R & M No.3833

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PROCUREMENT EXECUTIVE, MINISTRY OF DEFENCE

Aeronautical Research Council
Reports and Memoranda

ON INTERFACING STRUCTURAL INFORMATION AND LOADING DATA IN AEROELASTIC ANALYSES

by

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London: Her Majesty's Stationery Office
£8 NET

UDC 533.6.013.42 : 533.6.048.1

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May 1977

SUMMARY

The development of a systematic means of interfacing structural and loading information in aeroelastic analyses is presented, and a computer implementation, with particular application to plate-like structures, is described. Various numerical examples of the use of the method are given, and the overall accuracy of the procedure advocated is critically examined.

* Replaces RAE Technical Report 77074 - ARC 37619

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Introduction

In seeking to model mathematically an aeroelastic problem of a flight vehicle, the aeroelastician has formally to describe the interaction of the loading mechanism, be it aerodynamic, propulsive, inertial or gravitational, with the deformation of the structure. By virtue of the very nature of both the structure and the loading, it is usually only possible to describe either in a numerical sense, such a description being basically discrete in character. In this Report attention is restricted to those structural models in which the elastic characteristics are referred to a discrete set of points. It is likely that the loading on and the elastic characteristics of the structure will be most readily quantifiable at different sets of points over the structure. The analyst therefore must interface various data defined with respect to differing sets of points. Some systematic means of bringing about this interface is obviously desirable particularly when one realizes that for coverage of a wide range of flight conditions, diverse aerodynamic theories may need to be employed, each probably yielding data at an especial set of points.

In a mathematical description of the aeroelastic problem pertinent to the continuum, both the structural characteristics and the loading are each everywhere defined and the problem of interfacing one with the other does not exist. However, in practice, it is necessary to restrict the number of degrees of freedom of the structure in order to obtain a solution of the governing equation of aeroelastic equilibrium, and an interface problem arises in the reconciliation of the discrete representation of the structure with diverse loadings. The basis for a general method of effecting the interface is developed in section I.2 by considering the evolution of the equation of aeroelastic equilibrium for a system with a finite number of degrees of freedom from that appropriate to the continuum. A practical form of such an interface is propounded in section I.3 which in effect provides a framework within which any particular interface can be developed.

In many aeroelastic investigations it is acceptable to make gross simplifying assumptions concerning the elastic characteristics of the structure, and the nature of the loading. In Part II a common idealisation, namely, a plate-like structure subject to transverse loading, is considered. In this instance the interface problem is shown to reduce to a transformation of the structural information. Various aspects of the computer program ALFI, developed to implement these transformations are described. Some numerical examples are presented, and the accuracy of the proposed means of effecting the interface discussed.

Part I

A GENERAL METHOD OF EFFECTING THE INTERFACE

I.1 GENERAL NOTATION

The notation given below is relevant to Part I. Symbols appearing in both parts of the Report represent similar physical entities in each, but are described below in the general terms appropriate to Part I. The particular meanings appropriate to Part II and definitions of other symbols peculiar to Part II are given in the List of Symbols on Page 64.

$\mathbf{A}^i$	a matrix which when post-multiplied by δ yields δ^i
$\mathbf{F}$	a column vector of generalised forces acting at the points of Σ_1
$\mathbf{F}_0$	a static equilibrium value of $\mathbf{F}$
$\mathbf{F}_{d_0}$	that part of the loading $\mathbf{F}_0$ that arises from effects other than aerodynamic
$\mathbf{F}^i$	a column vector of the contributions to the generalised forces for the degrees of freedom δ^i from the region R_i
$\mathbf{G}^i$	for some assumed polynomial representation of the state of displacement over R_i, the matrix $\mathbf{G}^i$ is such that when post-multiplied by a column of the constants of the polynomial expression it gives the deformation parameters of R_i (equation (I-24))
$\mathbf{H}(x)$	a matrix used when expressing the polynomial representations of the state of displacement over R_i in matrix form (equation (I-21))
$\mathbf{I}_n$	the unit matrix of order n
$\mathbf{J}_{\ell_r}$	a column vector of ℓ_r elements, each of which is unity
$\mathbf{N}(x)$	a matrix which when post-multiplied by δ gives the displacement of the structure (equations (I-28) and (I-29))
$\mathbf{N}^i(x)$	a matrix which when post-multiplied by δ^i gives the displacement at a point $\mathbf{x}$ associated with R_i
$\mathbf{Q}_0(x)$	a representation of the static equilibrium loading on the deformed flight vehicle
$\mathbf{Q}_{d_0}(x)$	that part of the loading $\mathbf{Q}_0(x)$ which is independent of the deformation
$\mathbf{Q}_0^r(x)$	a description of the static equilibrium loading for the reduced order system
$\mathbf{S}$	A flexibility matrix relating deformation parameters and force type parameters at the set of points Σ_1, obtained by some undefined method
$\mathbf{U}^i$	a matrix defined by equation (I-36)
$\mathbf{Z}_r^i$	a column vector of the forces in the Ox_r direction that are included in $\mathbf{F}^i$

a^i	a column vector of constants in the polynomial representations of the state of displacement over R_i (equation (I-21))
a_r^i	sub-matrix of a^i, a_r^i being the constants in the polynomial expression for Δ_r^i
$h_r(x)$	sub-matrix of $H(x)$, defined by equation (I-21a)
o_u	a null row vector of u elements
$p(x)$	a description of the loading on the vehicle
$p^i(x)$	that portion of $p(x)$ acting over R_i
$q(x)$	the deformation of the body
$q_0(x)$	the static equilibrium value of $q(x)$
$q_{d_0}(x)$	that part of $q_0(x)$ that arises from non-aerodynamic loads
$q^r(x)$	the deformation of the reduced order system
$q_0^r(x)$, $q_{d_0}^r(x)$	are related to $q^r(x)$ in the same manner as $q_0(x)$, $q_{d_0}(x)$ are related to $q(x)$
$q^{r^*}(x)$	a virtual displacement of the structure, corresponding to δ^*
w_r^i	a vector of the ℓ_r linear displacements in the direction Ox_r, that are included in δ^i
x	the position vector of a general point with respect to the axis system $Ox_1x_2x_3$
$x(\delta_j^i)$	the position vector of the point with which δ_j^i is associated
$\Delta(x)$	a vector describing the displacement of the vehicle derived from a finite representation of the flexibility characteristics
$\Delta^i(x)$	a vector describing the state of displacement over R_i, derived from δ^i
δ	a column vector of generalised coordinates associated with the points of Σ_1
δ_0	the static equilibrium value of δ
δ^*	a column vector, corresponding to δ, of virtual displacements in the generalised coordinates
δ^i	a column vector of generalised coordinates at the n_i characteristic points of R_i
ξ	the position vector of a general point with respect to the axis system $Ox_1x_2x_3$

A^i_{jk}	a general element of $\mathbf{A}^i$
$0x_1x_2x_3$	a set of rectangular body attached axes whose origin 0 is a definite material point of the body. This axis system is rigorously defined in Ref 1, Part I, section 2.2
$0_E x_E y_E z_E$	a set of earth axes
Q	a general force on the vehicle
Q_d	a disturbing force independent of the deformation
Q_{d_0}	a particular static equilibrium value of Q_d
R	the number of regions which together comprise the domain over which the description of the flexibility characteristics of the structure are applicable
R_i	the ith region
a^i_k	the kth element of $\mathbf{a}^i$
k_r	the number of terms in the algebraic expression for Δ^i_r
ℓ_r	the number of elements in each of the vectors $\mathbf{w}^i_r$ and $\mathbf{z}^i_r$
n_i	the number of characteristic points of R_i
q	a generalised elastic displacement definable for any part of the vehicle
q_0	the static equilibrium value of q
s_1	the order of the matrix $\mathscr{S}$
t_i	the number of deformation parameters associated with R_i
$\Gamma(\mathbf{x};\xi)$	a second-order tensor of flexibility influence functions
$\Gamma^r(\mathbf{x};\xi)$	the derived structural influence function tensor, appropriate to the reduced order system
$\Delta^i_r(\mathbf{x})$	the rth component of $\Delta^i(\mathbf{x})$, detailing the deformation in the direction $0x_r$ over R_i
Σ_1	the set of points with respect to which the available flexibility data are defined
$\alpha_k, \beta_k, \gamma_k$	powers of x_1, x_2, x_3 in the algebraic expressions for Δ^i_r (equation (I-20))
$\delta(\mathbf{x})$	the Dirac delta function
δ^i_j	the jth component of δ^i
$\kappa^j(\mathbf{x})$	a function which has the value 1 if the point $\mathbf{x}$ is associated with R_j, and is otherwise 0

σ_1 — the number of points in the set Σ_1

$\tilde{\mathscr{A}}$ — an aerodynamic operator

$\tilde{\mathscr{G}}_j$ — a linear operator which when operating on Δ^i gives a continuous description, over R_i, of the particular deformation parameter which takes the value δ^i_j at the point whose position vector is $\mathbf{x}(\delta^i_j)$

$\tilde{\mathscr{I}}$ — the inertial operator

$\tilde{\mathscr{K}}$ — the structural stiffness operator

$\mathscr{S}$ — an exact flexibility matrix for the structure, derived from the structural properties of the continuum

$\mathscr{N}^r(\mathbf{x})$ — is a matrix of functions describing the nature of the deformation of the continuum (equation (I-10)) under certain loadings

I.2 A MATHEMATICAL DESCRIPTION OF AEROELASTIC EQUILIBRIUM

I.2.1 Statement of equilibrium in terms of aeroelastic operators

The total motion of an unrestrained flexible flight vehicle will ultimately be referred to a set of earth axes $0_E x_E y_E z_E$, but it is convenient to refer the deformation of the vehicle and the loads arising from that deformation to a set of rectangular body-attached axes $0x_1x_2x_3$ whose origin, 0, is a definite material point of the body (see Ref 1, Part I, section 2.2). For the flexible body we define a structural stiffness operator[2], $\tilde{\mathscr{K}}$, such that a general distributed force Q is related to the general elastic displacement q by the following expression

$$Q = \tilde{\mathscr{K}}(q) \quad . \tag{I-1}$$

Both Q and q are referred to the body-attached axes $0x_1x_2x_3$. Symbolically we may write equation (I-1) in the inverse form as

$$q = \tilde{\mathscr{K}}^{-1}(Q) \quad . \tag{I-2}$$

In an aeroelastic problem the distributed force Q may be compounded from two types of loading - one including those aerodynamic and inertial loadings which are due to the generalised displacement q, and the other comprising loading systems which are independent of the generalised displacement. Hence we may write equation (I-1) as

$$\tilde{\mathscr{K}}(q) = Q_d + \tilde{\mathscr{A}}(q) + \tilde{\mathscr{I}}(q) \tag{I-3}$$

and equation (I-2) as

$$q = \tilde{\mathscr{K}}^{-1}[Q_d + \tilde{\mathscr{A}}(q) + \tilde{\mathscr{I}}(q)] \quad \text{(I-4)}$$

where $\tilde{\mathscr{A}}$ is an aerodynamic operator and $\tilde{\mathscr{A}}(q)$ represents the continuously distributed aerodynamic loading; $\tilde{\mathscr{I}}$ is the inertial operator; and Q_d is a disturbing force independent of the deformation.

In static aeroelastic problems, the inertia term $\tilde{\mathscr{I}}(q)$ is zero so that such problems may be described formally by

$$\tilde{\mathscr{K}}(q_0) = Q_{d_0} + \tilde{\mathscr{A}}(q_0) \quad \text{(I-5)}$$

or its inverse

$$q_0 = \tilde{\mathscr{K}}^{-1}[Q_{d_0} + \tilde{\mathscr{A}}(q_0)] \quad \text{(I-6)}$$

where the subscript 0 indicates the static equilibrium values. The effects of steady acceleration can be included in a quasi-static aeroelastic analysis of a steady manoeuvre by adding an inertial term independent of time to Q_{d_0} in equation (I-6).

The inverse operator may be expressed as a volume integral

$$\tilde{\mathscr{K}}^{-1}(\ ) = \int_V \Gamma(\mathbf{x};\xi)(\ )dV$$

so that a deformation vector $\mathbf{q}_0$ may be written

$$\mathbf{q}_0(\mathbf{x}) = \int_V \Gamma(\mathbf{x};\xi)\mathbf{Q}_0(\xi)dV \quad \text{(I-7)}$$

where $\Gamma(\mathbf{x};\xi)$ is a second-order influence function tensor, related to the body-attached axes system $Ox_1x_2x_3$ which describes the deformation of any point in the structure in terms of displacements in the coordinate directions due to unit forces applied, in turn, in each of the coordinate directions at each point of the structure. For an elastic structure, Γ satisfies the reciprocity condition

$$\Gamma(\mathbf{x};\xi) = \Gamma(\xi;\mathbf{x}) \quad .$$

$Q_0(\xi)$ is the loading on the deformed flexible body and is formally given by

$$Q_0(\xi) = Q_{d_0}(\xi) + \tilde{\mathscr{A}}(q_0(\xi)) \quad . \tag{I-8}$$

Equations (I-7) and (I-8) may thus be taken to represent formally quasi-static aeroelastic problems in an infinite number of degrees of freedom, ie they are equations pertinent to a continuous system. From equations (I-7) and (I-8) the equilibrium equation can be written in terms of the deformation q_0 or the load Q_0 , viz

$$q_0(x) = q_{d_0}(x) + \int_V \Gamma(x;\xi)\tilde{\mathscr{A}}(q_0(\xi))dV \tag{I-9a}$$

or

$$Q_0(x) = Q_{d_0}(x) + \tilde{\mathscr{A}}\left(\int_V \Gamma(x;\xi)Q_0(\xi)dV\right) \quad . \tag{I-9b}$$

In the continuum there is no problem of interfacing the loading and the description of the flexibility characteristics because each is everywhere defined. However, because of the non-uniform character of an aircraft structure and the nature of the aerodynamic theories available to predict aerodynamic loading on bodies of arbitrary shape, it will not usually be possible to specify the structural characteristics and aerodynamic operator in exact analytical form. It is therefore unlikely that an exact solution applicable to the continuous system can be found. Instead one must seek to approximate the continuous system by a system with a finite number of degrees of freedom and to specify the operators in a purely numerical sense. As regards the former, it is instructive to consider the specification of such a system in the light of the structural properties of the continuum which, for the moment, we shall assume to be known, since this provides a basic insight into the way in which a discrete description of the flexibility characteristics of a structure can be used to provide, in a self consistent manner, deformation data for diverse loadings.

I.2.2 A description for some semi-rigid representation of the structure

The actual structure can respond uniquely to an infinity of loading actions but for a finite set of s_1 distinct loadings the resulting deformation, $q^r(x)$, is definable everywhere in terms of s_1 generalised coordinates, δ , which are associated with σ_1 points of the structure which comprise the set Σ_1 . Thus for this restricted loading we have

$$\mathbf{q}^r(\mathbf{x}) = \mathscr{N}^r(\mathbf{x})\delta \tag{I-10}$$

where $\mathscr{N}^r(\mathbf{x})$ is a matrix of functions describing the relationship between $\mathbf{q}^r(\mathbf{x})$ and δ .

The corresponding distributed loading, $\mathbf{Q}^r(\mathbf{x})$, is given via equation (I-1)

$$\mathbf{Q}^r(\mathbf{x}) = \widetilde{\mathscr{K}}(\mathbf{q}^r(\mathbf{x})) \quad . \tag{I-11}$$

We choose s_1 discrete force-type parameters, $\mathbf{F}$, acting at the points of Σ_1 and corresponding to δ such that the work done by $\mathbf{Q}^r(\mathbf{x})$ during a virtual displacement, δ^* is given by $\delta^{*T}\mathbf{F}$, (the superscript T denotes transpose) *ie*

$$\delta^{*T}\mathbf{F} = \int_V \mathbf{q}^{r*^T}(\mathbf{x})\mathbf{Q}^r(\mathbf{x})dV$$

$$= \int_V \delta^{*T}(\mathbf{x})\mathscr{N}^{r^T}(\mathbf{x})\mathbf{Q}^r(\mathbf{x})dV \qquad \text{by equation (I-10).}$$

Since this latter equation must hold for all such virtual displacements δ^* , it can be concluded that

$$\mathbf{F} = \int_V \mathscr{N}^{r^T}(\mathbf{x})\mathbf{Q}^r(\mathbf{x})dV \quad .$$

By virtue of equations (I-11) and (I-10) we may, for $\widetilde{\mathscr{K}}(\ )$ a linear operator, write

$$\mathbf{F} = \int_V \mathscr{N}^{r^T}(\mathbf{x})\widetilde{\mathscr{K}}(\mathscr{N}^r(\mathbf{x}))dV\,\delta$$

or

$$\delta = \mathscr{S}\mathbf{F} \tag{I-12}$$

where
$$\mathscr{S}^{-1} = \int_V \mathscr{N}^{r^T}(\mathbf{x})\widetilde{\mathscr{K}}(\mathscr{N}^r(\mathbf{x}))dV$$

is a constant stiffness matrix. The flexibility matrix, $\mathscr{S}$, of equation (I-12) describes the flexibility characteristics of a structure with s_1 degrees of freedom which for loadings that are wholly describable in terms of the distributed loads which are the elements of $\tilde{\mathscr{K}}(\mathscr{N}^r(\mathbf{x}))$ responds as the continuum would to that loading. The reduced order system will respond to any general loading $\mathbf{p}(\mathbf{x})$, which gives rise to generalised forces in the degrees of freedom of that system. How closely the deformation of the reduced order system resembles that of the continuum depends upon the particular loading $\mathbf{p}(\mathbf{x})$. In general, for any loading $\mathbf{p}(\mathbf{x})$, which would induce in the continuum a deformation which is of higher order or of greater complexity than that represented by equation (I-10), the resulting deformations of the reduced order system and the continuum are likely to be very different. Therefore in selecting an approximative reduced order system in a particular analysis due consideration should be given to the loading environment to which the system is to be subjected and whether in those circumstances the resulting deformations are likely to be of acceptable accuracy.

The response of the reduced order system to an arbitrary loading $\mathbf{p}(\mathbf{x})$ can be found via the functions $\mathscr{N}^r(\mathbf{x})$ since the generalised forces $\mathbf{F}_p$ in these modes resulting from the loading $\mathbf{p}(\mathbf{x})$ are, based on the principle of virtual work, given by

$$\mathbf{F}_p = \int_V \mathscr{N}^{r^T}(\mathbf{x})\mathbf{p}(\mathbf{x})dV \quad . \qquad \text{(I-13)}$$

$\mathbf{F}_p$ represents those components of $\mathbf{p}(\mathbf{x})$ which give rise to a deformation, of the reduced order system, which is characterised by

$$\delta_p = \mathscr{G}\mathbf{F}_p \quad . \qquad \text{(I-14)}$$

The deformation throughout the system is given by

$$\mathbf{q}_p^r(\mathbf{x}) = \mathscr{N}^r(\mathbf{x})\delta_p \quad . \qquad \text{(I-15)}$$

Equations (I-13) and (I-15) are termed collectively the interface equations since they enable us to find the displacement of the reduced order system, at any point, due to an arbitrary loading via the discrete description $\mathscr{G}$ of the flexibility characteristics. Combining equations (I-15), (I-14) and (I-13) we may write

$$\mathbf{q}_p^r(\mathbf{x}) = \int_V \mathscr{N}^r(\mathbf{x})\mathscr{G}\mathscr{N}^{r^T}(\xi)\mathbf{p}(\xi)dV \quad .$$

On comparison with equation (I-7) we identify a structural influence function tensor for the reduced order system as

$$\Gamma^r(x;\xi) = \mathcal{N}^r(x)\mathcal{S}\mathcal{N}^{r^T}(\xi) \quad . \tag{I-16}$$

For the reduced order system equations (I-9a) and (I-9b) take the form

$$q_0^r(x) = q_{d_0}^r(x) + \int_V \Gamma^r(x;\xi)\tilde{\mathcal{A}}(q_0^r(\xi))dV \tag{I-17a}$$

and

$$Q_0^r(x) = Q_{d_0}^r(x) + \tilde{\mathcal{A}}\left(\int_V \Gamma^r(x;\xi)Q_0^r(\xi)dV\right) \quad . \tag{I-17b}$$

Equations (I-17a) and (I-17b) appear as intractable as did (I-9a) and (I-9b) but a solution of the former can be easily obtained via the solution of a set of simultaneous equations for either $\mathbb{F}_0$ or δ_0. The equation in $\mathbb{F}_0$ is found by premultiplying (I-17b) by $\mathcal{N}^{r^T}(x)$ and integrating over the volume as

$$F_0 = F_{d_0} + \int_V \mathcal{N}^{r^T}(x)\,\tilde{\mathcal{A}}\,[\mathcal{N}^r(x)\,\mathcal{S}F_0]dV \quad . \tag{I-17c}$$

Premultiplication of (I-17c) by $\mathcal{S}$ gives

$$\delta_0 = \delta_{d_0} + \int_V \mathcal{S}\mathcal{N}^{r^T}(x)\,\tilde{\mathcal{A}}\,[\mathcal{N}^r(x)\delta_0]dV \quad . \tag{I-17d}$$

If equation (I-17c) is solved for F_0 then δ_0 is found from

$$\delta_0 = \mathcal{S}F_0 \quad .$$

The deformation $q_0^r(x)$ is found via equation (I-10)

$$q_0^r(x) = \mathcal{N}^r(x)\delta_0 \quad .$$

The generalised forces F_0 represent those loads, $Q_0^r(x)$, which occasion deformation of the reduced order system. An estimate of the loading on the continuum, $Q_0(x)$, is given by $Q_0^r(x)$ plus the loading $(Q_{d_0}(x) - Q_{d_0}^r(x)$ which does no work on the reduced order system, *ie*

$$Q_0(x) = Q_{d_0}(x) + \tilde{\mathscr{A}}(\mathscr{N}^r(x)\delta_0) .$$

The precise form of the operator $\tilde{\mathscr{A}}()$ of equations (I-17) is governed by the chosen description of the aerodynamic loading. Aerodynamic theories are formulated in terms of incidence distributions over the surface of the body or directly in terms of the surface ordinates. In the above development the operand of $\tilde{\mathscr{A}}$ has been assumed to be $\ddot{q}_0(x)$ but other possible operands are readily deducible from it since $q_0(x)$ is everywhere defined. The particular description of the aerodynamic load will likewise depend upon the chosen theory but for the purposes of the above development, it has been assumed to be a loading distributed throughout the volume and to be everywhere defined. Surface pressure distributions, discrete loads etc are merely especial forms of the aforementioned loading distribution.

It is evident from the above that the functions $\mathscr{N}^r(x)$ afford the realization of a ready interface between an arbitrary loading distribution; a discrete description of the flexibility characteristics, $\mathscr{S}$; and a deformation pattern which is everywhere defined. However, for an assumed set of distinct loadings, $\mathscr{S}$ and its associated functions $\mathscr{N}^r(x)$ for the reduced order system cannot, in general, be obtained precisely via the structural properties of the continuum. Nonetheless, a flexibility matrix S is often calculable under certain assumptions about the way in which the structure deforms under load; and for a known system of restraints on the idealised structure relates generalised coordinates and corresponding generalised forces. If S is to be used in an aeroelastic analysis in place of $\mathscr{S}$ the constrained deformation pattern used in the derivation of S should, logically, be used in the specification of functions akin to $\mathscr{N}^r(x)$ above. If, for a given matrix S, the compatible deformation pattern is unknown, then the aeroelastician must furnish a deformation pattern such that his analysis based on S is one of acceptable accuracy. The analyst also finds himself in a similar situation when the given flexibility matrix has been determined experimentally. In the next section, consideration will be given to the specification of the constrained deformation pattern in terms of the known flexibility characteristics.

I.3 THE GENERAL INTERFACE

I.3.1 Preliminaries

It is assumed that there exists a symmetric flexibility matrix $\mathbf{S}$ which for a known system of restraints* on the structure relates s_1 generalised coordinates, δ, and corresponding generalised forces $\mathbf{F}$ by

$$\delta = \mathbf{SF} \quad . \tag{I-18}$$

Further it is assumed that $\mathbf{S}$ is referred to a set of points Σ_1 and that an appropriate system of constraints is not specified. We assume that there is a domain of the original structure for which the displacement of any point in that domain, under a concentrated load at any point in that domain, is little different from the displacement of the semi-rigid structure under the same loading. This domain is in some sense characterised by the σ_1 points of Σ_1. A constraint system can then be formulated solely on the basis of the given information. For a number of reasons it was decided to set up the constraint system on a local basis - that is the constraints are to be constructed piecemeal throughout the structure. This enables the analyst, basic data permitting, to tailor his assumed deformation characteristics in a manner appropriate to a given problem. A global form of constraint is not thereby excluded since it can be accommodated by assuming that the 'localness' extends over the whole structure. However, techniques which involve high order polynomial fits to deformation data can lead to a rapidly varying deformation pattern between the data points. Within the main body of points this effect can be controlled by careful selection of the polynomial but the predicted deformation can still exhibit undesirable characteristics in areas adjoining but outside the main block of data points. This is particularly relevant when only experimentally determined flexibility data is available, since it may be difficult to obtain data at the extremities of the structure. Thus by adopting a constraint system formulated on a local basis the analyst can exercise control over the deformation in such areas.

I.3.2 Local constraint functions

Suppose that the domain of Σ_1 is divided into a finite number of regions, say R, the ith region, R_i being characterised by n_i $(i = 1,2,\ldots,R)$ points of Σ_1 and the generalised displacements associated with the n_i points of R_i are

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* To avoid ambiguity 'restraints' are those forces and moments that prevent movement of the body as a whole, whereas 'constraints' are forces and moments that in some sense restrict movement of one point of the structure relative to another.

denoted by δ_j^i , $j = 1, 2, \ldots, t_i$ where $t_i \geqslant n_i$, or in matrix notation δ^i . (Note that δ^i is a subset of δ .) It is assumed that the constraints are formulated for each region in terms of functions $N^i(x)$ such that the state of displacement at any point of R_i , $\Delta^i(x)$, whose components are the displacements in the coordinate directions* may be written in terms of δ^i in the matrix form

$$\Delta^i(x) = N^i(x)\delta^i \quad . \tag{I-19}$$

In general, $N^i(x)$ is a matrix of order $3 \times t_i$ where $3 \leqslant t_i$, the elements of which are functions of the position vectors of the n_i points of the ith region in addition to the general position vector, x .

In general, the particular functions $N^i(x)$ chosen to describe the state of displacement over R_i will depend upon the number and type of deformation parameters associated with that region. Thus in general the particular choice of $N^i(x)$ will depend upon the type of flexibility information available and on the particular regional structure chosen.

$\Delta_j^i(x)$, a component of $\Delta^i(x)$, is given by the inner product of the jth row of $N^i(x)$ with δ^i and it is assumed that

$$\left.\begin{aligned}
\Delta_1^i(x) &= \sum_{k=1}^{k_1} x_1^{\alpha_k} x_2^{\beta_k} x_3^{\gamma_k} a_k^i \\
\Delta_2^i(x) &= \sum_{k=k_1+1}^{k_1+k_2} x_1^{\alpha_k} x_2^{\beta_k} x_3^{\gamma_k} a_k^i \\
\Delta_3^i(x) &= \sum_{k=k_1+k_2+1}^{k_1+k_2+k_3} x_1^{\alpha_k} x_2^{\beta_k} x_3^{\gamma_k} a_k^i
\end{aligned}\right\} \tag{I-20}$$

where α_k, β_k and γ_k are to be interpreted as the powers of x_1, x_2 and x_3 ,
a_k^i , $k = 1, \ldots, t_i$ respectively, are constants,

and k_1, k_2, k_3 are integers such that $t_i = k_1 + k_2 + k_3$.

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* In the following development displacements in all three coordinate directions are allowed, and further it is assumed that each δ^i contains relevant generalised displacements.

Collectively we may write the assumptions regarding the form of $\Delta^i(x)$ as

$$\Delta^i(x) = H(x)a^i \qquad \text{(I-21)}$$

where non-zero elements of $H(x)$ are some polynomial functions of x_1, x_2 and x_3 and a^i is a vector of t_i elements which are constants as yet unknown.

Clearly equation (I-21) can be written in partitioned form, viz:

$$\begin{Bmatrix} \Delta_1^i(x) \\ \Delta_2^i(x) \\ \Delta_3^i(x) \end{Bmatrix} = \begin{pmatrix} h_1(x) & o_{k_2} & o_{k_3} \\ o_{k_1} & h_2(x) & o_{k_3} \\ o_{k_1} & o_{k_2} & h_3(x) \end{pmatrix} \begin{Bmatrix} a_1^i \\ a_2^i \\ a_3^i \end{Bmatrix} \qquad \text{(I-21a)}$$

where o_u is a null row vector of u elements

k_j is the number of terms in the algebraic expression for Δ_j^i ,

and a_j^i is the vector of constants associated with Δ_j^i .

The constants a^i can be determined from the requirement that the chosen description of the state of displacement over R_i (equation (I-21)) shall yield the given δ^i at the characteristic points of that region. Now, in general δ_j^i is given by

$$\delta_j^i = \tilde{\mathscr{G}}_j(\Delta^i(x))\big|_{x(\delta_j^i)} \qquad \text{(I-22)}$$

where $\tilde{\mathscr{G}}_j$ is an appropriate linear operator,

and $x(\delta_j^i)$ is the position vector of the point with which δ_j^i is associated.

By use of equation (I-21), equation (I-22) may be written as

$$\delta_j^i = \tilde{\mathscr{G}}_j(H(x)a^i)\big|_{x(\delta_j^i)}$$

or since $\tilde{\mathscr{G}}_j$ is a linear operator

$$\delta_j^i = \tilde{\mathscr{G}}_j(H(x))\big|_{x(\delta_j^i)}a^i \quad . \qquad \text{(I-23)}$$

Expressions for all δ_j^i, $j = 1, 2, \ldots, t_i$, typified by equation (I-23) may be written collectively as the matrix equation

$$\delta^i = G^i a^i \quad . \qquad \text{(I-24)}$$

The matrix G^i is square and under the assumption that the distribution of the points associated with the region and the algebraic expressions describing the state of displacement over the region are such that the inverse of G^i exists, we may write

$$a^i = G^{i^{-1}} \delta^i \quad . \tag{I-25}$$

From equations (I-21) and (I-25) we may write

$$\Delta^i(x) = H(x) G^{i^{-1}} \delta^i \quad .$$

Reference to equation (I-19) shows that $H(x)G^{i^{-1}}$ is to be identified with $N^i(x)$, ie

$$N^i(x) = H(x) G^{i^{-1}} \quad . \tag{I-26}$$

The above procedure is applied to each of the R regions so that the state of displacement at all points of the structure is prescribed.

I.3.3 The global interface

The global interface is a relationship between the state of displacement at any part of the structure, $\Delta(\xi)$, and $p(x)$. This can be written in terms of the local quantities of section I.3.2 as follows. A function $\kappa^i(x)$ is defined such that

$$\kappa^i(x) = \begin{cases} 1 & \text{if } x \text{ is associated with region } R_i \\ 0 & \text{otherwise} \end{cases}$$

whence

$$\Delta(\xi) = \sum_{i=1}^{R} \kappa^i(\xi) \Delta^i(\xi) \quad . \tag{I-27}$$

$\Delta^i(\xi)$ is given by equation (I-19) wherein since δ^i is a subset of δ , $\Delta(\xi)$ may be written as

$$\Delta(\xi) = \sum_{i=1}^{R} \kappa^i(\xi) N^i(\xi) A^i \delta$$

where A^i is a matrix of order $(t_i \times s_1)$ whose elements are such that if δ_k is the jth deformation parameter of region R_i then $A^i_{jk} = 1$, otherwise $A^i_{jk} = 0$.

Alternatively, if

$$\mathbf{N}(\xi) = \sum_{i=1}^{R} \kappa^{i}(\xi)\mathbf{N}^{i}(\xi)\mathbf{A}^{i} \qquad \text{(I-28)}$$

then

$$\Delta(\xi) = \mathbf{N}(\xi)\delta \qquad \text{(I-29)}$$

which is the global equivalent of equation (I-19).

Following section I.2 we write

$$\mathbf{F}_p = \int_V \mathbf{N}^T(\mathbf{x})\mathbf{p}(\mathbf{x})dV \qquad \text{(I-30)}$$

and on combining equations (I-29), (I-18) and (I-30) we have

$$\Delta(\xi) = \int_V \mathbf{N}(\xi)\mathbf{S}\mathbf{N}^T(\mathbf{x})\mathbf{p}(\mathbf{x})dV \qquad \text{(I-31)}$$

which describes the deformation of the structure at any point for an arbitrary loading via a discrete description $\mathbf{S}$ of its flexibility characteristics.

I.4 CONSEQUENCES OF THE REGIONAL SPECIFICATION OF DEFORMATION

By virtue of the piecewise specification of $\mathbf{N}(\mathbf{x})$, the generalised forces appropriate to a general loading are also apportioned on a regional basis. This is readily seen by substituting equation (I-28) for $\mathbf{N}(\mathbf{x})$ in equation (I-30) to give

$$\mathbf{F}_p = \int_V \sum_{i=1}^{R} \mathbf{A}^{i^T} \mathbf{N}^{i^T}(\mathbf{x})\kappa^{i}(\mathbf{x})\mathbf{p}(\mathbf{x})dV \quad .$$

Now corresponding to equation (I-27) we have

$$\mathbf{p}(\mathbf{x}) = \sum_{i=1}^{R} \kappa^{i}(\mathbf{x})\mathbf{p}^{i}(\mathbf{x})$$

where $p^i(x)$ is that portion of $p(x)$ acting over the region R_i , ie

$$p^i(x) = \kappa^i(x)p(x)$$

so that the above expression for F_p can be written

$$F_p = \sum_{i=1}^{R} A^{i^T} \int_{R_i} N^{i^T}(x)p^i(x)dV \quad .$$

We identify

$$F^i = \int_{R_i} N^{i^T}(x)p^i(x)dV \qquad \text{(I-32)}$$

as contributions, from the region R_i , to the generalised forces in the degrees of freedom δ^i , acting at the n_i characteristic points of the region.

By virtue of equation (I-26), equation (I-32) can be written as

$$F^i = \int_{R_i} G^{i^{T^{-1}}} H^T(x)p^i(x)dV \quad . \qquad \text{(I-33)}$$

If, for R_i , a displacement in the Ox_r-direction is among the generalised coordinates, δ^i , by a judicious choice of the functions $H(x)$ it is relatively simple to ensure that both F^i and $p^i(x)$ yield the same total load in the coordinate directions. A sufficient condition for equality of loads in the Ox_r-direction from the two loading systems F^i and $p^i(x)$ is that $h_r(x)$ shall contain an element having the value unity. This is easily demonstrated by assuming, without loss of generality, a particular ordering of the elements of δ^i and F^i viz

$$\delta^i = \begin{Bmatrix} w_1^i \\ w_2^i \\ w_3^i \\ \begin{Bmatrix} \text{parameters of} \\ \text{derivative type} \end{Bmatrix} \end{Bmatrix} \qquad \text{(I-34a)}$$

and

$$F^i = \begin{Bmatrix} z_1^i \\ z_2^i \\ z_3^i \\ \{\text{moments etc}\} \end{Bmatrix} \qquad \text{(I-34b)}$$

where w_r^i, z_r^i are vectors having ℓ_r elements of linear displacements, forces in the Ox_r-direction at ℓ_r distinct characteristic points of R_i. If each $h_r(x)$ has an element unity then for some arrangement of the elements of a^i, it is possible to write $H(x)$ in the partitioned form

$$H(x) = [I_3 \,\vdots\, H_{12}(x)] \quad . \qquad \text{(I-35)}$$

In addition, by virtue of the assumed ordering of δ^i, (equation (I-34a)), G^i may be partitioned

$$G^i = \left[U^{iT} \,\vdots\, G_{12}^i \right] \qquad \text{(I-36)}$$

where $U^i = \begin{bmatrix} J_{\ell_1}^T & o_{\ell_2} & o_{\ell_3} & o_p \\ o_{\ell_1} & J_{\ell_2}^T & o_{\ell_3} & o_p \\ o_{\ell_1} & o_{\ell_2} & J_{\ell_3}^T & o_p \end{bmatrix}$

with J_k defined as a column vector of k elements each equal to unity and $p = t_i - \sum_j \ell_j$.

Equation (I-33) can be written as

$$G^{iT} F^i = \int_{R_i} H^T(x) p^i(x) dV \qquad \text{(I-37)}$$

which by virtue of equations (I-35) and (I-36) yields

$$U^i F^i = \int_{R_i} p^i(x) dV$$

which shows that the total loads in the coordinate directions for the loading systems F^i and $p^i(x)$ are the same. In the general three-dimensional case, it is not possible to ensure, for the chosen form of $H(x)$, by the mere inclusion of certain terms in the algebraic expressions for $\Delta^i_j(x)$, and that the total moments about any line of the two loading systems F^i and $p^i(x)$ are equal. However, in problems where loads in one coordinate direction only need be considered, there exist forms of $\Delta^i_j(x)$ that are sufficient to ensure conservation of moments for the loading systems F^i and $p^i(x)$. This is demonstrated in Part II of this Report in which such a problem is considered.

If the flexibility information and the chosen regional structure are such that the above procedure can be followed for all regions then F_p effectively contains a redistribution of the loading $p(x)$ among the set of points Σ_1. A direct consequence of this is that the solution of the equation of aeroelastic equilibrium written in terms of the generalised forces (equation (I-17c)) yields total loading information directly.

I.5 DISCUSSION

A systematic method of utilising structural information related to a particular set of points to calculate the deformed shape of the structure under an arbitrary loading has been formulated. The method involves dividing the structure into a number of regions over which the displacement is approximated by polynomial functions of position. This method was preferred to the use of a high-order polynomial fitted over all the data points since it affords a certain freedom for the analyst to tailor the deformation characteristics to suit the particular problem. One can also cope (obviously to a limited extent) with the prediction of structural information at points which lie outside the main body of points with respect to which the available flexibility information is defined, in a controlled and readily quantifiable manner. The distributed loading is apportioned among the generalised forces of the given flexibility matrix by invoking the principle of virtual work. The material in this Report provides the basic framework within which any particular interface between structural and loading data can be effected.

The degree of complexity necessary in the regional structure is of course problem-dependent. As far as static aeroelastic problems are concerned the quality and/or applicability of other data in the whole problem-solving process is likely to be such that fairly simple regions and displacement functions can be employed with a good expectation of an acceptable result. This is particularly so in view of the likely deformed shape under load. In Part II we examine the procedure advocated when particular simplifying assumptions are made with regard to the determination of aeroelastic equilbrium.

Part II

THE INTERFACE FOR PLATE-LIKE STRUCTURES SUBJECT TO TRANSVERSE LOAD

(The precise meaning of the notation of Part I in the context of Part II and the definitions of other symbols relevant to this part of the report are given in the List of Symbols on page 64.)

II.1 A PARTICULAR SIMPLIFICATION OF THE EQUATION OF AEROELASTIC EQUILIBRIUM

II.1.1 The structural characteristics

The elastic characteristics of the structure are referred to a rectangular body-attached axis system $Ox_1x_2x_3$, whose origin 0 is a definite material point of the body, and the simplifying assumptions considered here are that the loading acts in one direction, say Ox_3 , and that only the deformation in that direction need be considered. Further it is assumed that the deformation in the Ox_3 direction is a function of the (x_1,x_2) coordinates only.

This idealised system may be regarded as a plate whose mid-plane, when the plate is unloaded, lies in the Ox_1x_2 plane. We postulate that, for a known system of restraints on the plate, there exists a flexibility matrix (*cf* equation (I-18)) which relates the displacement, at a set of points Σ_1 of the mid-plane of the plate, in the direction Ox_3 , to discrete loads in the same direction via the matrix equation*

$$\mathbf{w}_{\Sigma_1} = \mathbf{S}_{11}\mathbf{L}_{\Sigma_1} \qquad \text{(II-1)}$$

where $\mathbf{w}_{\Sigma_1}$ is a column vector of displacements in the Ox_3 direction at σ_1 points of the Ox_1x_2 plane which constitute the set Σ_1

and $\mathbf{L}_{\Sigma_1}$ is a column vector of discrete loads in the Ox_3 direction applied at the points of Σ_1 .

II.1.2 The aerodynamic loading

We here consider those descriptions of the aeroelastic problem in which the theory used to predict the aerodynamic loading on the vehicle is cast in terms of either:

(i) displacements in the Ox_3 direction at a set of points Σ_2 , which will be denoted $\mathbf{w}_{\Sigma_2}$

or

(ii) spatial derivatives of displacements at a set of points Σ_2 , which will be denoted $\mathbf{w}^{I}_{\Sigma_2}$.

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* To be consistent with Part I $\mathbf{w}$ and $\mathbf{L}$ should have a subscript 3, but here since only loads and deflections in the Ox_3 direction are considered the subscript can be dropped without ambiguity. It is however convenient to introduce a subscript to denote the set of points with respect to which a vector (or matrix) is defined.

It is further assumed that the loading is described by a set of discrete loads, in the direction Ox_3 , applied at a set of points Σ_3 . For case (i) we have, in the notation of Part I,

$$\tilde{\mathscr{A}}(w(\xi)) = \sum_{i=1}^{\sigma_3} \delta(\xi - \xi_i) Z_i^a \tag{II-2}$$

where Z_i^a is an element of the vector $\mathbf{Z}_{\Sigma_3}^a$ of aerodynamic loads due to the deformation $w(\xi)$,

ξ denotes the coordinates (ξ_1, ξ_2) of a general point,

and ξ_i defines the position of the ith point of Σ_3 .

If the aerodynamic load calculation is described by a process $\mathscr{P}$ we may write symbolically*

$$Z_{\Sigma_3}^a = \mathscr{P}(w_{\Sigma_2}) \quad . \tag{II-3}$$

II.1.3 <u>The simplified equation of aeroelastic equilibrium</u>

Under the assumptions made in sections II.1.1 and II.1.2 an equation of aeroelastic equilibrium for the idealised structure may be written** (*cf* equation (I-17a))

$$w_0^r(\mathbf{x}) = w_{d_0}^r(\mathbf{x}) + \int_S \Gamma^r(\mathbf{x};\xi)\tilde{\mathscr{A}}(w_0^r(\xi))dS \tag{II-4}$$

where $w_0^r(\mathbf{x})$ is the total deformation of the plate in the direction Ox_3 at the point (x_1, x_2)

$\Gamma^r(\mathbf{x};\xi)$ is the derived influence function tensor and is to be calculated from $\mathbf{S}_{11}$ in the form (*cf* equation (I-31))

$$\Gamma^r(\mathbf{x};\xi) = \mathbf{N}(\mathbf{x})\mathbf{S}_{11}\mathbf{N}^T(\xi)$$

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* The equation for case (ii) is similar, with a superscript I attached to w_{Σ_2} .

** The development presented first assumes the aerodynamic load to be a function of w_{Σ_2} . The other case is considered later.

where $\mathbf{N}(x)$ is a matrix of order $(1 \times \sigma_1)$,
and $w^r_{d_0}(x)$ is the deformation due to a non-aerodynamic load $Z_{d_0}(x)$ viz

$$w^r_{d_0}(x) = \int_S \Gamma^r(x;\xi) Z_{d_0}(\xi) dS$$

$$= \mathbf{N}(x)\mathbf{S}_{11} \int_S \mathbf{N}^T(\xi) Z_{d_0}(\xi) dS$$

$$= \mathbf{N}(x)\mathbf{S}_{11}\mathbf{Z}_{d_{0_{\Sigma_1}}} \tag{II-5}$$

where $\mathbf{Z}_{d_{0_{\Sigma_1}}}$ is a set of discrete loads acting at the points of Σ_1, appropriate to $Z_{d_0}(x)$.

Substituting equation (II-2) into equation (II-4) and integrating, we obtain

$$w^r_0(x) = w^r_{d_0}(x) + \mathbf{N}(x)\mathbf{S}_{11}\mathbf{N}^T_{31}\mathbf{Z}^a_{0_{\Sigma_3}} \tag{II-6}$$

where $\mathbf{N}_{31}$ is a matrix whose ith row is the vector $\mathbf{N}(\xi)$, evaluated at the ith point of Σ_3.

Using equation (II-3), equation (II-6) may be written in two distinct discrete forms, one in terms of the discrete total displacements $\mathbf{w}_{0_{\Sigma_2}}$, and the other in terms of $\mathbf{w}_{0_{\Sigma_1}}$ viz:

$$\mathbf{w}_{0_{\Sigma_2}} = \mathbf{w}_{d_{0_{\Sigma_2}}} + \mathbf{N}_{21}\mathbf{S}_{11}\mathbf{N}^T_{31}\mathscr{P}\left(\mathbf{w}_{0_{\Sigma_2}}\right) \tag{II-7a}$$

or, by virtue of equation (I-29)

$$\mathbf{w}_{0_{\Sigma_1}} = \mathbf{w}_{d_{0_{\Sigma_1}}} + \mathbf{S}_{11}\mathbf{N}^T_{31}\mathscr{P}\left(\mathbf{N}_{21}\mathbf{w}_{0_{\Sigma_1}}\right) \tag{II-7b}$$

which corresponds to equation (I-17d).

The formulation (II-7a) is a more attractive form of the aeroelastic equilibrium equation since the interface problem is divorced from the aerodynamic solution process, whereas in (II-7b) it is an integral part of the solution.

The equation of aeroelastic equilibrium may alternatively be cast in terms of discrete forces. Substituting for $w_{0_{\Sigma_2}}$ from equation (II-7a) in equation (II-3) gives

$$z^a_{0_{\Sigma_3}} = \mathscr{P}\left(w_{d_{0_{\Sigma_2}}} + N_{21}S_{11}N^T_{31}z^a_{0_{\Sigma_3}}\right) \qquad \text{(II-7c)}$$

which is a convenient representation if the process $\mathscr{P}$ is an iterative one. The total load on the flexible vehicle is in this case given by

$$Z_0(x) = Z_{d_0}(x) + \sum_{i=1}^{\sigma_3} \delta(x - x_i)Z^a_{0_i} \quad .$$

From equation (II-5) we may write

$$w_{d_{0_{\Sigma_2}}} = N_{21}S_{11}z_{d_{0_{\Sigma_1}}}$$

and substituting this expression in equation (II-7c)

$$z^a_{0_{\Sigma_3}} = \mathscr{P}\left(N_{21}S_{11}\left(z_{d_{0_{\Sigma_1}}} + N^T_{31}z^a_{0_{\Sigma_3}}\right)\right) \quad .$$

Pre-multiplying by N^T_{31}, and recognising that $z^a_{0_{\Sigma_1}} = N^T_{31}z^a_{0_{\Sigma_3}}$ gives

$$z^a_{0_{\Sigma_1}} = N^T_{31}\mathscr{P}\left(N_{21}S_{11}\left(z_{d_{0_{\Sigma_1}}} + z^a_{0_{\Sigma_1}}\right)\right) \quad .$$

The discrete forces acting at the points of Σ_1 which are equivalent to the total loading on the body are given by

$$z_{0_{\Sigma_1}} = z_{d_{0_{\Sigma_1}}} + z^a_{0_{\Sigma_1}}$$

and hence we have

$$z_{0_{\Sigma_1}} = z_{d_{0_{\Sigma_1}}} + N_{31}^T \mathscr{P}\left(N_{21} S_{11} z_{0_{\Sigma_1}}\right) \tag{II-7d}$$

which corresponds to equation (I-17c).

If the aerodynamic load calculation is instead defined in terms of $w_{\Sigma_2}^I$ equation (II-3) becomes

$$z_{\Sigma_3}^a = \mathscr{P}\left(w_{\Sigma_2}^I\right) \quad . \tag{II-8}$$

and consequently equation (II-6) takes the form

$$w_0^r(x) = w_{d_0}^r(x) + N(x) S_{11} N_{31}^T \mathscr{P}\left(w_{0_{\Sigma_2}}^I\right) \quad . \tag{II-9}$$

Two alternate discrete forms of equation (II-9) can be written. If it is evaluated directly at the points of Σ_2 we have

$$w_{0_{\Sigma_2}} = w_{d_{0_{\Sigma_2}}} + N_{21} S_{11} N_{31}^T \mathscr{P}\left(w_{0_{\Sigma_2}}^I\right) \tag{II-10a}$$

which gives the deformation of the structure. If instead equation (II-9) is differentiated appropriately and then evaluated at the points of Σ_2 we have

$$w_{0_{\Sigma_2}}^I = w_{d_{0_{\Sigma_2}}}^I + N_{21}^I S_{11} N_{31}^T \mathscr{P}\left(w_{0_{\Sigma_2}}^I\right) \tag{II-10b}$$

which allows an iterative solution of the aeroelastic equilibrium equation in terms of $w_{0_{\Sigma_2}}^I$.

We may also use equation (II-9) and a differentiated form of equation (I-29) to write (*cf* equation (II-7b))

$$w_{0_{\Sigma_1}} = w_{d_{0_{\Sigma_1}}} + S_{11} N_{31}^T \mathscr{P}\left(N_{21}^I w_{0_{\Sigma_1}}\right) \quad . \tag{II-10c}$$

As with equations (II-7a) and (II-7b) it may be noted that in equation (II-10a) and (II-10b) the interface may be extracted in the form of a transformation of the flexibility information.

Manipulation of equations (II-10a), (II-10b) and (II-10c) using equations (II-5) and (II-8) leads to two further equations, corresponding to (II-7c) and (II-7d), but with aerodynamic load defined in terms of $w^I_{\Sigma_2}$ viz:

$$z^a_{0_{\Sigma_3}} = \mathscr{P}\left(w^I_{d_{0_{\Sigma_2}}} + N^I_{21} S_{11} N^T_{31} z^a_{0_{\Sigma_3}}\right) \qquad \text{(II-10d)}$$

and

$$z_{0_{\Sigma_1}} = z_{d_{0_{\Sigma_1}}} + N^T_{31} \mathscr{P}\left(N^I_{21} S_{11} z_{0_{\Sigma_1}}\right) \quad . \qquad \text{(II-10e)}$$

Again, in (II-10d) we can identify a transformation of flexibility information that represents the interface.

II.1.4 The form of the interface

From the development in section II.1.3 it is evident that the most general transformations of flexibility information likely to be encountered take the form

$$S_{23} = N_{21} S_{11} N^T_{31} \qquad \text{(II-11)}$$

and

$$S^I_{23} = N^I_{21} S_{11} N^T_{31} \quad . \qquad \text{(II-12)}$$

S_{23} and S^I_{23} relate certain discrete deformation characteristics - linear displacements in the case of S_{23} and spatial derivatives in the case of S^I_{23} - to discrete loads applied at the points of the set Σ_3 *. All other forms of the interface used in section II.1.3 involve one or more of the constituent matrices of the transformations in equations (II-11) and (II-12).

The kth row of N_{i1} is given by $N_{i1}(x_k)$ where x_k is the kth point of Σ_i . The derivation of the matrix $N(x)$ was discussed in general terms in Part I, section I.3, and is formally described by equations (I-26) and (I-28). In order to write the forms of these equations for the particular idealisation under consideration, it is useful to state relevant forms of equations used in

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* Although in section II.1.3 Σ_3 was specifically associated with loads of aerodynamic origin, it can in the present context be any set of points with respect to which a set of discrete loads is defined.

the development of $N(x)$. The polynomial expression detailing the displacement in the Ox_3 direction over region R_i is written (*cf* equation (I-21)*

$$w^i(x) = h(x)a \quad . \qquad \text{(II-13)}$$

Since the only deformation parameters involved are linear displacements the operator $\tilde{\mathscr{G}}$ of equation (I-22) is an identity operator and the matrix G^i of equation (I-24) can be written

$$G^i = \begin{bmatrix} h(x_1) \\ h(x_2) \\ \vdots \\ h(x_{t_i}) \end{bmatrix} \qquad \text{(II-14)}$$

and

$$N(x) = \sum_{j=1}^{R} \kappa^j(x) N^j(x) A^j$$

where $N^i(x) = h(x) G^{i^{-1}}$.

Therefore we may write the kth row of N_{i1} as

$$\text{kth row } (N_{i1}) = \sum_{j=1}^{R} \kappa^j(x_k) N^j(x_k) A^j \quad . \qquad \text{(II-15)}$$

In the following sections we shall concentrate on the forms of the interface given by equations (II-11) and (II-12) where N_{i1} is evaluated via equation (II-15).

How closely the deformation characteristics of the idealised structure are represented by S_{23} or S^I_{23} must be judged in the light of the proposed use of the derived information, and is best illustrated numerically. We shall return to this point in section II.5, but we must first devise some suitable means of performing the transformation.

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* Strictly h and a^i should have a subscript 3 attached to them but following the convention adopted elsewhere in this Part of the Report, the subscript is dropped.

One relevant observation can, however, be made here. The matrix S_{11} is, for a physically realisable structure, symmetric, and either positive definite or positive semi-definite. If following equation (II-11) we define a matrix

$$S_{22} = N_{21} S_{11} N_{21}^T$$

which purports to relate displacement and load at the set of points Σ_2 *, it is shown in Appendix A that S_{22} is also either positive semi-definite or positive definite, and therefore also relates to some realisable structure, though not necessarily the one to which S_{11} relates.

II.2 PHILOSOPHY OF COMPUTER IMPLEMENTATION

The above transformations involve, for any real problem, a considerable effort to effect, and it is desirable that the process should be implemented in some way on a digital computer. We must therefore decide how much of the work can reasonably be included in a computer program, and how much is best left to the analyst, whilst keeping in mind the limitations imposed by program size and complexity, ease of use, and particularly the capability to accommodate a wide variety of problems.

Initial examination indicates that implementation of the method can be divided into four separate parts:

1. Division of the domain of Σ_1 into smaller regions, each having a set of characteristic points and an associated function $h(x)$.
2. The association of each point of Σ_2 (and/or Σ_3) with one of these regions.
3. The determination of the individual $N^i(x)$ and hence the appropriate matrices N_{21}, N_{21}^I etc.
4. Matrix multiplication to derive the new flexibility data (equations (II-11), (II-12).

The computer program written in ICL 1900 Fortran to effect the transformations of flexibility information described in section II.1.4 has been christened ALFI (ALternative Flexibility Information) and the various general aspects of the implementation will now be discussed in greater detail.

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* This does not constitute a restriction, since if we consider Σ_2 to be in fact the agglomeration of Σ_2 and Σ_3 , the matrix S_{23} is obtainable from S_{22} by deleting various rows and columns.

The advent of finite element structural analysis programs has led to the development of 'mesh generation' techniques to perform automatically the subdivision of an arbitrary shape into smaller regions of specified type. It is possible that these could be adapted to perform the first part of the proposed implementation. The computer programs currently available tend, however, to be fairly large and complex, and the required modifications might well exacerbate this. Also this subdivision is an area in which the analyst needs considerable freedom of choice, so that, if necessary, especial features of the data can be dealt with in an appropriate way. For these reasons, it has been decided to leave the subdivision into what will hereafter be called the 'regional structure' outside the program.

We must, however, decide on a suitable framework within which the regional structure can be defined. For any region we must specify its extent, number of characteristic points, and the function $\mathbf{h}(x)$. These could, in theory, be specified by the analyst, but the associated computer implementation would be very complex. It is, though, relatively easy to offer the analyst the choice of a number of pre-programmed options. This latter course of action has been adopted, and will be discussed in section II.3.

The association of points with regions is another task that could be left outside the program, but it is possible to formulate it in a manner suitable for inclusion. To simplify this task we introduce the restriction that the agglomeration of all regions must form a simply connected domain in the Ox_1x_2 plane. Any point of the structure must lie within a region or on a region boundary, or be external to the regional structure. For simplicity, if a point of Σ_2 or Σ_3 lies on a common boundary between two regions, we will arbitrarily associate it with one of them. Anomalies may arise from our arbitrary choice of one region, and these aspects will be discussed in more detail in section II.3 for the options currently included in the program.

The calculation of the individual $\mathbf{N}^i(x)$, and assembly of matrices of the type $\mathbf{N}_{i1}$ is mainly algebraic manipulation. The form of the matrices $\mathbf{N}^I_{i1}$ must however be considered. Nominally these are obtained by differentiating the rows of $\mathbf{N}(x)$, which effectively means differentiation of the appropriate $\mathbf{N}^i(x)$. This could be achieved algebraically, but it is more convenient to employ a finite difference approximation. If two sets of points Σ_i^+ and Σ_i^- are defined, each point of Σ_i^+ and Σ_i^- being respectively a distance d either side of the corresponding point of Σ_i , and the corresponding matrices $\mathbf{N}^+_{i1}$ and $\mathbf{N}^-_{i1}$ calculated, then the required matrix is given by

$$N_{i1}^{I} = \frac{1}{2d}\left[N_{i1}^{+} - N_{i1}^{-}\right] \quad . \qquad \text{(II-16)}$$

The last part of the implementation is simple matrix algebra, and is easily included in a computer program. The development described above yields matrices N_{21} etc, that are sparse in character, and use is made of this fact to reduce program size and running time.

II.3 THE REGIONAL STRUCTURE INCORPORATED IN THE COMPUTER PROGRAM ALFI

II.3.1 Location and extent of a region

In any reference to a region we have so far only mentioned the points of Σ_1 that characterize the region, its spatial position being undefined. From the standpoint of computer implementation a simple method of defining the boundary of a region is to use a series of straight line segments joining some points of Σ_1 to form a closed loop; the points so joined for any one region can then be used as its characteristic points. Thus in ALFI a region R_i having n_i characteristic points will be specified by n_i points of Σ_1 , $P_1, P_2, \ldots, P_{n_i}$ say; and the boundary then taken to be the line segments $P_1P_2, P_2P_3, \ldots, P_{n_i-1}P_{n_i}, P_{n_i}P_1$. Note that the order in which the points are specified is important*. Since the coordinates of the points P_j are known, the region is located in space and defined in extent. For reasons that will be explained later* the individual regions must not overlap and the boundary line segments must not cross.

II.3.2 Choice of displacement approximation

In the mathematical development of section II.1.4 the function $\mathbf{h}(\mathbf{x})$ is used in conjunction with $\mathbf{a}^i$ to specify the transverse displacement over a region R_i . It has been assumed (in Part I) that $\mathbf{h}(\mathbf{x})$ has exactly n_i components. For a 3-point region we assume that the variation of transverse displacement over a region is given by

$$w(\mathbf{x}) = a_1 + a_2x_1 + a_3x_2 \qquad \text{(II-17)}$$

so that $\mathbf{h}(\mathbf{x})$ is the row vector $(1, x_1, x_2)$.

The use over a region of a displacement approximation with more terms in $\mathbf{h}(\mathbf{x})$ requires more characteristic points for that region. Included in ALFI are 4- and 6-point regions and for each region the simplest polynomial expressions

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* See section II.4.

with the appropriate number of terms in them are used to approximate the transverse displacement. Thus we have:

$$\left.\begin{aligned} \mathbf{h}(\mathbf{x}) &= (1,\ x_1,\ x_2) && \text{for the 3-point region} \\ &\quad (1,\ x_1,\ x_2,\ x_1x_2) && \text{for the 4-point region} \\ &\quad (1,\ x_1,\ x_2,\ x_1x_2,\ x_1^2,\ x_2^2) && \text{for the 6-point region.} \end{aligned}\right\} \quad \text{(II-18)}$$

They are the only three so far included in the computer program, and will for the remainder of this Report be denoted by the identifiers L3, Q4 and P6 respectively. This choice of $\mathbf{h}(\mathbf{x})$ is not entirely arbitrary for it preserves total transverse load on the structure and at its moments about Ox_1 and Ox_2. This follows immediately from the form of equation (I-37) in this particular application. The forms of $\mathbf{h}(\mathbf{x})$ given in equations (II-18) imply that $\mathbf{G}^i$ (equation (II-14)) and $\mathbf{H}(\mathbf{x})$ can be written as

$$\mathbf{G}^i = \left[\mathbf{J}_{n_i} \quad \mathbf{x}_1^i \quad \mathbf{x}_2^i \;\middle|\; \mathbf{C} \right]$$

where $\mathbf{J}_{n_i}$ is a column vector with n_i components, each of which is unity,

$\mathbf{x}_j^i$ is the column vector $(x_{j_1}^i, \ldots, x_{j_{n_i}}^i)$ where $(x_{1_k}^i, x_{2_k}^i)$ are the coordinates of the kth characteristic point of the region R_i

and $\mathbf{C}$ is a sub-matrix of $\mathbf{G}^i$ and does not exist if $n_i = 3$,

and

$$\mathbf{H}(\mathbf{x}) = \mathbf{h}(\mathbf{x}) = [1 \quad x_1 \quad x_2 \;|\; \mathbf{c}]$$

where $\mathbf{c}$ is a sub-matrix of $\mathbf{h}(\mathbf{x})$ and does not exist if $n_i = 3$. For a single load L_0 acting at the point $\mathbf{x}_0$ associated with region R_i, $\mathbf{p}^i(\mathbf{x})$ takes the form

$$\mathbf{p}^i(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0) L_0$$

whence equation (I-37) yields

$$\begin{bmatrix} \mathbf{J}_{n_i}^T \\ \\ \mathbf{x}_1^{i\,T} \\ \\ \mathbf{x}_2^{i\,T} \end{bmatrix} \mathbf{F}^i = \begin{bmatrix} L_0 \\ \\ x_{1_0} L_0 \\ \\ x_{2_0} L_0 \end{bmatrix}$$

which shows that the total local loads in the Ox_3-direction, and moments of those loads about Ox_1 and Ox_2 are the same for the loading systems F^i and L_0 . The free term in $h(x)$ ensures conservation of total load while the x_1 and x_2 terms ensure conservation of moments about Ox_2 and Ox_1 respectively. Similarly, inclusion of the x_1^2 and x_2^2 terms in $h(x)$ for P6 preserves the second moments about Ox_2 and Ox_1 . This preservation of load and moment is a physically attractive concept, and is the reason for adopting the three simple formulae of equation (II-18).

II.3.3 Continuity of the displacement characteristics across region boundaries

Ideally the displacement $w(x)$, and all its derivatives, should be continuous across a region boundary, so that if a point ξ of Σ_2 or Σ_3 lies on the common boundary of two regions R_i and R_j , then any related function may be derived from either $w^i(\xi)$ or $w^j(\xi)$, ie the point may arbitrarily be associated with one of the two regions. This situation does not, however, prevail.

Since the transformation is developed to calculate only displacement, or a spatial derivative of the displacement, a less restrictive requirement than the above one is that these two quantities shall be continuous across a region boundary. A consequence of using the finite difference formulation, equation (II-15), to calculate derivatives is that if the displacement is continuous, so is the spatial derivative. Here we require only displacement continuity.

For the simplest region incorporated in ALFI, the L3 type, we note that along any boundary line segment of such a region the variation of transverse displacement with distance along that line is linear, and is determined solely by its values at the end points. The transverse displacement $w(x)$ is therefore continuous across the boundary between any two such regions.

The other two region types, Q4 and P6, are included in ALFI specifically for the case where the points of Σ_1 are regularly distributed. (As is likely to be the case when using measured structural data) and for these regions displacement continuity is obtained only in special circumstances. For adjoining Q4 regions, the common boundary line must be parallel to Ox_1 or Ox_2 , and for P6 regions, the adjoining regions must have three common points which are collinear. If for Q4 and P6 regions these restrictions are not satisfied and the displacement is discontinuous, the spatial derivative must be calculated from one region only.

In these cases of discontinuity, anomalies arise from the arbitrary choice of region with which to associate a point on the boundary, and the existence of these must be indicated by the program. It is, however, unlikely that the overall result of any one particular transformation will be noticeably affected by a few such anomalies.

II.3.4 Degenerate interpolation

The discussion in sections II.1.4 and II.3.2 left undecided the problem of whether or not the matrix G^i is non-singular, and therefore invertible. We now return to this, for the three region types that are currently included in the computer program. G^i is obviously singular when there is spatial coincidence of two of the points defining the region, since there are then two identical rows in the matrix. Another cause of singularity is that which arises from a particular geometrical arrangement of the points defining the region. Consider as a simple example a region of type L3, defined by the points whose coordinates are $(x1_1, x2_1)$, $(x1_2, x2_2)$ and $(x1_3, x2_3)$. The matrix G^i is, from equations (II-14) and (II-18)

$$G^i = \begin{bmatrix} 1 & x1_1 & x2_1 \\ 1 & x1_2 & x2_2 \\ 1 & x1_3 & x2_3 \end{bmatrix}$$

which is singular if the determinant is zero. This is precisely the condition that the three points are collinear. Similar cases can be found for the other two region types. In general, the failure of the transformation method in these circumstances cannot be detected until, as in the example, the determinant is calculated. In most cases it can be avoided by changing the regional structure.

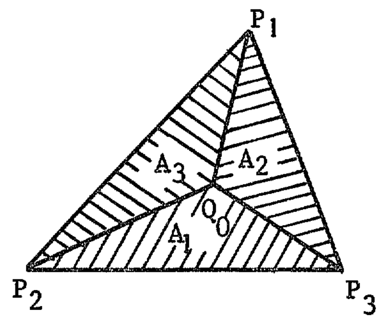
The reader may, however, find it helpful to visualise the physical circumstances associated with these singularities. They represent attempts to define a surface through a set of points when the disposition of the points renders any particular solution non-unique. Referring again to the example above, a multiplicity of planes may be defined if the only requirement is that they each pass through three collinear points.

II.4 ASSOCIATION OF POINTS WITH REGIONS IN ALFI

II.4.1 Algorithm for points interior to regions

The simplest situation is when a point of Σ_2 or Σ_3 lies inside a particular region. This region is obviously the one with which the point should be associated. We therefore need a method for determining whether or not a given point is within a region.

Consider first a triangle $P_1P_2P_3$, of area A , and a point, Q_0 , within the triangle. Given the magnitude of the areas A_1, A_2, A_3 shown in the sketch, the following relationships hold:



$A_1 + A_2 + A_3 = A$ if Q_0 is inside the triangle or on the boundary

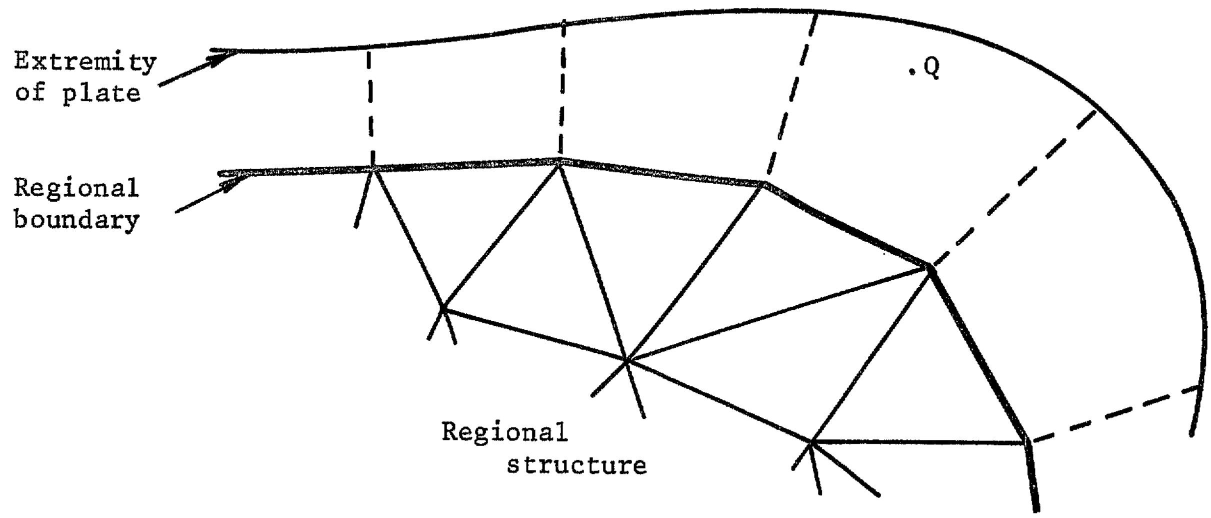
$A_1 + A_2 + A_3 > A$ if Q_0 is outside the triangle.

These may be used as a test to determine whether or not Q_0 is within the triangle $P_1P_2P_3$. This test may be applied immediately to an L3 region. The extension to Q4 and P6 regions is achieved by subdividing these more complex shapes into 2 and 4 simple triangles respectively (Figs 1a and 1b). By use of this technique all points of Σ_2 and Σ_3 , that lie within some region will be associated with that region. It does however introduce restrictions on the method of definition of the regional structure (section II.3.1). If the interior of a region is to be properly defined the lines used to subdivide it must lie within the region. This is impossible if the boundary line segments cross. Also certain geometrical arrangements of the characteristic points may lead to a violation of this condition (Fig 2a). The order in which the characteristic points are given is also important, as a different ordering may lead to the specification of a different interior (Fig 2b).

II.4.2 Algorithm for points outside any region

Occasion may arise when we require flexibility information for a point outside the regional structure. This eventuality can be accommodated if the point is associated with some region, and the displacement approximation over that region used for extrapolation, rather than interpolation as hitherto. The part of the structure outside all regions may be subdivided by the internal bisectors of the angles between pairs of successive line segments which define part of the boundary* of the regional structure (see following sketch).

* The method used in ALFI to determine which points of Σ_1 lie on the outline of the regional structure may fail if the regions overlap, and it is only for this reason that overlaps are prohibited (section II.3.1).



A point Q of either Σ_2 or Σ_3 lying between two such bisectors and external to the regional structure may then be associated with the region that has as part of its boundary the line joining the two bisectors. Effectively then those regions which may be said to provide a part of the boundary of the regional structure are extended towards the extremity of the plate. In certain circumstances the variation of transverse displacement across the boundary between two such extensions (the dashed lines of the sketch) is continuous but in general this is unlikely.

In principle a point such as Q may be an unlimited distance from the region with which it is associated, but intuition suggests that beyond some limit the results of the extrapolation might well be of little value. Logically, such limits should be linked in some way to the overall dimensions of the particular region chosen. This is achieved if the limits are established by examination of the numerical values of the elements of the appropriate $\mathbf{N}^i$. As an illustration we consider the region type L3.

For any point $\mathbf{x}$ associated with such a region we write $\mathbf{N}^i(\mathbf{x}) = (N_1^i, N_2^i, N_3^i)$ and define n_{max} and n_{min} to be the largest and smallest of the components of $\mathbf{N}^i(\mathbf{x})$. If the point $\mathbf{x}$ is inside, or on the boundary of, the region it can be shown that

$$0 \leqslant n_{min}, n_{max} \leqslant 1 \qquad \text{(II-22)}$$

and that the equality $n_{min} = 0$ applies if the point is on the boundary. Further if the point $\mathbf{x}$ is one of the three corner points $n_{max} = 1$. If the point (x) is outside the region, the inequality (II-22) no longer holds, and n_{max} and n_{min} increase and decrease respectively as the point is moved further away from the region. Numerical examination of the vector $\mathbf{N}^i(x)$ for the region types Q4 and P6 indicates that similar increase and decrease of n_{max} and n_{min} occurs*. Fig 3 depicts, for each region type, the limits on the range of extrapolation that are currently included in ALFI**.

II.5 EXAMPLE TRANSFORMATIONS USING ALFI

II.5.1 General remarks

The particular transformations discussed in the following sections have been effected using ALFI and have been artificially contrived in such a way that the accuracy of the transformation process in various circumstances may be assessed. We shall consider flexibility matrices appropriate to a flat constant-thickness cantilever plate which is skewed or swept at an angle of 45^o to the line of fixing. In contrast to the beam example of a previous Memorandum[3], there is no general analytic solution for the transverse displacement of such a plate under load and so it is necessary to resort to the use of numerical techniques in order to obtain the basic flexibility data. The flexibility characteristics of the plate were determined using an available finite element program[4], with a fairly large number of elements. The plate together with the finite element arrangement used are shown diagrammatically in Fig 4. To avoid a proliferation of numerical data we will restrict our attention to transformations of the type $\mathbf{S}_{22} = \mathbf{N}_{21}\mathbf{S}_{11}\mathbf{N}_{21}^T$. It is convenient to assume that a flexibility matrix relating transverse displacement and load, $\mathbf{S}_\Pi$, appropriate to the set of 325 points, Π , comprised of the vertices of the elemental triangles shown in Fig 4, has been calculated. With the fine elemental structure chosen, it is likely that the flexibility characteristics so calculated will be very close to the exact solution for the set of points Π and hereafter we will assume that the use of $\mathbf{S}_\Pi$ with a vector of discrete loads at the points Π will yield a 'true' displacement pattern over the plate for that loading. We can take as specific examples of original flexibility matrices (*cf* $\mathbf{S}_{11}$ of section II-2) those appropriate to various subsets Π_i of Π , extracting the matrices from $\mathbf{S}_\Pi$.

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* Here n_{max} and n_{min} refer to the largest and smallest components of the appropriate $\mathbf{N}^i(x)$, not to the vector appropriate to a region of type L3.

** These limits can, if necessary, be overridden.

In any particular application of the transformation process, the flexibility information contained within the original matrix has to some extent been misrepresented in the derived matrix. Whether or not the accuracy of any derived flexibility matrix is acceptable can only be judged in the context of the proposed use of the data. We will examine the displacement pattern obtained from a derived matrix for some representative loading. We choose to derive a matrix for a subset, Π_d , of Π so that $\mathbf{S}_{\Pi}$ may be used in conjunction with that same representative loading to calculate a 'true' displacement pattern, thus providing a yardstick against which the displacements obtained from our derived flexibility data may be judged. The loading, $\mathbf{L}_{\Pi_L}$, used is applied at 18 points (which collectively must be a subset of Π_d) on the plate whose disposition is shown in Fig 4. The loading data are given in Table 1. If we regard the plate as some idealisation of an aircraft wing, the loading pattern of Table 1 is typical of that expected in a discrete representation of subsonic aerodynamic load.

II.5.2 Derived flexibility matrices relating transverse displacement and load

II.5.2.1 Transformations involving interpolation only

From the matrix $\mathbf{S}_{\Pi_1}$ appropriate to the 45 points of Π_1 shown in Fig 5, we derive a matrix appropriate to the 36 points of Π_d which are also shown in Fig 5. The disposition of the points of Π_1 is typical of that which might be used in the calculation of the flexibility characteristics of the structure by the finite element method, in that information is available for points at the extremities of the structure. In such cases there is likely to be little difficulty in defining a regional structure which extends over the whole structure. This is indeed possible in our example and two alternative regional structures are detailed in Fig 5. Thus the transformation of flexibility data will involve interpolation only.

As regards the points of Π_d which lie on the boundaries of regions it should be noted that since, in the P6 regional structure (Fig 5), the adjoining regions have three common points which are collinear, the assumed variation of transverse displacement over each region (equations (II-13) and (II-18)) leads to a displacement pattern which is continuous across the region boundaries. Therefore it is immaterial with which of the adjoining regions a point on the boundary is associated. Matrices appropriate to Π_d were derived using the two alternative regional structures of Fig 5 and the displacements, at the points of Π_d , due to $\mathbf{L}_{\Pi_L}$ evaluated in each case. These displacements together with the corresponding 'true' displacements are tabulated in Table 2. The displacements calculated using the derived flexibility data are in good agreement with the 'true' displacements. The agreement is especially close when the matrix derived from the P6 regional structure is considered. In common with the findings in the beam example of Ref 3, such discrepancies as there are are larger in absolute terms for points closest to the root area. However as far as the loading $\mathbf{L}_{\Pi_L}$ is concerned either of the derived matrices yields results of acceptable accuracy.

II.5.2.2 Transformations involving some extrapolation

Three flexibility matrices appropriate to Π_d are derived from a matrix $\mathbf{S}_{\Pi_2}$ appropriate to the set of points Π_2 using the alternative regional structures shown in Fig 6. Basically there are 18 points of Π_2 (Fig 6) but in one instance these are augmented by three points at the station $x_2 = 0$ to make possible a regional structure consisting of P6 regions only. The spatial distribution of the points of Π_2 is typical of that which might be employed in an experimental determination of flexibility data. The regional structures based on Π_2 do not extend over the whole structure and data at points external to the regional structure must be produced by extrapolation. Following the ideas of section II.4.2, that part of the plate external to the regional structure has been divided to form the extensions to boundary regions illustrated in Fig.6 The precise regions with which points of Π_d external to the regional structure are associated is then obvious from an inspection of that figure.

The derived matrices appropriate to Π_d are used to calculate the transverse displacements due to the loading $\mathbf{L}_{\Pi_L}$ and these are given in Table 3. As regards the results for the L3 regional structure, the calculated displacements are again in good agreement with the 'true' solution with the exception of that for the point (3,2) labelled Q in Fig 6. This is a direct consequence of the point Q having been associated with the region R_2 whereas perhaps an association with R_1 would have been more appropriate since this would have involved extrapolation in the Ox_1 direction utilizing displacement data at the points P_1 and P_2 (Fig 6). This type of physical argument might well be useful in the selection of a suitable regional structure for any given problem. For instance if we adopt the Q4 regional structure of Fig 6, extrapolation to the points external to the regional structure is always in the chordwise sense. This is a logical procedure to adopt given our particular arrangement of points of Π_2 . The Q4 regional structure does in fact lead to transverse displacements, due to $\mathbf{L}_{\Pi_L}$, at points in the root and tip areas, which agree more closely with the 'true' results than those obtained using the L3 regional structure (Table 3).

As regards the P6 regional structure the displacements, due to L_{Π_L} (see Table 3) are in general very close to the 'true' results. However we note that the calculated displacement varies quite rapidly in the vicinity of the points external to the regional structure which suggests that when extrapolation is involved in the transformation process, the results should be critically examined to ensure that the extrapolation has not been carried to extremes.

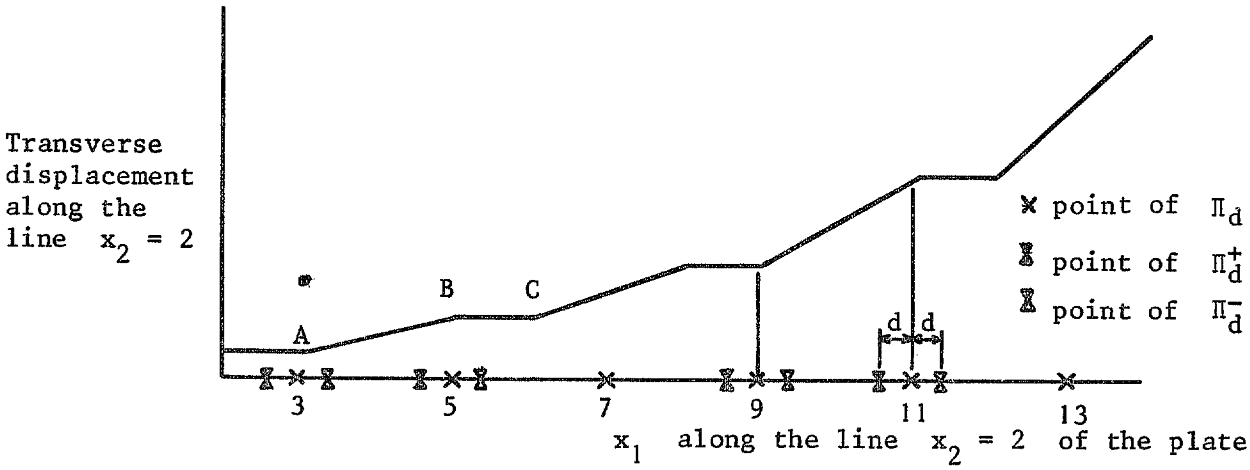
II.5.3 Derived flexibility matrices relating spatial derivatives and load

II.5.3.1 Transformations involving interpolation only

The regional structures based on the set of points Π_1 shown in Fig 5 have been employed in the derivation, from S_{Π_1}, of matrices which purport to relate $\frac{\partial}{\partial x_1}$ (transverse displacement) and transverse load appropriate to the set of points Π_d. (*cf* S^I_{23} of equation (II-12)). These matrices were used in conjunction with the loading L_{Π_L} and the resulting spatial derivatives may be compared with the 'true' results by reference to Table 4. In general there is a fair measure of agreement between them, those data obtained via the P6 regional structure being the closer to the quoted 'true' results. Once again, the discrepancies are numerically greatest in the root area.

As regards data obtained via the L3 regional structure, it can be seen, from an inspection of Table 4, that the spatial derivatives at the points (3,2) and (5,2) take the same numerical value. (A similar situation exists with regard to the points (9,2) and (11,2).) This is due to the variation of transverse displacement along the line $x_2 = 2$ which is implicitly assumed in the transformation process.

From Fig 5 we see that along this line (from $x_1 = 2$ to $x_1 = 14$) we pass alternately through regions with two points and one point fixed. Now the transverse displacement over an L3 region having two characteristic points which are fixed is constant along lines parallel to the region boundary line joining the two fixed points. This accounts for segments such as that marked BC in the following sketch.



The points (3,2) and (5,2) lie on region boundaries (Fig 5) and correspond to breakpoints, A and B respectively, of the transverse displacement sketched above. In this situation, the procedure adopted for evaluating spatial derivatives (equation (II-16)), leads, for sufficiently small d, to identical values at the two points in question. Other seemingly equal entries in the Table are the result of rounding.

Turning our attention to the data obtained via the P6 regional structure, it should be noted that apart from entries for the pairs of points (3,2), (5,2) and (9,2), (11,2) all other occurrences of equal entries in Table 4 are the result of rounding. *A propos* the aforementioned pairs of points, we note that each pair lies within a P6 region having three collinear fixed characteristic points (Fig 5). The state of transverse displacement, equations (II-13) and (II-18), in such a situation reduces to

$$w^i(x) = a_3x_2 + a_4x_1x_2 + a_6x_2^2 \quad .$$

Thus along any line x_2 = constant of such a region, we have a linear variation of transverse displacement along that line. Therefore points such as (3,2) and (5,2) have an identical value of the spatial derivative under consideration.

II.5.3.2 <u>Transformations involving some extrapolation</u>

The spatial derivatives of Table 5 were produced in an analogous manner to those of Table 4, but starting with a flexibility matrix appropriate to S_{Π_2} and using the regional structures based on Π_2 which are shown in Fig 6. Equal entries in the columns of data from the various derived matrices may be explained using similar arguments to those used in section II.5.3.1 in similar circumstances, the particular region with which any point of Π_d is associated being determined from Fig 6.

In judging the results from the derived matrices one must bear in mind the paucity of information points in the original flexibility data. The results, apart from those in the root area are remarkably good. With regard to the root area, it is to be hoped that, in the event of certain features of the local deformation pattern being significant in a particular problem, the distribution of the points for which the original flexibility matrix is defined would be such as to permit an adequate description of that local deformation.

II.6 RELATIONSHIP BETWEEN ACCURACY AND LOADING PATTERN

In section II.5 we used derived flexibility matrices to calculate certain features of the deformation pattern due to a particular loading. As regards the likely accuracy to be expected from such matrices used in conjunction with other loads patterns, we note that the eigenvectors of any flexibility matrix can be considered to constitute a basis for any loads pattern which may be used with that matrix. Therefore it is pertinent to consider loading patterns in the shape of each eigenvector of a derived matrix and to compare the displacements obtained via the derived matrix with those of the 'true' solution for the same loading. In this way we can determine the character of those loading patterns which, when used in conjunction with the derived matrix, give rise to displacements which are grossly in error.

We take as a specific example the flexibility matrix (displacement/load) derived from $\mathbf{s}_{\Pi_2}$ using the Q4 regional structure (section II.5.2.2) and use it to calculate the displacements due to a total load* of four units of force distributed in the shape of specific eigenvectors of the derived matrix. The results for loadings in the shape of the eigenvectors corresponding to the higher eigenvalues or flexibilities, $\phi_1, \phi_2, \ldots, \phi_9$ $(\phi_1 > \phi_2 > \ldots > \phi_9)$, are shown in Figs 7, 8 and 9 as spot heights on the contour plots of the corresponding 'true' solutions. While there are some local differences between the 'true' and calculated displacements of Figs 7, 8 and 9, overall, there are no major discrepancies. Obviously the detailed agreement between the two sets of data deteriorates as modes of lower and lower flexibilities are considered. The illustrations corresponding to ϕ_{10}, ϕ_{11} and ϕ_{12} (Fig 10) show the emergence of an identifiable characteristic, viz that the 'true' displacements towards the

* The phrase 'total load' is used here to mean the sum of the individual discrete loads.

free end of the plate are large compared with those calculated via the derived matrix. This characteristic persists for loading patterns in the shape of eigenvectors corresponding to eigenvalues which are numerically less than ϕ_{12} . (These displacement patterns are not illustrated because in the broad they are similar to that corresponding to ϕ_{12} .) To understand this trend it is necessary to appreciate that when the elastic characteristics of the structure are represented by a flexibility matrix, A , say, a loading in the pattern of an eigenvector of A gives rise to displacements in the same pattern but when that same loading is used in conjunction with another flexibility matrix, B , the eigenvectors of B then constitute a basis for the loading and the displacements calculated via B will have components in the shape of *each* eigenvector of B *. For our problem we identify the derived matrix with A and the matrix S_{II} with B . For loadings in the shape of eigenvectors corresponding to eigenvalues less than or equal to ϕ_{12} , the 'true' displacement as calculated via S_{II} is, over the outer half of the plate, dominated by component displacements in the graver modes of S_{II} . Thus on the basis of the criterion used heretofore to judge accuracy of data obtained via the derived matrix, we must regard the displacements from the derived matrix as being incorrect for these loadings. In this way, we have a quantitative feel for the type of loadings for which the derived matrix will yield incorrect displacements. In any particular problem, the precise level at which such effects can be expected to manifest themselves is dependent upon the number and distribution of the points of the sets with which the original and derived matrices are associated.

In using the above technique to assess the accuracy of a derived matrix with some general loading pattern, it must be borne in mind that the principal flexibilities of the higher modes are small in relation to those of the graver modes. A displacement calculated via the derived matrix is thus only significantly in error if the loading vector corresponds to loads in the modes of the derived matrix with relatively small principal flexibilities.

As an illustration, the composition of the loading L_{II_L} in terms of the total loads in each of the modes of the derived matrix considered in this section, and the flexibilities of these modes are given in Table 6. The errors produced by the inclusion of each of the first 12 modes can then be deduced by examining Figs 7 to 10. From Table 6 it is evident that the deformation due to the loading

* The set of points with respect to which A is referred is assumed to be a subset of that associated with B .

${}^{L}\Pi_{L}$ consists in the main of contributions from the modes of lower flexibilities and the contributions from the higher modes are very small in comparison.

II.7 CONCLUDING REMARKS

The equation of aeroelastic equilibrium has been developed from the general presentation of Part I when particular simplifying assumptions in respect of both the structural characteristics of and the loading on the structure are made. The interface between the structural information and the loading data in this particular instance is shown to reduce to some transformation of the flexibility information, the derived structural information taking the form of discrete flexibility information (possibly of a different form to the original) referred to a different set of points.

A Fortran computer program ALFI has been written to implement such transformations occurring in aeroelastic analyses. The main features of the program have been described in this Report. Briefly the user specifies a regional structure which is compounded from a number of pre-programmed region types. ALFI can then be used, in conjunction with flexibility data of displacement/ load form referred to a set of points Σ_1 to produce a matrix which purports to relate either displacement or a spatial derivative of displacement at a set of points Σ_2, to loads at a set of points Σ_3.

The program has been used to calculate flexibility matrices related to one set of points for a swept cantilever plate of constant thickness from a number of different matrices related to other sets of points using diverse regional structures. The derived matrices were used to calculate the displacement of the plate under a particular loading which, if the plate were to be regarded as a structural idealisation of an aircraft wing, would be typical of that expected in a discrete representation of subsonic aerodynamic load. The displacements and spatial derivatives so obtained were compared with a 'true' solution obtained directly from a structural finite element analysis. In general good agreement was obtained which suggests that for this particular loading at least the derived matrices are of an acceptable accuracy. A greater density of information points in the original set Σ_1 is reflected in greater accuracy of the deformation as calculated from the derived matrices when viewed overall. Flexibility matrices derived from one appropriate to a relatively coarse distribution of points in Σ_1 also gives, for our representative loading, deformation data of acceptable accuracy. This, to a certain extent, is a reflection on the nature of the 'true' deformation pattern for that loading, in that it can be reasonably described by the displacements over a coarse grid of points.

With regard to other loading patterns, it was noted that the eigenvectors of any flexibility matrix can be considered to constitute a basis for any loads pattern used with that matrix. Upon examination of the eigenvectors of a derived matrix, it was found that, for transformations involving a comparable number of points in each set, about half the eigenvectors of the derived matrix are valid in that the derived displacements due to a loading pattern in the shape of the eigenvector can be considered accurate. The valid eigenvectors are associated with the higher flexibilities and although the remainder are invalid, in the sense described above, they are however associated with the lower flexibilities. It is concluded that, so long as the displacements, as calculated from the derived matrix, do not resemble those of an eigenvector of that matrix which is associated with a relatively low flexibility (viewed in relation to the number of, and distribution of points in the two sets) then they are likely to be of an acceptable accuracy.

Appendix A

ON THE CHARACTERISTICS OF A DERIVED FLEXIBILITY MATRIX

A square matrix (order s_2), $\mathbf{S}_{22}$ which relates displacements at the set of points, Σ_2 , due to loads at the same set of points can be derived from a square matrix (order s_1), $\mathbf{S}_{11}$ defined with respect to a set of points Σ_1 , via the transformation*

$$\mathbf{S}_{22} = \mathbf{N}\mathbf{S}_{11}\mathbf{N}^T \quad . \tag{A-1}$$

The following discussion will be based on transformations of the type given by equation (A-1). This does not represent a restriction on the generality of the discussion since, if necessary, we may regard the set here designated Σ_2 as the union of the sets Σ_2 and Σ_3 of section II.1.4, the particular transformed matrix of equation (II-11) being some sub-matrix of that of equation (A-1). In addition, we assume that $\mathbf{S}_{11}$ is symmetric and can, therefore, be written as

$$\mathbf{S}_{11} = \mathbf{V}\Lambda\mathbf{V}^T \tag{A-2}$$

where Λ is a diagonal matrix, the diagonal elements of which are the eigenvalues of $\mathbf{S}_{11}$ and are termed principal flexibilities,

and $\mathbf{V}$ is an orthogonal matrix, the columns of which are the associated eigenvectors of $\mathbf{S}_{11}$ normalised such that their lengths are unity.

Physically the eigenvectors of a flexibility matrix, $\mathbf{S}$, may be interpreted as follows. When a vector of discrete loads, the components of which are proportional to the corresponding components of an eigenvector of $\mathbf{S}$, is used in conjunction with $\mathbf{S}$, the resulting displacements are equal to the product of the load vector and the corresponding principal flexibility.

$\mathbf{S}_{22}$ is also symmetric and may in an analogous manner be written as

$$\mathbf{S}_{22} = \mathbf{W}\Phi\mathbf{W}^T \quad . \tag{A-3}$$

For the matrix $\mathbf{S}_{22}$ derived from $\mathbf{S}_{11}$ by the transformation of equation (A-1), the rank of $\mathbf{S}_{22}$, $R(\mathbf{S}_{22})$, satisfies the inequality**

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* To be consistent with the notation of Part II, $\mathbf{N}$ of equation (A-1) should carry the subscript 21 but since attention in this Appendix is restricted to this one transformation, the subscript can be dropped.

** The rank theorems utilised in this Appendix are listed in Appendix B.

$$R(\mathbf{S}_{22}) \leqslant \min(R(\mathbf{N}),\ R(\mathbf{S}_{11})) \quad .$$

In particular $R(\mathbf{S}_{22})$ cannot exceed $R(\mathbf{S}_{11})$ which means that although the transformation process may impose certain additional constraints on the original structural model such that some modes of flexibility may be suppressed and others misrepresented in the derived matrix $\mathbf{S}_{22}$, the number of modes of flexibility cannot be increased.

$\mathbf{S}_{11}$ is said to be positive definite if and only if all its eigenvalues, λ_{ii}, $i = 1, 2, \ldots, s_1$, satisfy the inequality: $\lambda_{ii} > 0$. Further $\mathbf{S}_{11}$ is positive semi-definite if and only if $\lambda_{ii} \geqslant 0$ and the strict equality holds for at least one value of i. As regards the derived matrix, $\mathbf{S}_{22}$, we can write in view of equations (A-3), (A-1) and (A-2)

$$\Phi = \mathbf{B}\Lambda\mathbf{B}^T \tag{A-4}$$

where $\mathbf{B} = \mathbf{W}^T\mathbf{N}\mathbf{V}$.

An element $\phi_{i\ell}$ of Φ is given by

$$\phi_{i\ell} = \sum_{k=1}^{s_1}\sum_{j=1}^{s_1} b_{ij}\lambda_{jk}b_{\ell k}$$

where b_{ij} is an element of $\mathbf{B}$ and λ_{jk} is an element of Λ.

But $\lambda_{jk} = 0$ for $j \neq k$ and $\phi_{i\ell} = 0$ for $i \neq \ell$ and we can therefore write ϕ_{ii}, an eigenvalue of $\mathbf{S}_{22}$ as

$$\phi_{ii} = \sum_{j=1}^{s_1} b_{ij}^2\lambda_{jj}$$

whence we see that for either $\lambda_{jj} > 0$, $j = 1, \ldots, s_1$, or $\lambda_{jj} \geqslant 0$, $j = 1, \ldots, s_1$, it may be asserted that $\phi_{ii} \geqslant 0$, $i = 1, \ldots, s_2$. Thus, if $\mathbf{S}_{11}$ has no negative eigenvalues (ie $\mathbf{S}_{11}$ positive definite or positive semi-definite) the same will be true of $\mathbf{S}_{22}$.

Whether $\mathbf{S}_{22}$ is positive definite or positive semi-definite depends upon the precise form of $\mathbf{N}$ in the particular instance and the orders of the two matrices $\mathbf{S}_{22}$ and $\mathbf{S}_{11}$. Necessary and sufficient conditions for $\mathbf{S}_{22}$ to be positive definite given $\mathbf{S}_{11}$ positive definite or positive semi-definite can be written in terms of $\mathbf{N}$ and the characteristics of $\mathbf{S}_{11}$. For if $\mathbf{S}_{11}$ is of rank s_R, positive definite ($s_R = s_1$) or positive semi-definite ($s_R < s_1$), without loss of generality we can write equation (A-4) as

$$\Phi = \mathbf{B}_{11} \Lambda_{s_R} \mathbf{B}_{11}^T \tag{A-5}$$

where Λ_{s_R} is a diagonal matrix of the s_R non-zero eigenvalues of $\mathbf{S}_{11}$ and

$$\mathbf{B}_{11} = \mathbf{W}^T \mathbf{N} \mathbf{V}_{11} \tag{A-6}$$

where $\mathbf{V}_{11}$ is a matrix of order $s_1 \times s_R$ whose columns are the eigenvectors associated with Λ_{s_R}.

Now for $\mathbf{S}_{22}$ to be positive definite we must have

$$R(\mathbf{S}_{22}) = R(\Phi) = s_2 \quad ,$$

which by virtue of equation (A-5) implies the two necessary conditions, viz

$$R(\mathbf{B}_{11}) \geqslant s_2 \tag{A-7}$$

and

$$R(\Lambda_{s_R}) \geqslant s_2 \quad . \tag{A-8}$$

But $\mathbf{B}_{11}$ is of order $s_2 \times s_R$ and its rank cannot therefore exceed s_2 so that the condition expressed by equation (A-7) may be sharpened to

$$R(\mathbf{B}_{11}) = s_2 \tag{A-9}$$

or by virtue of equation (A-6) in which $\mathbf{W}$ is a non-singular square matrix

$$R(\mathbf{N}\mathbf{V}_{11}) = s_2 \quad . \tag{A-10}$$

Since Λ is of order s_1 and rank s_R condition (A-8) may be expressed as

$$s_2 \leqslant s_R \leqslant s_1 \quad . \tag{A-11}$$

That conditions (A-10) and (A-11) are sufficient to ensure $\mathbf{S}_{22}$ positive definite is evident from the following discussion. In the light of the above development we may write

$$\phi_{ii} = \sum_{j=1}^{s_R} b_{ij}^2 \lambda_{jj} \quad .$$

Therefore ϕ_{ii} is zero if and only if b_{ij} , $j = 1, \ldots, s_R$, is zero or, if we define $\mathbf{b}_i = (b_{i1}, b_{i2}, \ldots, b_{is_R})$, $\mathbf{b}_i$ is null.

Now

$$\mathbf{b}_i = \mathbf{w}_i^T \mathbf{N} \mathbf{V}_{11}$$

where $\mathbf{w}_i$ is the ith column of $\mathbf{W}$ and is of rank 1.

From theorems listed in Appendix B on the bounds of the rank of product of two matrices we have

$$R(\mathbf{b}_i) \geq R(\mathbf{w}_i) + R(\mathbf{N}\mathbf{V}_{11}) - s_2$$

and

$$R(\mathbf{b}_i) \leq \min(R(\mathbf{w}_i), R(\mathbf{N}\mathbf{V}_{11}))$$

and since $R(\mathbf{N}\mathbf{V}_{11}) = s_2$, we conclude that $R(\mathbf{b}_i) = 1$. Therefore $\mathbf{b}_i$ is not null and $\phi_{ii} \neq 0$. Therefore the condition $R(\mathbf{N}\mathbf{V}_{11}) = s_2$ is sufficient to ensure $\mathbf{S}_{22}$ positive definite given $s_2 \leq s_R \leq s_1$. An immediate corollary is that given $\mathbf{S}_{11}$ positive definite, $\mathbf{S}_{22}$ is positive definite if $s_2 \leq s_1$ and $R(\mathbf{N}) = s_2$.

Since quantities like $R(\mathbf{N})$ and $R(\mathbf{N}\mathbf{V}_{11})$ clearly depend upon the particular transformation being attempted, it is impossible to make further general statements about the character of $\mathbf{S}_{22}$. Among the factors figuring prominently in the constitution of the matrices $\mathbf{N}$ and $\mathbf{N}\mathbf{V}_{11}$ we may cite the relative numbers of and distribution of information points in the grids concerned; the regional structure and the suitability of the chosen functions $\mathbf{N}$. The first of these may well be outside the control of the aeróelastician, but among the diverse options available in respect of the last two factors there may well be an arrangement which leads to compliance with conditions such as $R(\mathbf{N}\mathbf{V}_{11}) = s_2$.

Appendix B

RANK THEOREMS USED IN APPENDIX A

1. The rank of a product of two matrices is not greater than the rank of either factor, ie if $\mathbf{AB}$ exists then

$$R(\mathbf{AB}) \leqslant \min(R(\mathbf{A}), R(\mathbf{B})) \quad .$$

Ref 5, Theorem 5.6.2

2. The rank of a matrix remains unchanged if the matrix is premultiplied or postmultiplied by a non-singular square matrix.

Ref 5, Theorem 5.6.3

3. If the product $\mathbf{AB}$ exists, then

$$R(\mathbf{AB}) \geqslant R(\mathbf{A}) + R(\mathbf{B}) - n$$

where n is the number of columns in $\mathbf{A}$ (and of rows in $\mathbf{B}$).

Ref 5, Theorem 5.6.5

Table 1

THE LOADING, $\mathbb{L}_{\Pi_L}$, USED IN THE ASSESSMENT OF DERIVED FLEXIBILITY DATA

Coordinates of the points of Π_L (units of length)		The components of the loading $\mathbb{L}_{\Pi_L}$ (units of force)
x_2	x_1	
2	3	0.4362
	7	0.2109
	11	0.1204
6	7	0.4922
	11	0.2068
	15	0.1058
10	11	0.5155
	15	0.2058
	19	0.1010
14	15	0.5217
	19	0.1998
	23	0.0928
18	19	0.5084
	23	0.1781
	27	0.0720
22	23	0.4443
	27	0.1082
	31	0.0336

Table 2

TRANSVERSE DISPLACEMENTS AT THE POINTS OF Π_d DUE TO L_{Π_L} :

THE TRANSFORMATION OF FLEXIBILITY DATA INVOLVING INTERPOLATION ONLY

(Reference to Fig 5 may be advantageous)

Coordinates of the points of Π_d		Displacements due to L_{Π_L} as calculated by matrices derived from S_{Π_1} using a regional structure comprising:		'True' displacements due to L_{Π_L}
x_2	x_1	L3 regions	P6 regions	
2	3	1.7×10^{-4}	0.4×10^{-4}	1.1×10^{-4}
	5	4.1	2.1	2.1
	7	5.9	3.0	3.5
	9	7.7	5.6	5.4
	11	13.7	9.6	8.3
	13	18.5	14.7	13.5
6	7	29.3	28.8	28.8
	9	40.0	40.0	40.0
	11	53.6	53.0	53.0
	13	68.7	68.1	68.0
	15	85.4	85.1	85.2
	17	104.8	104.0	104.1
10	11	119.6	116.8	117.3
	13	143.5	140.5	140.6
	15	166.0	164.6	164.9
	17	192.2	189.7	190.2
	19	218.8	216.4	216.5
	21	244.8	243.4	243.7
14	15	256.8	255.6	255.5
	17	287.6	285.2	285.3
	19	317.1	315.5	315.4
	21	347.0	345.6	345.6
	23	378.2	376.0	376.1
	25	408.8	406.9	406.9
18	19	420.9	419.4	419.5
	21	453.1	451.5	451.5
	23	485.2	483.3	483.4
	25	517.2	515.1	515.2
	27	549.2	546.8	547.0
	29	581.3	578.6	578.8
22	23	594.7	592.2	592.5
	25	627.1	624.4	624.6
	27	659.4	656.4	656.6
	29	691.6	686.3	688.6
	31	723.8	720.3	720.5
	33	755.9×10^{-4}	752.2×10^{-4}	752.4×10^{-4}

Table 3

TRANSVERSE DISPLACEMENTS AT THE POINTS OF Π_d DUE TO L_{Π_L} :
THE TRANSFORMATION OF FLEXIBILITY DATA INVOLVING SOME EXTRAPOLATION

(Reference to Fig 6 may be advantageous)

Coordinates of the points of Π_d		Displacements due to L_{Π_L} as calculated by matrices derived from S_{Π_2} using regional structure comprising:			'True' displacements due to L_{Π_L}
x_2	x_1	L3 regions	Q4 regions	P6 regions	
2	3 (E)	-4.9×10^{-4}	0.9×10^{-4}	2.2×10^{-4}	1.1×10^{-4}
	5	2.3	2.3	1.5	2.1
	7	3.7	3.7	2.9	3.5
	9	5.9	5.9	5.9	5.4
	11	9.0	9.0	9.0	8.3
	13 (E)	12.1	12.2	12.2	13.5
6	7 (E)	27.9	27.8	29.0	28.8
	9	40.7	40.8	40.0	40.0
	11	53.6	53.8	53.0	53.0
	13	68.5	68.8	68.0	68.0
	15	85.4	85.8	85.1	85.2
	17 (E)	102.3	102.8	104.3	104.1
10	11 (E)	116.6	116.8	116.9	117.3
	13	140.7	141.0	140.9	140.6
	15	164.8	165.3	165.3	164.9
	17	189.9	190.4	189.9	190.2
	19	215.9	216.6	216.1	216.5
	21 (E)	241.9	242.7	244.5	243.7
14	15 (E)	254.9	255.5	255.7	255.5
	17	284.6	285.4	285.4	285.3
	19	314.4	315.2	315.4	315.4
	21	344.4	345.3	345.6	345.6
	23	374.6	375.6	376.2	376.1
	25 (E)	404.7	405.9	407.1	406.9
18	19 (E)	418.6	419.6	419.8	419.5
	21	450.2	451.3	451.6	451.5
	23	481.7	482.9	483.5	483.4
	25	513.1	514.5	515.2	515.2
	27	544.5	546.0	547.0	547.0
	29 (E)	575.9	577.5	579.1	578.8
22	23 (E)	591.0	592.4	592.9	592.5
	25	622.6	624.1	624.7	624.6
	27	654.1	655.7	656.7	656.6
	29	685.6	687.4	688.7	688.6
	31	717.1	719.0	720.7	720.5
	33 (E)	748.6×10^{-4}	750.6×10^{-4}	753.5×10^{-4}	752.4×10^{-4}

NOTE: (E) in the x_1 coordinate column indicates that the point is external to the regional structure.

Table 4

SPATIAL DERIVATIVES IN THE Ox_1 DIRECTION OF THE TRANSVERSE DISPLACEMENT AT THE POINTS OF Π_d DUE TO L_{Π_L} : THE TRANSFORMATION OF FLEXIBILITY DATA INVOLVING INTERPOLATION ONLY

Coordinates of the points of Π_d		$\frac{\partial}{\partial x_1}$ (transverse displacement) due to L_{Π_L}		
		Calculated via S_{Π_2} (Fig 5)		'True' values
x_2	x_1	L3 regions	P6 regions	
2	3	0.61×10^{-4}	0.81×10^{-4}	0.44×10^{-4}
	5	0.61	0.81	0.59
	7	1.79	0.37	0.78
	9	1.51	2.00	1.10
	11	1.51	2.00	1.93
	13	4.77	3.32	3.03
6	7	5.37	5.17	5.18
	9	6.08	6.06	6.00
	11	6.78	6.96	7.01
	13	8.35	8.07	8.07
	15	9.03	8.98	9.03
	17	9.71	9.89	9.89
10	11	10.25	11.56	11.38
	13	12.44	11.92	11.91
	15	12.64	12.19	12.43
	17	12.16	13.15	12.91
	19	13.72	13.36	13.37
	21	13.92	13.62	13.80
14	15	14.51	14.69	14.87
	17	14.71	14.95	14.96
	19	15.47	15.21	15.08
	21	15.02	15.06	15.20
	23	15.23	15.32	15.33
	25	15.74	15.54	15.47
18	19	16.10	16.07	16.06
	21	16.06	15.98	15.96
	23	16.03	15.88	15.92
	25	16.01	15.88	15.89
	27	16.02	15.89	15.90
	29	16.03	15.91	15.92
22	23	16.24	16.15	16.11
	25	16.15	16.02	16.02
	27	16.12	15.98	15.98
	29	16.09	15.98	15.97
	31	16.09	15.98	15.96
	33	16.09×10^{-4}	15.96×10^{-4}	15.96×10^{-4}

Table 5

SPATIAL DERIVATIVES IN THE Ox_1 DIRECTION OF THE TRANSVERSE DISPLACEMENT AT THE POINTS OF Π_d DUE TO L_{Π_L} : THE TRANSFORMATION OF FLEXIBILITY DATA INVOLVING SOME EXTRAPOLATION

Coordinates of the points of Π_d		$\frac{\partial}{\partial x_1}$ (transverse displacement) due to L_{Π_L}			
		Calculated via S_{Π_2} (Fig 6)			'True' values
x_2	x_1	L3 regions	Q4 regions	P6 regions	
2	3 (E)	6.44×10^{-4}	0.70×10^{-4}	-0.85×10^{-4}	0.44×10^{-4}
	5	0.70	0.70	0.18	0.59
	7	0.70	0.70	1.21	0.78
	9	1.55	1.56	1.56	1.10
	11	1.55	1.56	1.56	1.93
	13 (E)	1.55	1.56	1.56	3.03
6	7 (E)	6.44	6.50	4.95	5.18
	9	6.44	6.50	5.98	6.00
	11	6.44	6.50	7.01	7.01
	13	8.45	8.51	8.04	8.07
	15	8.45	8.51	9.08	9.03
	17 (E)	8.45	8.51	10.11	9.89
10	11 (E)	12.06	12.12	11.94	11.38
	13	12.06	12.12	12.09	11.91
	15	12.06	12.12	12.23	12.43
	17	13.00	13.06	12.61	12.91
	19	13.00	13.06	13.64	13.37
	21 (E)	13.00	13.06	14.66	13.80
14	15 (E)	14.87	14.94	14.77	14.87
	17	14.87	14.94	14.92	14.96
	19	14.87	14.94	15.06	15.08
	21	15.09	15.15	15.21	15.20
	23	15.09	15.15	15.35	15.33
	25 (E)	15.09	15.15	15.50	15.47
18	19 (E)	15.76	15.82	15.88	16.06
	21	15.76	15.82	15.92	15.96
	23	15.76	15.82	15.95	15.92
	25	15.69	15.75	15.82	15.89
	27	15.69	15.75	15.97	15.90
	29 (E)	15.69	15.75	16.12	15.92
22	23 (E)	15.78	15.84	15.91	16.11
	25	15.78	15.84	15.95	16.02
	27	15.78	15.84	15.98	15.98
	29	15.74	15.80	16.02	15.97
	31	15.74	15.80	16.05	15.96
	33 (E)	15.69×10^{-4}	15.80×10^{-4}	16.73×10^{-4}	15.96×10^{-4}

NOTE: (E) in the x_1 coordinate column indicates that the point is external to the regional structure

Table 6

COMPOSITION OF THE LOADING L_{Π_L} IN TERMS OF TOTAL LOADS IN THE PATTERN OF THE EIGENVECTORS OF THE MATRIX DERIVED FROM S_{Π_2} USING Q4 REGIONS

Mode number i	Eigenvalue ϕ_i	Total load in that mode	Running total load	Displacements × 10^4 at point (33,22)	at point (21,10)
1	5.1334×10^{-1}	1.994159	1.994159	795.7	237.4
2	1.6706×10^{-2}	1.567728	3.561887	749.7	251.3
3	6.6936×10^{-3}	-0.312103	3.249784	748.1	243.5
4	2.1750×10^{-3}	0.496509	3.746293	750.7	243.2
5	7.1895×10^{-4}	-0.031265	3.715028	750.8	242.6
6	6.1439×10^{-4}	0.173855	3.888883	750.5	242.5
7	2.5547×10^{-4}	0.140891	4.029774	750.6	242.7
8	2.1790×10^{-4}	0.021525	4.051299	750.6	242.6
9	1.3448×10^{-4}	0.068650	4.119949	750.6	242.7
10	9.7778×10^{-5}	-0.009129	4.110820	750.6	242.7
11	6.2652×10^{-5}	0.001296	4.112116	750.6	242.7
12	4.8832×10^{-5}	-0.033165	4.078951	750.6	242.7
13	3.5973×10^{-5}	-0.094590	3.984361	750.6	242.7
14	3.5152×10^{-5}	0.533708	4.518069	750.6	242.7
15	3.2376×10^{-5}	0.032124	4.550193	750.6	242.7
16	3.0135×10^{-5}	-0.000401	4.549792	750.6	242.7
17	2.2829×10^{-5}	0.032100	4.581892	750.6	242.7
18	1.2971×10^{-5}	-0.028636	4.553256	750.6	242.7

LIST OF SYMBOLS

A^i	a matrix which when post-multiplied by w_{Σ_1} yields the transverse displacements at the n_i characteristic points of R_i
B	the matrix product $W^T N V$ (Appendix A)
B_{11}	the matrix product $W^T N V_{11}$ (Appendix A)
C	a sub-matrix of G^i
F^i	a vector of forces acting at the n_i characteristic points of R_i , and equivalent to the single load L_0 at x_0
G^i	an assembly of particular vectors of the type $h(x)$, defined by equation (II-14)
I_n	the unit matrix of order n
J_{n_i}	a column matrix of n_i elements, each of which is unity
L_{Σ_i}	a column vector of discrete loads in the Ox_3 direction acting at the set of points Σ_i
L_{Π_L}	a particular column vector of discrete loads in the Ox_3 direction applied at the points of Π_L
N	a particular $N_{k\ell}$ (Appendix A)
$N(x)$	a matrix, which when post-multiplied by the column vector of displacements at the set of points Σ_1 gives the displacement in the Ox_3 direction at the point x
$N^i(x)$	a row vector which when post-multiplied by a column vector of the displacements at the n_i characteristic points of R_i , gives the displacement in the Ox_3 direction at a point x associated with R_i
N_{k1}	a matrix whose jth row is $N(x_j)$, where x_j is the jth point of Σ_k , and which when post-multiplied by a column vector of the displacements at Σ_1 gives a column vector of displacements at the points of Σ_k
N^I_{21}	a matrix relating a spatial derivative of the displacement at the set of points Σ_2 to the displacement in the Ox_3 direction at the points of Σ_1
N^+_{21}, N^-_{21}	particular forms of N_{k1} , relating to the sets of points Σ^+_2, Σ^-_2 respectively
$O_{i,j}$	a zero matrix of order $(i \times j)$
S	a general flexibility matrix

LIST OF SYMBOLS (continued)

$\mathbf{S}_{k\ell}$	a flexibility matrix relating displacements at the set of points Σ_k to loads applied at the points of the set Σ_ℓ
$\mathbf{S}^{I}_{23}$	a matrix relating a spatial derivative of the displacement at the set of points Σ_2 to loads applied at the set of points Σ_3
$\mathbf{S}_{\Pi}$	a flexibility matrix relating load and displacement at the set of points Π
$\mathbf{V}$	an orthogonal matrix whose columns are the suitably normalised eigenvectors of $\mathbf{S}_{11}$ (Appendix A)
$\mathbf{V}_{11}$	a sub-matrix of $\mathbf{V}$ whose columns are those eigenvectors of $\mathbf{S}_{11}$ which are associated with non-zero eigenvalues (Appendix A)
$\mathbf{W}$	an orthogonal matrix whose columns are the suitably normalised eigenvectors of $\mathbf{S}_{22}$ (Appendix A)
$\mathbf{Z}^{a}_{\Sigma_3}$	a column vector of discrete loads in the Ox_3 direction acting at the set of points Σ_3 which represent the aerodynamic loading on the plate
$\mathbf{Z}^{a}_{\Sigma_1}$	a column vector of forces acting at the set of points Σ_1, which are equivalent to $\mathbf{Z}^{a}_{\Sigma_3}$
$\mathbf{Z}^{a}_{0_{\Sigma_i}}$	a particular static equilibrium value of $\mathbf{Z}^{a}_{\Sigma_i}$
$\mathbf{Z}_{0_{\Sigma_1}}$	a column vector of discrete forces acting at the points of Σ_1 which are equivalent to the total load on the plate
$\mathbf{Z}_{d_{0_{\Sigma_1}}}$	that part of $\mathbf{Z}_{0_{\Sigma_1}}$ that is non-aerodynamic
$\mathbf{a}$	a column vector of constants used in the representation of the state of displacement over a region
$\mathbf{b}_i$	ith row of $\mathbf{B}_{11}$
$\mathbf{c}$	a sub-matrix of $\mathbf{h}(\mathbf{x}_0)$
$\mathbf{h}(\mathbf{x})$	a row vector used in the representation of the state of displacement over a region
$\mathbf{w}_i$	the ith column of $\mathbf{W}$
$\mathbf{w}_{\Sigma_i}$	a column vector of displacements in the Ox_3 direction at the set of points Σ_i
$\mathbf{w}_{0_{\Sigma_i}}$	a particular static equilibrium value of $\mathbf{w}_{\Sigma_i}$

LIST OF SYMBOLS (continued)

$w^I_{\Sigma_2}$	a column vector of spatial derivatives of displacement evaluated at the set of points Σ_2
$w^I_{0_{\Sigma_2}}$	the particular static equilibrium value of $w^I_{\Sigma_2}$
$w_{d0_{\Sigma_i}}$	that part of $w_{0_{\Sigma_i}}$ attributable to non-aerodynamic loads
x	the coordinate pair (x_1,x_2)
x_0	a particular point at which the load L_0 is applied
x_k	the position of the kth point of the appropriate set
x^i_r	a column vector whose jth element is $x^i_{r_j}$
Λ	a diagonal matrix of the eigenvalues of S_{11}
Λ_{S_R}	a diagonal matrix of the non-zero eigenvalues of S_{11}
Φ	a diagonal matrix of the eigenvalues of S_{22}
ξ	the coordinate pair (ξ_1,ξ_2), corresponding to (x_1,x_2)
L_0	a single discrete load in the Ox_3 direction acting at x_0
N^i_j	the jth element of $N^i(x)$
$Ox_1x_2x_3$	a rectangular body attached axis system whose origin 0 is a definite material point of the body
R_i	the ith region
Z^a_i	an element of $Z^a_{\Sigma_3}$
$Z^a_{0_i}$	an element of $Z^a_{0_{\Sigma_3}}$
$Z_0(x)$	the total static equilibrium load on the flexible vehicle
$Z_{d0}(x)$	the non-aerodynamic part of $Z_0(x)$
a_i	the ith element of a
b_{ij}	an element of B
d	the interpolation distance used in the finite difference calculation of the spatial derivative of the displacement (equation (II-16))

LIST OF SYMBOLS (continued)

n_i	the number of characteristic points of R_i
n_{max}, n_{min}	the largest and smallest values respectively of the individual components of a particular $\mathbf{N}^i(\mathbf{x})$ as the point $\mathbf{x}$ moves within a defined limit
s_1, s_2	the orders of the matrices $\mathbf{S}_{11}$, $\mathbf{S}_{22}$ respectively
s_R	the number of non-zero eigenvalues of $\mathbf{S}_{11}$
$w(\mathbf{x})$	the displacement of the plate in the direction Ox_3
$w^i(\mathbf{x})$	the displacement of the plate in the direction Ox_3 at the point $\mathbf{x}$ associated with R_i
$w_0^r(\mathbf{x})$	the static equilibrium deflection in the direction Ox_3 of the constrained plate
$w_{d_0}^r(\mathbf{x})$	the non-aerodynamic part of $w_0^r(\mathbf{x})$
$x_{r_j}^i$	the x_r coordinate of the jth point of R_i
$\Gamma^r(\mathbf{x};\xi)$	the flexibility influence function tensor for the constrained plate
Π	a set of points in the Ox_1x_2 plane used to calculate the matrix $\mathbf{S}_\Pi$ for the plate considered in the example. The points lie at the vertices of the triangles shown in Fig 4
Π_i	a subset of Π for which a particular matrix $\mathbf{S}_{\Pi_i}$, of the form of $\mathbf{S}_{11}$, is calculated
Π_d	a subset of Π at which the displacement of the plate considered in the example is calculated
Π_d^+, Π_d^-	sets of points related to Π_d in a manner similar to the relation of Σ_2^+, Σ_2^- to Σ_2
Π_L	a subset of Π, defined in Fig 4, at which the representative loading $\mathbf{L}_{\Pi_L}$ is applied to the plate used as an example
Σ_1	a set of points in the Ox_1x_2 plane for which flexibility data are available.
Σ_2	a set of points in the Ox_1x_2 plane at which either displacements, or spatial derivatives of displacements are required
Σ_3	a set of points at which the plate will be loaded

LIST OF SYMBOLS (concluded)

Σ_2^+ , Σ_2^-	two sets of points which are a distance d either side of the points of Σ_2 , in a specified direction, all points of Σ_2^+ (or Σ_2^-) being on the same relative side of Σ_2
$\kappa^j(x)$	a function which has the value 1 if the point x is associated with the region R_i , and the value 0 otherwise
λ_{jk}	an element of Λ
ϕ_i	the ith eigenvalue of a particular derived matrix
$\phi_{i\ell}$	an element of Φ
σ_i	the number of points in the set Σ_i
$\tilde{\mathscr{A}}$	an aerodynamic operator
$\mathscr{P}$	a formal representation of the aerodynamic solution process, defined by equation (II-3)

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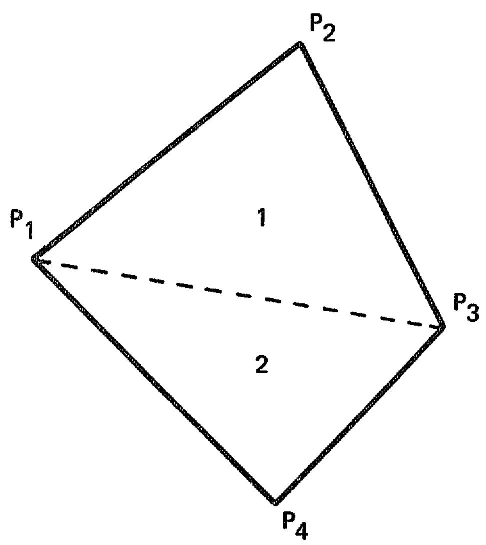


Characteristic points specified in the order $P_1P_2P_3P_4$

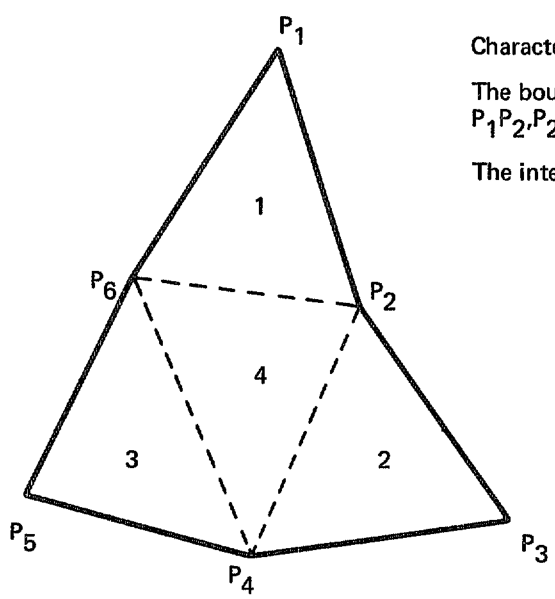
The boundary of the region is defined by the line segments $P_1P_2, P_2P_3, P_3P_4, P_4P_1$

The interior of Q4 is the 2 simple triangles

1 $P_1P_2P_3$

2 $P_3P_4P_1$

Fig 1a Decomposition of Q4 region into simple triangles



Characteristic points specified in the order $P_1P_2P_3P_4P_5P_6$

The boundary of the region is defined by the line segments $P_1P_2, P_2P_3, P_3P_4, P_4P_5, P_5P_6, P_6P_1$

The interior of P6 is the 4 simple triangles

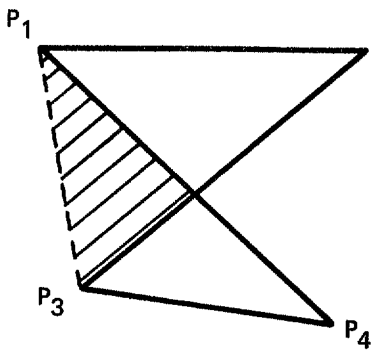
1 $P_1P_2P_6$

2 $P_2P_3P_4$

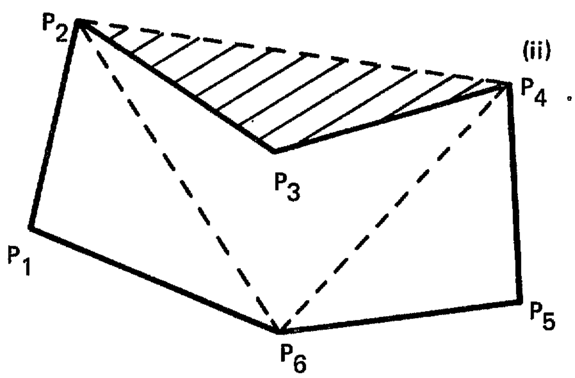
3 $P_4P_5P_6$

4 $P_2P_4P_6$

Fig 1b Decomposition of P6 region into simple triangles

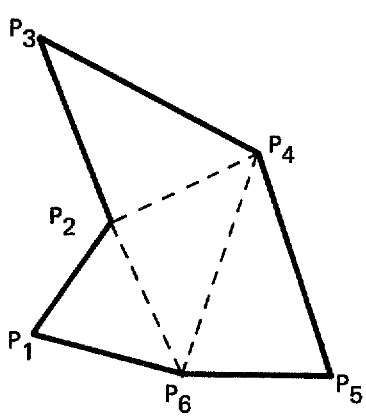


(i) Q4 Characteristic points specified in the order $P_1P_2P_3P_4$. For the arrangement shown $P_1P_2P_3$ and $P_3P_4P_1$ do not correspond to the region interior. Also the boundary lines P_1P_4 and P_2P_3 cross

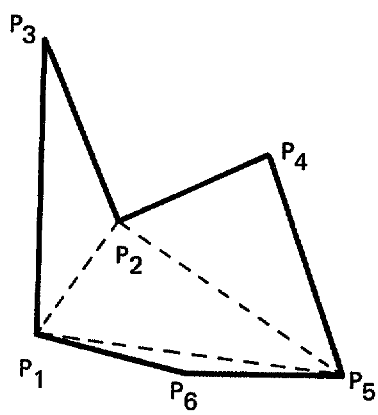


(ii) P6 Characteristic points specified in the order $P_1P_2P_3P_4P_5P_6$. The triangle $P_2P_3P_4$ is assumed to be part of the interior, but for this arrangement of points should not be so

Fig 2a Examples of non-correspondence of the interior of a region with assumed simple triangles



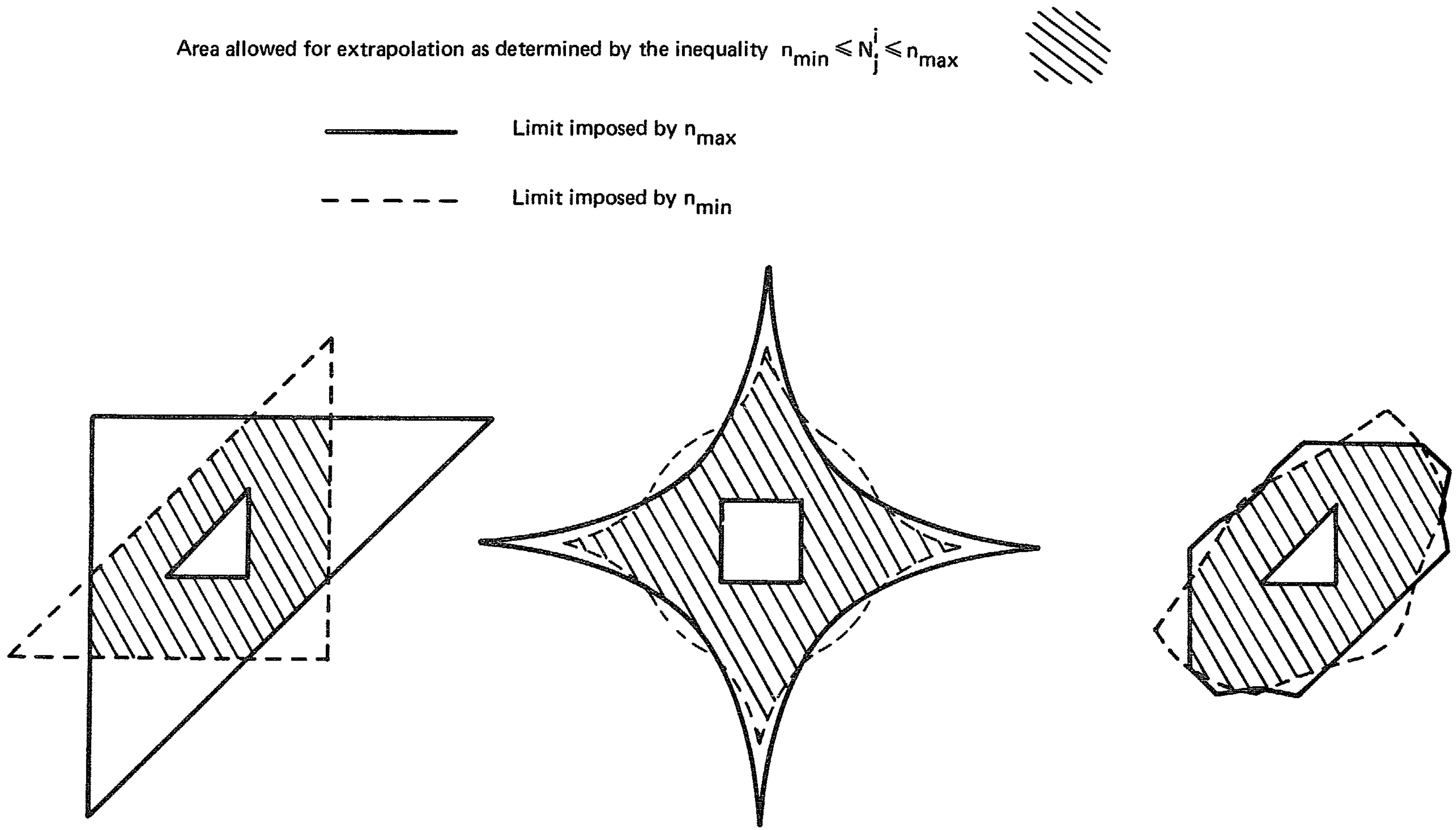
(i) Order of points $P_1P_2P_3P_4P_5P_6$



(ii) Order of points $P_3P_2P_4P_5P_6P_1$

Fig 2b Difference in region interior due to different ordering of the same characteristic points

Fig 3



Region type L3
$n_{max} = 2$
$n_{min} = -1$

Region type Q4
$n_{max} = 2$
$n_{min} = -1$

Region type P6
$n_{max} = 4$
$n_{min} = -6$

Fig 3 Areas external to a region within which, in ALFI, extrapolation is deemed acceptable

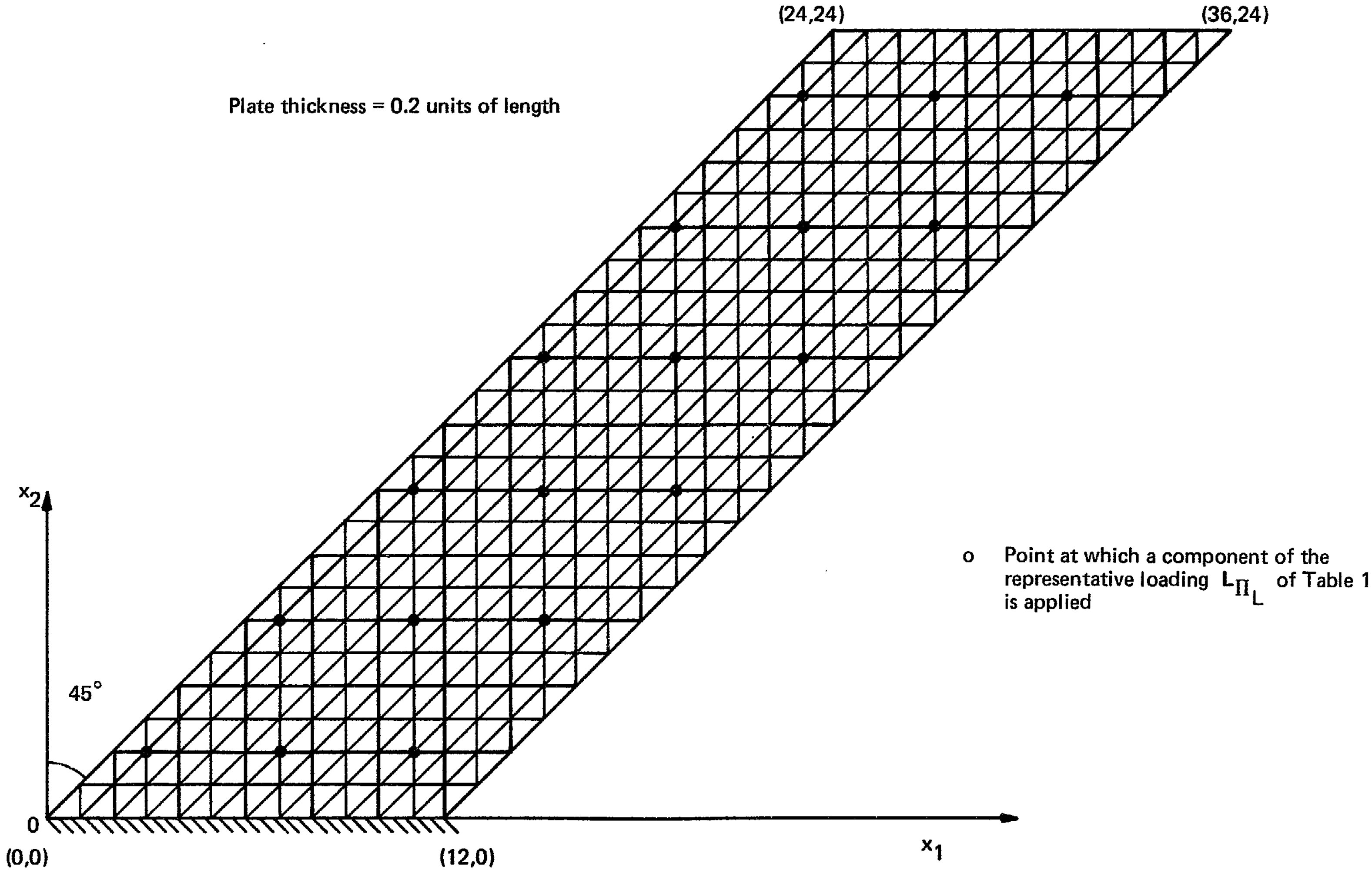


Fig 4 General arrangement of plate and the finite-element structure used to calculate S_{Π}

Fig 5

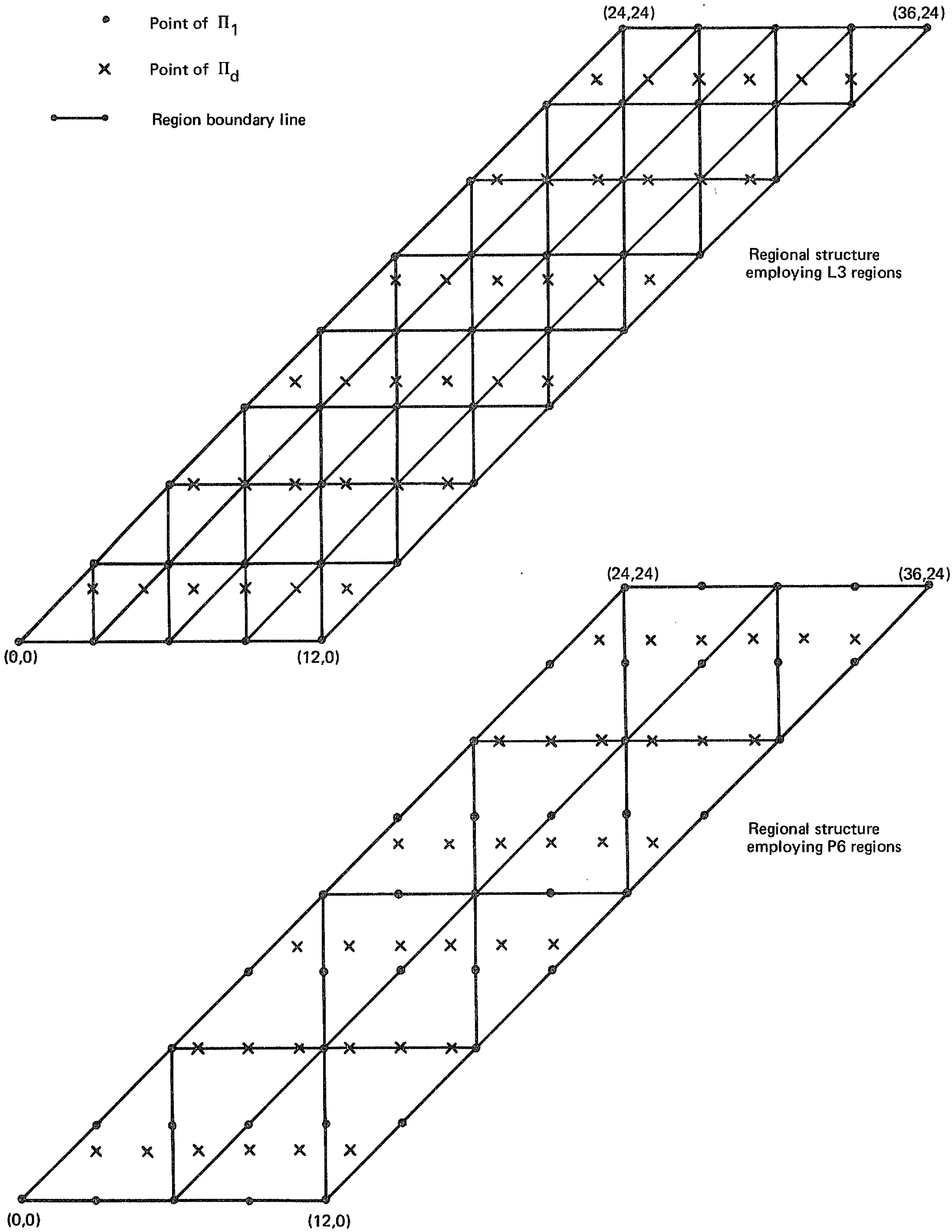


Fig 5 Alternative regional structures utilizing the 45 points of Π_1

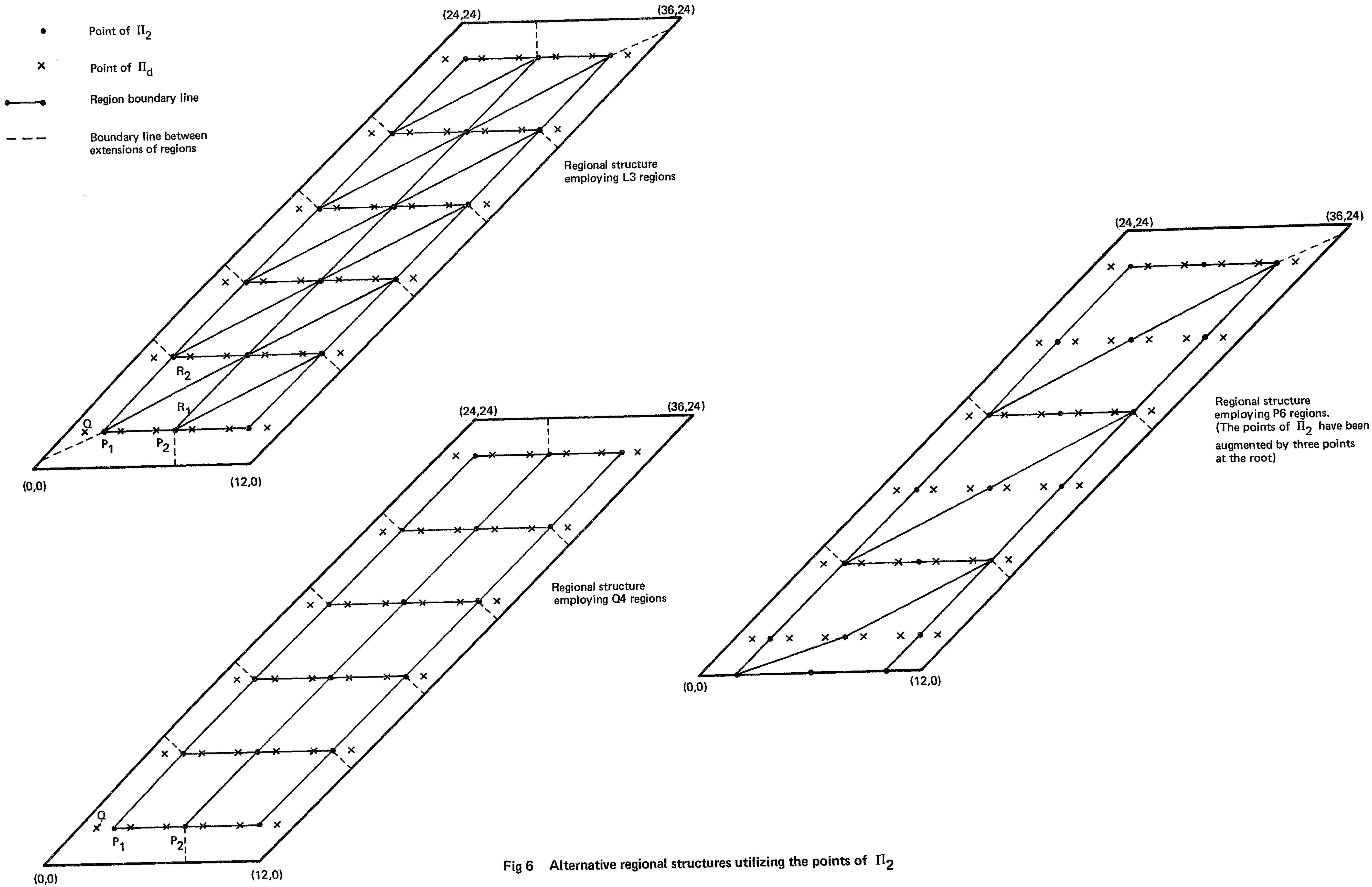


Fig 6 Alternative regional structures utilizing the points of Π_2

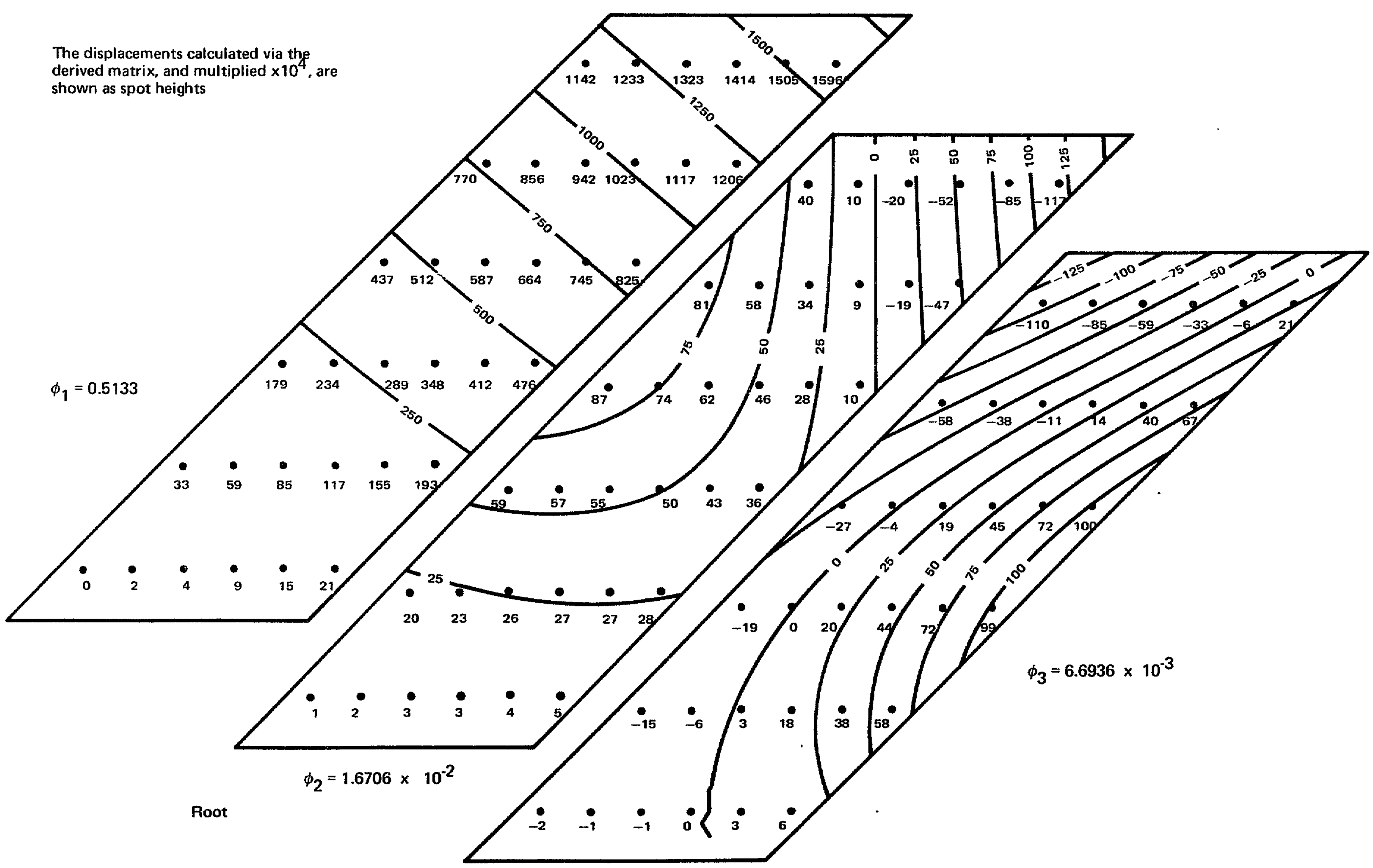


Fig 7 Contour plots of the 'true' transverse displacement x10^4 of the plate under four units of total load distributed in the shape of the eigenvectors corresponding to the eigenvalues ϕ_1, ϕ_2 and ϕ_3 of a particular derived matrix

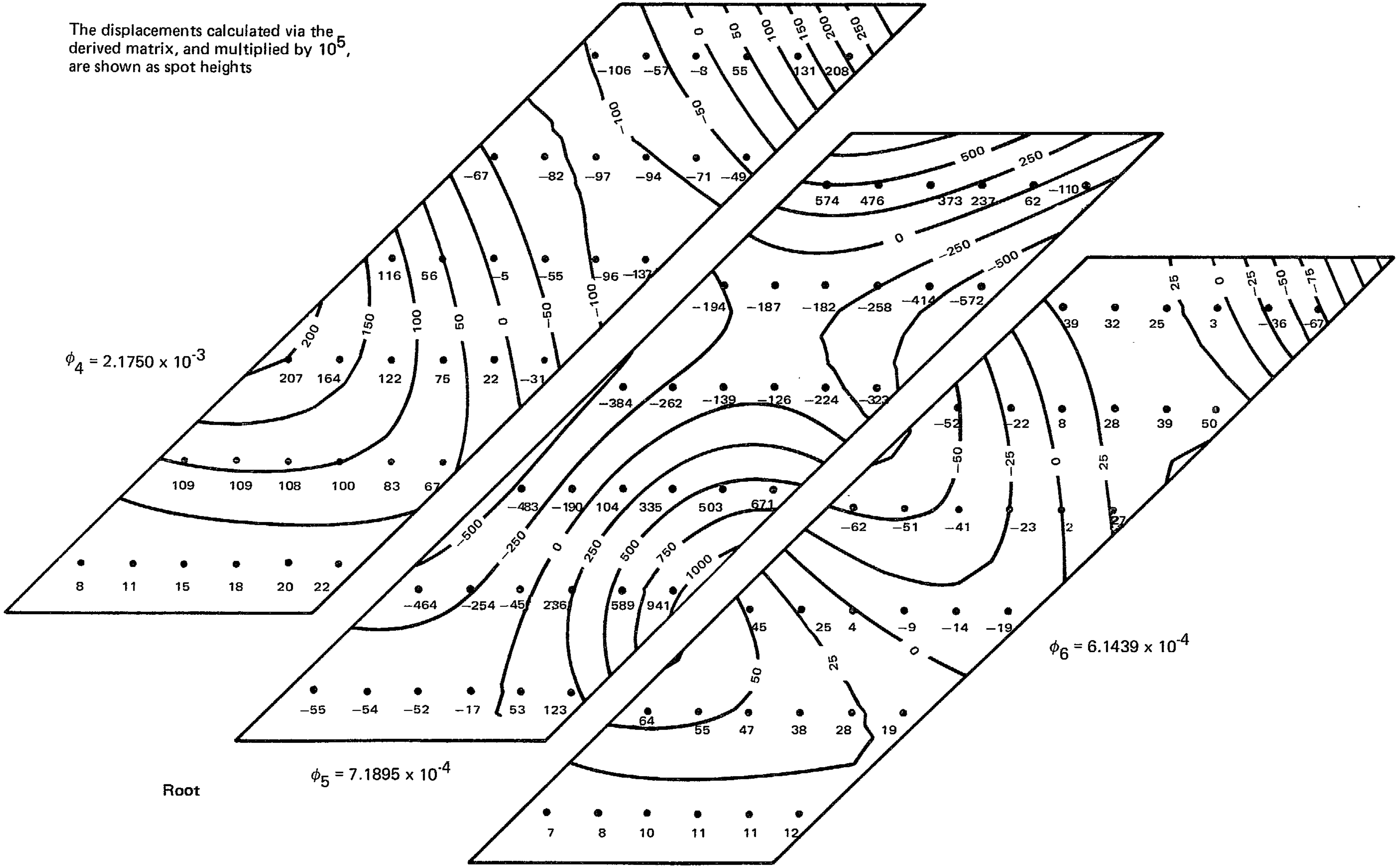


Fig 8 Contour plots of the 'true' transverse displacement $\times 10^5$ of the plate under four units of total load distributed in the shape of the eigenvectors corresponding to the eigenvalues of ϕ_4, ϕ_5 and ϕ_6 of a particular derived matrix

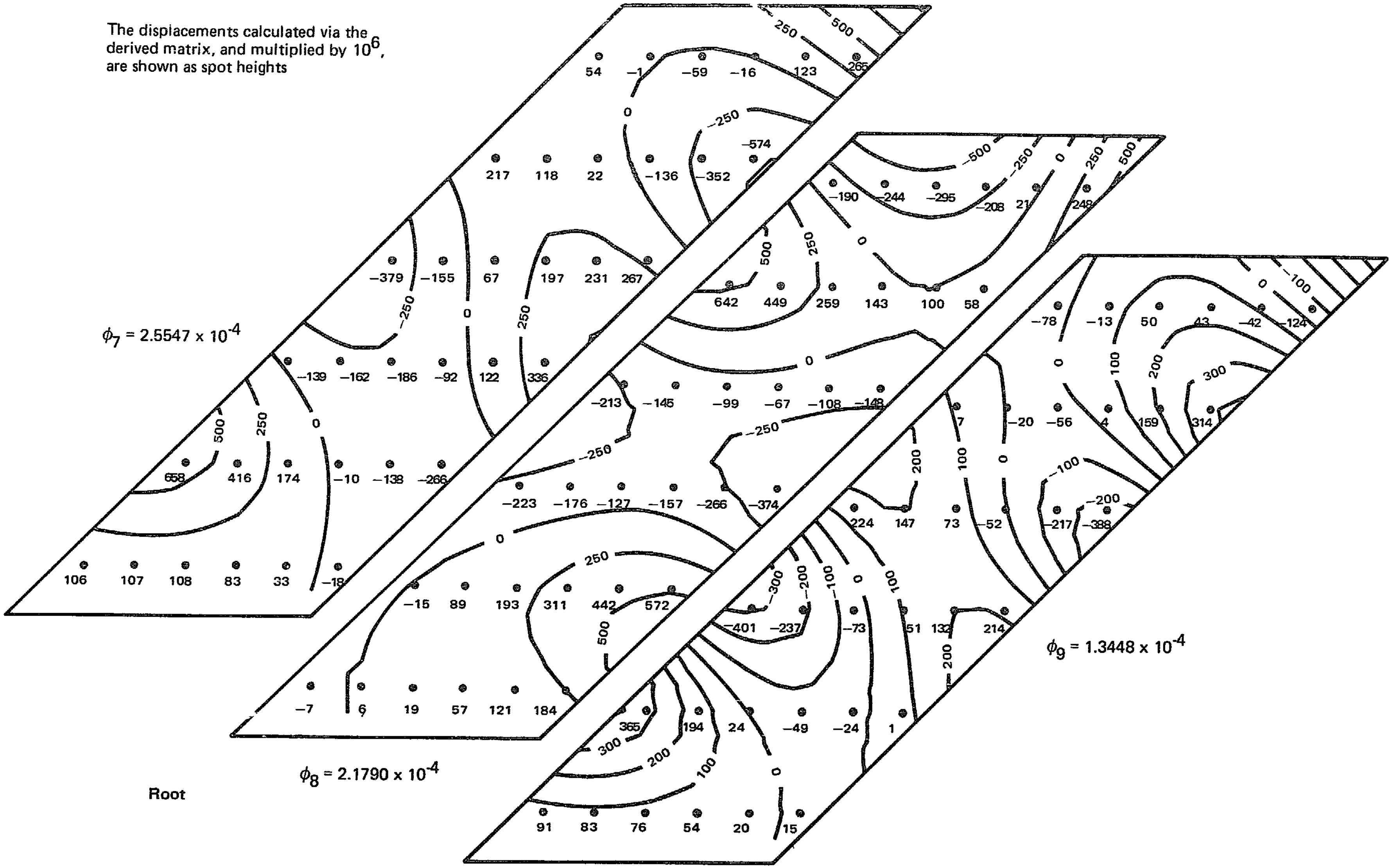


Fig 9 **Contour plots of the 'true' transverse displacement x10^6 of the plate under four units of total load distributed in the shape of the eigenvectors corresponding to the eigenvalues of ϕ_7, ϕ_8 and ϕ_9 of a particular derived matrix**

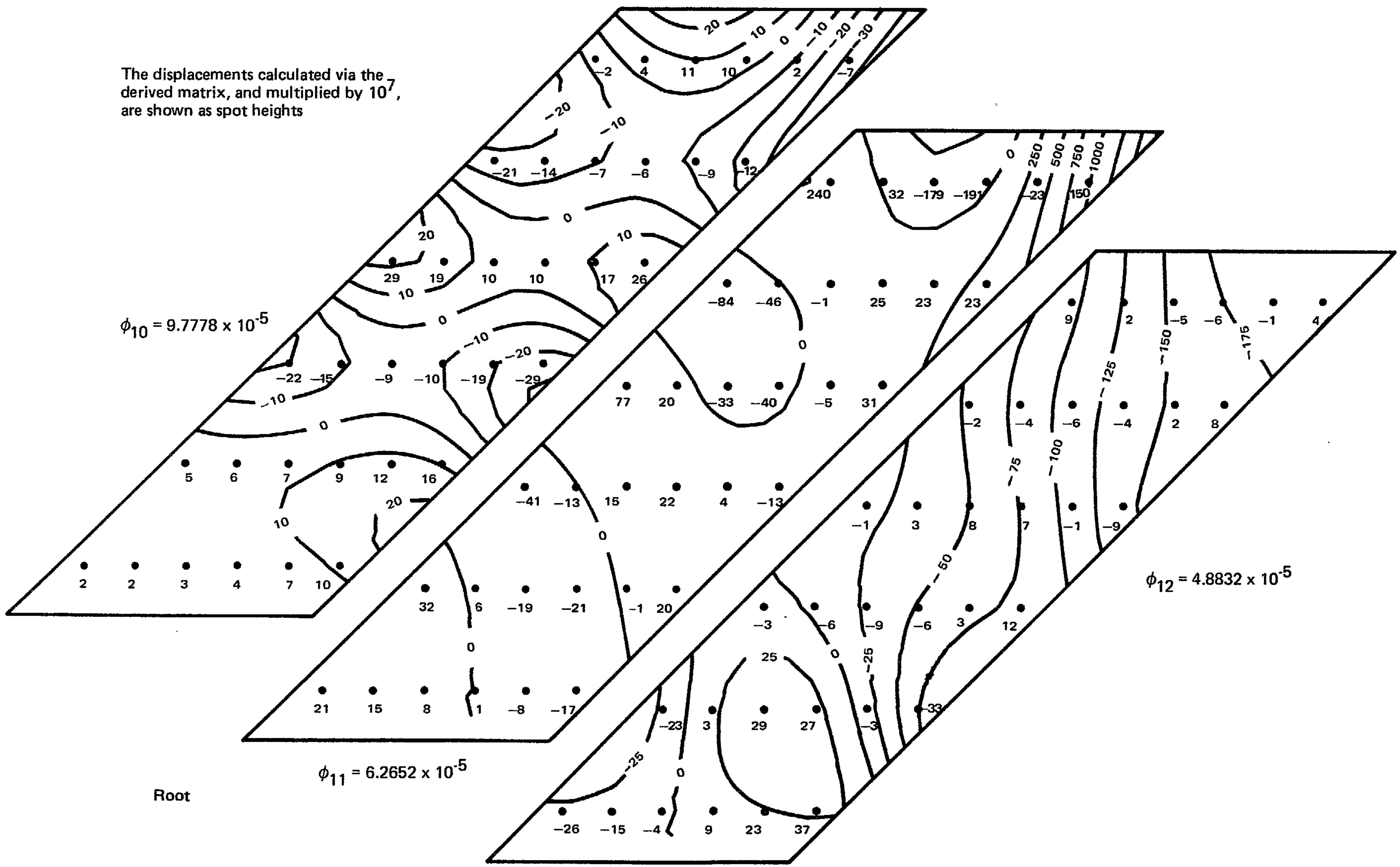


Fig 10 **Contour plots of the 'true' transverse displacement $\times 10^7$ of the plate under four units of total load distributed in the shape of the eigenvectors corresponding to the eigenvalues of ϕ_{10}, ϕ_{11} and ϕ_{12} of a particular derived matrix**

Printed in England for Her Majesty's Stationery Office by the Royal Aircraft Establishment, Farnborough. Dd.586386 K4 3/79.

First published 1979

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