R. & M. No. 3770

R. & M. No. 3770

PROCUREMENT EXECUTIVE, MINISTRY OF DEFENCE

AERONAUTICAL RESEARCH COUNCIL

REPORTS AND MEMORANDA

# The Refraction of Sound by a Shear Layer made up of Discrete Vortices

By ANN P. DOWLING

Aerodynamics Dept., R.A.E., Farnborough

LONDON: HER MAJESTY'S STATIONERY OFFICE

1975

PRICE £2·90 NET

# The Refraction of Sound by a Shear Layer made up of Discrete Vortices

By Ann P. Dowling*

Aerodynamics Dept., R.A.E., Farnborough

*Reports and Memoranda No.* 3770†

*January,* 1974

**Summary**

Rays from a moving sound source are traced through a regular array of convected vortices (representing a turbulent jet) and also through the corresponding mean shear flow. The far-field intensity is compared in the two cases and significant differences are found.

---

* Cambridge University vacation student during summer 1973.

† Replaces R.A.E. Technical Report 74001—A.R.C. 35 489.

LIST OF CONTENTS

## 1. Introduction

In obtaining a solution of Lighthill's equation for sound generated aerodynamically, refraction is usually allowed for (if at all) by taking account of the mean shear flow in the jet and ignoring the time-dependent part of the velocity field (*cf.* Pao[1] and Lilley[2] who consider only first-order terms in the time-dependent motion). However in general this method predicts an intensity upstream which is less than the observed experimental values.

Experiment has shown that large 'puffs' and turbulent 'slugs' are produced in outlet pipes (I. J. Wyganski and F. H. Champagne[3]) and it seems reasonable to suppose that these regions of large vorticity are still present in the exhaust gas. Woolridge and Wooten[4] report the results of measurements in the initial region of a subsonic jet. They suggest that the observed pressure field might be produced by vortex rings propagating away from the jet lip. Further light has also been shed on the nature of these large vortices by the analysis of Damms and Küchemann,[5] who consider in some detail flow downstream of a splitter plate between two streams of unequal velocity where in general the time-dependent terms in the velocity are not small.

This Report uses ray-theory approximations to present an analysis of the refraction of noise by a regular structure of convected vortices and compares the result with the refraction produced by the corresponding mean shear flow. The noise producing mechanism itself is not considered here, where for simplicity we merely assume the existence of a localised source which is convected with the free stream of the jet. Furthermore the jet itself is represented in a very idealised way (a two-dimensional, single shear layer). Nevertheless the essential features are believed to be representative of a real jet, and it is therefore of interest to compare the behaviour of the rays through this time-dependent flow containing powerful discrete vortices with that through a steady shear layer.

## 2. The Velocity Field of the Convected Vortices

The analysis is restricted to two dimensions. One half of the jet flow is modelled by an infinite number of vortices arranged (as shown in Fig. 1) with their centres lying along the $x$-axis and at a distance $d$ apart. These vortices are considered to be travelling downstream with a velocity $(U/2, 0)$ and are of such a strength that the fluid far above them has a velocity $(U, 0)$; hence by symmetry the fluid far below is at rest. This model of a turbulent jet is postulated by Lau, Fisher and Fuchs[6] and they show that it agrees well with experimental results. In order to study what effect this arrangement has on the noise produced by an aircraft, a simple line-source of sound is considered in the fluid, being convected downstream with it at a height $d$ above the line of centres of the vortices (there is no special reason for choosing this height although it should be some length of this order).

A single stationary vortex is assumed to produce a velocity distribution of rigid-body rotation inside a core of radius $a$ and potential flow outside, so that

$$\left.\begin{aligned} U_\theta &= Kr \qquad & r \leqslant a \\ U_\theta &= \frac{Ka^2}{r} \qquad & r \geqslant a \end{aligned}\right\} \tag{1}$$

where $K$ = constant (to be determined) and $r$ = radial distance from the centre of the vortex.

This velocity distribution implies neglect of compressibility in the basic flow but if the true compressible flow-field were known and used, the main features of the results would not be expected to change significantly. Relative to a frame of reference moving with the vortex centres, the largest value reached by the local Mach number is about 0·5 in the examples given later.

In this reference frame, the total velocity distribution $(v_1, v_2)$ is obtained by superimposing contributions from all the vortices. Hence

$$\begin{aligned} v_1(x, y) &= -Ky - \sum_{\substack{-\infty \\ n \neq n_0}}^{\infty} \frac{Ka^2 y}{y^2 + (x - nd)^2} \qquad \text{if } y^2 + (x - n_0 d)^2 \leqslant a^2 \\ &\qquad\qquad \text{i.e. if } (x, y) \text{ lies within the } n_0\text{th vortex core} \\ &= -\sum_{-\infty}^{\infty} \frac{Ka^2 y}{y^2 + (x - nd)^2} \qquad \text{if } (x, y) \text{ is outside all the vortex cores,} \end{aligned} \tag{2}$$

$$v_2(x, y) = K(x-n_0d) + \sum_{\substack{-\infty \\ n\neq n_0}}^{\infty} \frac{Ka^2(x-n_0d)}{y^2+(x-nd)^2} \qquad \text{if } y^2+(x-n_0d)^2 \leqslant a^2,$$

$$= \sum_{-\infty}^{\infty} \frac{Ka^2(x-nd)}{y^2+(x-nd)^2} \qquad \text{if } (x, y) \text{ is outside all the vortex cores.}$$

Evaluating the summations we find

$$v_1(x, y) = -Ky + \frac{Ka^2y}{y^2+(x-n_0d)^2} - \frac{Ka^2\pi}{d}\frac{\sinh 2\pi y/d}{\cosh 2\pi y/d - \cos 2\pi x/d} \qquad \text{if } y^2+(x-n_0d)^2 \leqslant a^2,$$

$$= -\frac{Ka^2\pi}{d}\frac{\sinh 2\pi y/d}{\cosh 2\pi y/d - \cos 2\pi x/d} \qquad \text{if } (x, y) \text{ is outside all the vortex cores,}$$

$$v_2(x, y) = K(x-n_0d) - \frac{Ka^2\,(x-n_0d)}{y^2+(x-n_0d)^2} + \frac{K\pi a^2}{d}\frac{\sin 2\pi x/d}{\cosh 2\pi y/d - \cos 2\pi x/d} \qquad \text{if } y^2+(x-n_0d)^2 \leqslant a^2,$$

$$= \frac{K\pi a^2}{d}\frac{\sin 2\pi x/d}{\cosh 2\pi y/d - \cos 2\pi x/d} \qquad \text{if } (x, y) \text{ is outside all the vortex cores.}$$

We want

$$\text{as } y \to \infty, \qquad (v_1, v_2) \to \left(\frac{U}{2}, 0\right),$$

and

$$\text{as } y \to -\infty, \qquad (v_1, v_2) \to \left(-\frac{U}{2}, 0\right);$$

thus

$$-\frac{Ka^{2\circ}}{d} = \frac{U}{2}, \qquad K = -\frac{Ud}{2\pi a^2}.$$

Hence the velocity field is

$$v_1(x, y) = \frac{Ud}{2\pi a^2}y - \frac{Ud}{2\pi}\frac{y}{y^2+(x-n_0d)^2} + \frac{U}{2}\frac{\sinh 2\pi y/d}{\cosh 2\pi y/d - \cos 2\pi x/d} \qquad \text{if } y^2+(x-n_0d)^2 \leqslant a^2,$$

$$= \frac{U}{2}\frac{\sinh 2\pi y/d}{\cosh 2\pi y/d - \cos 2\pi x/d} \qquad \text{if } (x, y) \text{ is outside all vortex cores,} \tag{3}$$

$$v_2(x, y) = -\frac{Ud}{2\pi a^2}(x-n_0d) + \frac{Ud(x-n_0d)}{2\pi y^2+(x-n_0d)^2} - \frac{U}{2}\frac{\sin 2\pi x/d}{\cosh 2\pi y/d - \cos 2\pi x/d} \qquad \text{if } y^2+(x-n_0d)^2 \leqslant a^2,$$

$$= -\frac{U}{2}\frac{\sin 2\pi x/d}{\cosh 2\pi y/d - \cos 2\pi x/d} \qquad \text{if } (x, y) \text{ is outside all vortex cores.} \tag{4}$$

In a Lilley[2] or Schubert[7] approximation one considers the path of the ray through the *mean* shear flow. Since the strength of the vortices and the convection velocity are invariant with time, a time average in the observer's frame is given by an average over the $x$-component in the frame of the stationary vortices.

where $c$ = local speed of sound, $\mathbf{v}$ = local flow velocity, $\mathbf{k}$ = wave number, $k = |\mathbf{k}|$ and $(x_1, x_2) = (x, y)$ is a point on the ray.

In general the frequency will depend on the reference frame; the frequency parameter $\omega$ is defined in that frame for which the flow field does not change with time and is connected with the wave number and velocity by the relation

$$\omega = v_i k_i + kc. \tag{7}$$

Lighthill[8] shows that provided changes in the velocity field are small on a wavelength scale, $\omega$ is constant along a ray.

These equations are greatly simplified when the flow variables are functions of only one space co-ordinate. In the case of this shear flow we have $\mathbf{v} = (v_1(y), 0)$, and, since the flow is considered to be at a uniform temperature, $c$ is taken to be constant over all space.

Equation (6) gives

$$k_1 = \text{constant} = K_c \text{ (say)}$$

and so

$$k_2 = -K_c \tan \psi,$$

where $\psi$ is the angle of the wave-normal below the horizontal.

From equation (7)

$$\omega = \bar{v}_1 K_c + cK_c(1 + \tan^2 \psi)^{\frac{1}{2}},$$

$$\frac{\omega}{cK_c} = \frac{\bar{v}_1}{c} + \sec \psi \tag{8}$$

and, since $\omega/cK_c$ is constant along a ray, it follows that $\bar{v}_1/c + \sec \psi$ is also constant along a ray. Furthermore the ray paths can be traced, using equation (5), to give

$$\frac{dx}{dy} = -\operatorname{cosec} \psi \frac{\bar{v}_1(y)}{c} - \cot \psi, \tag{9}$$

where $\psi$ is determined from equation (8).

(This illustrates that the ray direction does not lie along the wave-normal when the surrounding fluid is not at rest.)

Equation (9) can be evaluated to give the path of the ray through the shear layer. However any displacement of the ray in this layer is unimportant in the far field where the *angle* at which the ray emerges from the shear layer will be of dominant importance in evaluating the $x$ components for large negative $y$.

For a given ray let $\psi$ = initial angle of wave-normal below the horizontal and $\lambda$ = final angle of wave-normal below the horizontal.

Then equation (8) implies that

$$\sec \lambda = \sec \psi + M \tag{10}$$

since, when the ray has passed through the shear layer, it is in a region of still air. In this region

$$\frac{dx}{dy} = -\cot \lambda.$$

Therefore at a large vertical distance $h$ below the shear layer $(h \gg d)$

$$x = h \cot \lambda + 0(d) \tag{11}$$

where $x$ is the horizontal distance from the source.

There is a minimum value of $\lambda$, corresponding to $\psi=0$. For this value $\lambda_{min}$ (say) we have $\sec\lambda_{min}=1+M$, by equation (10). For $M=0$ we have $\lambda_{min}=0$ and rays from the source reach all values of $x$.

For $M\neq 0$

$$\begin{aligned} x_{max} &= h[(1+M)^2-1]^{-\frac{1}{2}} \\ &= \frac{h}{(2M+M^2)^{\frac{1}{2}}}, \end{aligned} \tag{12}$$

and no sound rays are found further downstream from the source than this.

One further point is that for aircraft noise from a half-shielded source (i.e. noise emitted at the instant of crossing the jet exit plane, *see* Fig. 2) the emerging rays will have wave numbers such that $-\pi/2<\psi<\pi/2$.

Hence $\sec\lambda$ is positive for all rays, and the directions of all rays have a positive downstream component. Consequently the only noise heard upstream of the source will be due to diffraction around the lower edge of the shielding and hence will be necessarily small.

### 3.2. Through the vortices

In the frame in which the vortices are at rest we have again the case where the dispersion relationship $\omega(\mathbf{x},\mathbf{k},t)$ has no explicit dependence on time $t$.

Hence equations (5) and (6) apply, namely

$$\frac{dx_i}{dt}=v_i+c\frac{k_i}{k},$$

$$\frac{dk_i}{dt}=-k_j\frac{\partial v_j}{\partial x_i}-k\frac{\partial c}{\partial x_i},$$

where $\mathbf{v}$ is as given by equations (3) and (4), and $c$ is again taken to be constant.

We introduce non-dimensional co-ordinates

$$X=\frac{x}{d},\quad Y=\frac{y}{d},\quad M=\frac{U}{c},\quad T=\frac{tc}{d}$$

$$s=\frac{d}{a},\quad \mathbf{K}=\mathbf{k}d,\quad \boldsymbol{\kappa}=\frac{1}{K}\mathbf{K},$$

and the equations for a ray inside the $n_0$th vortex core then became (*see* Appendix A).

$$\begin{aligned} \frac{dX}{dT} &= \frac{Ms^2Y}{2\pi}-\frac{MY}{2\pi\{Y^2+(X-n_0)^2\}}+\frac{M}{2}\frac{\sinh 2\pi Y}{(\cosh 2\pi Y-\cos 2\pi X)}+\kappa_1, \\ \frac{dY}{dT} &= -\frac{Ms^2}{2\pi}(x-n_0)+\frac{M}{2\pi}\frac{X-n_0}{\{Y^2+(X-n_0)^2\}}-\frac{M}{2}\frac{\sin 2\pi X}{(\cosh 2\pi Y-\cos 2\pi X)}+\kappa_2, \\ \frac{d\kappa_i}{dT} &= \frac{1}{K}\frac{dK_i}{dT}-\kappa_i\left(\kappa_1\frac{1}{K}\frac{dK_1}{dT}+\kappa_2\frac{1}{K}\frac{dK_2}{dT}\right),\qquad (i=1,2) \end{aligned} \tag{13}$$

$$\frac{1}{K}\frac{dK_1}{dT}=-\kappa_1\left[\frac{MY(X-n_0)}{\pi\{Y^2+(X-n_0)^2\}^2}-M\pi\frac{\sinh 2\pi Y\sin 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right]-$$

$$-\kappa_2\left[-\frac{Ms^2}{2\pi}+\frac{M}{2\pi}\frac{Y^2-(X-n_0)^2}{\{Y^2+(X-n_0)^2\}^2}-M\pi\frac{\cosh 2\pi Y\cos 2\pi X-1}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right]$$

and

$$\frac{1}{K}\frac{dK_2}{dT}=-\kappa_1\left[\frac{Ms^2}{2\pi}-\frac{M}{2\pi}\frac{(X-n_0)^2-Y^2}{\{Y^2+(X-n_0)^2\}^2}+M\pi\frac{1-\cosh 2\pi Y\cos 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right]-$$

$$-\kappa_2\left[-\frac{M}{\pi}\frac{(X-n_0)Y}{\{Y^2+(X-n_0)^2\}^2}+M\pi\frac{\sin 2\pi X\sinh 2\pi Y}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right].$$

And for $(X, Y)$ outside all the vortex cores

$$\frac{dX}{dT}=\frac{M}{2}\frac{\sinh 2\pi Y}{\cosh 2\pi Y-\cos 2\pi X}+\kappa_1,$$

$$\frac{dY}{dT}=-\frac{M}{2}\frac{\sin 2\pi X}{\cosh 2\pi Y-\cos 2\pi X}+\kappa_2,$$

$$\frac{d\kappa_i}{dT}=\frac{1}{K}\frac{dK_i}{dT}-\kappa_i\left(\kappa_1\frac{1}{K}\frac{dK_1}{dT}+\kappa_2\frac{1}{K}\frac{dK_2}{dT}\right), \qquad (i=1,2) \tag{14}$$

$$\frac{1}{K}\frac{dK_1}{dT}=\kappa_1 M\pi\frac{\sinh 2\pi Y\sin 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}+\kappa_2 M\pi\frac{\cosh 2\pi Y\cos 2\pi X-1}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}$$

and

$$\frac{1}{K}\frac{dK_2}{dT}=-\kappa_1 M\pi\frac{1-\cosh 2\pi Y\cos 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}-\kappa_2 M\pi\frac{\sin 2\pi X\sinh 2\pi Y}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}.$$

These equations (13) and (14) are in a form suitable for numerical integration. To solve them a fourth order Runge–Kutta method was used with a step length in $T$ of $1/000$; in most cases the results agreed to three significant figures with those obtained with a step length of $1/10$.

The paths for several different rays are sketched in Figs. 3 to 5, with initial conditions

$$X=0, \qquad Y=1,$$

$$s=4,$$

$$\kappa_1=\cos\psi, \qquad \kappa_2=-\sin\psi$$

for various different values of the Mach number $M$. From these sketches it is observed that the region of potential flow produces very little change in the direction of a ray, but that once it enters a vortex core the ray is considerably deflected. Using the approximate value obtained in Appendix B for the deflection produced by the flow in a vortex core and neglecting any deflection produced by the potential flow, we obtain

$$\lambda=\psi-\frac{Md}{\pi a}\cos(\phi+\psi)$$

where $\phi$ represents the point at which the ray enters the $n_0$th vortex core, in the sense that

$$x-n_0 d=a\cos\phi$$

and

$$y=a\sin\phi.$$

This approximate solution agrees well with the results obtained from solving the equations exactly.

The sketches also show that it is possible for rays to cross, and Fig. 6 shows in detail the paths of intersecting rays. The crossing of rays in the far field means that $d\lambda/d\psi$ has changed sign, and so between intersecting rays there must be some $\psi$ such that

$$\frac{d\lambda}{d\psi}=0.$$

One further point illustrated by these sketches is that for a half-shielded source (such as in Fig. 2) which produces only rays with wave numbers such that $-\pi/2<\psi<\pi/2$, it is possible for rays to be refracted through a sufficiently large angle for them to travel into the upstream region. This is in contrast to the result obtained for the mean shear flow, where no sound from the shielded source is refracted upstream.

## 4. The intensity

### 4.1. The Evaluation of the Intensity

Along a ray tube $EuA$ = constant (*see* Lighthill[8]), where $E$ = density of *excess* energy per unit length due to sound source, $u$ = group velocity and $A$ = cross-sectional area of the ray tube.

Lighthill points out that there has been much confusion over what to take as $E$ in a moving fluid, and shows that a correction factor should be included, giving $E$ as the average of

$$c^2\rho^{-1}\rho_s^2\left(1+v_i\frac{k_i}{kc}\right), \tag{15}$$

where $c$ = local velocity of sound, $\rho$ = density of the fluid and $\rho_s$ = excess density due to the sound waves.

The average of $c^2\rho^{-1}\rho_s^2$ is the value of $E$ in a stationary fluid.

The source is taken to be at a height $d$ above the line of centres of the vortices. Then the velocity of the source in the listener's frame of reference is given by $U$ in the mean shear flow and by $\frac{1}{2}U(1+0(e^{-2\pi}))+U/2$ in the case of flow through the vortices.

We now apply equation (15) to a ray tube starting at the source between two rays with directions making downward angles $\theta$, and $\theta+d\theta$ with the horizontal and with wave normals making angles $\psi$, and $\psi+d\psi$. Near the source

$$E=\frac{P}{2\pi r}(1+M\cos\psi),$$

where $P$ represents the power output per unit length of the source, since in a fluid at rest $E=P/2\pi r$ (the source in a stationary medium is assumed uniform in all directions), $1+M\cos\psi$ is the directional factor[8] caused by the moving fluid. Now

$$u=\{(U+c\cos\psi)^2+c^2\sin^2\psi\}^{\frac{1}{2}}$$

and

$$A=r\delta\theta.$$

Hence

$$EuA=\frac{P}{2\pi}\delta\theta\{U^2+2Uc\cos\psi+c^2\}^{\frac{1}{2}}(1+M\cos\psi).$$

The ray direction is given by the resultant of the fluid velocity and velocity $c$ in the direction of the normal to the wave, so

$$\sin(\psi-\theta)=M\sin\theta.$$

This gives

$$\tan\theta=\frac{\sin\psi}{M+\cos\psi},$$

and on differentiating and rearranging we obtain

$$\frac{d\theta}{d\psi}=\frac{1+M\cos\psi}{M^2+2M\cos\psi+1}.$$

Hence near the source

$$\begin{aligned}EuA&=\frac{Pc}{2\pi}(M^2+2M\cos\psi+1)^{\frac{1}{2}}(1+M\cos\psi)\frac{d\theta}{d\psi}d\psi\\&=\frac{Pc}{2\pi}\frac{(1+M\cos\psi)^2}{(M^2+2M\cos\psi+1)^{\frac{1}{2}}}d\psi.\end{aligned} \tag{16}$$

The value of $E$ at the listener's position gives a measure of the intensity of the noise heard there. In the far field

$$u = c,$$
$$x = h \cot \lambda$$
$$\delta x = -h \operatorname{cosec}^2 \lambda \, d\lambda.$$

Hence

$$A = |\delta x| \sin \lambda = h \operatorname{cosec} \lambda \, \delta\lambda,$$
$$EuA = Ech \operatorname{cosec} \lambda \, \delta\lambda, \tag{17}$$

where $E$ = energy density at the listener's position, due to the sound source, $\lambda$ = angle of normal to wave below the horizontal after passing through the vortices.

Equating expressions (16) and (17) we obtain

$$E = \frac{P}{2\pi h} \frac{\delta\psi}{\delta\lambda} \frac{(1+M\cos\psi)^2 \sin\lambda}{(M^2+2M\cos\psi+1)^{\frac{1}{2}}}. \tag{18}$$

### 4.2. The Intensity Distribution Produced by the Shear Layer

Equation (10) gives

$$\sec\lambda = \sec\psi + M, \tag{19}$$

and hence

$$\frac{d\psi}{d\lambda} = \frac{\sec\lambda \tan\lambda}{\sec\psi \tan\psi}. \tag{20}$$

From equations (18), (19) and (20) we have

$$E = \frac{P}{2\pi h} \frac{(1+M\cos\psi)^2}{\sin\psi} \frac{((M^2-1)\cos^2\psi + 2M\cos\psi + 1)}{(M^2+2M\cos\psi+1)^{\frac{1}{2}}} \tag{21}$$

and

$$x = h \cot \lambda \qquad \text{as in equation (11).}$$

Corresponding to $\psi = 0$ we have $x = h/(2M+M^2)^{\frac{1}{2}}$ as given by equation (12) and for this value of $\psi$ the intensity given by equation (21) is infinite. However the ray with $\psi = 0$ is the limiting case of rays which can pass through the layer. The energy expressed in the form $(2\pi h/P)E$ is shown plotted against $x$ for different values of the Mach number in Fig. 7.

### 4.3. The Intensity Distribution Produced by the Vortices

In the case of the vortices $d\psi/d\lambda$ has to be evaluated numerically.

It was observed in Section 3.2 that the rays frequently cross (from the approximate equation (9) one expects that crossing is possible at each vortex), and associated with intersecting rays is the existence of a ray such that $d\lambda/d\psi = 0$. This gives local infinities for the value of $E$, the *excess* energy density in the far field, and hence these points are of great importance in determining the noise heard. In practice these infinities mean that at the singular points neighbouring rays have become parallel in the far field and hence in two dimensions there is no attenuation of the energy (other than that due to viscosity and the limitations of ray theory), and so the peak value is of the same order as the intensity in the near field. This can be extended into three-dimensions to a more realistic model of a jet. Considering an axisymmetric jet flow, these results suggest that along certain ray paths an attenuation like $1/r$ is to be expected instead of the usual $1/r^2$ decrease.

A further consequence of the crossing of the rays is that rays emitted from the source in different directions may arrive at the same point. Since the far-field approximation implies an interest in the behaviour over length scales of order $d$ or greater, and since any fluctuations that may arise from differences in phase will be on a length scale of $1/k$ which is small compared with $d$, energy is conserved on average, and it follows that the intensities propagating in the same direction from different initial paths should be added.

The intensity was evaluated for rays with initial angles $\psi$ at intervals of 0·01. The derivative $d\psi/d\lambda$ was found by taking $\delta\psi = 0{\cdot}001$ and evaluating the corresponding difference $\delta\lambda$.

The intensity distribution for a partly shielded source of the type illustrated in Fig. 2 is sketched in Fig. 8, and the existence of several 'infinities' is observed; it may be noted that these 'infinities' are instantaneous values and a fixed observer would hear a succession of peaks as the source and different vortices pass downstream. It may be noted also that the intensity is not zero for negative $x$ upstream of the source as it would be for a mean shear layer.

## 5. Conclusions

There are significant differences between the refraction produced by the vortex core and that produced by the shear layer. Firstly the vortices produce a completely different intensity distribution, having a number of points in the far field at which the intensity is large, and is, in fact, of the same order as that of the near field. The second point of interest is that even for a half-shielded source the vortices refract the sound so that it can be heard upstream. The shear layer produces no such refraction.

Application of this to the noise produced by a real aircraft in three-dimensions with an axisymmetric jet suggests that there are directions of large intensity in the far field where the intensity is only decaying as $1/r$. A further consequence is that methods considering only the mean shear and ignoring large-scale turbulence would be expected to predict a lower upstream intensity than is actually observed.


## Acknowledgments

I wish to thank Mr. E. G. Broadbent and Prof. J. Cooke for all their help and advice.

## LIST OF SYMBOLS

| | |
|---|---|
| $a$ | Radius of a vortex core |
| $c$ | Velocity of sound |
| $d$ | Distance between adjacent vortices |
| $E$ | Excess energy density due to sound source |
| $h$ | Vertical distance of listener below turbulent layer |
| $\mathbb{k}$ | Wave-number vector |
| $\mathbb{K}$ | Non-dimensional wave-number vector |
| $M$ | Mach number of the jet |
| $P$ | Power output per unit length of the line source |
| $r$ | Polar co-ordinate |
| $s$ | Ratio $d/a$ |
| $t$ | Time |
| $T$ | Non-dimensional time |
| $u$ | Group velocity |
| $U$ | Jet velocity |
| $\mathbb{v}$ | Fluid-velocity vector |
| $\bar{\mathbb{v}}$ | Mean-fluid-velocity vector |
| $x$ | Cartesian co-ordinate |
| $y$ | Cartesian co-ordinate |
| $X$ | Non-dimensional cartesian co-ordinate |
| $Y$ | Non-dimensional cartesian co-ordinate |
| $\theta$ | Angle of ray direction below the horizontal |
| $\boldsymbol{\kappa}$ | $\dfrac{1}{\lvert K \rvert}\mathbb{K}$ |
| $\lambda$ | Final angle of wave-front normal below the horizontal |
| $\psi$ | Initial angle of wave-front normal below the horizontal |
| $\omega$ | Frequency |

## REFERENCES


| *No.* | *Author* | *Title, etc.* |
|---|---|---|
| 1 | S. P. Pao | Developments of a generalised theory of jet noise. *J. A.I.A.A., Vol. 10,* Part 5, pp. 596–602 (1971) |
| 2 | G. M. Lilley | Sound generation in shear flow turbulence *Fluid Dynamics Transactions* (Poland) Vol. 6, Part II (1971) |
| 3 | I. J. Wyganski and F. H. Champagne | On transition in a pipe. Part I The origin of puffs and slugs and the flow in a turbulent slug. *J. Fluid Mech, Vol. 59,* Part 2, pp. 281–335 (1973) |
| 4 | C. E. Wooldridge and D. C. Wooten | A study of the large scale eddies of jet turbulence producing jet noise. A.I.A.A. Paper No. 71-154 (1971) |
| 5 | S. Dammns and D. Küchemann | On a vortex-sheet model for the mixing between two parallel streams R.A.E. Technical Report 72139 A.R.C. 34267 (1872) |
| 6 | J. C. Lau, M. J. Fisher and H. V. Fuchs | The intrinsic structure of turbulent jets. *J. Sound Vibration,* 22, 4, pp. 379–406 |
| 7 | L. K. Schubert | Numerical study of sound refraction by a jet flow. *J. Acoustical Soc. of America, Vol. 51,* No. 2 (Part 1) (1971) |
| 8 | M. J. Lighthill | The Fourth Annual Fairey Lectures: the propagation of sound through moving fluids. *J. Sound Vibration, Vol. 24,* Part 4, pp. 471–492 (1972) |

## APPENDIX A

### The Ray Tracing Equations in the Case of the Vortices

At a point inside the $n_0$th vortex core $y^2+(x-n_0d)^2 \leq a^2$, and

$$\frac{\partial v_1}{\partial x}=\frac{Ud}{\pi}\frac{(x-n_0d)y}{\{y^2+(x-n_0d)^2\}^2}-\frac{U\pi}{d}\frac{\sinh 2\pi y/d \sin 2\pi x/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2},$$

$$\frac{\partial v_1}{\partial y}=\frac{Ud}{2\pi a^2}-\frac{Ud}{2\pi}\frac{(x-n_0d)^2-y^2}{\{y^2+(x-n_0d)^2\}^2}+\frac{U\pi}{d}\frac{1-\cos 2\pi x/d \cosh 2\pi y/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2},$$

$$\frac{\partial v_2}{\partial x}=-\frac{Ud}{2\pi a^2}+\frac{Ud}{2\pi}\frac{y^2-(x-n_0d)^2}{\{y^2+(x-n_0d)^2\}^2}-\frac{U\pi}{d}\frac{\cosh 2\pi y/d \cos 2\pi x/d-1}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2}$$

and

$$\frac{\partial v_2}{\partial y}=\frac{Ud}{\pi}\frac{(x-n_0d)y}{\{y^2+(x-n_0d)^2\}^2}+\frac{U\pi}{d}\frac{\sin 2\pi x/d \sinh 2\pi y/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2}.$$

At a point $(x, y)$ outside all the vortex cores:

$$\frac{\partial v_1}{\partial x}=-\frac{U\pi}{d}\frac{\sinh 2\pi y/d \sin 2\pi x/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2},$$

$$\frac{\partial v_1}{\partial y}=\frac{U\pi}{d}\frac{1-\cos 2\pi x/d \cosh 2\pi y/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2},$$

$$\frac{\partial v_2}{\partial x}=\frac{U\pi}{d}\frac{1-\cosh 2\pi y/d \cos 2\pi x/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2}$$

and

$$\frac{\partial v_2}{\partial y}=\frac{U\pi}{d}\frac{\sinh 2\pi x/d \sinh 2\pi y/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2}.$$

Hence one obtains the equations:

(i) for a point $(x, y)$ inside the $n_0$th vortex, i.e. $(x-n_0d)^2+y^2 \leq a^2$

$$\frac{dx}{dt}=\frac{Udy}{2\pi a^2}-\frac{Ud}{2\pi}\frac{y}{\{y^2+(x-n_0d)^2\}}+\frac{U}{2}\frac{\sinh 2\pi y/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}}+c\frac{k_1}{k},$$

$$\frac{dy}{dt}=-\frac{Ud(x-n_0d)}{2\pi a^2}+\frac{Ud}{2\pi}\frac{(x-n_0d)}{\{y^2+(x-n_0d)^2\}}-\frac{U}{2}\frac{\sin 2\pi x/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}}+c\frac{k_2}{k},$$

$$\frac{dk_1}{dt}=-k_1\left[\frac{Ud}{\pi}\frac{y(x-n_0d)}{\{y^2+(x-n_0d)^2\}^2}-\frac{U\pi}{d}\frac{\sinh 2\pi y/d \sin 2\pi x/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2}\right]-$$
$$-k_2\left[-\frac{Ud}{2\pi a^2}+\frac{Ud}{2\pi}\frac{y^2-(x-n_0d)^2}{\{y^2+(x-n_0d)^2\}^2}-\frac{U\pi}{d}\frac{\cosh 2\pi y/d \cos 2\pi x/d-1}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2}\right]$$

and

$$\frac{dk_2}{dt}=-k_1\left[\frac{Ud}{2\pi a^2}-\frac{Ud}{2\pi}\frac{(x-n_0d)^2-y^2}{\{y^2+(x-n_0d)^2\}^2}+\frac{U\pi}{d}\frac{1-\cos 2\pi x/d \cosh 2\pi y/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2}\right]-$$
$$-k_2\left[-\frac{Ud}{\pi}\frac{(x-n_0d)y}{\{y^2+(x-n_0d)^2\}^2}+\frac{U\pi}{d}\frac{\sin 2\pi x/d \sinh 2\pi y/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2}\right].$$

(ii) Outside the vortices

$$\frac{dx}{dt}=\frac{U}{2}\frac{\sinh 2\pi y/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}}+c\frac{k_1}{k},$$

$$\frac{dy}{dt}=-\frac{U}{2}\frac{\sin 2\pi x/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}}+c\frac{k_2}{k},$$

$$\frac{dk_1}{dt}=\frac{k_1U\pi}{d}\frac{\sinh 2\pi y/d\sin 2\pi x/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2}+\frac{k_2U\pi}{d}\frac{\cosh 2\pi y/d\cos 2\pi x/d-1}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2},$$

$$\frac{dk_2}{dt}=\frac{k_1U\pi}{d}\frac{1-\cos 2\pi x/d\cosh 2\pi y/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2}-\frac{k_2U\pi}{d}\frac{\sin 2\pi x/d\sinh 2\pi y/d}{\{\cosh 2\pi y/d-\cos 2\pi x/d\}^2}.$$

We introduce non-dimensional co-ordinates

$$X=\frac{x}{d},\qquad M=\frac{U}{c},\qquad T=\frac{tc}{d},\qquad K_1=k_1d\quad\text{and}\quad s=\frac{d}{a}.$$

Hence we have inside the $n_0$th vortex core: $Y^2+(X-n_0)^2<1/s^2$

$$\frac{dX}{dT}=\frac{Ms^2Y}{2\pi}-\frac{MY}{2\pi\{Y^2+(X-n_0)^2\}}+\frac{M}{2}\frac{\sinh 2\pi Y}{(\cosh 2\pi Y-\cos 2\pi X)}+\frac{K_1}{K},$$

$$\frac{dY}{dT}=-\frac{Ms^2(X-n_0)}{2\pi}+\frac{M}{2\pi}\frac{(X-n_0)}{\{Y^2+(X-n_0)^2\}}-\frac{M}{2}\frac{\sin 2\pi X}{(\cosh 2\pi Y-\cos 2\pi X)}+\frac{K_2}{K},$$

$$\frac{dK_1}{dT}=-K_1\left[\frac{M}{\pi}\frac{(X-n_0)Y}{\{Y^2+(X-n_0)^2\}^2}-M\pi\frac{\sinh 2\pi Y\sin 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right]-$$
$$-K_2\left[-\frac{Ms^2}{2\pi}+\frac{M}{2\pi}\frac{Y^2-(X-n_0)^2}{\{Y^2+(x-n_0)^2\}^2}-M\pi\frac{\cosh 2\pi Y\cos 2\pi X-1}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right],$$

$$\frac{dK_2}{dT}=-K_1\left[\frac{Ms^2}{2\pi}-\frac{M}{2\pi}\frac{(X-n_0)^2-Y^2}{\{Y^2+(X-n_0)^2\}^2}+M\pi\frac{1-\cosh 2\pi Y\cos 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right]-$$
$$-K_2\left[-\frac{M}{\pi}\frac{(X-n_0)Y}{\{Y^2+(X-n_0)^2\}^2}+M\pi\frac{\sin 2\pi X\sinh 2\pi Y}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right].$$

For $(X, Y)$ outside the cores of any of the vortices

$$\frac{dX}{dT}=\frac{M}{2}\frac{\sinh 2\pi Y}{(\cosh 2\pi Y-\cos 2\pi X)}+\frac{K_1}{K},$$

$$\frac{dY}{dT}=-\frac{M}{2}\frac{\sin 2\pi X}{(\cosh 2\pi Y-\cos 2\pi X)}+\frac{K_2}{K},$$

$$\frac{dK_1}{dT}=K_1M\pi\frac{\sinh 2\pi Y\sin 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}+K_2M\pi\frac{\cosh 2\pi Y\cos 2\pi X-1}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}$$

and

$$\frac{dK_2}{dT}=-K_1M\pi\frac{1-\cosh 2\pi Y\cos 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}-K_2M\pi\frac{\sin 2\pi X\sinh 2\pi Y}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}.$$

The magnitude of $(K_1, K_2)$ is eliminated from these equations by introducing $\boldsymbol{\kappa}=(1/K)(K_1, K_2)$ and hence

$$\begin{aligned}\frac{d\kappa_i}{dT}&=\frac{d}{dT}\left(\frac{K_i}{K}\right)\\&=\frac{1}{K}\frac{dK_i}{dT}-\frac{K_i}{K^2}\frac{dK}{dT}\\&=\frac{1}{K}\frac{dK_i}{dT}-\frac{K_i}{K}\left(\frac{K_1}{K}\frac{1}{K}\frac{dK_1}{dT}+\frac{K_2}{K}\frac{1}{K}\frac{dK_2}{dT}\right).\end{aligned}$$

Thus we have finally; for a point $(X, Y)$ inside the $n_0$th vortex $X^2+(Y-n_0)^2\leq 1/s^2$

$$\begin{aligned}\frac{dX}{dT}&=\frac{Ms^2Y}{2\pi}-\frac{MY}{2\pi\{Y^2+(X-n_0)^2\}}+\frac{M}{2}\frac{\sinh 2\pi Y}{(\cosh 2\pi Y-\cos 2\pi X)}+\kappa_1,\\ \frac{dY}{dT}&=-\frac{Ms^2}{2\pi}(X-n_0)+\frac{M}{2\pi}\frac{X-n_0}{\{Y^2+(X-n_0)^2\}}-\frac{M}{2}\frac{\sin 2\pi X}{(\cosh 2\pi Y-\cos 2\pi X)}+\kappa_2,\\ \frac{d\kappa_i}{dT}&=\frac{1}{K}\frac{dK_i}{dT}-\kappa_i\left(\kappa_1\frac{1}{K}\frac{dK_1}{dT}+\kappa_2\frac{1}{K}\frac{dK_2}{dT}\right)\qquad(i=1,2),\end{aligned}\tag{13}$$

$$\begin{aligned}\frac{1}{K}\frac{dK_1}{dT}=&-\kappa_1\left[\frac{M}{\pi}\frac{Y(X-n_0)}{\{Y^2+(X-n_0)^2\}^2}-M\pi\frac{\sinh 2\pi Y\sin 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right]-\\&-\kappa_2\left[-\frac{Ms^2}{2\pi}+\frac{M}{2\pi}\frac{Y^2-(X-n_0)^2}{\{Y^2+(X-n_0)^2\}^2}-M\pi\frac{\cosh 2\pi Y\cos 2\pi X-1}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right],\end{aligned}$$

$$\begin{aligned}\frac{1}{K}\frac{dK_2}{dT}=&-\kappa_1\left[\frac{Ms^2}{2\pi}-\frac{M}{2\pi}\frac{(X-n_0)^2-Y^2}{\{Y^2+(X-n_0)^2\}^2}+M\pi\frac{1-\cosh 2\pi Y\cos 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right]-\\&-\kappa_2\left[-\frac{M}{\pi}\frac{(X-n_0)Y}{\{Y^2+(X-n_0)^2\}^2}+M\pi\frac{\sin 2\pi X\sinh 2\pi Y}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}\right].\end{aligned}$$

For $(X, Y)$ outside all the vortex cores we have

$$\begin{aligned}\frac{dX}{dT}&=\frac{M}{2}\frac{\sinh 2\pi Y}{(\cosh 2\pi Y-\cos 2\pi X)}+\kappa_1,\\ \frac{dY}{dT}&=-\frac{M}{2}\frac{\sin 2\pi X}{(\cosh 2\pi Y-\cos 2\pi X)}+\kappa_2,\end{aligned}$$

$$\frac{d\kappa_i}{dT}=\frac{1}{K}\frac{dK_i}{dT}-\kappa_i\left(\kappa_1\frac{1}{K}\frac{dK_1}{dT}+\kappa_2\frac{1}{K}\frac{dK_2}{dT}\right)\qquad(i=1,2),$$

$$\frac{1}{K}\frac{dK_1}{dT}=\kappa_1M\pi\frac{\sinh 2\pi Y\sin 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}+\kappa_2M\pi\frac{\cosh 2\pi Y\cos 2\pi X-1}{\{\cosh 2\pi Y-\cos 2\pi X\}^2},$$

$$\frac{1}{K}\frac{dK_2}{dT}=-\kappa_1M\pi\frac{1-\cosh 2\pi Y\cos 2\pi X}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}-\kappa_2M\pi\frac{\sin 2\pi X\sinh 2\pi Y}{\{\cosh 2\pi Y-\cos 2\pi X\}^2}.$$

APPENDIX B

## An Approximate Expression for the Deflection of a Ray Passing Through a Vortex

The exact velocity field inside a vortex is given by:

$$v_1=\frac{Ud}{2\pi a^2}y+\frac{U}{2}\frac{\sinh 2\pi y/d}{\cosh 2\pi y/d-\cos 2\pi x/d}-\frac{Ud}{2\pi}\frac{y}{x^2+y^2},$$

that is

$$v_1=\frac{Udy}{2\pi a^2}\left[1+0\left(\frac{a^2}{d^2}\right)\right],$$

and similarly

$$v_2=-\frac{Udx}{2\pi a^2}\left[1+0\left(\frac{a^2}{d^2}\right)\right].$$

Hence a reasonable approximation inside a vortex is

$$v_1=\frac{Udy}{2\pi a^2}$$

and

$$v_2=-\frac{Udx}{2\pi a^2}.$$

Then the equations for ray tracing give

$$\frac{dx}{dt}=py+\frac{k_1}{k}c, \tag{B-1}$$

$$\frac{dy}{dt}=-px+\frac{k_2}{k}c, \tag{B-2}$$

$$\frac{dk_1}{dt}=k_2p, \tag{B-3}$$

$$\frac{dk_2}{dt}=-k_1p, \tag{B-4}$$

where

$$p=\frac{Ud}{2\pi a^2}.$$

It now follows that by multiplying equation (B-3) by $k_1$ and equation (B-4) by $k_2$, that $dk^2/dt=0$ and hence $k=\text{constant}$.

Let $q=c/k$ which is therefore constant.

If we write

$$z=x+iy \quad \text{and} \quad l=k_1+ik_2,$$

then equations (B-1), (B-2), (B-3) and (B-4) lead to

$$\frac{dz}{dt} = -ipz + ql$$

and

$$\frac{dl}{dt} = -ipl.$$

Hence

$$l = A \exp(-ipt)$$

and

$$z = D \exp(-ipt) + tqA \exp(-ipt).$$

Suppose a ray enters the vortex at a time $t = 0$ and at a point ($a \cos \phi$, $a \sin \phi$) and with the wave normal making an angle $\alpha$ below the horizontal.

Then

$$D = ae^{i\phi}$$

and

$$A = ke^{-i\alpha},$$

and so

$$z = a \exp i(\phi - pt) + qkt \exp i(-\alpha - pt).$$

hence

$$x = a \cos(\phi - pt) + qkt \cos(-a - pt)$$

and

$$y = a \sin(\phi - pt) + qkt \sin(-\alpha - pt).$$

When $x^2 + y^2 = a^2$ we have

$$2aqkt[\cos(\phi - pt)\cos(-\alpha - pt) + \sin(\phi - pt)\sin(-\alpha - pt)] + q^2k^2t^2 = 0,$$

giving

$$t = 0 \quad or \quad t = -\frac{2a}{qk}\cos(\phi - \alpha).$$

Hence the ray leaves the vortex core at a time

$$t = -\frac{2a}{qk}\cos(\phi + \alpha).$$

Hence on exit we have

$$k_1 + ik_2 = k \exp\left\{i\left(-\alpha + \frac{2ap}{qk}\cos(\phi + \alpha)\right)\right\}$$

and so the angle of the wave normal below the horizontal on leaving the vortex is

$$\alpha - \frac{Ud}{\pi ca}\cos(\phi + \alpha).$$

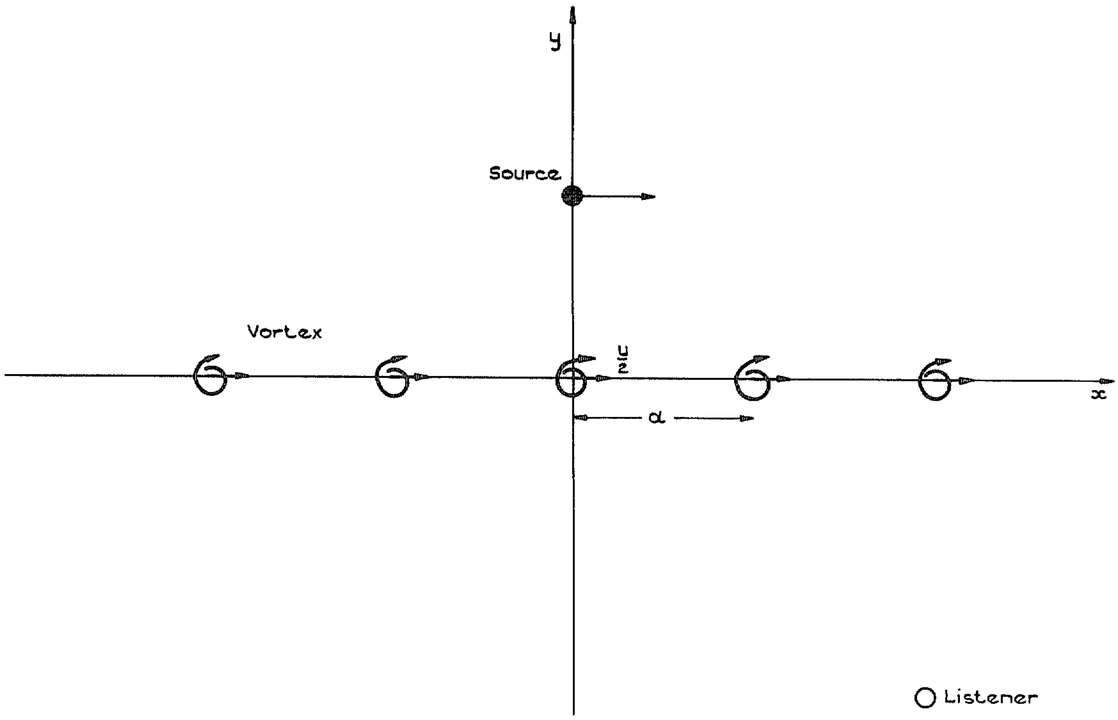


FIG. 1. The arrangement of the vortices.

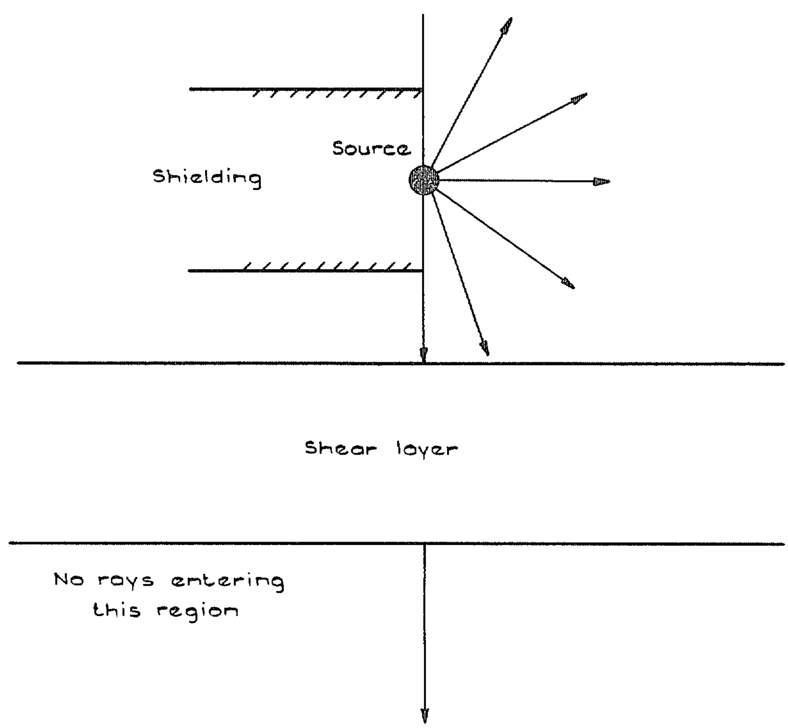


FIG. 2. Half shielded source.

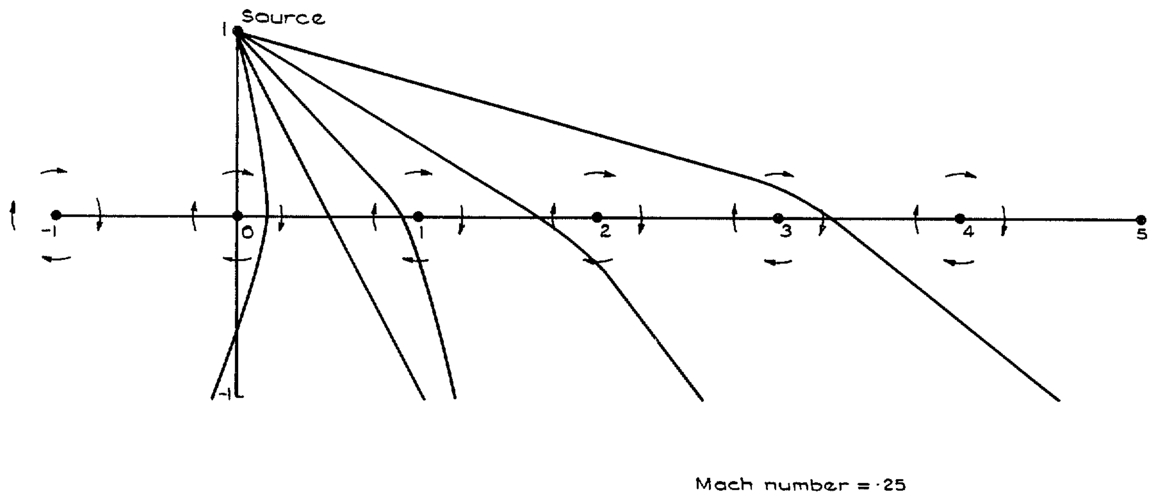


FIG. 3. Paths of rays through the vortices.

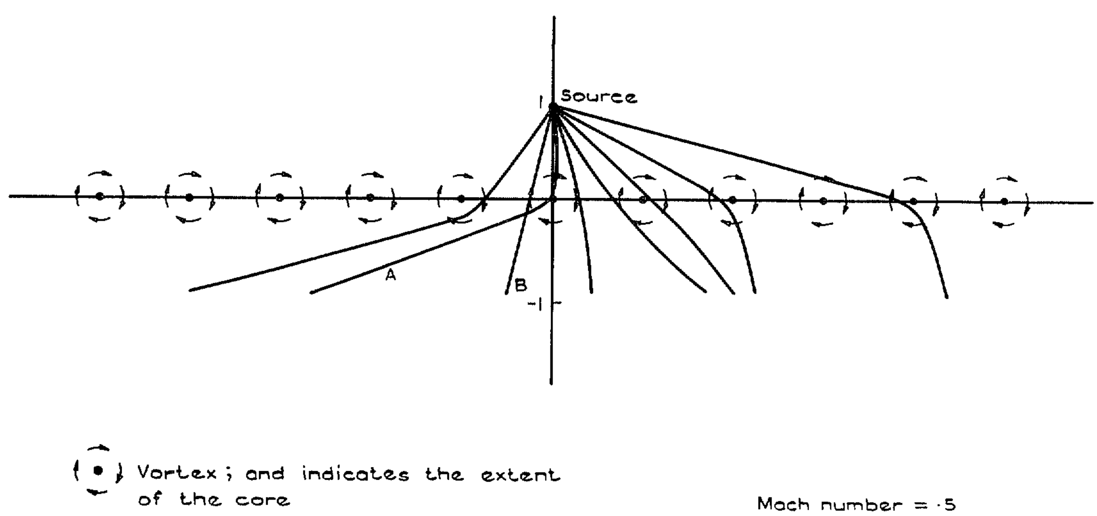


FIG. 4. Paths of rays through the vortices.

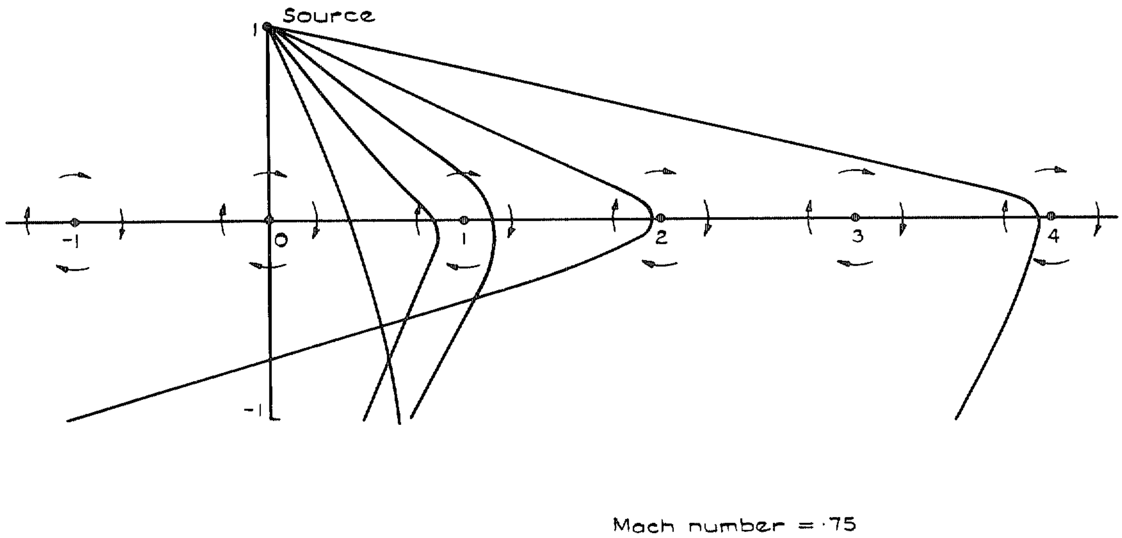


FIG. 5. Paths of rays through the vortices.

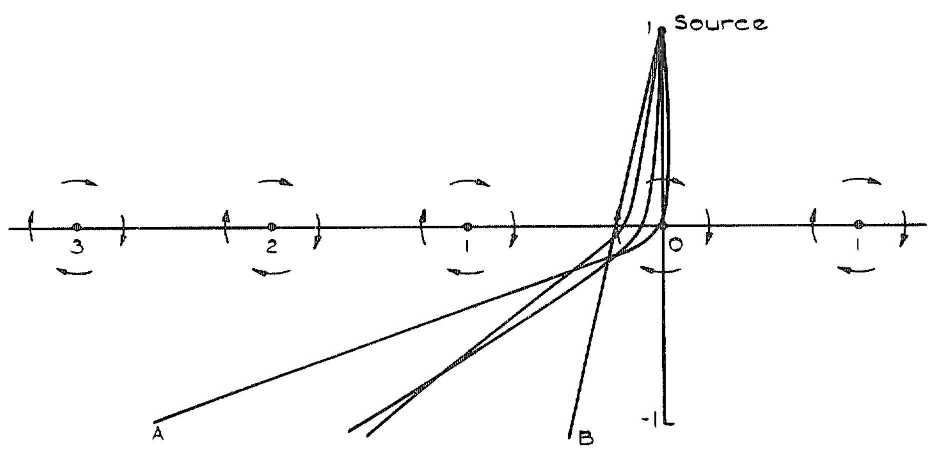


FIG. 6. Enlargement showing detail of rays between A and B in Fig. 4.

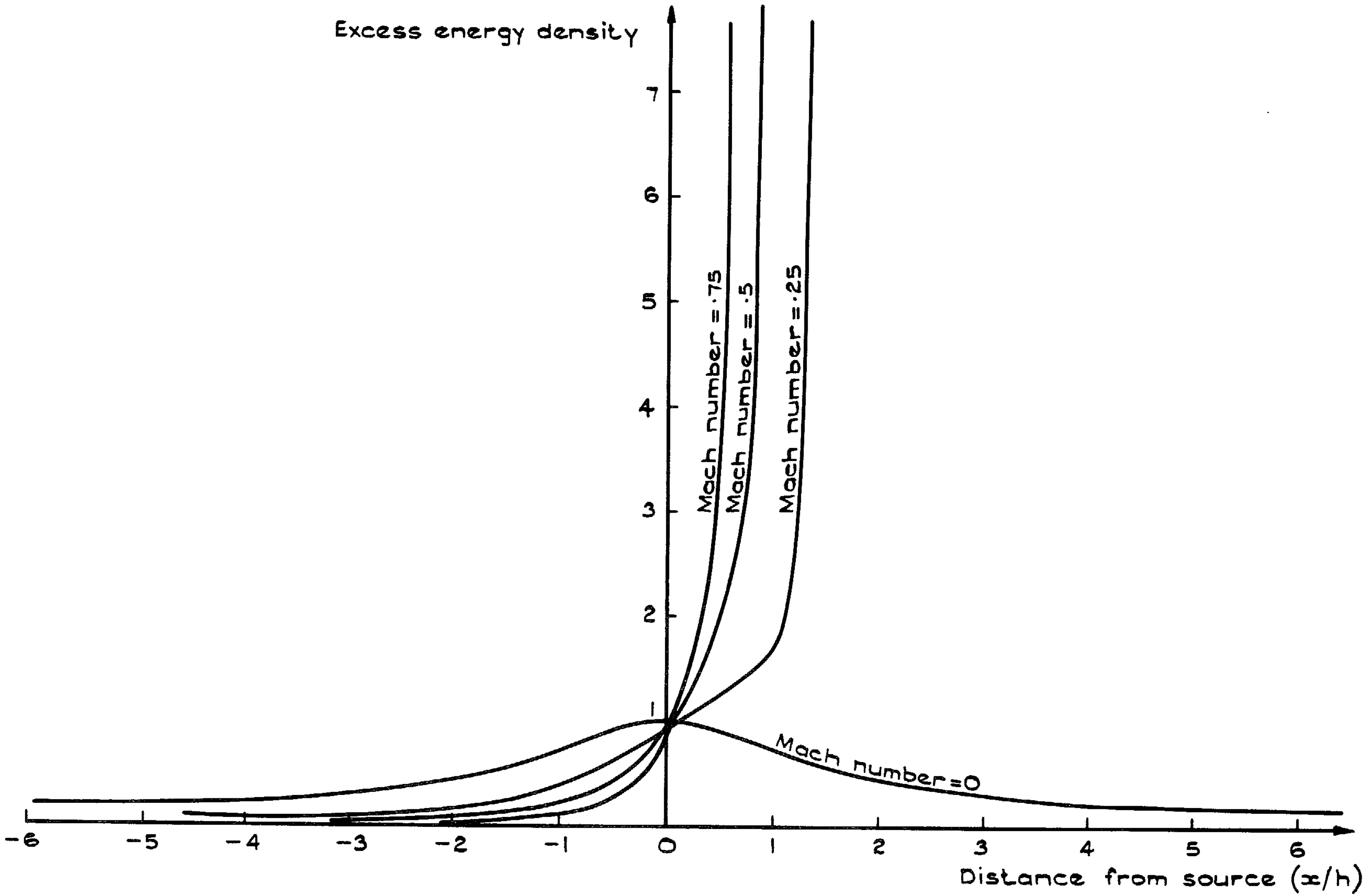


FIG. 7. Intensity distribution produced by the shear layer.

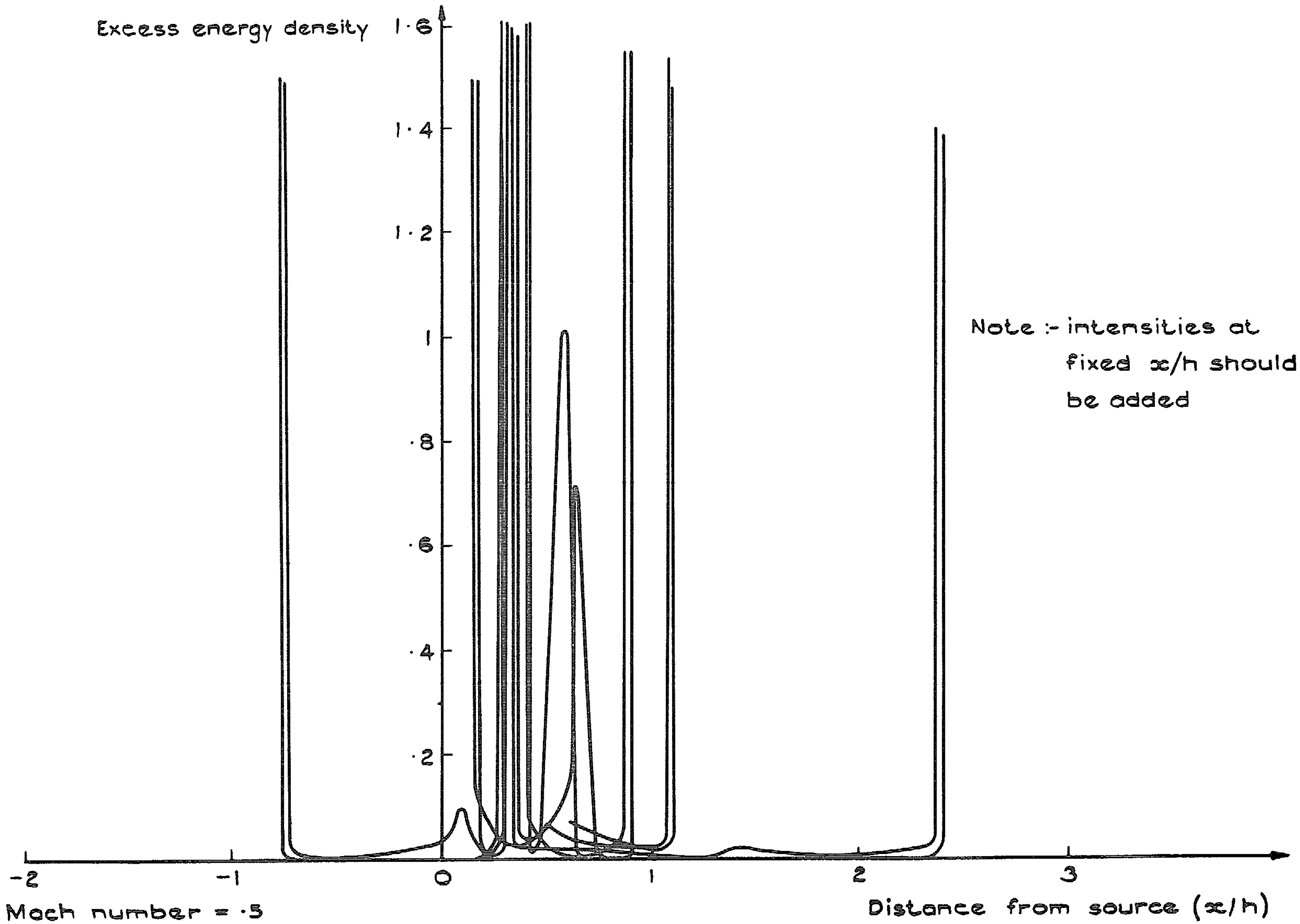


FIG. 8. Intensity distribution produced by vortices for half shielded source.



Printed in England for Her Majesty's Stationery Office by J. W. Arrowsmith Ltd., Bristol BS3 2NT
Dd. 505374 K5 6/75

HER MAJESTY'S STATIONERY OFFICE

*Government Bookshops*

49 High Holborn, London WC1V 6HB
13a Castle Street, Edinburgh EH2 3AR
41 The Hayes, Cardiff CF1 1JW
Brazennose Street, Manchester M60 8AS
Southey House, Wine Street, Bristol BS1 2BQ
258 Broad Street, Birmingham B1 2HE
80 Chichester Street, Belfast BT1 4JY

*Government Publications are also available through booksellers*

R. & M. No. 3770

ISBN 0 11 470915 7