies of computation, and partly to the fact that the leading edge values used for the $\omega = 1$ case were almost certainly more accurate than those for the other two cases. The results are probably a much better guide to the first derivative of separation distance and they show that at $\omega = 1$ the proportional rate of change of separation distance with ω,

$$\frac{d}{d\omega}\left[1 - \left(\frac{u_1}{u_d}\right)_{sep.}\right] \Big/ 1 - \left(\frac{u_1}{u_d}\right)_{sep.}$$

is in the region of -0.23.

The time required to compute the case with $\omega = 0.9$ (including the computation of the leading edge values) was about 24 hours. A slide rule, not a calculating machine, was used as great accuracy was not required.

3. Cooled Wall Cases with $\omega = \sigma = 1$

Finally two cases with $\omega = \sigma = 1$ and the wall temperature everywhere kept equal to the datum, free stream, temperature θ_d were considered: (i) $M_d = 4$, and constant adverse external velocity gradient (ii) $M_d = 2$, and constant adverse pressure gradient. Case (i) was computed first roughly by the author and subsequently much more accurately by the Mathematics Division, N.P.L. Case (ii) was entirely computed by the Mathematics Division.

The author's computation of case (i) was as follows: Solutions for q and I were obtained by the iteration process at the position $u_1/u_d = 0.925$, the leading edge values having been computed from Table 1 and $I_{\eta 1}$ from equation (20). The relation used for the first approximation to the enthalpy distribution was of the form

$$I(x,\eta) = I(0,\eta) + x\,I_x(0,\eta) + G\eta + H\eta^2$$

where/

where the constants G and H were chosen to give the correct values of I_1 and $I_{\eta 1}$ at x. The second approximation for I did not differ greatly from the first, and the third was almost identical with the second, so the convergence was very satisfactory.

The results are given in Table 4: they were obtained with about 30-40 hours' work. They show that the position $u_1/u_d = 0.925$ must be a long way from separation. For q probably varies in very roughly the same parabolic way with x downstream of the station of integration as upstream of it, and the extrapolated curve of q at the wall does not become zero until u_1/u_d reaches the region of 0.75.

The Mathematics Division's estimate of u_1/u_d at separation, obtained by using many steps of integration, is 0.78 ± 0.03. It is of course fortuitous that this result should be so close to the author's rough estimate.

For case (ii) Mathematics Division find that separation occurs at a pressure ratio $p/p_d = 2.2 \pm 0.3$.

Thus in both cases separation is very greatly delayed as compared with the corresponding insulated wall cases. For if these are computed by a combination of the methods of Stewartson and Thwaites (Refs.4 and 5) it is found that separation occurs at $u_1/u_d = 0.94$ and $p/p_d = 1.65$ respectively. As is to be expected, the difference between the insulated and cooled wall cases is greater at the higher Mach number.

Acknowledgements

As already stated, much of the work described above was carried out under the supervision of Dr. L. Fox and Mr. C. W. Clenshaw by the Mathematics Division of the N.P.L.

The author's share of the work was performed whilst at Imperial College in receipt of a generous award from the Trust for Beit Fellowships for Scientific Research. Much helpful criticism and advice was given by Dr. B. Thwaites. An account of the work formed part of a thesis submitted for the degree of Ph.D. in the University of London.

References

No.	Author(s)	Title, etc.
1	L. Crocco	A characteristic transformation of the equations of the boundary layer in gases. (Atti di Guidonia, No.7, 10th May, 1939, XVII.) Translated by M. Flint. Air Ministry Translation No.1030. A.R.C.4582.
2	S. Goldstein	On laminar boundary layer flow near a position of separation. Quarterly Journal of Mechanics and Applied Mathematics, Vol.1, p.43, 1948.

3/

Table 2

Incompressible Case with $u_1 = u_d - \beta x$, β constant.
Calculated and True Values of q at Stations of Integration.
(3 Step and 1 Step Calculations).

Station $\left(1 - \frac{u_1}{u_d}\right)$	0.0375		0.0750			0.1000	
η	3 step	True	1 step	3 step	True	3 step	True
0	0.06388	0.0640	0.0248	0.02717	0.0284	0.00840	0.0096
0.095	0.07029	0.0708		0.03906	0.0400	0.02340	0.0247
0.190	0.07555	0.0757	0.0471	0.04929	0.0503	0.03627	0.0370
0.285	0.07863	0.0789		0.05672	0.0576	0.04584	0.0465
0.380	0.07865	0.0793	0.0592	0.06058	0.0617	0.05146	0.0530
0.475	0.07490	0.0759		0.06029	0.0619	0.05270	0.0545
0.570	0.06697	0.0671	0.0546	0.05550	0.0567	0.04926	0.0511
0.665	0.05476	0.0548		0.04621	0.0471	0.04128	0.0431
0.760	0.03878	0.0390	0.0308	0.03296	0.0345	0.02936	0.0313
0.855	0.02057	0.0210		0.01747	0.0190	0.01484	0.0168
0.950	0.00425	0.0045	0.0038	0.00380	0.0038	0.00370	0.0034

Table 3/

Table 3

Cases with $M_d = 4, \sigma = 1,$
no Heat Transfer, $u_1 = u_d - \beta x$, β constant

η	$\omega = 1$ q	$\omega = 1$ $\frac{-q_x u_d}{u_{1x}}$	$\omega = 0.9$ q	$\omega = 0.9$ $\frac{-q_x u_d}{u_{1x}}$	$\omega = 0.7$ q	$\omega = 0.7$ $\frac{-q_x u_d}{u_{1x}}$	$\omega = 1$ q	0.9 q	0.7 q
0	0.1102	-0.751	0.1011	-0.602	0.0811	-0.413	0.0316	0.0335	0.0309
0.190	0.1090	+0.760	0.1005	+0.711	0.0806	+0.573	0.1279	0.1168	0.0936
0.380	0.1015	1.780	0.0930	1.610	0.0751	1.259	0.1858	0.1671	0.1321
0.570	0.0806	2.000	0.0751	1.813	0.0615	1.432	0.1817	0.1635	0.1302
0.760	0.0452	1.294	0.0433	1.219	0.0365	0.989	0.1113	0.1005	0.0818
0.950	0.0050	0.157	0.00504	0.150	0.00445	0.1309	0.0128	0.0118	0.0102

At Leading edge (columns $\omega = 1$, 0.9, 0.7 with $\frac{-q_x u_d}{u_{1x}}$); At $1 - \frac{u_1}{u_d} = 0.040$ (last three columns)

Table 4

Heat Transfer Case with $M_d = 4$, $\omega = \sigma = 1$, $I_w = I_d$, $u_1 = u_d - \beta x$,
β constant. Station of integration at $1 - \frac{u_1}{u_d} = 0.075$
Leading edge values as per Table 1.

η	Approximations for I: $\frac{I}{I_d}^{(1)}$	$\frac{I_\eta}{I_d}$	$\frac{I}{I_d}^{(2)}$	$\frac{I_\eta}{I_d}$	$\frac{I}{I_d}^{(3)}$	$\frac{I_\eta}{I_d}$	Approximations for q: (1) q	(2) q
0	1	4.17	1	4.58	1	4.61	0.1958	0.1878
0.190	1.665	2.72	1.703	2.83	1.705	2.83	0.2724	0.2661
0.380	2.043	1.28	2.083	1.23	2.086	1.23	0.3309	0.3275
0.570	2.160	-0.14	2.184	-0.19	2.185	-0.19	0.3121	0.3105
0.760	2.003	-1.40	2.025	-1.50	2.027	-1.46	0.1897	0.1887
0.950	1.618	-3.19	1.622	-2.80	1.622	-2.82	0.0222	0.0217

Guess (first two columns); Computed (remaining columns)

JDS.

C.P. No. 312
(15,101)
A.R.C. Technical Report

Printed and published by
HER MAJESTY'S STATIONERY OFFICE

To be purchased from
York House, Kingsway, London W.C.2
423 Oxford Street, London W.1
P.O. Box 569, London S.E.1
13A Castle Street, Edinburgh 2
109 St Mary Street, Cardiff
39 King Street, Manchester 2
Tower Lane, Bristol 1
2 Edmund Street, Birmingham 3
80 Chichester Street, Belfast
or through any bookseller

Printed in Great Britain

S.O. Code No. 23-9010-12

C.P. No. 312