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Secondary Flow in Cascades—The Effect of Compressibility

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Summary

Kelvin's circulation theorem is applied to compressible flow through a cascade of blades and expressions are derived for the three streamwise components of vorticity at exit from the cascade. Calculations show that for a decelerating flow, such as a compressor cascade, the distributed secondary vorticity increases as the inlet Mach number increases. However, for an accelerating flow, such as a turbine cascade, there is a wide range of inlet flow angles for which compressibility has little effect on the secondary flow.

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1. Introduction

When a non-uniform flow passes through a linear cascade of blades, a streamwise component of vorticity is produced and the exit flow angle varies along the span of the blades. The problem of secondary flow in cascades has been the subject of many papers during the past twenty-five years and an excellent review of previous work has been given by Horlock and Lakshminarayana.[1] For many years, the theory of secondary flows in cascades has been based on a relatively complex analysis, but recently, Came and Marsh[2] have shown that a simple derivation for the three streamwise components of vorticity can be obtained by applying Kelvin's circulation theorem to the flow through a many-bladed cascade. This new approach has led to a new expression for the trailing shed vorticity and removed several inconsistencies from secondary-flow theory.

In this report, Kelvin's circulation theorem is applied to a non-uniform compressible flow through a cascade and expressions are derived for the three streamwise components of secondary vorticity at exit from the cascade. The analysis can be compared with the earlier work of Loos,[3] who considered the effect of compressibility on secondary flows. The analysis given by Loos is based on a binomial expansion which requires that the inlet Mach number is small. However, in spite of this limitation, there is good agreement between the example quoted by Loos and the more exact analysis developed here. As the inlet Mach number increases towards unity, it is found that the effect of compressibility is to increase the distributed secondary vorticity at exit from a compressor cascade, but in a row of turbine nozzles, compressibility has little effect on secondary flow.

2. Kelvin's Circulation Theorem

For an inviscid incompressible flow, Kelvin's circulation theorem states that the circulation around any closed material curve is invariant with time,

$$\frac{D\Gamma}{Dt}=0. \tag{1}$$

For a compressible inviscid fluid, the more general form of Kelvin's circulation theorem states that the rate of change of circulation around any closed material curve C is given by

$$\frac{D\Gamma_c}{Dt}=-\oint_c \frac{\nabla p}{\rho}\cdot d\bar{r}, \tag{2}$$

where the line integral is evaluated for the contour C. If the entropy is constant throughout the fluid, then the density at any point is a function of the pressure alone and the line integral is zero, so that equation (2) reduces to equation (1). For compressible flow through a cascade, the non-uniformity in the inlet flow usually takes the form of a wall boundary layer and in this situation, the entropy is not constant throughout the flow field and the density at any point is not a function of the pressure alone.

When applying Kelvin's circulation theorem to a compressible non-homentropic flow, it is convenient to introduce temperature and entropy. For any process, the change of entropy can be expressed as

$$T\,ds=dh-\frac{1}{\rho}\,dp,$$

so that

$$\oint_c \frac{\nabla p}{\rho}\,.\,d\bar{r}=\oint_c \nabla h\,.\,d\bar{r}-\oint_c T\nabla s\,.\,d\bar{r}$$

or, using the identity

$$\oint_c \nabla h\,.\,d\bar{r}=0,$$

we have

$$\oint_c \frac{\nabla p}{\rho} \,.\, d\bar{r} = -\oint_c T\nabla s \,.\, d\bar{r}.$$

Kelvin's circulation theorem may then be written as

$$\frac{D\Gamma_c}{Dt} = \oint_c T\nabla s \,.\, d\bar{r}$$

which shows that in a flow field where there is a gradient of entropy, the circulation around a closed material curve may change with time.

3. The Time Difference for Flow over the Blade Surfaces

Consider a non-uniform flow passing through a cascade, as shown in Fig. 1, and assume that the primary flow takes place in planes which are normal to the span of the blades. This is the usual approximation for secondary-flow theory in which the secondary-flow perturbations are sufficiently small for there to be no significant rotation of the stream surfaces. If the flow originates from a reservoir with uniform stagnation enthalpy, then from the steady-flow energy equation, the stagnation enthalpy is uniform throughout the flow field. The non-uniform flow at inlet to the cascade is caused by a loss of stagnation pressure which leads to a variation of entropy along the span of the blades,

$$s = s(z). \tag{4}$$

If there are no changes of entropy produced while passing through the cascade, then the flow in a plane at a constant spanwise position z is an isentropic flow. For any closed material curve C lying in the plane $z = \text{constant}$, we have

$$\oint_c T\nabla s \,.\, d\bar{r} = 0$$

and Kelvin's circulation theorem for this curve C reduces to

$$\frac{D\Gamma_c}{Dt} = 0. \tag{5}$$

Hence, for the flow in a plane normal to the span, the circulation around any closed material curve does not vary with time.

If the flow upstream of the cascade has no component of vorticity in the spanwise direction, then there is zero circulation around any reducible closed circuit lying in a plane normal to the span of the blades. It therefore follows that for compressible flow through a cascade, there is no spanwise component of vorticity and at all points in the flow field

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{6}$$

This equation is based on the assumption that the real compressible flow can be modelled by isentropic flow on a series of parallel planes, so that entropy is a function of z alone.

Equation (6) can be integrated across the blade passage shown in Fig. 2 to obtain

$$u_b - u_a = s \cos^2 \beta \frac{d\bar{v}}{dx}, \tag{7}$$

where s is the pitch and $\bar{v}$ is the passage averaged value of the velocity component v. For any blade in the cascade, the difference in the time taken for fluid particles to travel over the pressure and suction surfaces of the blade is given by

$$\Delta t = \int_{\text{le}}^{\text{te}} \frac{dl}{q_a} - \int_{\text{le}}^{\text{te}} \frac{dl}{q_b}$$

$$= \int_{\text{le}}^{\text{te}} \left(\frac{1}{q_a} - \frac{1}{q_b}\right) dl \tag{8}$$

where q is the local velocity and l is the distance measured along the blade surface. At any point on the blade surface $u = q \cos \beta$, so that equation (8) can be written as

$$\Delta t = \int_{\text{le}}^{\text{te}} \left(\frac{1}{u_a} - \frac{1}{u_b}\right) \cos \beta \, . \, dl,$$

or from equation (7),

$$\Delta t = \int_{\text{le}}^{\text{te}} \frac{s \cos^2 \beta}{u_a u_b} \frac{d\bar{v}}{dx} \, . \, dx. \tag{9}$$

As the blade pitch, s, is reduced, the cascade becomes a many-bladed cascade with the flow angle α equal to the blade angle β and $\bar{v} = \bar{u} \tan \alpha$. Furthermore, for the many-bladed cascade the variation of u across the blade passage is infinitesimal, so that

$$u_a u_b = \bar{u}^2$$

where $\bar{u}$ is the average value of u in the blade passage. The expression for the time difference Δt in the many-bladed cascade is then given by

$$\Delta t = \int_{\text{le}}^{\text{te}} \frac{s \cos^2 \alpha}{\bar{u}^2} \, . \, \frac{d}{dx}(\bar{u} \tan \alpha) \, . \, dx, \tag{10}$$

which is the expression derived by Came and Marsh[2] for incompressible flow through cascades.

When the flow is compressible, there is a change of density across the cascade and the axial velocity at exit may differ from that at inlet to the cascade. If the primary flow is assumed to take place in stream surfaces of constant thickness which are normal to the span of the blades, then the passage averaged continuity equation is

$$\frac{d}{dx}(\overline{\rho u}) = 0, \tag{11}$$

or approximating $\overline{\rho u}$ by $\bar{\rho} \, . \, \bar{u}$ for the many-bladed cascade,

$$\frac{d}{dx}(\bar{\rho} \, . \, \bar{u}) = 0. \tag{12}$$

If the flow at inlet to the cascade is denoted by the subscript 1, then

$$\bar{\rho} \, . \, \bar{u} = \rho_1 \, . \, u_1. \tag{13}$$

The axial velocity $\bar{u}$ in equation (10) can now be eliminated to obtain an expression for Δt in terms of the variations of density and flow angle

$$\Delta t = \frac{s}{\rho_1 \, . \, u_1} \int_{\text{le}}^{\text{te}} \left[\rho \frac{d\alpha}{dx} - \sin \alpha \cos \alpha \frac{d\rho}{dx}\right] dx \tag{14}$$

or, integrating the second term by parts,

$$\Delta t = \frac{s}{u_1} \left\{ \int_{\text{le}}^{\text{te}} \frac{\rho}{\rho_1} [1 + \cos 2\alpha] \, d\alpha - \frac{\rho_2}{\rho_1} \frac{\sin 2\alpha_2}{2} + \frac{\sin 2\alpha_1}{2} \right\}. \tag{15}$$

In general, there is no simple expression for Δt in a compressible flow, but for incompressible flow, equation (15) reduces to

$$\Delta t = \frac{s}{u_1}(\alpha_2 - \alpha_1), \tag{16}$$

which is the expression derived by Came and Marsh.[2]

For a cascade with many blades of negligible thickness, the normal width of the blade passage is $s \cos \alpha$, so that if the inlet flow angle α_1, density ρ_1, and Mach number M_1 are defined, then at any other angle α, the density and Mach number may be found from tables of compressible flow functions, such as Ref. 4. Equation (15) can then be integrated numerically to determine the time difference Δt for fluid particles passing over the two surfaces of the blades. Fig. 3 shows the effect of inlet Mach number on the time difference Δt for a compressor cascade with an inlet flow angle of 40 degrees, an exit angle of 0 degrees and a gas with $\gamma = 1{\cdot}4$. For this example, the non-dimensional time difference increases rapidly as the inlet Mach number increases above 0·4. The time difference $u_1 \Delta t/s$ has been calculated for several cascades and it has been found that the effect of compressibility is important for compressor cascades, decelerating flows, with a high inlet Mach number.

The analysis relies on the use of the many-bladed cascade as a mathematical model for the flow in a real cascade. In practice, the real blades have thickness, the pitch is non-zero and for an entirely subsonic flow, the inlet Mach number must be less than about 0·7. However, in spite of this limitation, the model of a many-bladed cascade should provide a useful guide to the way in which compressibility affects the flow in the real cascade.

4. Kelvin's Circulation Theorem Applied to Cascades

So far, Kelvin's circulation theorem has only been applied to the flow in planes which are normal to the span of the blades. However, in secondary-flow theory it is necessary to consider the circulation around circuits which do not lie in a stream surface. Fig. 4 shows a compressible shear flow which passes through a many bladed cascade. Capital letters are used to denote fluid particles upstream of the cascade and small letters are the same particles at some later time when they have passed through the cascade. Fig. 4 shows the flow at a spanwise position z and the corresponding points on a parallel plane at a position $z + \delta z$ are denoted by a prime suffix.

Consider the circulation around the path $add'a'$ at exit from the cascade. At some earlier time, this circuit is $ADD'A'$ in the flow upstream of the cascade. For a compressible flow, the circulations around the paths $add'a'$ and $ADD'A'$ are not equal, the change in circulation being given by equation (3),

$$\frac{D\Gamma_c}{Dt} = \oint_c T\nabla s \,.\, dr.$$

The line integral may be determined by considering the contour at some intermediate time when it is at a position $a_i d_i d_i' a_i'$. For a compressible shear flow passing through a cascade, there is a variation of entropy along the span of the blades, $s = s(z)$, but no variation of entropy within the stream surfaces, so that

$$\left.\begin{aligned}
&\int_{a_i}^{d_i} T\nabla s \,.\, dr = 0, \\
&\int_{d_i}^{d_i'} T\nabla s \,.\, dr = T_{di}(s_{d_i'} - s_{d_i}), \\
&\int_{d_i'}^{a_i'} T\nabla s \,.\, dr = 0 \\
\text{and}\quad &\int_{a_i'}^{a_i} T\nabla s \,.\, dr = T_{a_i}(s_{a_i} - s_{a_i'}).
\end{aligned}\right\} \tag{17}$$

But the stream surfaces are surfaces of constant entropy,

$$s_{d_i'} = s_{a_i'} = s_{A'}$$

and

$$s_{d_i} = s_{a_i} = s_A$$

and the complete line integral is then

$$\oint_c T\nabla s \, . \, dr = (T_{a_i} - T_{d_i})(s_A - s_{A'}). \tag{18}$$

Now for any process, the change of entropy is given by

$$T\,ds = dh - \frac{1}{\rho}\,dp$$

and if the flow at inlet to the cascade has a uniform pressure and stagnation enthalpy, then

$$T_1 \frac{ds}{dz} = \frac{dh_1}{dz}$$

$$= \frac{d}{dz}\left(h_0 - \frac{q_1^2}{2}\right)$$

$$= -q_1 \frac{dq_1}{dz} \tag{19}$$

$$= -q_1 \xi_n, \tag{20}$$

where ξ_n is the component of vorticity normal to the flow at inlet. The entropy difference between the particles A and A' is therefore given by

$$s_A - s_{A'} = \frac{q_1 \xi_n}{T_1}\,\delta z \tag{21}$$

and is determined by the flow upstream of the cascade.

The local temperature difference $(T_{a_i} - T_{d_i})$ can be determined by applying the steady-flow energy equation and assuming that the working fluid is a perfect gas,

$$T_{a_i} - T_{d_i} = \frac{1}{2c_p}(q_{d_i}^2 - q_{a_i}^2). \tag{22}$$

By combining equations (21) and (22), Kelvin's circulation theorems for compressible flow through a cascade becomes

$$\frac{D\Gamma}{Dt} = \frac{q_1 \xi_n \delta z}{2c_p T_1}(q_{d_i}^2 - q_{a_i}^2). \tag{23}$$

The change of circulation around the path in the fluid is

$$\Gamma_{add'a'} - \Gamma_{ADD'A'} = \frac{q_1 \xi_n \delta z}{2c_p T_1}\left\{\int_{t_D}^{t_d} q_{d_i}^2\,dt - \int_{t_A}^{t_a} q_{a_i}^2\,dt\right\} \tag{24}$$

where the two integrals are taken over the time for the circuit to move from $ADD'A'$ to $add'a'$. It is convenient to write the two integrals as

$$\left.\begin{aligned} \int_{t_D}^{t_d} q_{d_i}^2\,dt &= \int_D^d q_{d_i} dl_d \\ \text{and} \qquad \int_{t_A}^{t_a} q_{a_i}^2\,dt &= \int_A^a q_{a_i} dl_a, \end{aligned}\right\} \tag{25}$$

where dl_d and dl_a are elementary distances along the paths followed by the particles, D to d and A to a. The change of circulation can then be expressed as

$$\Gamma_{add'a'} - \Gamma_{ADD'A'} = \frac{q_1 \xi_n \delta z}{2c_p T_1} \left\{ \int_D^d q_{d_i} \,.\, dl_d - \int_A^a q_{a_i} \,.\, dl_a \right\} \tag{26}$$

which is the form of Kelvin's circulation theorem most useful for secondary flow analysis.

5. Distributed Secondary Vorticity

Hawthorne[5] has identified three components of vorticity in the direction of flow at exit from a cascade. One component is distributed in the flow while the other two components form the vortex sheet leaving the blade trailing edge. The first component is the distributed secondary vorticity derived by Squire and Winter[6] and is caused by turning the shear flow in the cascade. Hawthorne has called the second component the trailing filament vorticity, this being caused by the stretching of the vortex filaments as they move over the surface of the blades. The third component is the trailing shed vorticity which is caused by the variation of circulation along the span of the blades. Expressions for the three components of streamwise vorticity can be obtained by applying Kelvin's circulation theorem to the flow through a many-bladed cascade.

Fig. 5a shows a shear flow passing through a many-bladed cascade. As before, capital letters denote fluid particles upstream of the cascade and small letters are the same particles at some later time when they have passed through the cascade. The particles A, B, C and D pass through the same blade passage while E passes through a neighbouring passage. Fig. 5a shows the flow in a plane at a spanwise position z and following the notation used earlier, corresponding points on a parallel plane at $z + \delta z$ are given a prime suffix.

Consider the circulation around the path $add'a'$ in the fluid which lies normal to the flow at exit from the cascade, Fig. 5b,

$$[\text{circulation}]_{add'a'} = [\xi_{\text{sec}} \times ad \times \delta z] \tag{27}$$

where ξ_{sec} is the distributed secondary vorticity in the flow at exit from the cascade. Now from Kelvin's circulation theorem, in the form of equation (26), the change in circulation as the circuit $ADD'A'$ moves to $add'a'$ is given by

$$\Gamma_{add'a'} - \Gamma_{ADD'A'} = \frac{q_1 \xi_n \delta z}{2c_p T_1} \left\{ \int_D^d q_{d_i} dl_d - \int_A^a q_{a_i} dl_a \right\}. \tag{28}$$

The two integrals can be expressed symbolically as

$$\left.\begin{aligned} \int_D^d &= \int_D^C + \int_C^B + \int_B^G + \int_G^H + \int_H^d \\ \text{and} \quad \int_A^a &= \int_A^F + \int_F^J + \int_J^K + \int_K^a. \end{aligned}\right\} \tag{29}$$

For a many-bladed cascade, there are no variations in the flow upstream and downstream of the cascade, so that

$$\left.\begin{aligned} \int_A^F q_{a_i} dl_a &= \int_B^G q_{d_i} dl_d \\ \text{and} \quad \int_K^a q_{a_i} dl_a &= \int_H^d q_{d_i} dl_d \end{aligned}\right\} \tag{30}$$

and these four integrals cancel. The remaining five integrals may be obtained as follows,

$$\int_D^C q_{d_i} dl_d = q_1 \,.\, DC, \tag{31}$$

$$\int_B^C q_{d_i} dl_d = q_1 s \sin \alpha_1, \tag{32}$$

$$\int_J^K q_{a_i} dl_a = q_2 s \sin \alpha_2 \tag{33}$$

and

$$\int_G^H q_{d_i} dl_d - \int_F^J q_{a_i} dl_a = \int_{\text{le}}^{\text{te}} \frac{(u_{d_i} - u_{a_i})}{\cos^2 \alpha} dx,$$

and substituting from equation (7)

$$\int_H^G q_{d_i} dl_d - \int_F^J q_{a_i} dl_a = s \int_{\text{le}}^{\text{te}} \frac{d\bar{v}}{dx} . dx$$
$$= s(q_2 \sin \alpha_2 - q_1 \sin \alpha_1). \tag{34}$$

If equations (30) to (34) are substituted into the two integrals of equations (28) then many terms cancel and the change of circulation is given by

$$\Gamma_{add'a'} - \Gamma_{ADD'A'} = \frac{q_1 \xi_n \delta z}{2c_p T_1}(q_1 DC)$$
$$= \left(\frac{\gamma - 1}{2}\right) M_1^2 \xi_n . DC . \delta z \tag{35}$$

where γ is the ratio of specific heats for the gas. The change of circulation on passing through the cascade is therefore dependent on the gas, the Mach number and normal component of vorticity at inlet, and the geometry of the cascade, which determines the magnitude of DC.

If the flow at inlet to the cascade has vorticity components ξ_s and ξ_n along and normal to the flow, then the circulation around the path $ADD'A'$ is

$$\Gamma_{ADD'A'} = \left[\xi_s\left(\frac{AC}{AD}\right) + \xi_n\left(\frac{DC}{AD}\right)\right] AD . \delta z$$
$$= [\xi_s . AC + \xi_n . DC]\delta z. \tag{36}$$

Combining equations (27), (35) and (36), the distributed secondary vorticity at exit from the blade row is given by

$$\xi_{\text{sec}} = \frac{1}{ad}\left\{\xi_s . AC + \left[1 + \left(\frac{\gamma - 1}{2}\right) M_1^2\right] \xi_n . DC\right\}. \tag{37}$$

The distances ad, AC and DC can be expressed as

$$\left.\begin{aligned} ad &= s \cos \alpha_2, \\ AC &= s \cos \alpha_1 \\ \text{and} \quad DC &= \frac{[\Delta x]_{DC}}{\cos \alpha_1} \end{aligned}\right\} \tag{38}$$

where $[\Delta x]_{DC}$ is the difference in the x coordinates of C and D. However,

$$[\Delta x]_{DC} = [\Delta x]_{DB} - [\Delta x]_{BC}$$
$$= [\Delta x]_{DB} - s \sin \alpha_1 \cos \alpha_1$$

and

$$[\Delta x]_{DB} = u_1[\Delta t]_{DB},$$

where $[\Delta t]_{DB}$ is the time taken for a particle at D to travel to B in the upstream flow. Now the time taken for a particle to travel from D to B in the upstream flow is the same as the time taken for the particle d to travel to b in the downstream flow, so that

$$\begin{aligned}[\Delta x]_{DB} &= u_1[\Delta t]_{db} \\ &= \frac{u_1}{u_2}[\Delta x]_{db} \\ &= \frac{u_1}{u_2}\{[\Delta x]_{de} + [\Delta x]_{eb}\}.\end{aligned}$$

The two distances $[\Delta x]_{de}$ and $[\Delta x]_{eb}$ can be written in terms of the flow at exit and the time difference Δt for particles which travel over the pressure and suction surfaces of the blades

$$[\Delta x]_{de} = s \sin \alpha_2 \cos \alpha_2$$

and

$$[\Delta x]_{eb} = u_2\, \Delta t,$$

so that

$$[\Delta x]_{DC} = s\frac{u_1}{u_2} \sin \alpha_2 \cos \alpha_2 + u_1\, \Delta t - s \sin \alpha_1 \cos \alpha_1. \tag{39}$$

Equations (38) and (39) determine the distances ad, AC and DC and substituting into equation (37), an expression is obtained for the distributed secondary vorticity,

$$\xi_{sec} = \frac{\cos \alpha_1}{\cos \alpha_2} \cdot \xi_s + \frac{[1+([\gamma-1]/2)M_1^2]\xi_n}{\cos \alpha_1 \cos \alpha_2}\left\{\frac{u_1}{2u_2} \sin 2\alpha_2 - \frac{1}{2} \sin 2\alpha_1 + \frac{u_1}{s}\Delta t\right\}. \tag{40}$$

This is a general expression for the distributed secondary vorticity produced in compressible flow through a many-bladed cascade. An alternative form of equation (40) is obtained by substituting for the time difference Δt from equation (15),

$$\xi_{sec} = \frac{\cos \alpha_1}{\cos \alpha_2}\xi_s + \frac{[1+([\gamma-1]/2)M_1^2]\xi_n}{\cos \alpha_1 \cos \alpha_2}\left\{\int_{le}^{te} \frac{\rho}{\rho_1}(1+\cos 2\alpha)\, dx\right\}. \tag{41}$$

When the Mach number is low, $M_1 \approx 0$ and $\rho \approx \rho_1$, then equation (41) becomes

$$\xi_{sec} = \frac{\cos \alpha_1}{\cos \alpha_2}\xi_s + \frac{\xi_n}{\cos \alpha_1 \cos \alpha_2}\{\tfrac{1}{2}(\sin 2\alpha_2 - \sin 2\alpha_1) + (\alpha_2 - \alpha_1)\}, \tag{42}$$

which is the expression derived by Came and Marsh[2] for incompressible flow.

When the flow angles α_1 and α_2 are small and the streamwise component of vorticity in the inlet flow is zero, then

$$\xi_{sec} = 2\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right]\xi_n(\alpha_2 - \alpha_1) \tag{43}$$

which reduces to

$$\xi_{sec} = 2\xi_n(\alpha_2 - \alpha_1) \tag{44}$$

for incompressible flow. Equation (43) is therefore a more general form of the expression derived by Squire and Winter[6] for incompressible flow, equation (44). Equation (43) shows that when the flow angles are small, the distributed secondary vorticity increases as the inlet Mach number is increased.

6. Trailing Filament Vorticity

The vortex sheet leaving the trailing edge of the blade has two components, the trailing filament and the trailing shed vorticities. The trailing filament vorticity is caused by the stretching of vortex filaments over the surface of the blades while the trailing shed vorticity is associated with the variation of circulation along the span of the blades. Fig. 6 shows a path $abb'a'$ around a section of the trailing vortex sheet. The circulation around this path can be separated into three parts,

$$\Gamma_{abb'a'} = \Gamma_{abla} - \Gamma_{b'a'l'b'} + \Gamma_{albb'l'a'} \tag{45}$$

where the points l and l' lie at the blade leading edge. The first two terms represent the change of circulation along the blade, the shed circulation. The third term is the difference between the total trailing circulation and the shed circulation, namely the trailing filament circulation.

The circulation around the path $alb'l'a'a$ can be related to the circulation around the same material curve at some previous time, such as $ABB'A'$ in the upstream flow and from equation (26),

$$\Gamma_{albb'l'a'a} - \Gamma_{ABB'A'} = \frac{q_1 \xi_n \delta z}{2c_p T_1}\left\{\int_B^b q_{b_i} dl_b - \int_A^a q_{a_i} dl_a\right\}. \tag{46}$$

The fluid particles a and b have passed over the suction and pressure surfaces of the blade respectively. In the flow upstream of the cascade, the particles A and B must therefore be separated by a distance $q_1\,\Delta t$ where Δt is the difference in time for particles to travel over the two surfaces of the blade. The circulation around the path $ABB'A'$ in the upstream flow is then

$$\Gamma_{ABB'A'} = -q_1\,\Delta t\,.\,\xi_n \delta z. \tag{47}$$

The two integrals in equation (46) can be expressed symbolically as

$$\begin{aligned}\int_B^b - \int_A^a &= \int_B^l + \int_l^b - \int_A^B - \int_B^l - \int_l^a \\ &= \int_l^b - \int_l^a - \int_A^B,\end{aligned}$$

so that

$$\int_B^b q_{b_i} dl_b - \int_A^a q_{a_i} dl_a = \int_{\text{le}}^{\text{te}} \frac{(u_{b_i} - u_{a_i})}{\cos^2\alpha}\,dx - \int_A^B q_{a_i} dl_a.$$

The first integral on the right can be evaluated by substituting for $(u_{b_i} - u_{a_i})$ from equation (7), with a_i now on the suction surface,

$$\int_{\text{le}}^{\text{te}} \frac{(u_{b_i} - u_{a_i})}{\cos^2\alpha}\,dx = -s(q_2 \sin\alpha_2 - q_1 \sin\alpha_1), \tag{48}$$

and the second integral is obtained by noting that $AB = q_1 \Delta t$,

$$\int_A^B q_{a_i} dl_a = q_1(q_1 \Delta t). \tag{49}$$

The expressions derived in equations (47), (48) and (49) may now be substituted into equation (46) to determine the trailing-filament circulation,

$$\Gamma_{albb'l'a'a} = -\xi_n q_1 \Delta t \left[1 + \left(\frac{\gamma-1}{2}\right) M_1^2\right] \delta z - $$
$$- \left(\frac{\gamma-1}{2}\right) M_1^2 \xi_n \left[\frac{q_2}{q_1} \sin\alpha_2 - \sin\alpha_1\right] \delta z. \tag{50}$$

The strength of the vortex sheet associated with the trailing filament circulation is thus

$$\xi_{fil} = -\xi_n q_1 \Delta t \left[1 + \left(\frac{\gamma-1}{2}\right) M_1^2\right] - \xi_n s \left(\frac{\gamma-1}{2}\right) M_1^2 \left[\frac{q_2}{q_1} \sin\alpha_2 - \sin\alpha_1\right], \tag{51}$$

and this forms one component of the vortex sheet leaving the blade trailing edge.

When the Mach number is low, then equation (51) reduces to

$$\xi_{fil} = -\xi_n q_1 \Delta t$$

which is the expression derived by Hawthorne[5] and Came and Marsh.[2]

7. Trailing Shed Vorticity

The trailing shed circulation is given by the change of circulation around the blade between the spanwise positions z and $z + \Delta z$,

$$\begin{aligned} \text{trailing shed circulation } &= \Gamma_{abla} - \Gamma_{a'b'l'a'} \\ &= -\frac{d}{dz}(sq_2 \sin\alpha_2 - sq_1 \sin\alpha_1)\delta z \\ &= -s(\xi_{2n} \sin\alpha_2 - \xi_{1n} \sin\alpha_1)\delta z - sq_2 \cos\alpha_2 \frac{d\alpha_2}{dz}\delta z + sq_1 \cos\alpha_1 \frac{d\alpha_1}{dz}\delta z \end{aligned} \tag{52}$$

where ξ_n is the component of vorticity normal to the flow. For the many-bladed cascade, the outlet flow angle α_2 is independent of the spanwise position z,

$$\frac{d\alpha_2}{dz} = 0$$

and at inlet, the streamwise component of vorticity is given by

$$\xi_s = -q_1 \frac{d\alpha_1}{dz}$$

where the variation of α_1 is assumed to be small. The trailing shed circulation is therefore

$$\Gamma_{shed} = -s(\xi_{2n} \sin\alpha_2 - \xi_{1n} \sin\alpha_1)\delta z - s\xi_s \cos\alpha_1 \delta z \tag{53}$$

and this expression can only be determined after calculating the change in ξ_n across the cascade.

The component of vorticity normal to the flow at exit from the cascade may be obtained by applying Kelvin's circulation theorem, equation (26), to the flow shown in Fig. 7. The fluid particles A and D lie on the same streamline and pass through the cascade to a and d. From Kelvin's circulation theorem for compressible flow,

$$\begin{aligned} \Gamma_{add'a'} - \Gamma_{ADD'A'} &= \frac{q_1 \xi_{1n} \delta z}{2c_p T_1} \left\{ \int_D^d q_{d_i} dl_d - \int_A^a q_{a_i} dl_a \right\} \\ &= \frac{q_1 \xi_{1n} \delta z}{2c_p T_1} \{q_1 AD - q_2 ad\}. \end{aligned} \tag{54}$$

The time taken for a fluid particle to pass from D to A is the same as for a particle to pass from d to a, say δt, so that

$$AD = q_1 \delta t$$

and

$$ad = q_2 \delta t. \tag{55}$$

These two expressions for AD and ad are also required when evaluating the circulation around $add'a'$ and $ADD'A'$,

$$\left.\begin{aligned} \Gamma_{add'a'} &= \xi_{2n} \,.\, ad \,.\, \delta z_2 \\ &= \xi_{2n} \,.\, q_2 \delta t \,.\, \delta z_2 \\ \Gamma_{ADD'A'} &= \xi_{1n} \,.\, AD \,.\, \delta z_1 \\ &= \xi_{1n} \,.\, q_1 \delta t \,.\, \delta z_1 . \end{aligned}\right\} \tag{56}$$

If the secondary flow is caused by a weak shear flow, a small perturbation on a uniform flow, then $\delta z_1 \approx \delta z_2$ and the primary flow takes place on planes of constant thickness.

These expressions for the circulations and distances can now be substituted into equation (54) to obtain ξ_{2n},

$$q_2 \xi_{2n} - q_1 \xi_{1n} = \frac{q_1 \xi_{1n}}{2 c_p T_1} [q_1^2 - q_2^2] \tag{57}$$

or

$$q_2 \xi_{2n} = \frac{q_1 \xi_{1n}}{c_p T_1} \left[c_p T_1 + \frac{q_1^2}{2} - \frac{q_2^2}{2} \right].$$

But with constant stagnation enthalpy,

$$c_p T_1 + \frac{q_1^2}{2} = c_p T_2 + \frac{q_2^2}{2},$$

so that the two normal components of vorticity are related by

$$\frac{q_2 \xi_{2n}}{T_2} = \frac{q_1 \xi_{1n}}{T_1}, \tag{58}$$

or in terms of the inlet and exit Mach numbers

$$\xi_{2n} = \xi_{1n} \frac{M_1}{M_2} \left[\frac{1 + ([\gamma - 1]/2) M_1^2}{1 + ([\gamma - 1]/2) M_2^2} \right]^{\frac{1}{2}}. \tag{59}$$

For compressible flow through a cascade, there can be a large change in the Mach number and this can lead to a large change in the normal component of vorticity. Equations (58) and (59) are a generalisation of the analysis given by Came and Marsh[2] for the change in the normal component of vorticity in incompressible flow through a cascade.

For the purpose of this report, the more useful expression for ξ_{2n} is equation (57) in the form

$$\xi_{2n} = \frac{q_1}{q_2} \xi_{1n} + \left(\frac{\gamma - 1}{2} \right) M_1^2 \xi_{1n} \left[\frac{q_1}{q_2} - \frac{q_2}{q_1} \right] \tag{60}$$

and substituting into equation (53) we have

$$\Gamma_{\text{shed}} = -s \xi_{1n} \left[\frac{q_1}{q_2} \sin \alpha_2 - \sin \alpha_1 \right] \delta z + \left(\frac{\gamma - 1}{2} \right) M_1^2 s \xi_{1n} \sin \alpha_2 \left[\frac{q_2}{q_1} - \frac{q_1}{q_2} \right] \delta z - s \xi_s \cos \alpha_1 \delta z. \tag{61}$$

This equation shows how compressibility affects the trailing shed circulation.

The ratio of inlet to outlet velocity can be written as

$$\frac{q_1}{q_2}=\frac{u_1}{u_2}\frac{\cos\alpha_2}{\cos\alpha_1}$$

and substituting this expression into equation (61), the strength of the trailing vortex sheet caused by the trailing shed vorticity is

$$\xi_{\text{shed}}=-\frac{s\xi_n}{\cos\alpha_1}\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right]\left[\frac{u_1}{u_2}\frac{\sin 2\alpha_2}{2}-\frac{\sin 2\alpha_1}{2}\right]+$$
$$+\left(\frac{\gamma-1}{2}\right)M_1^2 s\xi_n\left[\frac{q_2}{q_1}\sin\alpha_2-\sin\alpha_1\right]-s\xi_s\cos\alpha_1 \tag{62}$$

where ξ_n is the normal component of vorticity at inlet. Comparison with equations (40) and (51) shows that the three streamwise components of vorticity have many terms in common.

For flows with a low Mach number, the trailing shed vorticity is

$$\xi_{\text{shed}}=-\frac{s\xi_n}{\cos\alpha_1}\left[\frac{\sin 2\alpha_2}{2}-\frac{\sin 2\alpha_1}{2}\right]-s\xi_s\cos\alpha_1, \tag{63}$$

which is the expression derived by Came and Marsh[2] for incompressible flow. Equation (63) differs from the expression for ξ_{shed} which has previously been used in secondary flow theory and this is discussed in detail in Ref. 2. It is shown that equation (63), which is based on vorticity dynamics, leads to a calculated secondary flow which is consistent with the strength of the trailing vortex sheet.

When the exit flow angle α_2 is zero, then the trailing shed vorticity is

$$\xi_{\text{shed}}=-s\xi_n\sin\alpha_1-s\xi_s\cos\alpha_1. \tag{64}$$

The inlet Mach number does not enter this equation and for $\alpha_2=0$, the trailing shed vorticity is determined by the inlet vorticity components and flow angle α_1. When the flow angles α_1 and α_2 are both small, then the trailing shed vorticity, as given by equation (62) becomes

$$\xi_{\text{shed}}=-s\xi_n(\alpha_2-\alpha_1)-s\xi_s \tag{65}$$

and again, the inlet Mach number does not enter the expression for ξ_{shed}. The results given by equations (64) and (65) suggest that compressibility has little effect on the trailing shed vorticity and this is discussed later in more detail.

8. The Total Secondary Circulation in the Downstream Flow

The theory of secondary flow developed in this report has been based on compressible flow through a many-bladed cascade and this has led to formulae for the three streamwise components of vorticity at exit from the cascade

$$\xi_{\text{sec}}=\frac{\cos\alpha_1}{\cos\alpha_2}\cdot\xi_s+\frac{\xi_n[1+([\gamma-1]/2)M_1^2]}{\cos\alpha_1\cos\alpha_2}\left\{\frac{u_1}{2u_2}\sin 2\alpha_2-\frac{1}{2}\sin 2\alpha_1+\frac{u_1\Delta t}{s}\right\}, \tag{66a}$$

$$\frac{\xi_{\text{fil}}}{s}=-\xi_n q_1\frac{\Delta t}{s}\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right]-$$
$$-\xi_n\left(\frac{\gamma-1}{2}\right)M_1^2\left[\frac{q_2}{q_1}\sin\alpha_2-\sin\alpha_1\right] \tag{66b}$$

and

$$\frac{\xi_{\text{shed}}}{s}=-\frac{\xi_n[1+([\gamma-1]/2)M_1^2]}{\cos\alpha_1}\left[\frac{u_1}{2u_2}\sin 2\alpha_2-\frac{1}{2}\sin 2\alpha_1\right]+$$

$$+\xi_n\left(\frac{\gamma-1}{2}\right)M_1^2\left[\frac{q_2}{q_1}\sin\alpha_2-\sin\alpha_1\right]-\xi_s\cos\alpha_1. \tag{66c}$$

The total secondary circulation per unit pitch in the downstream flow is obtained by considering the circulation around the circuit $acc'a'$ which lies normal to the flow at exit from the cascade, as shown in Fig. 8,

$$\frac{1}{s}\Gamma_{acc'a'}=\xi_{\text{sec}}\cos\alpha_2+\frac{\xi_{\text{fil}}}{s}+\frac{\xi_{\text{shed}}}{s}=0 \tag{67}$$

In the flow downstream of the many-bladed cascade, the total secondary circulation is always zero and there is no secondary flow. This result is consistent with the mathematical model in that for a many-bladed cascade, the pitch is infinitesimal, there is no variation of outlet flow angle along the span of the blades and there can be no secondary flow. Although it is possible to identify three streamwise components of vorticity at exit from a many-bladed cascade, the theory predicts that there is no net secondary circulation in the downstream flow. A similar conclusion was reached by Preston[7] in 1954 for a row of impulse blades, $\alpha_2=-\alpha_1$, and Came and Marsh[2] have shown that for incompressible flow through a many-bladed cascade, the total secondary circulation is always zero.

9. The Calculation of Secondary Flow in a Real Cascade

The analysis given here predicts that the total secondary circulation in the flow downstream of a many-bladed cascade is always zero and there is no secondary flow. Many experiments with cascades have shown the existence of secondary flow and the theory must therefore be re-examined to see whether it is possible to remove the restrictions imposed by the many-bladed-cascade model.

The concept of flow through a many-bladed cascade has been used when determining Δt, the time difference for particles to travel over the pressure and suction surfaces of the blades, and also in the derivation of the trailing shed vorticity where it was assumed that α_2 does not vary along the span. The general formula for the time difference is

$$\Delta t=-\oint\frac{dl}{q} \tag{68}$$

where the integral is taken from the leading edge of the blade along the suction surface to the trailing edge and then back along the pressure surface. It is only for the many-bladed cascade that the time difference can be expressed by equation (15), but this should remain a reasonable approximation for a real cascade with non-zero pitch which has the same inlet and outlet flow angles. It can therefore be argued that equations (66a) and (66b) should give reasonable approximations for ξ_{sec} and ξ_{fil} in a real cascade, with the time difference Δt defined by equation (68). If the integral in equation (68) cannot be evaluated, then the time difference Δt can be estimated from equation (15).

When deriving the trailing shed circulation, equation (53), it was assumed that there was no variation of outlet flow angle, α_2, along the span of the blades, an essential feature of the many-bladed cascade. In a real cascade, the pitch of the blades is not zero and there may then be a variation of α_2 along the span of the blades. When the outlet flow angle varies with z, then the strength of the trailing shed vortex sheet is

$$\begin{aligned}\frac{\xi_{\text{shed}}}{s}&=\frac{\xi_n[1+([\gamma-1]/2)M_1^2]}{\cos\alpha_1}\left[\frac{u_1}{2u_2}\sin 2\alpha_2-\frac{\sin 2\alpha_1}{2}\right]+\\&+\xi_n\left(\frac{\gamma-1}{2}\right)M_1^2\left[\frac{q_2}{q_1}\sin\alpha_2-\sin\alpha_1\right]-\\&-\xi_s\cos\alpha_1-q_2\cos\alpha_2\frac{d\alpha_2}{dz}.\end{aligned} \tag{69}$$

The trailing shed vortex sheet is therefore dependent on the variation of exit flow angle produced by the secondary flow.

A consistent method for calculating the secondary flow in a real cascade is obtained by using equations (66a) and (66b) to calculate ξ_{sec} and ξ_{fil} and equation (69) for ξ_{shed}. In Fig. 8, the total secondary circulation around the path $acc'a'$ in the downstream flow is then

$$\Gamma_{acc'a'} = [\xi_{\text{sec}} s \cos \alpha_2 + \xi_{\text{fil}} + \xi_{\text{shed}}]\delta z$$
$$= -sq_2 \cos \alpha_2 \frac{d\alpha_2}{dz} . \delta z. \tag{70}$$

If the secondary velocity components are v_n normal to the primary flow and w along the span of the blades, the circulation around the path $acc'a'$ is

$$\Gamma_{acc'a'} = -s \cos \alpha_2 \frac{d\bar{v}_n}{dz} . \delta z \tag{71}$$

where $\bar{v}_n$ is the mean value of v_n across the blade passage. From the geometry of the outlet flow, if $\bar{v}_n \ll q_2$, then

$$\frac{d\bar{v}_n}{dz} = q_2 \frac{d\alpha_2}{dz}$$

and equations (70) and (71) are therefore consistent with each other.

Following the work of Hawthorne and Armstrong[8] and Loos,[3] the secondary velocity components v_n and w in the plane normal to the downstream flow, Fig. 8, satisfy the first order continuity relationship

$$\frac{\partial v_n}{\partial t_n} + \frac{\partial w}{\partial z} = 0 \tag{72}$$

and are related to the distributed secondary vorticity by

$$\frac{\partial w}{\partial y_n} - \frac{\partial v_n}{dz} = \xi_{\text{sec}}. \tag{73}$$

The secondary velocity components are therefore determined by the distributed secondary vorticity together with the boundary conditions on w and v_n which are imposed by the cascade geometry. As first shown by Loos,[3] the problem of determining v_n and w is the same for both compressible and incompressible flows. The secondary velocity components calculated from equations (72) and (73) are consistent with the strength of the trailing vortex sheet as given by equations (66b) and (69).

10. The Variation of Pressure along the Span in the Downstream Flow

It has already been assumed that in the flow upstream of the cascade, the pressure is uniform, so that equation (20) is obtained

$$T_1 \frac{ds_1}{dz} = -q_1 \xi_{1n}. \tag{20}$$

At exit from the cascade

$$T_2 \frac{ds_2}{dz} = \frac{dh_2}{dz} - \frac{1}{\rho_2} \frac{dp_2}{dz}$$

and for a flow with constant stagnation enthalpy

$$T_2 \frac{ds_2}{dz} = -q_2 \frac{dq_2}{dz} - \frac{1}{\rho_2} \frac{dp_2}{dz}. \tag{74}$$

Now if the flow takes place in planes which are normal to the span of the blades then

$$\frac{ds_1}{dz} = \frac{ds_2}{dz}$$

and eliminating the entropy gradients between equations (20) and (74), we have

$$\frac{q_1\xi_{1n}}{T_1}=\frac{q_2\xi_{2n}}{T_2}+\frac{1}{\rho_2 T_2}\frac{dp_2}{dz}, \tag{75}$$

or, introducing equation (58), which follows from Kelvin's circulation theorem, then

$$\frac{dp_2}{dz}=0. \tag{76}$$

The analysis assumes that the flow takes place in planes normal to the span of the blade, and there is then no variation of pressure along the span in the downstream flow. The theory predicts that for the flow downstream of the cascade, the pressure is uniform.

11. Comparisons with Previous Work

The first paper to consider in detail the effect of compressibility on secondary flow was that of Loos.[3] Loos derived an equation for the distributed secondary vorticity in the form

$$\frac{\xi_{sec}}{\rho_2 q_2}=\frac{\xi_s}{\rho_1 q_1}+2\int_1^2\frac{[c_p\nabla T_0-T_0\nabla s]}{\rho q^2}\,d\alpha \tag{77}$$

and showed that if the stagnation temperature is uniform throughout the flow and the flow takes place on parallel planes, then equation (77) reduces to

$$\frac{\xi_{sec}}{\rho_2 q_2}=\frac{\xi_s}{\rho_1 q_1}+2q_1\xi_n\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right]\int_1^2\frac{1}{\rho q^2}d\alpha \tag{78}$$

By introducing the continuity relationship for a many-bladed cascade

$$\rho q\cos\alpha=\text{constant},$$

the equation for ξ_{sec} can be written

$$\xi_{sec}=\frac{\cos\alpha_1}{\cos\alpha_2}\xi_s+\frac{[1+([\gamma-1]/2)M_1^2]\xi_n}{\cos\alpha_1\cos\alpha_2}\int_{le}^{te}\frac{\rho}{\rho_1}(1+\cos 2\alpha)\,d\alpha,$$

which is identical with equation (41) of this report. Loos evaluated the integral by expressing the product ρq in terms of the Mach number and then assuming that $M_1 \ll 1$, he used a binomial expansion in M_1. The analysis given by Loos is very different to that used here, but the two approaches lead to the same general expression for the distributed secondary vorticity. Loos did not consider the trailing filament and trailing shed vortex sheets.

More recently, Lakshminarayana and Horlock[9] have given general expressions for the secondary vorticity in both incompressible and compressible flow. They have shown that the growth of secondary vorticity is governed by the equation

$$\frac{\xi_{sec}}{\rho_2 q_2}=\frac{\xi_s}{\rho_1 q_1}+\int_1^2\frac{2}{\rho\rho_0 q^2}\left(\frac{dp_0}{dz}\right)d\alpha \tag{79}$$

where ρ_0 and p_0 are the stagnation density and stagnation pressure. However, for any thermodynamic process

$$T_0\,ds=dh_0-\frac{1}{\rho_0}\,dp_0$$

and when the stagnation enthalpy is constant, then

$$\frac{1}{\rho_0}\frac{dp_0}{dz}=-T_0\frac{ds}{dz}.$$

If the flow takes place in planes which are normal to the span of the blades, then

$$\frac{ds}{dz} = \text{constant},$$

and from equation (20)

$$\frac{ds}{dz} = -\frac{q_1 \xi_n}{T_1}$$

so that

$$\begin{aligned}\frac{1}{p_0}\frac{dp_0}{dz} &= \frac{T_0}{T_1} q_1 \xi_n \\ &= \left[1 + \left(\frac{\gamma-1}{2}\right) M_1^2\right] q_1 \xi_n. \end{aligned} \qquad (80)$$

If this expression for the gradient of stagnation pressure is substituted into equation (79), then we have

$$\xi_{\text{sec}} = \frac{\cos\alpha_1}{\cos\alpha_2}\xi_s + \frac{\xi_n[1 + ([\gamma-1]/2)M_1^2]}{\cos\alpha_1 \cos\alpha_2}\int_{\text{le}}^{\text{te}} \frac{\rho}{\rho_1}(1 + \cos 2\alpha)\, d\alpha, \qquad (81)$$

which is again identical with equation (41). It is thus seen that the analysis given in this report is consistent with the general expression derived by Lakshminarayana and Horlock for the distributed secondary vorticity.

11.1. Numerical Solutions for the Distributed Secondary Vorticity

When there is no streamwise component of vorticity at inlet to the cascade, then the distributed secondary vorticity is given by

$$(\xi_{\text{sec}})_M = \frac{\xi_n[1 + ([\gamma-1]/2)M_1^2]}{\cos\alpha_1 \cos\alpha_2}\int_{\text{le}}^{\text{te}} \frac{\rho}{\rho_1}(1 + \cos 2\alpha)\, d\alpha \qquad (82)$$

and for incompressible flow this becomes

$$(\xi_{\text{sec}})_0 = \frac{\xi_n}{\cos\alpha_1 \cos\alpha_2}\int_{\text{le}}^{\text{te}} (1 + \cos 2\alpha)\, d\alpha. \qquad (83)$$

The effect of compressibility can be shown by comparing $(\xi_{\text{sec}})_M$ and $(\xi_{\text{sec}})_0$ and forming the ratio

$$\frac{(\xi_{\text{sec}})_M}{(\xi_{\text{sec}})_0} = \frac{[1 + ([\gamma-1]/2)M_1^2]\int_{\text{le}}^{\text{te}} (\rho/\rho_1)(1 + \cos 2\alpha)\, d\alpha}{\int_{\text{le}}^{\text{te}} (1 + \cos 2\alpha)\, d\alpha} \qquad (84)$$

or integrating the denominator,

$$\frac{(\xi_{\text{sec}})_M}{(\xi_{\text{sec}})_0} = \frac{[1 + ([\gamma-1]/2)M_1^2]\int_{\text{le}}^{\text{te}} (\rho/\rho_1)(1 + \cos 2\alpha)\, d\alpha}{\alpha_2 - \alpha_1 + \frac{1}{2}(\sin 2\alpha_2 - \sin 2\alpha_1)}. \qquad (85)$$

For any cascade geometry, defined by α_1 and α_2, and inlet Mach number M_1, the integral in equation (85) can be calculated by using tables of compressible flow functions, Ref. (4), to determine the variation of ρ with α for the cascade. This calculation has been made for several cascades with a range of inlet Mach numbers and the results are described below. It is convenient to divide the cascades into those for decelerating flows and those for accelerating flows. The impulse blade row, where $\alpha_2 = -\alpha_1$, is an example where there is no net change of velocity and this forms a limiting case for both decelerating and accelerating flows.

11.1.1. *Decelerating flows.* When the flow is turned from an inlet angle α_1 to an outlet angle α_2 where $|\alpha_2|<|\alpha_1|$, then the net result is a deceleration of the flow. The ratio $(\xi_{sec})_M/(\xi_{sec})_0$ has been calculated for a wide range of cascades with

$$\alpha_1 = 20, 40, \text{ and } 60 \text{ degrees}$$

$$-\alpha_1 < \alpha_2 < \alpha_1$$

and

$$M_1 = 0{\cdot}6, 0{\cdot}8 \text{ and } 1{\cdot}0,$$

and the results are shown in Figs. 9, 10 and 11. For a decelerating flow, the distributed secondary vorticity increases as the inlet Mach number increases. The effect of inlet Mach number can be inferred from equation (84) in that for a decelerating flow, the density is always greater than the density at inlet and hence

$$\frac{(\xi_{sec})_M}{(\xi_{sec})_0} > 1.$$

This confirms the general conclusion reached by Loos,[3] that compressibility has a significant effect on a decelerating flow, such as the flow in a compressor cascade. For example, with an inlet Mach number of 0·8, an inlet angle of 40 degrees, and an outlet angle of 20 degrees, the distributed secondary vorticity is 1·28 times the value for incompressible flow.

It is interesting to note that for a given inlet angle α_1, the ratio $(\xi_{sec})_M/(\xi_{sec})_0$ is exactly the same for cascades with $\alpha_2=0$ and $\alpha_2=-\alpha_1$ (impulse). The calculations therefore show that the effect of compressibility is extremely important for a cascade of impulse blades. For example, if a cascade of impulse blades has an inlet angle of 60 degrees and an inlet Mach number of 1·0, then at outlet from the cascade, the distributed secondary vorticity is 1·76 times the value for incompressible flow.

When the cascade deflects the flow through a very small angle, then there is a negligible change in density and

$$\lim_{\alpha_2\to\alpha_1}\left\{\frac{(\xi_{sec})_M}{(\xi_{sec})_0}\right\} = 1+\left(\frac{\gamma-1}{2}\right)M_1^2.$$

Figs. 9, 10 and 11 show that this limiting value for $(\xi_{sec})_M/(\xi_{sec})_0$ sets a lower bound on the effect of compressibility on a decelerating flow.

11.1.2. *Accelerating flows.* Loos[3] suggested that compressibility would have very little effect on the distributed secondary vorticity produced at exit from a cascade of turbine blades, an example of an accelerating flow. Calculations of the ratio $(\xi_{sec})_M/(\xi_{sec})_0$ have been made for several cascades with accelerating flow, $|\alpha_2|>|\alpha_1|$, and some results are shown in Fig. 12. For convenience, the results are plotted for a constant exit flow angle of -60 degrees, a variable inlet flow angle α_1 and three values for the exit Mach number M_2. It is seen that for a wide range of inlet flow angles, -40 degrees $<\alpha_1<+40$ degrees, compressibility has very little effect on the distributed secondary vorticity at exit. For a cascade of turbine blades with very small deflection

$$\lim_{\alpha_1\to\alpha_2}\left\{\frac{(\xi_{sec})_M}{(\xi_{sec})_0}\right\} = 1+\left(\frac{\gamma-1}{2}\right)M_2^2$$

which agrees with the corresponding result for a compressor cascade with small deflection.

When the inlet flow angle approaches +60 degrees, then the blades become an impulse design with $\alpha_2=-\alpha_1$, the ratio $(\xi_{sec})_M/(\xi_{sec})_0$ rises and for an exit Mach number of unity (corresponding to an inlet Mach number of unity) the ratio is 1·76. This is the same value for $(\xi_{sec})_M/(\xi_{sec})_0$ as that obtained by considering the cascade of impulse blades as the limiting case for decelerating flows.

11.2. A Comparison with the Calculations of Loos

Loos[3] calculated the effect of compressibility on the distributed secondary vorticity for a cascade where the passage width varied linearly with turning angle and the ratio of inlet to outlet width was 1·0 to 1·4. This example is equivalent to a compressor cascade which turns the flow from an inlet flow angle of 44·2 degrees to an outlet angle of 0 degrees. Loos calculated the ratio $(\xi_{sec})_M/(\xi_{sec})_0$ and Fig. 13 shows a comparison with the values obtained from equation (85) of this report. Loos used a binomial expansion in terms of the inlet Mach number which required that $M_1 \ll 1$, so that his calculations are likely to be less reliable as the Mach number increases toward unity. Fig. 13 shows that at low Mach numbers, there is close agreement between the results of Loos and those obtained from equation (85). However, as the inlet Mach number increases, the theory developed in this report shows a more rapid increase in the ratio $(\xi_{sec})_M/(\xi_{sec})_0$. For an inlet Mach number of 0·8, the distributed secondary vorticity is increased by 34 per cent compared with the value of 24 per cent quoted by Loos. However, in view of the approximation introduced by Loos, this level of agreement at a high Mach number is satisfactory.

12. The Effect of Compressibility on the Trailing Vortex Sheet

The trailing vortex sheet contains the trailing filament and trailing shed vorticities and for a many-bladed cascade these are given by

$$\frac{1}{s}\xi_{fil} = -\xi_n q_1 \frac{\Delta t}{s}\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right] - $$

$$-\xi_n\left(\frac{\gamma-1}{2}\right)M_1^2\left[\frac{q_2}{q_1}\sin\alpha_2 - \sin\alpha_1\right]$$

and

$$\frac{1}{s}\xi_{shed} = -\frac{\xi_n}{\cos\alpha_1}\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right]\left[\frac{u_1}{2u_2}\sin 2\alpha_2 - \frac{\sin 2\alpha_1}{2}\right] +$$

$$+\xi_n\left(\frac{\gamma-1}{2}\right)M_1^2\left[\frac{q_2}{q_1}\sin\alpha_2 - \sin\alpha_1\right],$$

where it is assumed that there is no streamwise component of vorticity at inlet to the cascade. These two expressions for ξ_{fil} and ξ_{shed} can be used to show the effect of compressibility on the trailing vortex sheet. Fig. 14 shows the variation of ξ_{fil} and ξ_{shed} with inlet Mach number for a many-bladed cascade where $\alpha_1 = 40$ degrees, and $\alpha_2 = 0$ degrees. For this cascade, ξ_{shed} is independent of the inlet Mach number and compressibility therefore affects only ξ_{sec} and ξ_{fil}. From Fig. 14, it is seen that the trailing filament vorticity increases rapidly as the inlet Mach number approaches unity.

A similar conclusion for the effect of compressibility on the trailing vortex sheet is obtained considering the three streamwise components of vorticity at exit from a many-bladed cascade where α_1 and α_2 are both small,

$$\xi_{sec} = 2\xi_n(\alpha_2-\alpha_1)\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right],$$

$$\frac{1}{s}\xi_{fil} = -\xi_n(\alpha_2-\alpha_1)[1+(\gamma-1)M_1^2]$$

and

$$\frac{1}{s}\xi_{shed} = -\xi_n(\alpha_2-\alpha_1).$$

These three equations suggest that the inlet Mach number has little effect on the trailing shed vorticity and that compressibility has more effect on the trailing filament vorticity than on the distributed secondary vorticity. It is also seen that the effect of compressibility is greater for a gas with a high value for γ, the ratio of specific heats.

13. Conclusions

The new approach to secondary-flow theory developed by Came and Marsh[2] has been extended to deal with the effect of compressibility. Kelvin's circulation theorem for compressible flow has been applied to a material circuit passing through a cascade and from this, expressions have been derived for the three streamwise components of vorticity at exit from the blades. It has been shown that for the many-bladed cascade, the total secondary circulation in the downstream flow is always zero. The behaviour of a real cascade with non-zero pitch has been considered and a consistent secondary-flow theory is obtained by modifying the trailing shed vorticity to allow for the variation of exit flow angle along the span of the blade.

The comparisons with the work of Loos[3] and Lakshminarayana and Horlock[9] shows that the expression derived for the distributed secondary vorticity is entirely consistent with earlier attempts to derive equations for the development of streamwise vorticity in compressible flow. The advance which has been made is that the new expression for ξ_{sec} can be integrated numerically to show the effect of compressibility on secondary flow in any cascade. In addition, the application of Kelvin's theorem for compressible flows leads to expressions for the strength of the trailing filament and trailing shed vortex sheets. These two components of the trailing vortex sheet are particularly important when considering the flow in a turbomachine, where it is necessary to calculate the streamwise and normal components of vorticity at entry to the following blade row.

The numerical results show that compressibility can lead to a significant increase in the distributed secondary vorticity at exit from a compressor cascade, or more generally a cascade with a decelerating flow. For a cascade with an accelerating flow, such as a row of turbine blades, the calculations show that for a wide range of inlet flow angles, compressibility has very little effect on the value of ξ_{sec}. With a cascade of impulse blades, the effect of compressibility is large and for an inlet angle of 60 degrees and an inlet Mach number of unity, the distributed secondary vorticity is 1·76 times the value for incompressible flow. The overall conclusion is therefore that compressibility can have a significant effect on the secondary flow produced at exit from a cascade of blades.

LIST OF SYMBOLS

h	Enthalpy
h_0	Stagnation enthalpy
M	Mach number
p	Pressure
p_0	Stagnation pressure
q	Local velocity
r	Distance around a material circuit
s	Entropy
t	Time
T	Temperature
T_0	Stagnation temperature
u, v	Velocity components in x and y directions
$\bar{u}$, $\bar{v}$	Passage-averaged velocity components
v_n, w	Secondary velocity components
x, y, z	Coordinates
y_n	Coordinate normal to the flow at exit
α	Flow angle
β	Blade angle
γ	Ratio of specific heats, c_p/c_v
ε	Deflection
ρ	Density
ρ_0	Stagnation density
ξ_n	Normal vorticity component at inlet
ξ_s	Streamwise vorticity at inlet
ξ_{fil}	Trailing filament vorticity
ξ_{sec}	Distributed secondary vorticity
ξ_{shed}	Trailing shed vorticity

Subscripts

1	Upstream
2	Downstream

Abbreviations

le	Leading edge
te	Trailing edge

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APPENDIX

The Effect of a Spanwise Variation in Stagnation Enthalpy

The analysis developed in this report assumes that the stagnation enthalpy is uniform, but it is not difficult to remove this restriction and allow a spanwise variation of stagnation enthalpy. It is assumed that the variations in flow and gas state are such that they can be regarded as small perturbations on a uniform flow and the primary flow then takes place on planes which are normal to the span of the blades. For this more general flow, equation (26) becomes

$$\Gamma_{add'a'} - \Gamma_{ADDA} = -\frac{\delta z}{2c_p}\left(\frac{ds}{dz}\right)_1 \left\{ \int_D^d q_{d_i} dl_d - \int_A^a q_{a_i} dl_a \right\} \tag{A-1}$$

and applying this form of Kelvin's circulation theorem to the cascade flow, the three streamwise components of vorticity at exit are

$$\xi_{sec} = \frac{\cos\alpha_1}{\cos\alpha_2}\xi_s + \frac{[\xi_n - (q_1/2c_p)(ds/dz)_1]}{\cos\alpha_1 \cos\alpha_2}\left\{\frac{u_1}{2u_2}\sin 2\alpha_2 - \frac{1}{2}\sin 2\alpha_1 + \frac{u_1 \Delta t}{s}\right\}, \tag{A-2}$$

$$\frac{1}{s}\xi_{fil} = -q_1\frac{\Delta t}{s}\left[\xi_n - \frac{q_1}{2c_p}\left(\frac{ds}{dz}\right)_1\right] - \frac{q_1}{2c_p}\left(\frac{ds}{dz}\right)_1\left[\frac{q_2}{q_1}\sin\alpha_2 - \sin\alpha_1\right] \tag{A-3}$$

and

$$\frac{1}{s}\xi_{shed} = \frac{[\xi_n - (q_1/2c_p)(ds/dz)_1]}{\cos\alpha_1}\left[\frac{u_1}{2u_2}\sin 2\alpha_2 - \frac{1}{2}\sin 2\alpha_1\right] - $$
$$-\frac{q_1}{2c_p}\left(\frac{ds}{dz}\right)_1\left[\frac{q_2}{q_1}\sin\alpha_2 - \sin\alpha_1\right] - \xi_n\cos\alpha_1 - q_2\cos\alpha_2\frac{d\alpha_2}{dz} \tag{A-4}$$

When there is no spanwise variation of stagnation enthalpy then equation (20) is valid,

$$\left(\frac{ds}{dz}\right)_1 = -\frac{q_1\xi_n}{T_1},$$

and equations (A-2), (A-3) and (A-4) are then identical with equations (66a), (66b) and (69).

An alternative expression for ξ_{sec} is obtained by using the relationship

$$T_1\left(\frac{ds}{dz}\right)_1 = \left(\frac{dh_0}{dz}\right)_1 - q_1\frac{dq_1}{dz} - \frac{1}{\rho_1}\frac{dp_1}{dz},$$

so that with no spanwise variation of pressure, equation (A-2) becomes

$$\xi_{sec} = \frac{\cos\alpha_1}{\cos\alpha_2}\xi_s + \frac{\{\xi_n[1 + ([\gamma-1]/2)M_1^2] - (q_1/2c_pT_1)(dh_0/dz)_1\}}{\cos\alpha_1\cos\alpha_2} \times$$
$$\times\left\{\frac{u_1}{2u_2}\sin 2\alpha_2 - \frac{1}{2}\sin 2\alpha_1 + \frac{u_1\Delta t}{s}\right\} \tag{A-5}$$

This expression shows more clearly the effect of a spanwise variation of enthalpy on the distributed secondary vorticity.

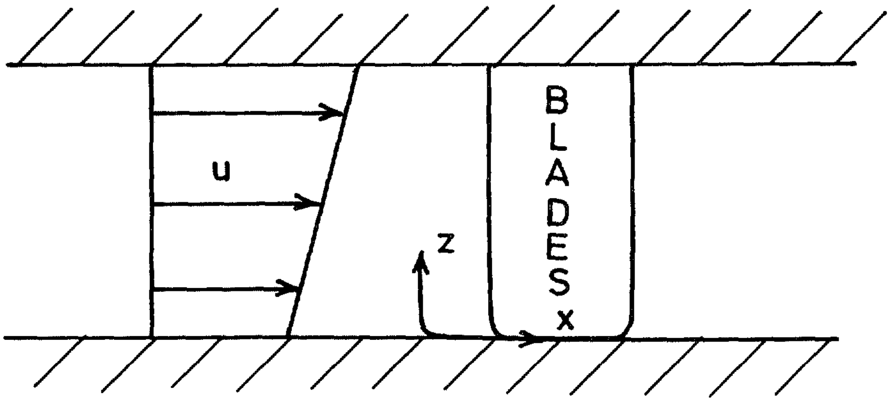


FIG. 1. Shear flow through a cascade.

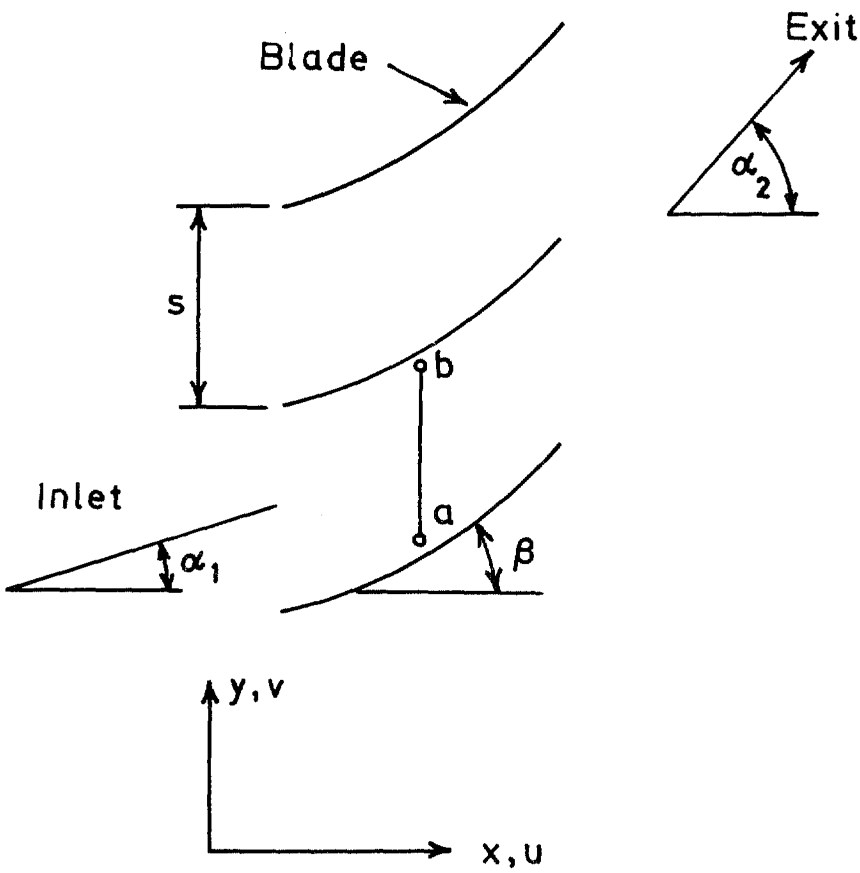


FIG. 2. Cascade of thin blades.

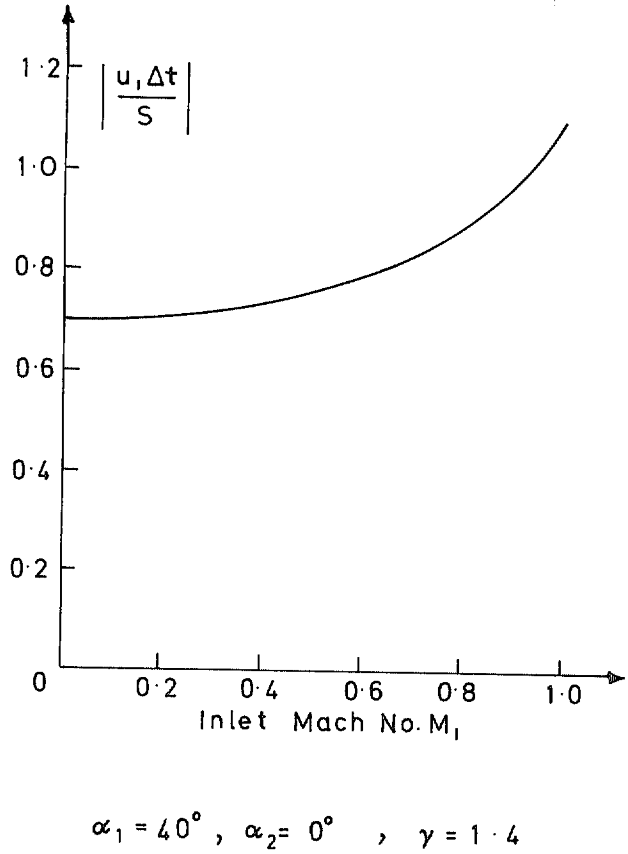


FIG. 3. The effect of Mach number on Δt.

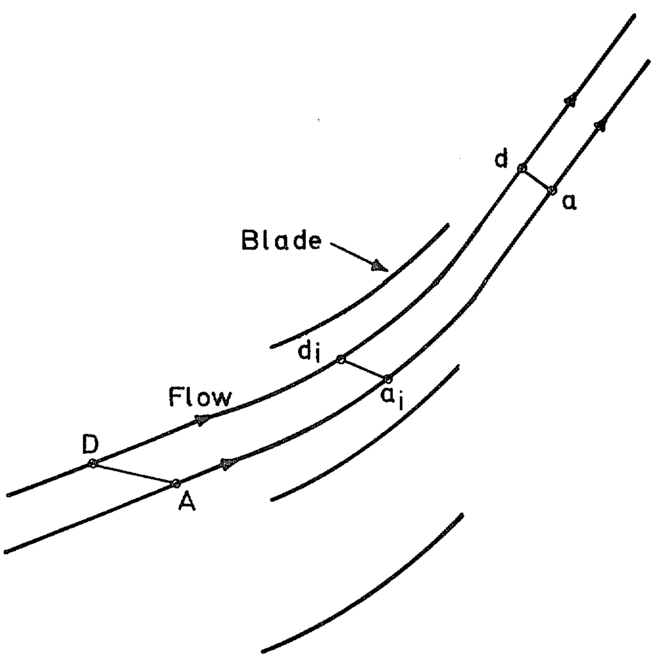


FIG. 4. Fluid particles passing through a cascade.

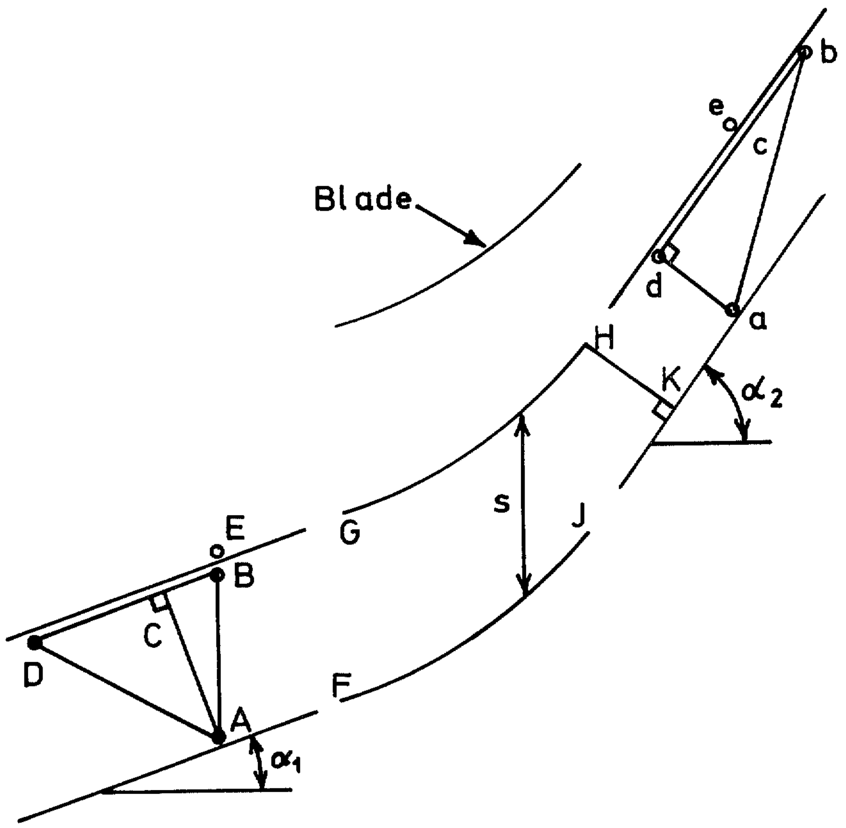


FIG. 5(a). Shear flow through a cascade.

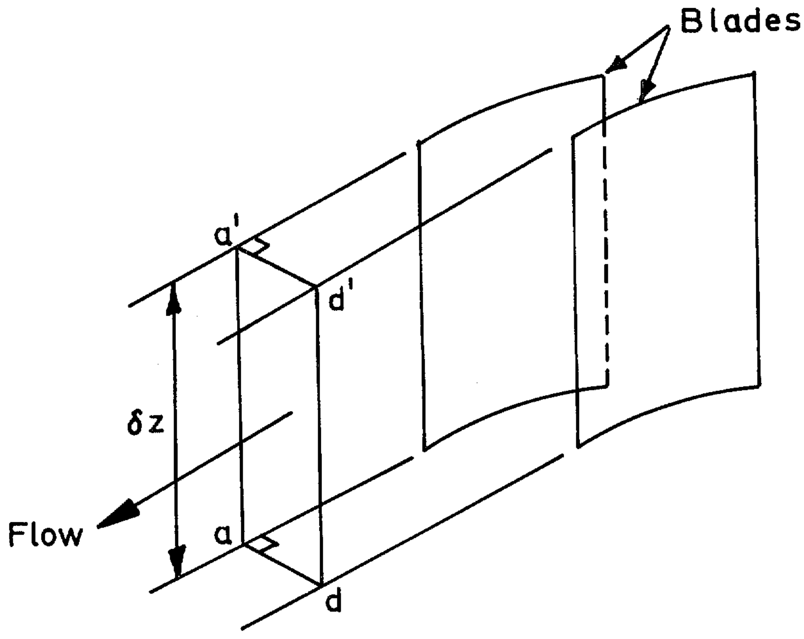


FIG. 5(b). Flow at exit from cascade.

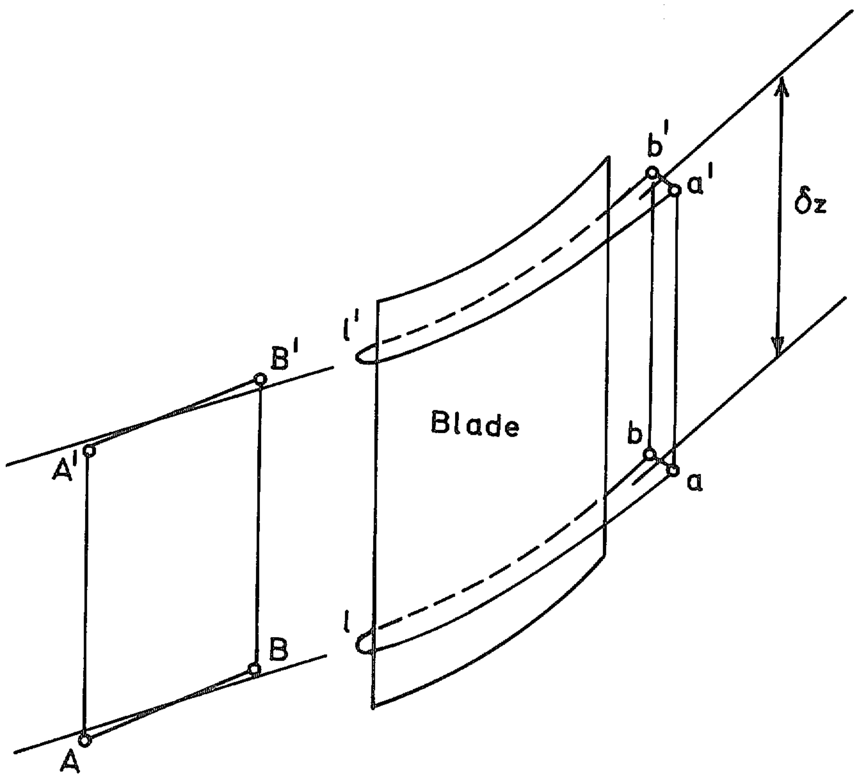


FIG. 6. Contours for trailing vorticity.

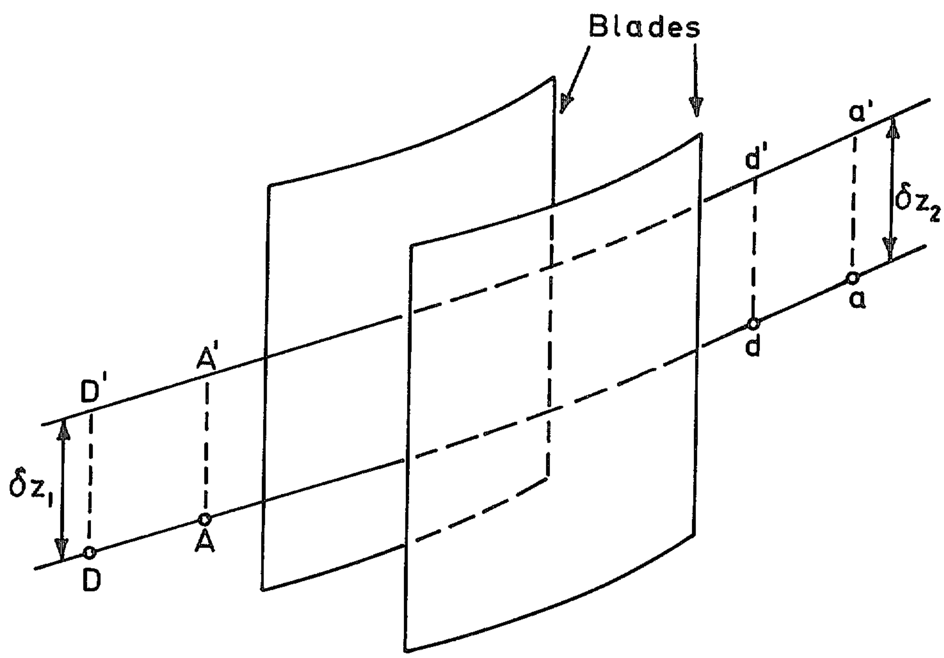


FIG. 7. Contours for the normal component of vorticity.

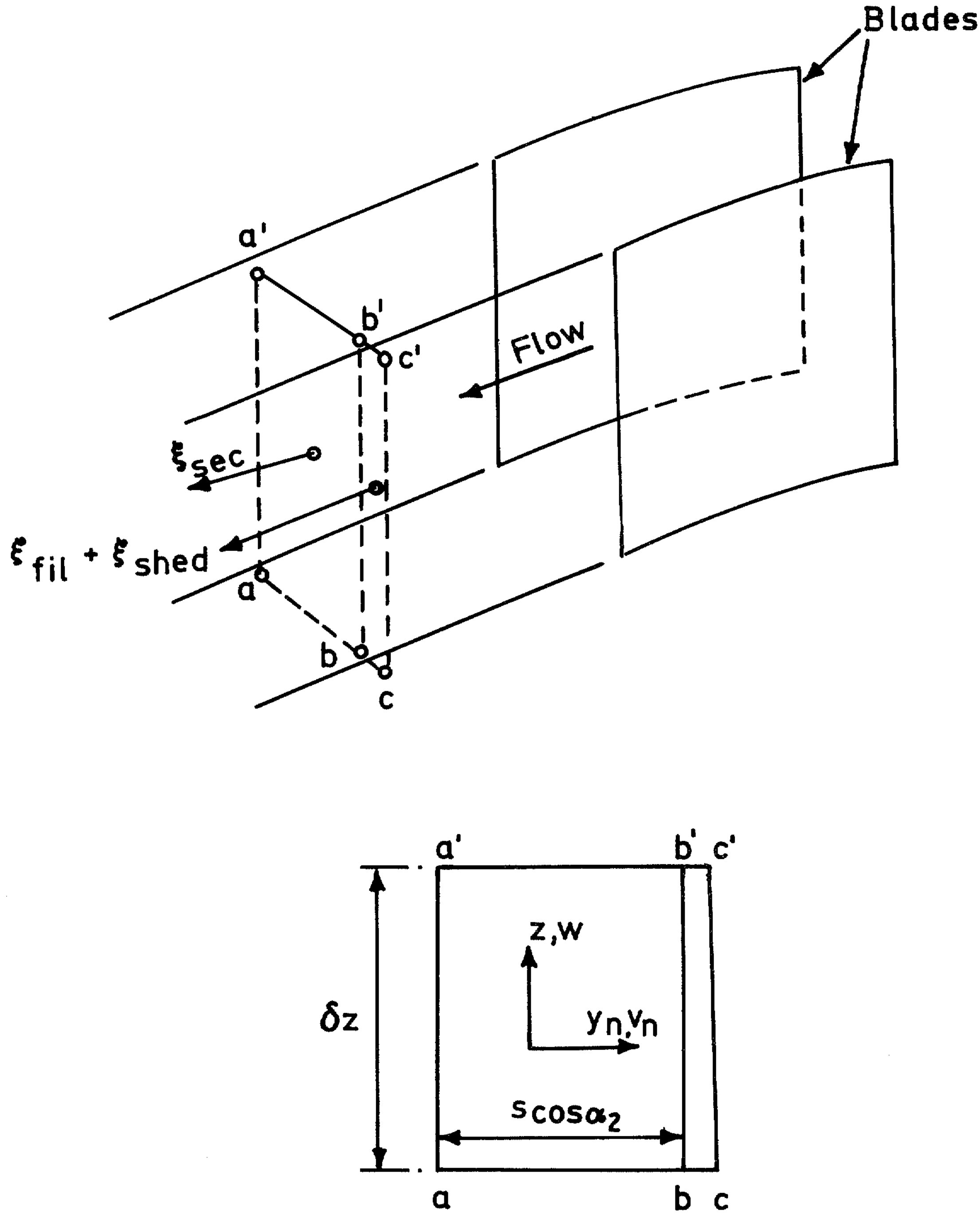


FIG. 8. Calculation plane for the secondary velocity components and total secondary circulation.

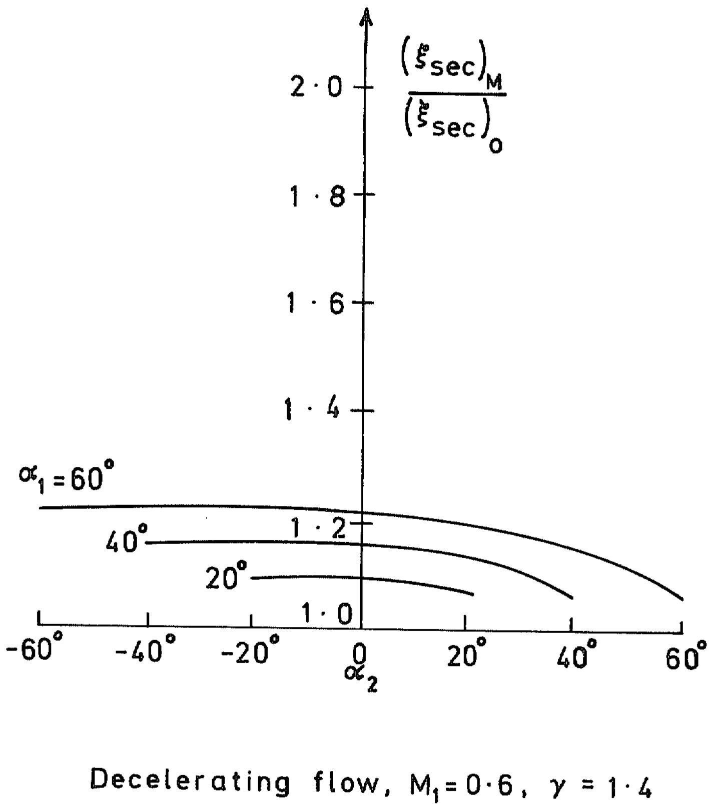


FIG. 9. The effect of compressibility.

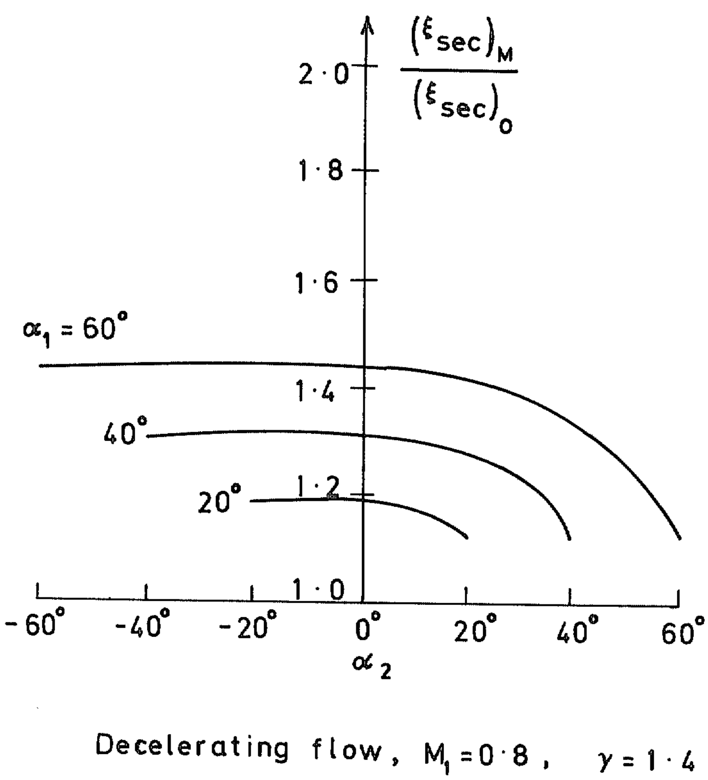
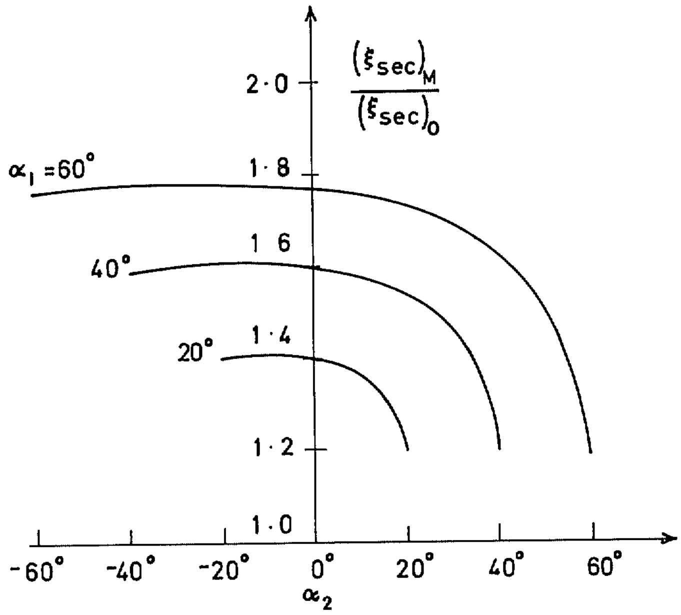
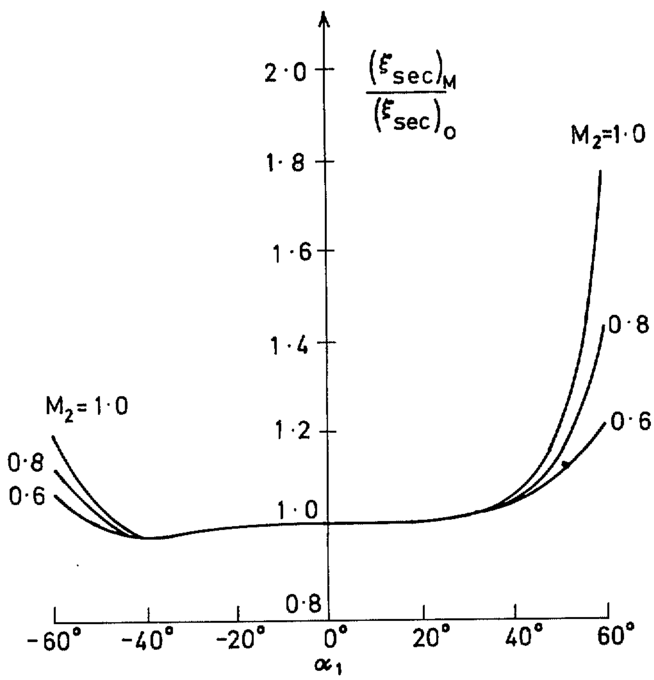


FIG. 10. The effect of compressibility.



Decelerating flow, $M_1 = 1{\cdot}0$, $\gamma = 1{\cdot}4$

FIG. 11. The effect of compressibility.



Accelerating flow, $\alpha_2 = -60°$, $\gamma = 1{\cdot}4$

FIG. 12. The effect of compressibility.

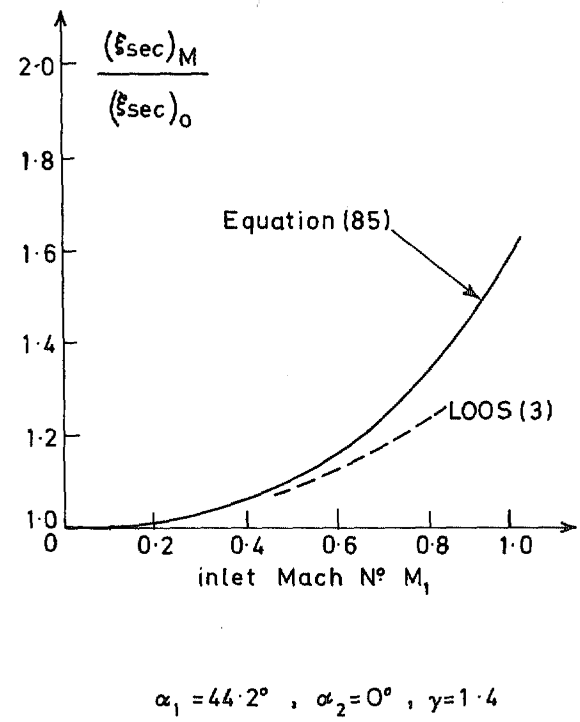


FIG. 13. Comparison with the work of Loos.[3]

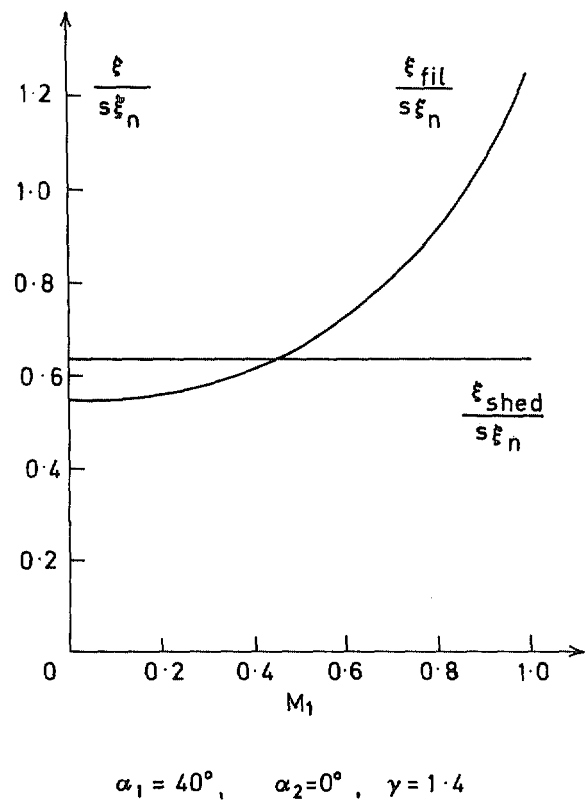


FIG. 14. The trailing vorticity components.

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