R. & M. No. 2616
(11,768)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

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1952
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Reports and Memoranda No. 2616*

September, 1948

Introduction and Summary.—The numerical method of characteristics is used to calculate the flow in a steady supersonic jet of air issuing from a slightly supersonic circular orifice into a vacuum. The calculations are entirely numerical, and no recourse is made to graphical methods.

The characteristic equations for steady supersonic flow with rotational symmetry are first derived, and then special consideration is given to the flow near the axis of symmetry, where the normal step-by-step numerical process breaks down.

In the calculation, the Mach angle in the plane of the orifice is taken as 85 deg to obviate the difficulties of a sonic orifice at which the initial characteristics would be perpendicular to the flow, and the potential equation parabolic. The results should be practically the same as for a sonic orifice. An alternative method of dealing with a sonic boundary-plane would have been the use of analytical solutions for the initial flow in this region (cf. Ref. 3).

The results of the calculations are presented in diagrams. The solution is a universal solution in so far that it applies to any similar jet, flowing into any external pressure, in that region bounded by the orifice and the first wave front which registers the existence of an external pressure outside the jet.

This fact allows the calculated pressure distribution along the axis of symmetry to be compared with experimental measurements in air jets with finite pressure ratios, and good agreement is obtained.

1. *The Characteristic Equations for Steady Supersonic-flow with Rotational Symmetry.*—Using cylindrical co-ordinates (z, r, ψ), with $r = 0$ as the axis of rotational symmetry, the motion is independent of ψ. Let $a_s u$, $a_s v$ be the components of fluid velocity in the z and r directions respectively, where a_s is a constant speed. Then, for irrotational adiabatic flow, a velocity potential ϕ may be defined so that,

$$\left.\begin{aligned} u &= P = \frac{\partial \phi}{\partial z} \\ v &= Q = \frac{\partial \phi}{\partial r} \end{aligned}\right\} \qquad (1.01)$$

* A.R.E. Report 30/48. Received 14th October, 1948.

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The speed of sound, Aa_s, is given by,

$$(Aa_s)^2 = \frac{dp}{d\rho} = \frac{\gamma p}{\rho} \cdot \quad \ldots \quad (1.02)$$

The equations of motion integrate to give Bernoulli's relation,

$$\tfrac{1}{2}a_s^2(P^2 + Q^2) + A^2 a_s^2/(\gamma - 1) = \text{constant}$$

$$= \frac{(\gamma + 1)\, a_s^2}{2(\gamma - 1)} \quad \ldots \quad (1.03)$$

if a_s is chosen to be the local speed of sound when the fluid is flowing with sonic velocity, *i.e.*, if $P^2 + Q^2 = 1$ when $A = 1$.
Hence,

$$P^2 + Q^2 - A^2 = (1 - A^2)/\lambda^2 \quad \ldots \quad (1.04)$$

where λ is defined by,

$$\lambda^2 = \frac{\gamma - 1}{\gamma + 1} \cdot \quad \ldots \quad (1.05)$$

The equation of continuity reduces to the potential equation,

$$(P^2 - A^2)R + 2PQS + (Q^2 - A^2)T = A^2 Q/r, \quad \ldots \quad (1.06)$$

where $R = \partial^2\phi/\partial z^2, \qquad S = \partial^2\phi/\partial r\,\partial z, \qquad T = \partial^2\phi/\partial r^2.$

The characteristic directions of this potential equation are real if the flow is supersonic, and co-incident if the flow is sonic. In the supersonic or hyperbolic case, the Mach angle μ may be defined by

$$V^2 = u^2 + v^2 = \frac{A^2}{\sin^2 \mu} \quad \ldots \quad (1.07)$$

and the differential equation of the characteristic curves may then be written,

$$\frac{dr}{dz} = \tan(\theta \pm \mu) \quad \ldots \quad (1.08)$$

where θ is the inclination of the flow to the z-axis, or

$$\tan \theta = v/u. \quad \ldots \quad (1.09)$$

The differential relations which hold along the characteristic curves (1.08) are respectively,

$$\frac{dV}{V} \mp \tan \mu \,.\, d\theta - \frac{\sin \theta \, \sin^2 \mu}{\sin(\theta \pm \mu)\cos \mu} \frac{dr}{r} = 0 \,. \quad \ldots \quad (1.10)$$

But, from (1.04) and (1.07)

$$V^2(\lambda^2 \cos^2 \mu + \sin^2 \mu) = 1 \quad \ldots \quad (1.11)$$

and therefore

$$\frac{dV}{V} + \frac{(1 - \lambda^2)\sin \mu \,.\, \cos \mu \,.\, d\mu}{(\lambda^2 \cos^2 \mu + \sin^2 \mu)} = 0 \,. \quad \ldots \quad (1.12)$$

Substituting this result, the differential relations (1.10) may be written

$$\nu/\lambda - \mu \pm \theta + \int^r \frac{\sin\theta \sin\mu}{\sin(\theta \pm \mu)} \frac{dr}{r} = \text{constant} \quad \ldots \quad (1.13)$$

where ν is an angle defined by,

$$\lambda \tan\nu = \tan\mu. \quad \ldots \quad (1.14)$$

The characteristic relations (1.13) are used in the calculations, but, as is done in the following Section, the limit of the integrand must be considered as the axis is approached and both r and $\sin\theta$ tend to zero.

In terms of the state of the fluid when flowing at sonic speed, namely pressure p_s, density ρ_s, speed of sound a_s, the speed of sound a at any point is given by,

$$a/a_s = A = \sin\nu; \quad \ldots \quad (1.15)$$

the resultant fluid velocity $a_s V$ is given by

$$V = \frac{\sin\nu}{\sin\mu}, \quad \ldots \quad (1.16)$$

with components,

$$u = \frac{\sin\nu \cos\theta}{\sin\mu}, \qquad v = \frac{\sin\nu \sin\theta}{\sin\mu}; \quad \ldots \quad (1.17)$$

the pressure p is given by,

$$p/p_s = (\sin\nu)^{1+1/\lambda^2}; \quad \ldots \quad (1.18)$$

and the density ρ is given by,

$$\rho/\rho_s = (\sin\mu)^{-1+1/\lambda^2}. \quad \ldots \quad (1.19)$$

2. *Flow near the Axis of Symmetry.*—Using the suffix 0 to denote values on the axis of symmetry, $r = 0$, it follows at once that

$$\theta_0 = 0 \quad \ldots \quad (2.01)$$

$$v_0 = Q_0 = 0 \quad \ldots \quad (2.02)$$

and therefore $$S_0 = \frac{dQ_0}{dz} = 0. \quad \ldots \quad (2.03)$$

Thus, from (1.15),

$$A_0 = \sin\nu_0 \quad \ldots \quad (2.04)$$

and from (1.16),

$$P_0 = u_0 = V_0 = \frac{\sin\nu_0}{\sin\mu_0} \quad \ldots \quad (2.05)$$

The potential equation (1.06) then gives,

$$\left\{\frac{\sin^2 \nu_0}{\sin^2 \mu_0} - \sin^2 \nu_0\right\} R_0 - \sin^2 \nu_0 \, T_0 = \sin^2 \nu_0 \underset{r\to 0}{\mathrm{Lt}} \left\{\frac{Q}{r}\right\} \quad \cdots \quad (2.06)$$

whence, neglecting the trivial case $\sin \nu_0 = 0$,

$$R_0 \cot^2 \mu_0 - T_0 = \underset{r\to 0}{\mathrm{Lt}} \left\{\frac{Q}{r}\right\} \quad \cdots \quad (2.07)$$

But

$$\underset{r\to 0}{\mathrm{Lt}} \left\{\frac{Q}{r}\right\} = \underset{r\to 0}{\mathrm{Lt}} \left\{\frac{S\,dz + T\,dr}{dr}\right\}$$

$$= T_0$$

and therefore, from (2.07)

$$\underset{r\to 0}{\mathrm{Lt}} \left\{\frac{Q}{r}\right\} = T_0 = \tfrac{1}{2} R_0 \cot^2 \mu_0. \quad \cdots \quad (2.08)$$

Hence, the limit of the integrand in the characteristic relations (1.13), as the axis, $r = 0$, is approached, is given by,

$$\underset{r\to 0}{\mathrm{Lt}} \left\{\frac{\sin\theta \, \sin\mu}{r \sin(\theta \pm \mu)}\right\} = \pm \frac{\sin \mu_0}{\sin \nu_0} \underset{r\to 0}{\mathrm{Lt}} \left\{\frac{Q}{r}\right\}$$

$$= \pm \frac{\sin \mu_0}{2 \sin \nu_0} \cdot \cot^2 \mu_0 . \, R_0$$

$$= \pm \frac{\cos^2 \mu_0}{2 \sin \nu_0 \, \sin \mu_0} \underset{r\to 0}{\mathrm{Lt}} \frac{\partial}{\partial z} \left\{\frac{\sin \nu \, \cos\theta}{\sin \mu}\right\}$$

$$= \pm \frac{\cos^2 \mu_0}{2 \sin \nu_0 \, \sin \mu_0} \frac{d}{dz} \left\{\frac{\sin \nu_0}{\sin \mu_0}\right\}$$

and, after some reduction, this may be expressed in the very useful form,

$$\underset{r\to 0}{\mathrm{Lt}} \left\{\frac{\sin\theta \, \sin\mu}{r \sin(\theta \pm \mu)}\right\} = \mp \frac{\cot \mu_0}{2} \frac{d}{dz} (\nu_0/\lambda - \mu_0) \, . \quad \cdots \quad (2.09)$$

The relation (2.09) thus enables the integral term in the characteristic relations (1.13) to be evaluated accurately as $r \to 0$ along a characteristic, by making use of the gradient of $(\nu/\lambda - \mu)$ along the axis of symmetry.

3. *The Calculation.*—In the calculation, the Mach angle in the plane of the circular orifice, was taken as 85 deg, and the external pressure zero. The familiar 'patching' method of characteristics was used, in which the known solution at two points P_1, P_2, leads to the solution at a point P_3 such that P_1P_3 and P_2P_3 are small arcs of characteristic curves. In the calculation the small arcs P_1P_3, P_2P_3 were taken to be arcs of circles, *i.e.*, the inclinations of the chords P_1P_3, P_2P_3 were taken to be the means of the characteristic inclinations at P_1, P_3, and P_2, P_3 respectively. Since the characteristic relations (1.13) involve the space co-ordinates and do not

integrate exactly, an iterative process is necessary. For convenience in the calculation, a table giving values of the angles μ and ν at equal small intervals of the function $(\nu/\lambda - \mu)$ was used. This was prepared with the aid of National Accounting Machines. The integral term in the relations (1.13) was evaluated numerically by the trapezoidal rule.

The calculation of the points and conditions along the axis of symmetry was accomplished by means of the relation (2.09). The iterative process, in this case, was found to be much more difficult, and it was harder to maintain accuracy in these arcs.

The results of the calculation, so far as it was taken, are shown in the diagrams. Fig. 1 shows the pattern of the characteristic network; Fig. 2 shows the distribution of velocity and Mach number along the axis of symmetry. It is worth remarking that the maximum possible value of the non-dimensional velocity V is $1/\lambda = \sqrt{6} = 2 \cdot 45$, for $\gamma = 1 \cdot 40$.

Fig. 3 shows the calculated axial distribution of pressure, and, for comparison, some experimental measurements of the pressure along the axes of supersonic jets with sonic orifices and finite pressure ratios are also plotted in this figure. This comparison is possible since the solution calculated here for an infinite pressure ratio is, in some sense, a universal solution, in so far as it applies to that area, of any similar jet with finite pressure ratio, which is bounded by the orifice and the first wave front from the orifice boundary which registers the existence of an external pressure outside the jet. The experimental measurements are of two kinds; Pitot measurements taken along the axis of a jet flowing through a 6-mm orifice from a tank at a pressure of 7 kg/sq cm, by Hartmann and Lazarus (Ref. 1); and interferometric measurements made by Ladenburg and others with a jet flowing through a 10-mm orifice from a tank at a pressure of 5 atmospheres (Ref. 2). The agreement is good, particularly when it is remembered that, for convenience in starting the calculation, a Mach angle of 85 deg was taken at the orifice.

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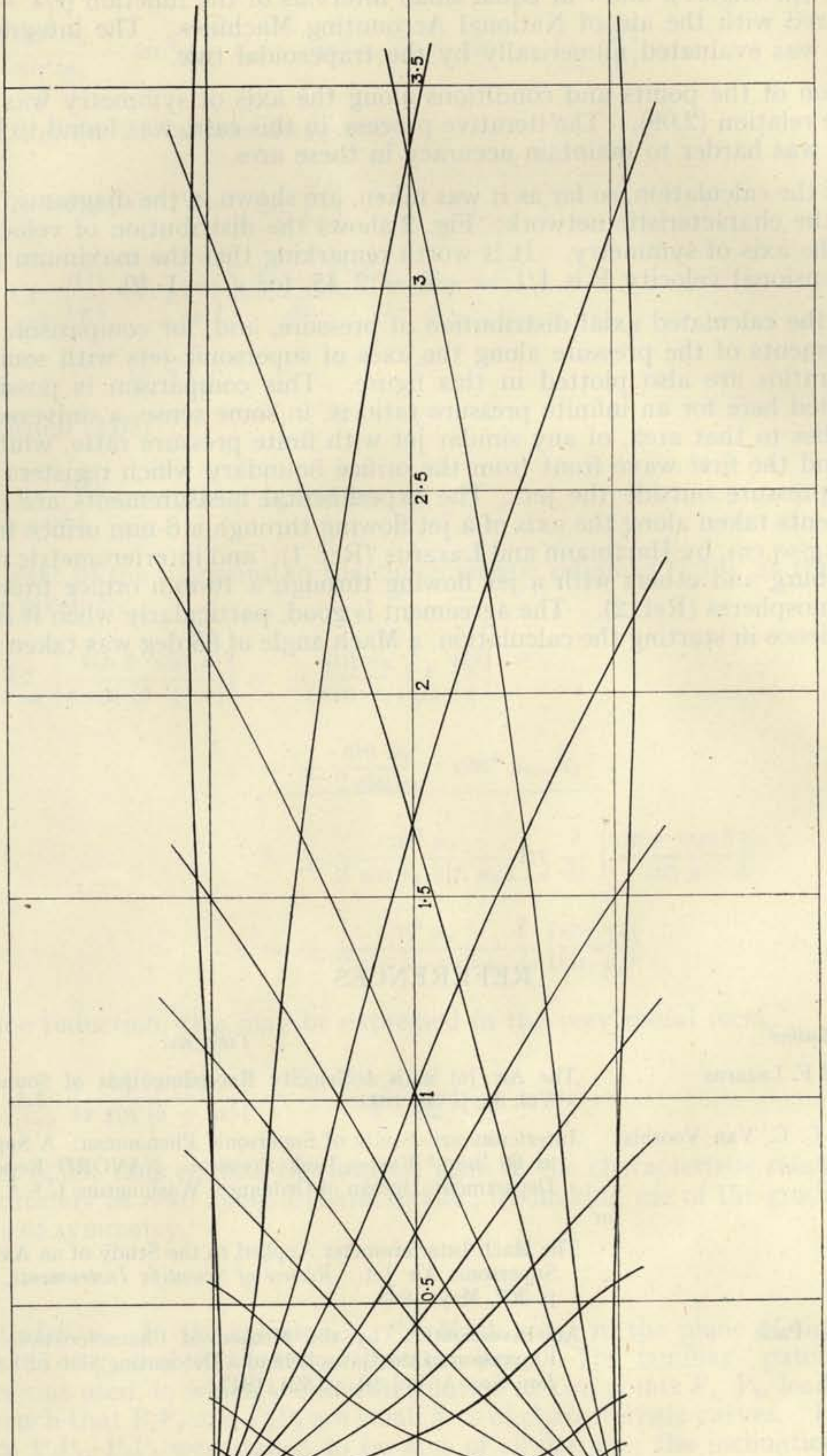
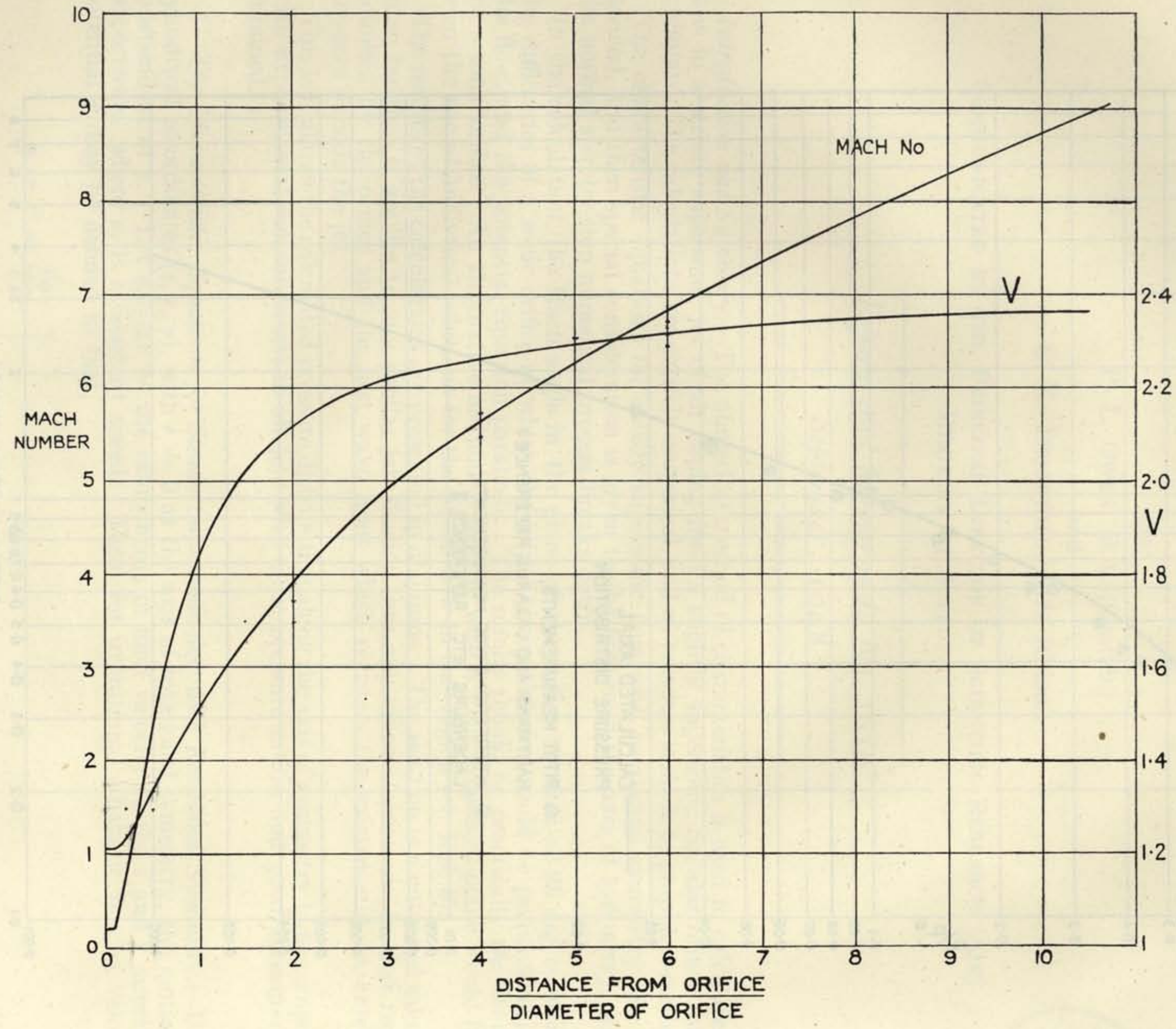


FIG. 1.



MACH NUMBER
MACH No
V
V
0
1
2
3
4
5
6
7
8
9
10
1·2
1·4
1·6
1·8
2·0
2·2
2·4
DISTANCE FROM ORIFICE
DIAMETER OF ORIFICE

FIG. 2.

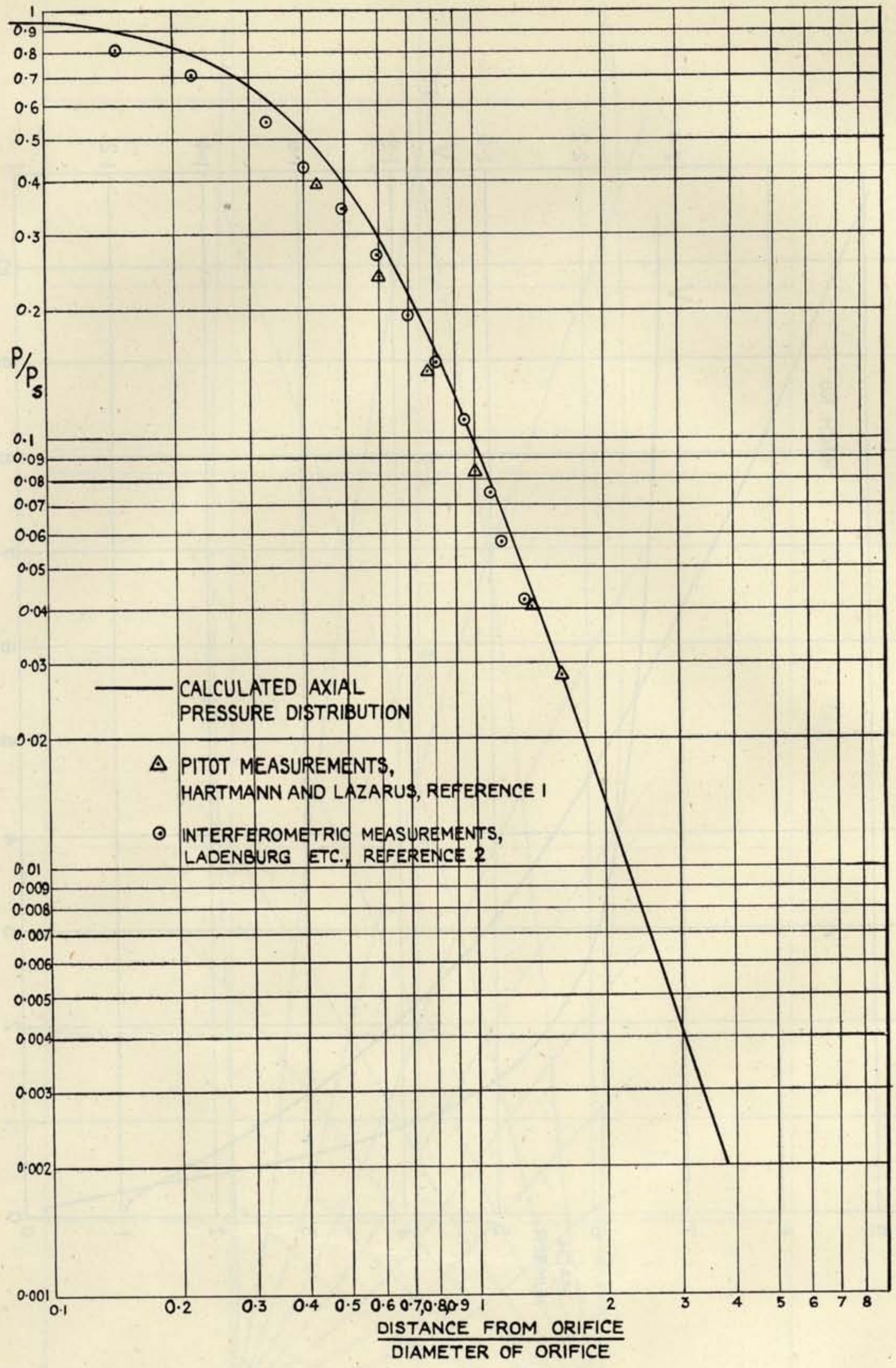


FIG. 3.

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